A basic result of enumerative combinatorics is that there is a bijection between the even-sized subsets and odd-sized subsets of a given set. In this paper, we give a generalization of this result to multisets, which we will treat as nonincreasing words. Depending on the given multiset, there may or may not be a bijection. Afterwards, we further generalize this result to arbitrary words, which are not necessarily nonincreasing.

Let $w$ be a nonincreasing word in the alphabet $[n]$. Then we have the following

**Theorem 1.** The following hold

1. If some letter of $w$ has odd multiplicity, then there is a one-to-one correspondence between the subwords of $w$ with even length and the subwords of $w$ with odd length.

2. If every letter of $w$ has even multiplicity, then we still have such a one-to-one correspondence except that exactly one even-length subword cannot be paired with an odd-length subword (in other words, there is one even-length subword more than the odd-length subwords).

We prove this theorem by constructing an explicit permutation $\phi$ of order 2 of the subwords of $w$ that swaps the even and odd lengthed subwords. For each letter of $w$ and $b$ a letter of $w$, let $m_w(b)$ denote the multiplicity of $b$ in $w$. We introduce a certain multiplicity pairing for each letter of $w$: $(0,1)$, $(2,3)$, $(4,5)$, $(6,7)$, and so on. Let $\max(w)$ and $\min(w)$ denote the greatest and least letter of $w$, respectively. For $v$ a subword of $w$, we define $\phi(v)$ in the following way:

Start by letting $c := \min(w)$, and apply the following procedure which utilizes this multiplicity pairing

1. If $m_v(c)$ is odd, then define $\phi(v)$ to be the subword obtained by removing one copy of $c$ from $v$.

2. If $m_v(c)$ is even and $m_v(c) < m_w(c)$, then define $\phi(v)$ to be the subword obtained by adding one copy of $c$ to $v$.

3. If $m_v(c) = m_w(c)$ is even and $c < \max(w)$, then set $c$ to be the next smallest letter and repeat the procedure.

4. If $m_v(c) = m_w(c)$ is even and $c = \max(w)$, then define $\phi(v) = v$.

**Remark 2.** Here is a more concise description of the above procedure: Pick the smallest letter $x$ of $w$ such that $m_v(x) < m_w(x)$ or $m_v(x)$ is odd. If $m_v(x)$ is odd, then remove a copy of $x$ from $v$. If $m_v(x)$ is even, then add a copy of $x$ to $v$. If no such letter $x$ exists, then fix $v$.

We see that $\phi(v) = v$ if and only if every letter of $w$ has even multiplicity and $v = w$, in which case $w$ is the only even-lengthed subword that cannot be paired with an odd-lengthed subword.
In fact, we can generalize Theorem 1. Let $u$ be a word of length $r$, not necessarily nondecreasing. Then we have

**Theorem 3.** There is an involution of the subwords of $u$ that maps the odd-lengthed subwords injectively to the even-lengthed subwords, and fixes those even-lengthed subwords that cannot be paired.

We prove this theorem by explicitly constructing the required involution $\hat{\phi}$. Label the segments of $u$ composed of copies of a single letter as $z_1, z_2, \ldots, z_l$. For example, if $u = 44353353$, then the segments are $z_1 = 4, z_2 = 3, z_3 = 5, z_4 = 33, z_5 = 5, z_6 = 3$. We order the segments as $z_1 < z_2 < \ldots < z_l$; we will use this order in the following procedure. For each segment $z_i$ denote the letter of $z_i$ by $z_i^*$, then the length of $z_i$ is the multiplicity of $z_i^*$ in $z_i$. We introduce this multiplicity pairing for each segment of $w$: $(0,1)$, $(2,3)$, $(4,5)$, $(6,7)$, and so on. Let $u'$ be a subword of $u$. Label the segments of $u'$ composed of copies of a single letter as $z'_1, z'_2, \ldots, z'_l$. Notice that $z_i \cap z'_i$ can only be $\emptyset, z_i, z'_i$. We define $\hat{\phi}(u')$ in the following way:

Start by letting $d := z_1$ and $g := z'_1$ ($d$ will always be a segment of $u$, and $g$ will always be a segment of $u'$), and apply the following procedure which utilizes this multiplicity pairing

1. If the multiplicity of $d^*$ in $g$ is odd and no larger than $|d|$, then define $\hat{\phi}(u')$ to be the subword obtained from $u'$ by removing a copy of $d^*$ from $g$.
2. If the multiplicity of $d^*$ in $g$ is even and less than $|d|$, then define $\hat{\phi}(u')$ to be the subword obtained from $u'$ by adding a copy of $d^*$ to $g$.
3. If the multiplicity of $d^*$ in $g$ is even and equal to $|d|$, then set $d$ to be the next smallest segment of $u$, set $g$ to be the next smallest segment of $u'$, and repeat the procedure.
4. If the multiplicity of $d^*$ in $g$ is greater than $|d|$, then
   
   (a) If $|d|$ is even, let $\tilde{d}$ be the next smallest segment of $u$ and then define $\hat{\phi}(u')$ to be the subword obtained from $u'$ by adding a copy of $d^*$ right after the segment $\tilde{d}$ in $g$.

   (b) If $|d|$ is odd, pick the smallest $k \geq 2$ such that the concatenation of the $k$ smallest (again, using our ordering above) $d^*$-segments in $u$ has length $L \geq |g|$. Label these segments as $h_1, h_2, \ldots, h_k$; $L$ is the total number of copies of $d^*$ in these $k$ segments. Apply the multiplicity pairing to $m_d(d^*) = |g|$:
   
   i. If $|g|$ is odd, then define $\hat{\phi}(u')$ to be the subword obtained from $u'$ by removing a copy of $d^*$ from $g$.
   
   ii. If $|g|$ is even and less than $L$, then define $\hat{\phi}(u')$ to be the subword obtained from $u'$ by adding a copy of $d^*$ to $g$.
   
   iii. If $|g| = L$ is even, then set $d$ to be the next segment of $u$ after $h_k$, set $g$ to be the next smallest segment of $u'$ (set $g$ to be the empty word if $u'$ has no next segment), and repeat the procedure.

If the above procedure does not terminate at $\raggedright{1. 2. 4(a) 4(b)(ii)\ 4(b)(ii)}$, then define $\hat{\phi}(u') = u'$.

**Example 4.**

1. If $u = 44333553$, then $\hat{\phi}$ acts by $\emptyset \rightarrow 4, 44 \rightarrow 443, 4433 \rightarrow 4433, 443353 \rightarrow 443353$, and so on. The only subword fixed by $\hat{\phi}$ is 4433353.

2. If $u = 445353$, then $\hat{\phi}$ acts by $\emptyset \rightarrow 4, 44 \rightarrow 445, 4455 \rightarrow 44553, 5533 \rightarrow 5533$, and so on. The only subword fixed by $\hat{\phi}$ is 44553.

3. If $u = 4433553$, then $\hat{\phi}$ acts by 443353 $\mapsto 44333$, 443353 $\mapsto 443333$, and so on. The only subword fixed is 4433553.
4. If \( u = 44355335 \), then \( \hat{\phi} \) acts by \( 4455 \mapsto 44355, 4433 \mapsto 44333 \), and so on.

5. If \( u = 353454323 \), then the subwords 3333 and 334433 are fixed by \( \hat{\phi} \).