Survey of Research Interests for Welcome Week

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Introduction

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- Something that skirts the boundary between theory-building and applications, being slightly more dominated by theory-building.
- Something that prepares me for a career in academia, but also allows me to entertain job offers in private industry upon graduating.
- Something that has the potential to unify different areas of mathematics. I think of myself of more of a "breadth" mathematician instead of a "depth" mathematician.

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Enter Computational Geometry and Optimization!

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Using a divide and conquer method, we can achieve $O(n \log n)$, and even $O(n \log \log n)$ deterministic algorithms. If we allow random algorithms, there are algorithms for this problem with expected completion time $O(n)$.

Helly-Type Theorems

When I'm feeling less computational, there are deep theorems that also warrant my attention.

Theorem (Helly, 1913)

Let $X_1, \ldots X_n$ be a finite collection of convex subsets of $\mathbf{R}^d$, with $n > d$. If the intersection of every $d+1$ of these sets is nonempty, then the whole intersection is nonempty, i.e.

$$\bigcap_{j=1}^{n} X_j \neq \emptyset$$

Recall that a set is convex if, given any two points in the set, the line connecting them also lies in the set.

This is an amazing result! Even if you gave me $2^{2^{2^{2^{2}}}}$ convex sets in the plane, I only need to check them in combinations of threes to guarantee non-empty intersection of all of them.

Example

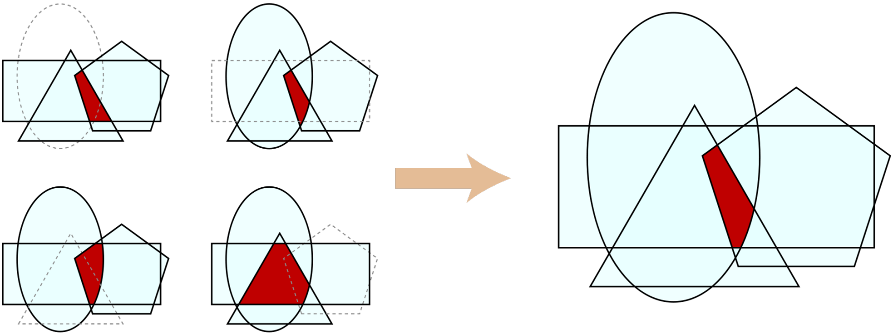

Figure: I shamelessly stole this graphic from wikipedia

Helly-Type Theorems

This result is so important that it spawned a class of theorems, which are called "Helly-Type".

Theorem (Helly-Type Theorem)

If every m members of a family of objects have property P, then the entire family has property P

For the classical Helly's theorem, m is $d+1$, the objects are convex sets, and P is the property "have non-empty intersection".

Doignon's Theorem

Recently, my favorite Helly-Type Theorem has been Doignon's Theorem.

Theorem (Doignon 1973)

Consider a polyhedron $A\mathbf{x} \leq \mathbf{b}$ where A is a $d \times n$ matrix and b is a vector of $\mathbf{R}^d$. If this polyhedron has no integer points in its interior, then there is a subset of rows of A of cardinality no more than 2^n such that this smaller polyhedron also has no integer points in its interior.

Note that this is the contrapositive of a Helly-type theorem.

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The idea is the following

- Given an integer linear program, there is a subset of the constraints of cardinality $2^n - 1$ with the same optimal objective value. (Doignon)
- Find it! Use random sampling to quickly grab constraints that are relevant and throw away others.
- Solve this subproblem. It is (hopefully) significantly smaller than the original.

Recent Developments

From the particular project that I am working on

Theorem (Aliev, De Loera, Louveaux, 2014)

Given an integer k, there exists a constant $c(k,n)$ depending only on the dimension n and k, such that if a polyhedron $\{x : Ax \leq b\}$ contains exactly k integer solutions, then there exists a subset of the rows of cardinality no more than $c(k,n)$, defining a polyhedron that contains exactly the same k integer solutions.

This is an amazing result! It extends Doignon's theorem, for which $k = 0$, to polyhedra with a fixed number of points inside.

Open Question

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- This would be extremely helpful in applications.
- Often times it is helpful to consider the "k best numerical options" instead of the just the "best numerical option".
- Implementing this algorithm and testing its success experimentally brings yet another perspective to this project.

Conclusion

Thank you!

References

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