Towards Reducing the HMZ Shuffle Conjecture

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Abstract

In her dissertation, Angela Hicks showed that, to reduce the HMZ shuffle conjecture, it is sufficient to find a certain bijection between two sets of two-part parking functions which, roughly speaking, either swaps the two parts of the parking function or transfers a car from the second part to the first part and which satisfies certain nice properties. I will consider a special case where this bijection, if it exists, must swap the two parts of the parking function, and I will give a simple operation which results in a bijection achieving this swap and satisfying the desired properties.

1 Background and Notations

First we define Dyck paths.

Definition 1 (Dyck path). Start with an \( n \times n \) grid with a main diagonal running from its southwest corner to its northeast corner; we will label the main diagonal 0, the diagonal above it 1, the diagonal above that 2, and so on. An \((n,n)\)-Dyck path is a series of north and east steps from the southwest corner to the northeast corner, such that the path never crosses the main diagonal.

Example 2. Here is a \((5,5)\)-Dyck path that occupies diagonal 0 and diagonal 1:

For parking functions, we use the following definition for \([1]\):

Definition 3 (Parking function). An \((n,n)\)-parking function is a labeled \((n,n)\)-Dyck path, whose north steps are labeled with the integers \(1, \ldots, n\) so that the labels of consecutive north steps are increasing from bottom to top. We require each such integer to lie in the cell to the right of the north step it labels, so that we can treat these integers as \(n\) “cars” parked along the north steps. If label \(i\) lies on diagonal \(d_i\), we say that \(d_i\) is the diagonal of car \(i\).

Definition 4 (Composition). The composition of a parking function is the composition \(c = (c_1, c_2, \ldots, c_k)\) of \(n\) where \(c_i\) is the number of north steps after the \(i\)th intersection of the Dyck
path with the main diagonal but before the \((i + 1)\)st intersection. Since its composition has \(k\) parts, we call this parking function a \(k\)-part parking function. For the \(i\)th part of this parking function, define anchor\((i)\) to be the unique car in this part which lies on the main diagonal, and define diag\((i)\) to be the set of diagonals occupied by the cars of this part.

In a parking function, we will call a maximal collection of consecutive north steps (which are all labeled with car numbers) a column of cars. Given a column \(\text{col}\) of the parking function, define diag\((\text{col})\) to be the set of diagonals occupied by the cars of \(\text{col}\). If \(u\) and \(v\) are the highest and lowest diagonals of \(\text{col}\) respectively, we say that \(\text{col}\) starts at \(u\) and ends at \(v\) respectively.

**Fact 5.** We can recover the parking function if given the ordered tuple

\[
((\text{col}_1, \text{diag}(\text{col}_1)), (\text{col}_2, \text{diag}(\text{col}_2)), \ldots, (\text{col}_m, \text{diag}(\text{col}_m)))
\]

of pairs consisting of the parking function’s columns and their corresponding diagonal sets.

**Definition 6** (dinv). Let \(x, y\) be two cars of a parking function \(\sigma\) where \(x < y\). We say that \(\{x, y\}\) form a primary dinv if they lie on the same diagonal and \(x\) is to the left of \(y\). We say that \(\{x, y\}\) form a secondary dinv if \(x\) is to the right of \(y\) and \(d_y = d_x + 1\). Denote by pdinv\((\sigma)\) and sdinv\((\sigma)\) the number of (unordered) pairs of cars of \(\sigma\) forming primary dinv and the number of (unordered) pairs of cars of \(\sigma\) forming secondary dinv, respectively. Define the dinv of \(\sigma\) by dinv\((\sigma) := \text{pdinv}(\sigma) + \text{sdinv}(\sigma)\).

**Definition 7** (reading word, ides). The reading word of a parking function \(\sigma\) is formed by reading cars by diagonals, starting with the diagonal farthest from the main diagonal, reading cars in a diagonal from northeast to southwest. The ides of \(\sigma\), denoted ides\((\sigma)\), is the set of \(r\) occurring after \(r + 1\) in the reading word.

**Example 8.** Here is a \((5, 5)\)-parking function of composition \((2, 3)\):

\[
\begin{array}{ccc}
1 & 4 & 2 \\
3 & & 5 \\
& 4 & \\
& & 2 \\
& & 1 \\
\end{array}
\]

Here cars 1 and 2 are in diagonal 0, cars 3 and 4 are in diagonal 1, and car 5 is in diagonal 2. Cars 1 and 2 form a primary dinv, while cars 4 and 2 form a secondary dinv. Thus, this parking function has dinv = \(1 + 1 = 2\). The reading word is 53421, and the ides is \(\{1, 2, 4\}\). We have anchor\((1) = 1\), anchor\((2) = 2\), diag\((1) = \{0, 1\}\), and diag\((2) = \{0, 1, 2\}\).

Given a composition \(c = (c_1, c_2, \ldots, c_k)\) of \(n\), we will denote by \(A_c\) the set of parking functions of composition \(c\).

The HMZ shuffle conjecture of Haglund, Morse, and Zabrocki states

**Conjecture 9.** For \(c\) a composition,

\[
\nabla C_c 1 = \sum_{PF \in A_c} t^{\text{area}(PF)} q^{\text{dinv}(PF)} Q^{\text{ides}(PF)}.
\]
Here the symmetric function operator $\nabla$ is defined by $\nabla \tilde{H}_u[X; q, t] = T_u \tilde{H}_u[X; q, t]$, where $\tilde{H}_u[X; q, t]$ is the modified Macdonald polynomial introduced by Garsia and Procesi, $T_\lambda = t^n(\lambda) q^{n(\lambda)}$, and $n(\lambda) = \sum_{i=1}^{\ell(\lambda)} \lambda_i (i - 1)$. $C_c$ is a modified Hall Littlewood operator defined by $C_c F[X] = C_{c_1} C_{c_2} \cdots C_{c_k} F[X]$ where

$$C_a F[X] = (\frac{-1}{q})^{a-1} \sum_{k \geq 0} F[X + \frac{1-q}{q} z] z^k h_{a+k}[X].$$

The quasi-symmetric functions

$$Q_S = \sum_{1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq n} x_{a_1} x_{a_2} \cdots x_{a_n}$$

are the elements of Gessel’s fundamental basis for the quasi-symmetric functions, where $S$ ranges through subsets of $[n-1]$. For details about these polynomials and operators, see [1].

As shown in [1], to reduce the shuffle conjecture from the composition case to the partition case it is sufficient to prove the following

**Conjecture 10.** For $a \leq b - 1$, there exists a bijection $f$

$$f : \mathcal{A}_{(a,b)} \cup \mathcal{A}_{(b-1,a+1)} \leftrightarrow \mathcal{A}_{(b,a)} \cup \mathcal{A}_{(a+1,b-1)}$$

with the following properties:

1. $f$ increases the dinv statistic by exactly one

2. $f$ does not change the diagonal of any car.

Note that $f$ is a bijection between two sets of two-part parking functions. Roughly speaking, this bijection either swaps the two parts of the parking function or transfers a car from the second part to the first part. In the remainder of this section, fix $a \leq b - 1$. I will study a special case where $f$ must swap the two parts of the parking function if $f$ exists, and I will give a simple invertible operation for achieving this swap with the desired properties.

## 2 Necessity of the Swap

We begin our search for the bijection $f$ by first analyzing a substantial subset of two-part parking functions where $f$ has no choice but to, roughly speaking, swap the two parts of the parking function and preserve the larger part (though not the cars) if $f$ exists. This special case may give us clues as to how to find $f$ in general.

Let us first consider the extreme case, which is the simplest to describe and will be useful in the analysis of the more general case. Let $\mathcal{A}_{(i,j)}^{c,c} \subset \mathcal{A}_{(i,j)}$ denote the set of parking functions $\sigma_{c,c}$ such that each part consists of a single column of cars; in other words, the cars of the first part of $\sigma_{c,c}$ occupies all the diagonals $0, \ldots, i-1$ and the cars of the second part of $\sigma_{c,c}$ occupies all the diagonals $0, \ldots, j-1$. Example [1] gives such a parking function.

**Proposition 11.** Suppose $g : \mathcal{A}_{(a,b)}^{c,c} \to \mathcal{A}_{(b',a')}^{c,c}$ has the property that $g$ preserves the diagonal of each car, where $b'$ and $a'$ are positive integers such that $b' + a' = n$ and $a < b' \leq b$. Then we must have $b' = b$, $a' = a$, and $g(\mathcal{A}_{(a,b)}^{c,c}) \subset \mathcal{A}_{(b,a)}^{c,c}$. 

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Proof. Let $\sigma_{c,c} \in \mathcal{A}^{c,c}_{(a,b)}$, so its second part is the longer column. $g$ must transfer some cars (sliding them to the left along their diagonals) from the second part to the first part of $\sigma_{c,c}$ so as to increase the first part of $\sigma_{c,c}$ by $l = b' - a$. We claim that the top $l$ cars of the second part must be among those transferred, and these are the only cars transferred that increase the size of the first part.

Consider diag(1) and diag(2) of $\sigma_{c,c}$. We clearly have diag(1) $\subseteq$ diag(2). Suppose car $i$ in the second part has $d_i \in$ diag(1). If $g$ moves car $i$ to the first part, then $g$ must move some car $i$ in the same diagonal from the first part to the second part, otherwise the resulting second part will be disconnected and no longer a Dyck path; in other words, $g$ must swap $i$ with $i$ which either leaves the sizes of the two parts unchanged or increases the size of the second part (the size can increase if $i = \text{anchor}(2)$). Therefore, each of the $l$ cars that make the difference under $g$ must have its diagonal in diag(2) − diag(1).

Suppose car $j$ in the second part has $d_j \in$ diag(2) − diag(1). If $g$ moves car $j$ to the first part, then it must move all the cars above car $j$ in the second part as well, otherwise the resulting second part will be disconnected. Therefore, $g$ must move the top $l$ cars of the second part to the first part. To ensure that the resulting first part is connected, we must have $l = b - a$. Therefore, the resulting first part of $g(\sigma_{c,c})$ must consist of a single column of height $a + l = b$. The resulting second part must consist of a single column of height $a$, as it consists of $a$ cars occupying diagonals $0, 1, \ldots, a - 1$. \hfill $\blacksquare$

For the same reason, we also have

**Proposition 12.** Suppose $g : \mathcal{A}^{c,c}_{(b,a)} \rightarrow \mathcal{A}_{(a',b')}$ has the property that $g$ preserves the diagonal of each car, where $a'$ and $b'$ are positive integers such that $a' + b' = n$ and $a < b' \leq b$. Then we must have $a' = a$, $b' = b$, and $g(\mathcal{A}^{c,c}_{(b,a)}) \subseteq \mathcal{A}^{c,c}_{(a,b)}$.

From these two propositions, we obtain the following

**Corollary 13.** If the bijection $f$ exists, then $f|\mathcal{A}^{c,c}_{(a,b)}$ must be a bijection between $\mathcal{A}^{c,c}_{(a,b)}$ and $\mathcal{A}^{c,c}_{(b,a)}$.

Let $\mathcal{A}^{c,c}_{(a,b)}$ denote the set of parking functions whose second [first] part is a single column while the first [second] part is arbitrary. We now use the two-column case described in Propositions 11 and 12 as a reference point to study the more general case of $\mathcal{A}^{c,c}_{(a,b)}$.

**Theorem 14.** Suppose $g : \mathcal{A}^{c,c}_{(a,b)} \rightarrow \mathcal{A}_{(b',a')}$ has the property that $g$ preserves the diagonal of each car, where $b'$ and $a'$ are positive integers such that $b' + a' = n$ and $a < b' \leq b$. Then we must have $b' = b$, $a' = a$, and $g(\mathcal{A}^{c,c}_{(a,b)}) \subseteq \mathcal{A}^{c,c}_{(b,a)}$.

**Proof.** Let $\sigma_{*,c} \in \mathcal{A}^{c,c}_{(a,b)}$, and denote the single column of the second part of $\sigma_{*,c}$ as $\text{col}_2$. $g$ must transfer some cars from $\text{col}_2$ to the first part to increase the first part by $l = b' - a$. Since $\text{col}_2$ is the highest column of $\sigma_{*,c}$ and contains anchor(2), we have $\text{diag}(\text{col}) \subseteq \text{diag}(\text{col}_2) = \text{diag}(2)$ for any column of the first part.

Suppose car $i$ in the second part has $d_i \in$ diag(1). If $g$ moves car $i$ to the first part, then $g$ must move some car $i$ in the same diagonal from the first part to the second part, otherwise the resulting second part will be disconnected and no longer a Dyck path; in other words, $g$ must swap $i$ with $i$ which either leaves the sizes of the two parts unchanged or increases the size of the second part (the size can increase if $i = \text{anchor}(2)$). Therefore, each of the $l$ cars that make the difference under $g$ must have its diagonal in diag(2) − diag(1).

Suppose car $j$ in the second part has $d_j \in$ diag(2) − diag(1). If $g$ moves car $j$ to the first part, then it must move all the cars above car $j$ to the first part as well, otherwise the resulting
second part will be disconnected. Therefore, \( g \) must move the top \( l \) cars of the second part to the first part. However, since the highest diagonal that any column of the first part reaches is \( \max(\text{diag}(1)) \), \( g \) must move at least the top \( b' - |\text{diag}(1)| \) cars to the first part.

After the top \( l \) cars of \( \text{col}_2 \) have been moved, the next \( b' - |\text{diag}(1)| - l = a - |\text{diag}(1)| \) cars of \( \text{col}_2 \) to be moved must be swapped with \( a - |\text{diag}(1)| \) cars of the first part in order to maintain the size difference of \( l \). For every one of these \( a - |\text{diag}(1)| \) cars of the first part that, upon being moved, decreases the highest diagonal of the first part, an additional car of the second part must be moved to the first part; otherwise these cars transferred from \( \text{col}_2 \) will not be able to connect with (i.e. end up exactly one diagonal above) a car of highest diagonal of the first part. Let \( i \) denote the first car transferred from \( \text{col}_2 \) that connects with a car of highest diagonal of the first part. It is straightforward to see that, by the time \( i \) is swapped with a corresponding car of the first part, the first part will have become a single column \( \text{col}'_2 \) (which contains anchor(1)) of height at most \( |\text{diag}(1)| \), and any subsequent car \( x \) transferred from \( \text{col}_2 \) must be swapped with a car of \( \text{col}'_2 \) of diagonal \( d_x \). Therefore, we eventually get a single column of height \( b \) for the first part of \( g(\sigma_{s,c}) \), so we have \( g(\sigma_{s,c}) \in \mathcal{A}_{(b,a)}^{c,*} \). \( \square \)

For the same reasons, we also have

**Theorem 15.** Suppose \( g : \mathcal{A}_{(b,a)}^{c,*} \rightarrow \mathcal{A}_{(a',b')}^{c} \) has the property that \( g \) preserves the diagonal of each car, where \( a' \) and \( b' \) are positive integers such that \( a' + b' = n \) and \( a < b' \leq b \). Then we must have \( a' = a, b' = b, \) and \( g(\mathcal{A}_{(b,a)}^{c,*}) \subset \mathcal{A}_{(a,b)}^{c,*} \).

From these two theorems, we obtain

**Corollary 16.** If the bijection \( f \) exists, then \( f|_{\mathcal{A}_{(a,b)}^{c,*}} \) must be a bijection between \( \mathcal{A}_{(a,b)}^{c,*} \) and \( \mathcal{A}_{(b,a)}^{c,*} \).

### 3 Simple Algorithm for Achieving the Swap

We now restrict to the case of \( \mathcal{A}_{(a,b)}^{c,*} \), on which \( f \) must act by swapping the two parts, by Corollary 16. We will give a simple algorithm on adjacent columns of cars that can be applied repeatedly until the swap is complete, through which the dinv increases by exactly one. The bijection resulting from the repeated application of this operation is simple in the sense that it leaves the configuration of columns unchanged after the swap (only the cars are changed), and the dinv is increased by one only at the last application of this operation (all previous applications produce no change in dinv).

We will define this operation \( \rho \) in general. Let \( \sigma \in \mathcal{A}_{(a,b)} \) be any parking function and let \( \text{col}_1, \text{col}_2 \) be two adjacent columns of \( \sigma \) such that \( \text{col}_2 \) is to the right of \( \text{col}_1 \) and \( \text{diag}(|\text{col}_1|) \subseteq \text{diag}(\text{col}_2) \).

**Definition 17.** Define \( \rho_{\sigma}(\text{col}_1, \text{col}_2) \) to be the parking function obtained from \( \sigma \) by swapping \( \text{col}_1 \) and \( \text{col}_2 \) in the following manner: Let \( \text{col}_1 \) denote the maximal subcolumn of \( \text{col}_1 \) that can properly fit in \( \text{col}_2 \) while displacing the corresponding subcolumn (which also has size \( |\text{col}_1| \) ) of \( \text{col}_2 \) to the right. \( \rho_{\sigma} \) acts by sliding \( \text{col}_1 \) into \( \text{col}_2 \) and displacing the corresponding subcolumn of \( \text{col}_2 \) to the right, and by moving the remaining cars of \( \text{col}_1 \) to the right of the corresponding cars of \( \text{col}_2 \).

**Remark 18.** Thus, \( \rho_{\sigma} \) transforms the adjacent columns \( \text{col}_1, \text{col}_2 \) to adjacent columns \( \text{col}'_1, \text{col}'_2 \), where \( \text{col}'_2 \) is to the right of \( \text{col}'_1 \), \( \text{col}'_2 \) contains \( \text{col}_1 \), \( |\text{col}'_1| = |\text{col}_2| \), and \( \text{col}'_2 \) contains the remaining cars of \( \text{col}_1 \) and the displaced subcolumn of \( \text{col}_2 \).
The following lemmas are straightforward. They state that $\rho_\sigma$ changes the dinv by only 1, $-1$, or 0.

**Lemma 19.** If $\text{col}_1, \text{col}_2$ start on the same diagonal, then $\text{dinv}(\rho_\sigma(\text{col}_1, \text{col}_2)) = \text{dinv}(\sigma) + 1$.

**Lemma 20.** If $\text{col}_1, \text{col}_2$ end on the same diagonal, then $\text{dinv}(\rho_\sigma(\text{col}_1, \text{col}_2)) = \text{dinv}(\sigma) - 1$.

**Lemma 21.** If $\text{col}_1, \text{col}_2$ neither start nor end on the same diagonal, then $\text{dinv}(\rho_\sigma(\text{col}_1, \text{col}_2)) = \text{dinv}(\sigma)$.

We now return to the case of $\mathcal{A}^*_{(a,b)}$ and define the relevant bijection.

**Definition 22.** To define $f^{*,c} : \mathcal{A}^*_{(a,b)} \rightarrow \mathcal{A}_{(b,a)}^*$, fix $\sigma^{*,c} \in \mathcal{A}_{(a,b)}^*$, let $\text{col}_{+}$ denote the single column of the second part, let $\text{col}^-$ denote the column of the first part containing anchor(1) (i.e. the first column), and let $\text{col}_1, \text{col}_2, \ldots, \text{col}_l$ denote the other columns of the first part in left-to-right order. We define $f^{*,c}(\sigma^{*,c})$ in the following manner:

1. First apply $\rho$ to $\sigma^{*,c}$ on the adjacent columns $\text{col}_{+}$ and $\text{col}_l$. This produces parking function $\sigma^{1,c}_{+}$ with adjacent columns $\text{col}_{+1}^+$ and $\text{col}_{l-1}^-$ where $|\text{col}_{+1}^+| = |\text{col}_+|$. This application of $\rho$ does not change the dinv.

2. Apply $\rho$ to $\sigma_{+}^{j,c}$ on the adjacent columns $\text{col}_{+}^j$ and $\text{col}_{l-j}$ for $1 \leq j \leq l-1$. This produces parking function $\sigma_{+}^{j+1,c}$ with adjacent columns $\text{col}_{+}^{j+1}$ and $\text{col}_{l-j-1}^-$ where $|\text{col}_{+}^{j+1}| = |\text{col}_{+}^j|$. This application of $\rho$ does not change the dinv.

3. Apply $\rho$ to $\sigma_{+}^{l,c}$ on the adjacent columns $\text{col}_{+}^l$ and $\text{col}_0 := \text{col}^-$. This produces parking function $f^{*,c}(\sigma^{*,c})$ and increases the dinv by 1.

**Remark 23.** Thus, the action of $f^{*,c}$ slides the single long column of the second part of $\sigma^{*,c}$ left past all the columns of the first part and swaps cars accordingly while leaving the configuration of the columns unchanged (other than the swap itself).

**Example 24.** We illustrate the above operation on the parking function $PF$, repeatedly applying it until the two parts of $PF$ are swapped to obtain $f^{*,c}(PF)$:
first gets transformed to

\[
\rho_{PF}((a), (b))
\]

which finally yields

\[
f^\ast\cdot c(PF)
\]

Notice that the operation on \(PF\) swaps columns (a) and (b) by displacing the subcolumn 6 of (b) with the subcolumn 7 of (a) and moving the remaining cars of (a) to the right, which produces no change in \(\text{dinv}\). The operation on \(\rho_{PF}((a), (b))\) swaps columns (c) and (d) by displacing the subcolumn \([4, 3]\) of (d) with the subcolumn \([5, 1]\) of (c), increasing the \(\text{dinv}\) of \(\rho_{PF}((a), (b))\) by 1. Thus, we have \(\text{dinv}(f^\ast\cdot c(PF)) = \text{dinv}(PF) + 0 + 1 = \text{dinv}(PF) + 1\).
Theorem 25. \( f^{*,c} \) thus defined is a bijection between \( A^{*,c}_{(a,b)} \) and \( A^{c,*}_{(b,a)} \) with the desired properties.

Proof. The operation \( \rho \) is clearly invertible. By Lemmas \[19\] and \[21\] \( f^{*,c} \) defined by repeated applications of \( \rho \) does increase the dinv by exactly 1.

4 Other Generalizations

As an immediate generalization, we can actually allow the second part of the parking function to have an arbitrary number of columns, so long as all these columns, with the exception of the first column, start on sufficiently high diagonals so that they do not interact with the columns of the first part. Let \( A^{*,c}_{(a,b)} \subset A_{(a,b)} \) denote the set of parking functions whose first part is arbitrary while the columns of the second part, except for the first column, start on diagonals at least two higher than \( \max(\text{diag}(1)) \). In this situation, \( \text{diag}(1) \) must be a proper subset of the diagonals occupied by the first column of the second part. The operation \( \rho \) and the bijection \( f^{*,c} : A^{*,c}_{(a,b)} \rightarrow A^{c,*}_{(b,a)} \) can be defined in the same way, except that we simply slide all the other columns of the second part along with the first column of the second part. We illustrate this bijection with an example.

Example 26. We demonstrate the action of \( f^{*,c} \) on the parking function \( PF \):

\[
\begin{array}{cccccccccc}
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& & & & & & & & & \\
\end{array}
\]

gets transformed (with no change in dinv) to
gets transformed (with no change in dinv) to

which finally yields (increasing the dinv by one)
References