Lower Bounds in an Extension of the Doignon-Bell-Scarf Theorem

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The Big Question

If we know that a polyhedron $\{x : Ax \leq b\}$ has exactly k integral points in its interior, what can we conclude about the structure of its facets?

Context and Motivation

Theorem (Doignon 1973 [6], Bell [3], Scarf [7])
If the problem $\{x : Ax \leq b, x \in Z^n\}$ is infeasible, then there is a subset S of the m rows of A, of cardinality no more than 2^n, with the property that the smaller integer program is also infeasible.

- Polyhedra with $k = 0$ are closely related to the Frobenius problem [8], which seeks to, given some set of coins, find the largest number which cannot be expressed using those coins. Similar problems can be posed for arbitrary k.
- Clarkson [4] used Doignon's Theorem to construct a randomized algorithm with an expected time linear in the number of constraints.
- Maximal lattice free bodies are useful in the generation of cutting planes when solving integer programs [5]. Characterizing bodies when $k \neq 0$ may have similar applications.

Upper Bounds on Facets

Theorem (Aliev, De Loera, Louveaux [1] 2014)
There exists a universal constant $c(n, k)$ depending only on k and n, such that if $\{x : Ax \leq b, x \in \mathbb{R}^n\}$ has exactly k integral solutions, then there is a subset of the rows of A of cardinality no more than $c(n, k)$ with the property that the smaller integer program has exactly the same k solutions.

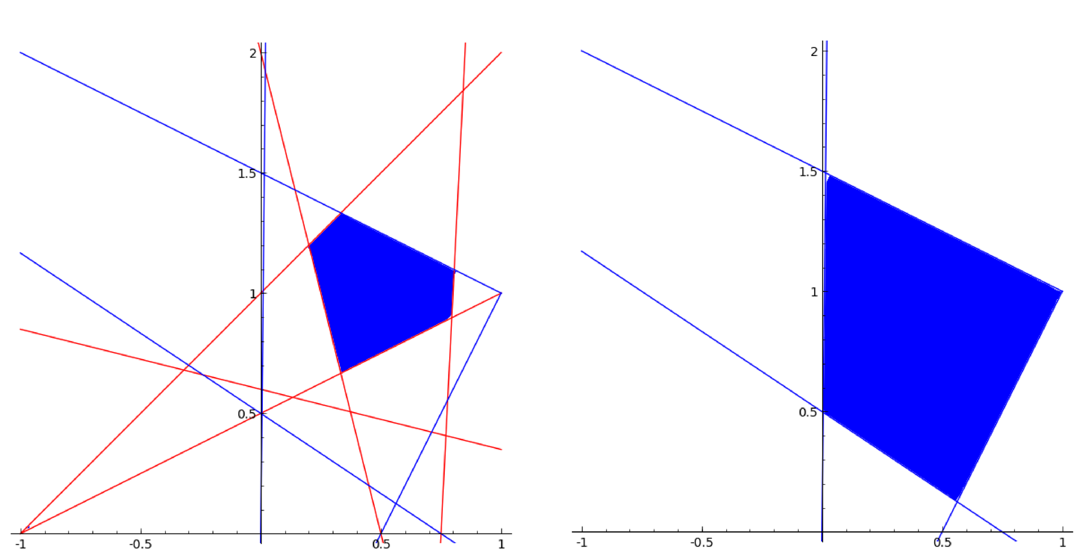

Figure 1: An Illustration of Doignon's Theorem

Lower Bounds on Facets

Both of the previous theorems provide upper bounds on the number of facets in a k-maximal polytope. But what about lower bounds?

- Given a polyhedron $P \subset \mathbb{R}^n$, we say that an inequality $I \subset P$ is *necessary* if the removal of I from P results in the inclusion of an additional integral point in the interior of P.
- In the case that $k = 0$, the theorem and bound are the same as Doignon's. Schrijver [9] provides an example that the bound of 2^n is the best possible, by exhibiting the system

$$\sum_{i\in I} x_i - \sum_{i\notin I} x_i \leq |I| - 1 \quad I \subseteq \{1, ..., n\}.$$

The Polyhedron

Given a set of natural numbers N, let $l_N := \min_{i\in N} i$ denote its least element. Define P to be $x \in \mathbb{R}^n$ that satisfies

$$\sum_{i=1}^{j-1} \frac{1}{2^i} x_i + x_j + \sum_{i=j+1}^{n} \frac{1}{2^{i-1}} x_i \leq 1;$$

$$-\sum_{i=1}^{j-1} \frac{1}{2^i} x_i - x_j - \sum_{i=j+1}^{n} \frac{1}{2^{i-1}} x_i \leq 1;$$

$$-\frac{x_{l_N}}{|N|} + \sum_{i\in N, i\neq l_N} \frac{x_i}{|N|} - \sum_{i\notin N} \frac{x_i}{|N|^n} \leq 1;$$

$$+\frac{x_{l_N}}{|N|} - \sum_{i\in N, i\neq l_N} \frac{x_i}{|N|} + \sum_{i\notin N} \frac{x_i}{|N|^n} \leq 1$$

as N ranges over $N \subseteq \{1, 2, .., n\}$ with $|N| \geq 2$ and j ranges from 1 to n.

Main Result

Theorem. There exists a polyhedron P in $\mathbb{R}^n$ that has exactly one interior integer point, $2(2^n - 1)$ facets and one integer point in the relative interior of each facet. Furthermore, all inequalities are necessary

- The $c(n, k)$ provided in [2] is $\lceil 2(k + 1)/3\rceil 2^n - 2\lceil 2(k + 1)/3\rceil + 2$, which implies that $2(2^n - 1)$ is the tightest possible upper bound for $c(n, 1)$.

The polyhedron P has the following important properties:

- Contains the origin
- The number of facets of P is in bijection with twice the nonempty subsets of $\{1, ..., n\}$, so that by construction P has $2(2^n - 1)$ facets.
- Each facet of P contains exactly one tight integer point in its relative interior

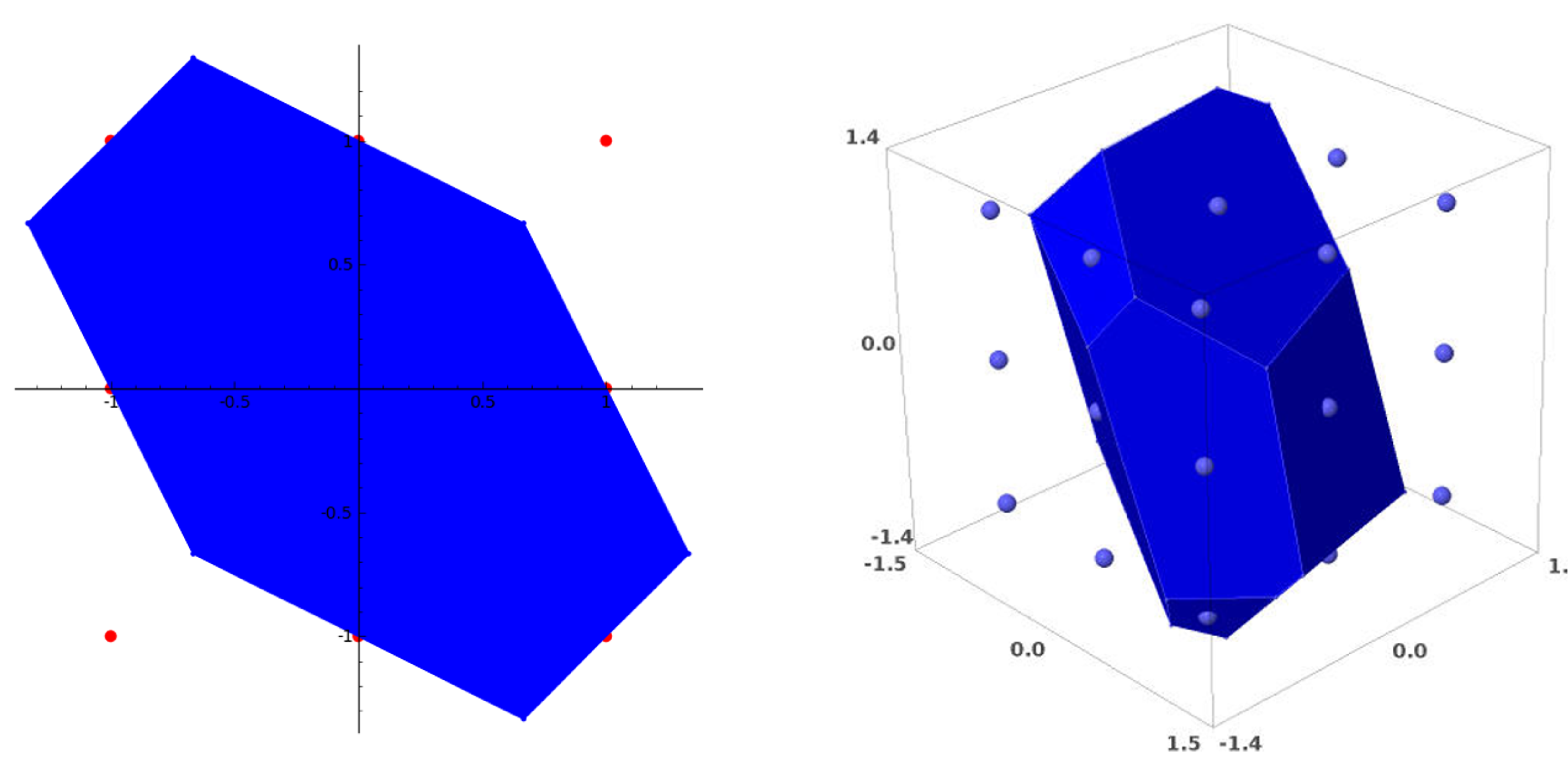

Figure 2: The Polyhedron P in 2 and 3 Dimensions

Components of the Proof

The proof that each of the inequalities in P is necessary can be broken down into the following steps

1. If an integral point y has an index j such that $|y_j| \geq 2$, then $y \notin P$.
2. A non-origin point $y \in \{-1, 0, 1\}^n$ is in P if and only if the sign of the left-most nonzero coordinate is not repeated.
   - $(-1, 1, 1)$ and $(1, 0, -1)$ are in P.
   - $(0, 1, 1)$ and $(1, -1, 1)$ are not in P.
3. There are $2(2^n - 1)$ of these points, each of which is tight on only one facet of P.

Extensions and New Directions

- The lower bound on the number of facets in a maximal k body for $k > 1$ remains open.
- The authors of [2] believe that their $c(n, k)$ bound is loose for $k > 3$, which provides a promising future direction.

References

[1] I. Aliev, J.A. De Loera, Q. Louveaux, "Integer Programs with Prescribed Number of Solutions and a Weighted Version of Doignon-Bell-Scarf's Theorem", to appear in *Proceedings of Integer Programming and Combinatorial Optimization, 17th International IPCO Conference*, Bonn Germany, June, 2014.

[2] I. Aliev, R. Bassett, J.A. De Loera, Q. Louveaux, "A Quantitative Doignon-Bell-Scarf Theorem", *Preprint*.

[3] D.E. Bell, "A theorem concerning the integer lattice". *Studies in Applied Mathematics*, 56(1), (1977).

[4] K.L. Clarkson, "Las Vegas algorithms for linear and integer programming when the dimension is small", *Journal of the ACM*, 42.2 (1995), 488-499.

[5] S.S. Dey, A. Tramontani, "Recent developments in multi-row cuts", *Optima 80* (2009), 2–8.

[6] J-P. Doignon, "Convexity in cristallographical lattices", *Journal of Geometry* 3.1 (1973), 71–85.

[7] H.E. Scarf, "An observation on the structure of production sets with indivisibilities", *Proceedings of the National Academy of Sciences* 74.9 (1977), 3637–3641.

[8] H.E. Scarf, D.F. Shallcross, "The Frobenius Problem and Maximal Lattice Free Bodies", *Mathematics of Operations Research* 18.3 (1993), 511-515.

[9] A. Schrijver, *Theory of Integer and Linear Programing*. Chichester: Wiley, 1998.