The Continuous Hirsch Conjecture

Robert Bassett

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Why Do We Care?

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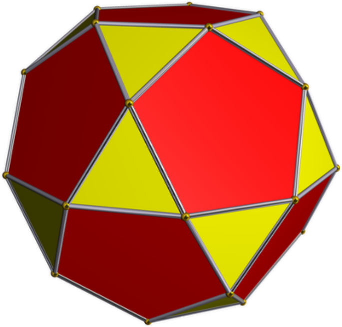
- i.e. any two vertices of the polytope can be reached by walking across $m - n$ edges.
- Provides a lower bound on the number of pivots of the simplex method.
- First put forth in a letter from Warren Hirsch to George Dantzig in 1957.
- First counter-example in 2010 by Francisco Santos.

Why Do We Care?



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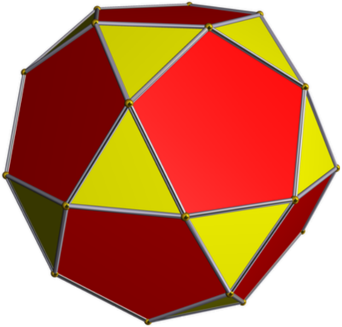


Question:

Is there a Hirsch-Like conjecture for solving linear programs via interior point methods?

The Central Path

Consider an polytope P defined by m inequalities.

$$P = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \leq \mathbf{b}\}$$

And the problem f of maximizing a linear functional $\mathbf{c}^T\mathbf{x}$ over P.

Next consider the family of approximate, unconstrained problems parametrized by μ.

$$f_\mu(\mathbf{x}) = \max_{x \in \mathbb{R}^n} \mathbf{c}^T\mathbf{x} + \mu \sum_{i=1}^{m} \ln(b_i - a_i\mathbf{x})$$

$\mu = 0$ gives the original objective.

By the strict convexity of the logarithmic barrier function, each f_μ has a unique solution $\mathbf{x}^*(\mu)$ in $\text{int}(P)$.

The Central Path

The *central path* is defined as the set $\{\mathbf{x}^* : \mu > 0\}$.

The central path is not associated with P itself, but depends on a linear objective $\mathbf{c}^T\mathbf{x}$.

An important fact: $\mathbf{x}^*(\mu)$ approaches the optimal value of the original problem as $\mu \to 0$.

Example

If f is

$$\max 3x$$

$$\text{subject to } 0 \leq x \leq 5$$

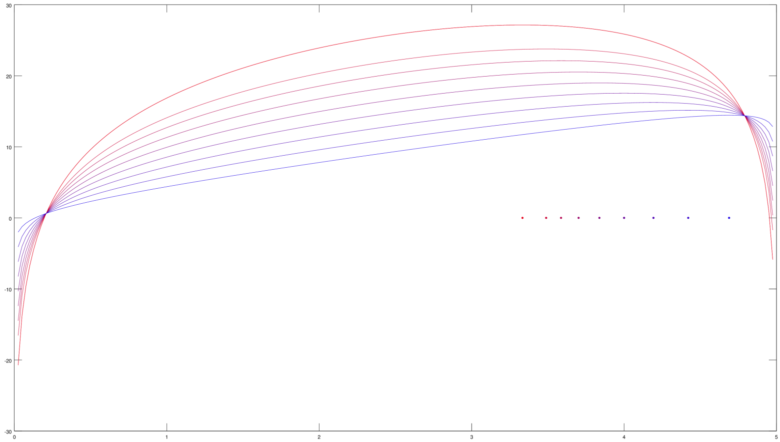
f_μ is

$$\max_{x \in \mathbb{R}} 3x + \mu \ln(x) + \mu \ln(5 - x)$$

Using calculus and a first derivative test

$$x^*(\mu) = \frac{-2\mu + 15 + \sqrt{(2\mu - 15)^2 + 60\mu}}{6}$$

Example



The Central Path

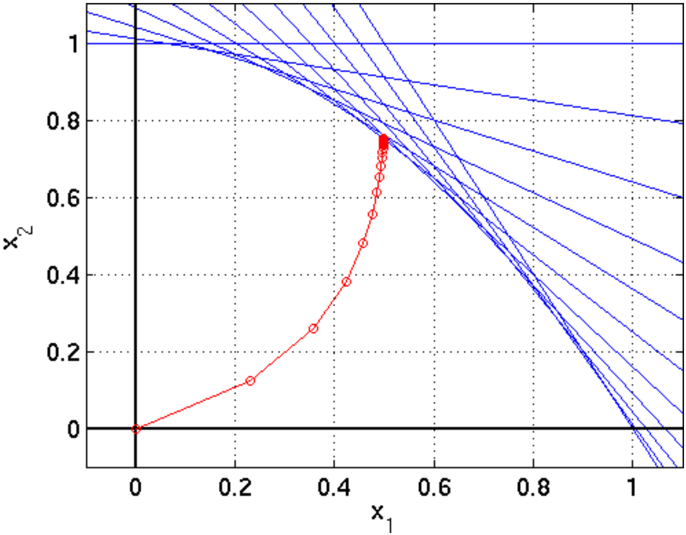


Figure: A 2-dimensional Example

Interior Point Methods

Interior point methods proceed by linearizing at a point along the central path and taking a small step in that direction.

Consider the usual linear program in equational form

$$\max \mathbf{c}^T \mathbf{x}$$

$$\text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

where A is $m \times n$ of rank m.

Introducing a barrier function for the inequality constraints, we get the approximate problems

$$\max \mathbf{c}^T \mathbf{x} + \mu \sum_{j=1}^{n} \ln x_j$$

$$\text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} > 0$$

Which we can use the method of Lagrange Multipliers to solve. Call the Lagrange multipliers $\mathbf{y}$.

Taking gradients, we have the system

$$A\mathbf{x} = \mathbf{b}$$
$$\mathbf{c} + \mu\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right) = A^T\mathbf{y}$$

We let $\mathbf{s} = \mu(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n})$, and impose the restriction, $\mathbf{s} \cdot \mathbf{x} = \mu\mathbf{1}$ to get the following

$$A\mathbf{x} = \mathbf{b}$$
$$A^T\mathbf{y} - \mathbf{s} = \mathbf{c}$$
$$\mathbf{s}^T\mathbf{x} = \mu\mathbf{1}$$

Optimality Equations

$$A\mathbf{x} = \mathbf{b}$$

$$A^T\mathbf{y} - \mathbf{s} = \mathbf{c}$$

$$\mathbf{s}^T\mathbf{x} = \mu\mathbf{1}$$

- When $\mu = 0$, the solution to this problem is primal and dual feasible.
- The set

$$\{(\mathbf{x}^*(\mu), \mathbf{y}^*(\mu), \mathbf{s}^*(\mu)) \in \mathbb{R}^{2n+m} : \mu > 0\}$$

is called the *primal-dual central path* of the linear program.

Interior Point Methods

Assume we start at a point $(\mathbf{x}, \mathbf{y}, \mathbf{s})$ on the central path, and want to find a step $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{s})$ along the path.

We want to solve

$$A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b}$$

$$A^T(\mathbf{y} + \Delta\mathbf{y}) + (\mathbf{s} + \Delta\mathbf{s}) = \mathbf{c}$$

$$((s_1 + \Delta s_1)(x_1 + \Delta x_1), \ldots, (s_n + \Delta s_n)(x_n + \Delta x_n)) = \mu\mathbf{1}$$

which, omitting nonlinear terms, gives use the system

Step Equations

$$A\Delta\mathbf{x} = \mathbf{0}$$

$$A^T\Delta y - \Delta\mathbf{s} = \mathbf{0}$$

$$(s_1\Delta x_1 + x_1\Delta s_1, \ldots, s_n\Delta x_n + x_n\Delta s_n) = \mu\mathbf{1} - (s_1x_1, \ldots, s_nx_n)$$

Interior Point Methods

Interior Point Algorithm

1. Find an initial point on the central path (can be hard). Set $\mu = 1$
2. Repeat steps 3 and 4 until approximately optimal
3. Replace μ with $\left(1 - \frac{1}{2\sqrt{n}}\right)\mu$.
4. Compute $\Delta\mathbf{x}$, $\Delta\mathbf{y}$, $\Delta\mathbf{s}$ as the solution of the step equations. Set $(\mathbf{x}, \mathbf{y}, \mathbf{s}) = (\mathbf{x} + \Delta\mathbf{x}, \mathbf{y} + \Delta\mathbf{y}, \mathbf{s} + \Delta\mathbf{s})$

But how good of an approximation is our linearization?

It seems like the performance of our algorithm would be directly affected by the accuracy of this step.

Total Curvature

The total curvature is a measure of how far off a curve is from being a straight line. Let $\phi : [a, b] \to \mathbb{R}^n$ be a curve, and $T(\alpha) = \frac{\dot{\phi}}{||\dot{\phi}||}$ be its unit tangent vector.

Recall that the curvature at a point α is $\kappa(\alpha) = \dot{T}(\alpha)$. Similarly, we define the *total curvature* of a curve to be

$$\int_a^b ||\kappa(s)||ds$$

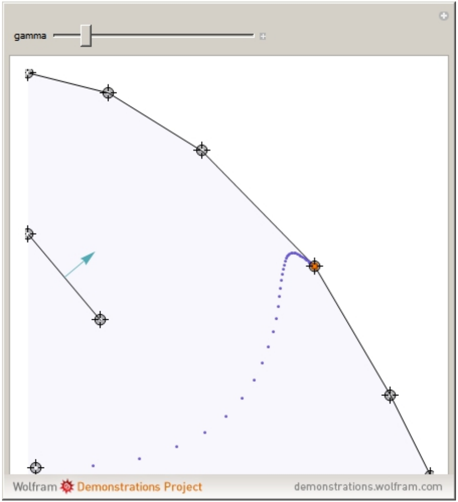
Let $\lambda^c(P)$ denote the total curvature of the central path corresponding to the linear optimization problem $\min\{c^T x : x \in P\}$. Define $\lambda(P)$ to be the largest of the $\lambda^c(P)$ over all c. This is called the *curvature* of the polytope.

The Continuous Hirsch Conjecture

Continuous Analogue of the Conjecture of Hisrch

The order of the curvature of a polytope defined by m inequalities in dimension n is m.

- This means that
$$\lim_{m\to\infty} \frac{\lambda(P)}{m} < \infty.$$
- It is known that the curvature of a polytope defined by m inequalities in dimension n is not greater than $2\pi m^n$ (Dedieu, Malajovich, Shub).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Wolfram Demonstrations Project
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Questions?