Equilibrium, uncertainty and risk in hydro-thermal electricity systems*

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August 20, 2013

Abstract

The correspondence of competitive partial equilibrium with a social optimum is well documented in the welfare theorems of economics. These theorems can be applied to single-period electricity pool auctions in which price-taking agents maximize profits at competitive prices, and extend naturally to standard models with locational marginal prices. In hydro-thermal markets where the auctions are repeated over many periods, agents seek to optimize their current and future profit accounting for future prices that depend on uncertain inflows. In this setting perfectly competitive partial equilibrium corresponds to a social optimum when all agents share common knowledge of the probability distribution governing future inflows. The situation is complicated when agents are risk averse. We illustrate some of the consequences of risk aversion on market outcomes using simple two-stage competitive equilibrium models in which agents are endowed with coherent risk measures. In this setting we show that welfare is optimized in a competitive market if there are enough traded market instruments to hedge inflow uncertainty but might not be if these are missing.

1 Introduction

Most industrialised regions of the world have over the last twenty years established wholesale electricity markets that take the form of an auction that matches supply and demand. The exact form of these auction mechanisms vary by jurisdiction, but they typically require offers of energy from suppliers at costs they are willing to supply, and clear a market by dispatching these offers in order of increasing cost. Day-ahead markets such as those implemented in most North American jurisdictions, seek to arrange supply well in advance of its demand, so that thermal units can be prepared in time. Since the demand cannot be predicted with absolute certainty, these day-ahead markets must be augmented with

*This project has been supported in part by the Air Force Office of Scientific Research, the Department of Energy, the National Science Foundation, and the USDA National Institute of Food and Agriculture.

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balancing markets to deal with the variation in load and generator availability in real time.

The market mechanisms are designed to be as efficient as possible in the sense that they should maximize the total welfare of producers and consumers. In a deterministic one-shot setting in which all agents act in perfect competition, the welfare theorems of microeconomics ensure that the auction designs lead to welfare maximization. We give a deriviation of such a result in section 2 for a simplified model of an electricity market. That welfare maximization can be compromised by the exercise of market power by strategic agents is well known, and many studies (see e.g. [2]) have been carried out to estimate the extent of the inefficiency caused by this.

Market inefficiencies can also be created by uncertainty. Each agent in an electricity market is faced with a dynamic stochastic optimization problem, to maximize their current and (possibly risk-adjusted) future profit. Ideally, an electricity market auction would provide every agent with a stochastic process of electricity prices with which they can perform this optimization, but in practice this is too difficult to arrange. Most markets operate in the short term with a day-ahead auction and a balancing market that is settled in real time. These markets are settled separately which can be inefficient in comparison with a single settlement procedure using stochastic programming, at least when agents act as price takers [11]. In some jurisdictions (like New Zealand) there is no day-ahead market and the market dispatch is computed close to real time for the next trading period and implemented as the day unfolds. Offers of generation and demand forecasts for future periods are used to forecast prices for future periods. These are used to guide each agent in what they offer. In this sense, the auction is iterating towards (but possibly never converging to) a set of prices that represent a realization of the stochastic process that each price-taking agent would want to have at their disposal.

A similar price discovery process occurs on a longer time scale for markets with stored hydro electricity. Generators with hydro-electric reservoirs face an inventory problem. They would like to optimize the release of water from reservoirs to maximize profits using a stochastic process of prices, but this process is not known, and must be deduced by each agent using current and future market conditions and hydrological models of future reservoir inflows. For an agent controlling releases from a hydro-electric reservoir, the marginal cost of supply in the current period involves some modeling of opportunity cost that includes possible high prices in future states of the world with low inflows.

In this paper we study the possible causes of market inefficiency that arise from uncertainty in reservoir inflows. To simplify this analysis, we assume that all agents are price-takers who do not act strategically. It is well-known that competitive electricity prices for a single trading period and single location can be computed as shadow prices from convex economic dispatch models that maximize total social welfare. These results remain true in the presence of a transmission network as long as the use of transmission assets is appropriately priced [18]. In other words the market must contain enough instruments to price transmission (or equivalently have locational marginal prices). Similarly when hydro reservoirs operated by different agents form a cascade on the same river system, as in the model studied by Lino *et al* [12], the market needs to be completed by an instrument that allows agents to trade water between reservoirs in order for a competitive equilibrium to correspond to the social optimum.

A similar incompleteness arises from uncertainty in future inflows. If all agents share the same view about future inflows in the sense that there is a single stochastic process of inflows that is common knowledge, and all agents maximize expected profit using the probability law determining these inflows then a competitive equilibrium will correspond to the welfare-optimizing solution computed by a social planner. If agents have different views about future inflows then such a welfare result might no longer be true. Indeed it is not clear what probability law the social planner should use in determining a welfare optimizing solution. This raises the question on how one might complete the market with suitable instruments to enable a welfare maximizing competitive equilibrium.

We provide a partial answer to this problem in the setting where there is a single stochastic process of inflows that is common knowledge, but agents have different attitudes to risk. Following previous work of [10] and [15], we assume that agents are endowed with coherent risk measures as defined by [1]. This means that each agent maximizes their expected profit using a worst-case probability distribution chosen from a well-defined set of distributions that is possibly different for each agent. As shown in [10] and [15], as long as the agents risk sets intersect, the addition of suitable instruments can provide a complete market for risk that yields a single probability law. In equilibrium, all agents share this law, and so a social planner might compute a welfare maximing solution using a stochastic optimization model.

Our paper is an important contribution to the theory of electricity market design under uncertainty. The "standard market design" using locational marginal pricing is now commonplace in many jurisdictions, and provides complete markets for transmission. The theory of investment from scarcity rents earned by suppliers when consumers shed load at price caps is now well understood, at least in a deterministic setting. However much of this theory is called into question in a probabilistic setting when investors are risk averse [6]. In the absence of a market to share risk, producers' investments will differ from the social optimum, and welfare will be lost. Our computational models confirm a similar effect for hydro releases in markets with hydro electricity. They also provide a first step towards understanding how to provide mechanisms to share the agents risk amongst market participants and thus get closer to an efficient market.

Our results should also be of particular interest to market regulators who seek a perfectly competitive counterfactual model to serve as a benchmark for market prices. A risk-neutral counterfactual solution is likely to incur some energy shortages and corresponding high prices, as these will happen occasionally to minimize the expected cost [14]. To avoid these shortages, offer prices are often marked up by risk premia by agents seeking to conserve water. These markups can also be interpreted as unilateral exercise of market power by hydro-generators [19]. Estimates of competitive risk premia (under some assumptions about completeness and agents' risk aversion) will go some way to establishing an appropriate benchmark for monitoring the efficiency of hydro-dominated markets.

Our paper is intended to be didactic rather than formal. For this reason we demonstrate the results in a two-stage setting with a finite number of inflow outcomes in the second stage. For simplicity, we also focus on a particular risk measure that is a convex combination of expectation and average value at risk, although the results are valid for any coherent risk measure. We illustrate the results using a computational example that has a single hydro plant, a single thermal plant and a single consumer. While the computational results are illustrated on pedagogical models, the approaches are usable on larger instances and available for general use [9].

The paper is laid out as follows. In section 2, to establish notation we present a stylized model of a hydro-thermal market with deterministic inflows. Section 3 extends this to a two-stage stochastic model with random inflows in the second stage. An example instance of the model with one consumer, one thermal plant and one reservoir is described to illustrate the notation. (Assuming a single hydro generator ensures that the stochastic system optimization problem is separable by agent.) This example will serve to illustrate the main ideas in this paper. In section 4, we turn attention to competitive equilibrium under risk. We describe coherent risk measures as defined by [1] and derive some results related to risk trading that we will illustrate in our model. When markets for risk are complete we show that a system risk measure may be defined using the construction of [10] and [15]. We then conclude the paper with some remarks about market completeness, and the effect of this on the equilibrium.

2 Preliminaries

Consider a model of several hydro and thermal players supplying a set of consumers with different demand curves. All agents are at the same location. The system optimization problem is

$$\begin{array}{lll} \text{NSP:} & \min & \sum_{j\in\mathcal{T}} C_j(v(j)) - \sum_{c\in\mathcal{C}} D_c(d(c)) - V(x) \\ \\ & \text{s.t.} & \sum_{i\in\mathcal{H}} U_i\left(u(i)\right) + \sum_{j\in\mathcal{T}} v(j) \geq \sum_{c\in\mathcal{C}} d(c), \\ & & x(i) = x_0(i) - u(i), \quad i \in \mathcal{H} \\ & & u(i), v(j), x(i) \geq 0. \end{array}$$

Here $d(c)$ is the consumption of consumer segment $c \in \mathcal{C}$, $u(i)$ is the water release of hydro reservoir $i \in \mathcal{H}$ and $v(j)$ is the thermal generation of plant $j \in \mathcal{T}$. The production function U_i converts water release to energy. We assume that U_i is strictly concave. The water level reservoir $i \in \mathcal{H}$ is denoted $x(i)$. We let $C_j(v(j))$ denote the cost of generation by thermal plant j in the current period, $D_c(d(c))$ the welfare accrued by consumer segment c, and $V(x)$ to be the future value of terminating the period with storage x. Here C_j, $j \in \mathcal{T}$, is assumed to be strictly convex, and D_c, $c \in \mathcal{C}$ and V are assumed to be strictly concave. Note that the derivative D_c' represents the inverse demand curve for consumer c, which is strictly decreasing by assumption.

In the model NSP, the function $V(x)$ derives from some future system cost given reservoir levels of x. This might not be separable by reservoir. For example, the reservoirs might form a cascade on the same river system, as in the model studied by Lino *et al* [12]. In these circumstances [12] show that there might be no set of marginal prices for electricity that will give a welfare maximizing equilibrium. The market needs to be completed by an instrument that allows agents to trade water between reservoirs.

In order to avoid such complexities, we will assume a future cost function that is separable by agent to give

$$\begin{array}{lll} \text{SSP:} & \min & \sum_{j\in\mathcal{T}} C_j(v(j)) - \sum_{c\in\mathcal{C}} D_c(d(c)) - \sum_{i\in\mathcal{H}} V_i(x(i)) \\ & \text{s.t.} & \sum_{i\in\mathcal{H}} U_i\left(u(i)\right) + \sum_{j\in\mathcal{T}} v(j) \geq \sum_{c\in\mathcal{C}} d(c), \\ & & x(i) = x_0(i) - u(i), \quad i \in \mathcal{H}, \\ & & u(i), v(j), x(i), d(c) \geq 0. \end{array}$$

where each V_i is a strictly concave univariate function. Such an assumption would be valid if, like in our examples, there were only one hydro generator, or hydro generators were decoupled by transmission constraints, so that their water values are essentially determined only by local conditions. This involves some loss of realism, but simplifies the analysis without losing the essence of the results that we wish to discuss.

We shall assume that all optimization problems throughout this paper satisfy some constraint qualification that guarantees the existence of Lagrange multipliers. In particular, this means that we can solve SSP by minimizing a Lagrangian with multipliers π to give

$$\begin{array}{lll} \text{LSSP:} & \min & \pi \sum_{c\in\mathcal{C}} d(c) - \sum_{c\in\mathcal{C}} D_c(d(c)) + \sum_{j\in\mathcal{T}} \left(C_j(v(j)) - \pi v(j)\right) \\ & & \quad - \sum_{i\in\mathcal{H}} \left(V_i(x(i)) + \pi U_i\left(u(i)\right)\right) \\ \text{s.t.} & & x(i) = x_0(i) - u(i), \quad i \in \mathcal{H}, \\ & & u(i), v(j), x(i), d(c) \geq 0. \end{array}$$

LSSP separates by agent. Here each hydro plant $i \in \mathcal{H}$ maximizes profit at prices π by solving

$$\begin{array}{lll} \text{HP}(i): & \max & \pi U_i\left(u(i)\right) + V_i(x(i)) \\ & \text{s.t.} & x(i) = x_0(i) - u(i) \\ & & u(i), x(i) \geq 0, \end{array}$$

and each thermal plant $j \in \mathcal{T}$ maximizes profit at prices π by solving

$$\begin{array}{lll} \text{TP}(j): & \max & \pi v(j) - C_j(v(j)) \\ \text{s.t.} & & v_1(j) \geq 0, \end{array}$$

and the consumers $c \in \mathcal{C}$ maximize their welfare by solving

$$\begin{array}{lll} \text{CP}(c): & \max & \sum_{c\in\mathcal{C}} D_c(d(c)) - \pi \sum_{c\in\mathcal{C}} d(c) \\ & \text{s.t.} & d(c) \geq 0. \end{array}$$

In a perfectly competitive (Walrasian) equilibrium, agents respond by optimizing production to maximize their benefits at the price π announced by the auctioneer. The price π that clears the market defines an equilibrium that is a solution to the following variational problem:

$$\begin{array}{lll} \text{CE:} & u(i), x(i) \in \arg\max \text{HP}(i), & i \in \mathcal{H}, \\ & v(j) \in \arg\max \text{TP}(j), & j \in \mathcal{T}, \\ & d(c) \in \arg\max \text{CP}(c), & c \in \mathcal{C}, \\ & 0 \leq \left(\sum_{i\in\mathcal{H}} U_i\left(u(i)\right) + \sum_{j\in\mathcal{T}} v(j)\right) - \sum_{c\in\mathcal{C}} d(c) \perp \pi \geq 0, & \end{array}$$

Since the strict convexity assumptions guarantee a unique solution to SSP, and a unique solution to each agent problem, the Lagrangian duality theorem gives the following result.

Proposition 1 *The (unique) welfare maximizing solution to SSP is the same as the (unique) competitive equilibrium solution to CE.*

2.1 Extended Mathematical Programming

All of the computational results in this paper are solved using the Extended Mathematical Programming (EMP) [9, 7, 3] features of the GAMS modeling system [4]. The EMP framework exists to enable formulations of problems that fall outside the standard framework within the modeling system. A high-level description of these extended models, along with tools to automatically create the different realizations or formulations possible, pass them on to the appropriate solvers, and interpret the results in the context of the original model, makes it possible to model more easily, to conduct experiments with formulations otherwise too time-consuming to consider, and to avoid errors that can make results meaningless or worse.

Multiple Optimization Problems with Equilibrium Constraints (MOPEC) involves a collection of agents $\mathcal{A}$ that determine their decisions $x_{\mathcal{A}} = (x_a,\, a \in \mathcal{A})$ by solving, independently, an optimization problem,

$$x_a \in \operatorname{argmax}_{x \in \mathbb{R}^{n_a}} f_a(p, x, x_{-a}), \qquad a \in \mathcal{A}, \tag{1}$$

where $f_a(p, \cdot, x_{-a}) : \mathbb{R}^{n_a} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is their criterion function, with $x_{-a} = (x_o, o \in \mathcal{A} \setminus \{a\})$ representing the decision of the other agents and $p \in \mathbb{R}^d$ being a parameter that may refer to prices in an economic application, stresses in mechanical systems, and environmental conditions in numerous other applications. This parameter and the decisions $x_{\mathcal{A}}$ satisfy a global equilibrium constraint, formulated as a *geometric variational inequality*,

$$F(p, x_{\mathcal{A}}) \in N_C(p), \tag{2}$$

with $N_C(p)$ the normal cone to C at p. We refer to (1)-(2) as a MOPEC, whose solution is a pair $(p, x_{\mathcal{A}})$ that satisfies the preceding inclusions. Even though (1) omits an explicit expression of constraints on $x_{\mathcal{A}}$, that possibility is handled herein by extended-value functions. Note that the model CE above is a MOPEC.

EMP provides the ability to describe a variational inequality within a modeling system. We annotate existing equations in the model, detailing which ones provide the function F, and which ones are part of the description of the underlying feasible set C. Note that there is no requirement that C is polyhedral, and the format generalizes both nonlinear equations and nonlinear complementarity systems. The main formulation of interest here is MOPEC, for example the problem described via (1) and (2). In this setting that variables x_a and p, and the functions f_a and F are defined with the usual model system, but an additional annotation is provided of the form:

```
equilibrium
max f_1 x_1
max f_2 x_2
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```
...
max f_k x_k
vi F p
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This describes a problem involving k agents, each of which solve an optimization problem whose objective function involves not only variables x_k but also other agents variables, and the price vector p. Similarly to above, the VI involving D and p nails down the values of p. In the GAMS implementation, the VI is converted into its KKT form, and then solved using the PATH solver [5, 8] - this allows problems of hundreds or thousands of variables to be processed.

Other features of EMP include stochastic programming and risk measures, hierarchical optimization, such as bilevel programming, extended nonlinear programming and disjunctive programming.

3 Two-stage hydro-thermal problems

We now wish to extend the analysis of the previous section to problems in which there is some uncertainty about future reservoir inflows. For simplicity we adopt in this section a two-stage structure, in which inflows in the first stage are known, and those in the second stage are random. Thus in stage 1, all generators know the storage level in the reservoirs and the inflows that will occur in this period. These are assumed to be available for release (or spill). The generators know only the probability distribution of inflows in the second stage, which consists of a finite number of scenarios ω_m, $m = 1, 2, \ldots, M$, each with probability μ_m. We assume without loss of generality that $\mu_m > 0$ for every $m = 1, 2, \ldots, M$. The inflow realization $h(\omega_m)$ is revealed to the generators at the end of stage 1 (after they have made their release and spill decisions).

Given electricity prices π_1 in the current period and random prices $\pi_2(\omega_m)$ in the next period, each hydro generator $i \in \mathcal{H}$ maximizes expected profit by solving

$$\begin{aligned}
\text{HP}(i,\mu): \quad \min \quad & -\pi_1 U_i(u_1(i)) \\
& - \textstyle\sum_{m=1}^{M} \mu_m [\pi_2(\omega_m) U_i(u_2(i,\omega_m)) + V_i(x_2(i,\omega_m))] \\
\text{s.t.} \quad & x_1(i) = x_0(i) - u_1(i) + h_1(i), \\
& x_2(i,\omega_m) = x_1(i) - u_2(i,\omega_m) + h_2(i,\omega_m), \qquad m = 1,2,\ldots,M, \\
& u_1(i), x_1(i) \geq 0, \quad u_2(i,\omega_m), x_2(i,\omega_m) \geq 0, \qquad m = 1,2,\ldots,M,
\end{aligned}$$

(We express this as a minimization of disbenefit for ease of notation in what follows.) Each thermal generator $j \in \mathcal{T}$ maximizes expected profit by solving

$$\begin{aligned}
\text{TP}(j.\mu): \quad \min \quad & -\pi_1 v_1(j) + C_j(v_1(j)) \\
& - \textstyle\sum_{m=1}^{M} \mu_m [\pi_2(\omega_m) v_2(j,\omega_m) + C_j\left(v_2(j,\omega_m)\right)] \\
\text{s.t.} \quad & v_1(j) \geq 0, \quad v_2(j,\omega_m) \geq 0, \qquad m = 1,2,\ldots,M.
\end{aligned}$$

The consumer group c maximizes expected welfare by solving

$$\begin{array}{lll}\mathrm{CP}(c,\mu): & \min & -\sum_{c\in\mathcal{C}} D_c(d(c)) + \pi \sum_{c\in\mathcal{C}} d_1(c) \\ & & -\sum_{m=1}^{M} \mu_m [\sum_{c\in\mathcal{C}} D_c(d_2(c,\omega_m)) - \pi_2(\omega_m) \sum_{c\in\mathcal{C}} d_2(c,\omega_m)] \\ & \text{s.t.} & d_1(c) \geq 0, \quad d_2(c,\omega_m) \geq 0, \qquad m = 1,2,\ldots,M.\end{array}$$

Observe that given current prices π_1 and random future price $\pi_2(\omega_m)$, the problems $\mathrm{TP}(j,\mu)$ and $\mathrm{CP}(c,\mu)$ are ex-post optimal (i.e. *wait-and-see* problems).

In order to simplify notation in what follows we write

$$(u_1(i), u_2(i,\cdot), x_1(i), x_2(i,\cdot)) \in \mathrm{HP}(i)$$

$$(v_1(j), v_2(j,\cdot)) \in \mathrm{TP}(j)$$

$$(d_1(c), d_2(c,\cdot)) \in \mathrm{CP}(c)$$

to denote that these sets of variables satisfy the constraints of $\mathrm{HP}(i,\mu)$, $\mathrm{TP}(j,\mu)$, and $\mathrm{CP}(c,\mu)$. (Observe that the constraints are not affected by the probability measure, and so we drop the argument μ from the notation.)

We define the competitive equilibrium to be a solution to the following variational problem.

$$\begin{array}{lll}\mathrm{CE}(\mu): & (u_1(i), u_2(i,\omega_m), x_1(i), x_2(i,\omega_m)) \in \arg\min \mathrm{HP}(i,\mu), & i \in \mathcal{H} \\ & (v_1(j), v_2(j,\omega_m)) \in \arg\min \mathrm{TP}(j,\mu), & j \in \mathcal{T} \\ & (d_1(c), d_2(c,\omega_m)) \in \arg\min \mathrm{CP}(c,\mu), & c \in \mathcal{C} \\ & 0 \leq \sum_{i\in\mathcal{H}} U_i(u_1(i)) + \sum_{j\in\mathcal{T}} v_1(j) - \sum_{c\in\mathcal{C}} d_1(c) \perp \pi_1 \geq 0, & \\ & 0 \leq \sum_{i\in\mathcal{H}} U_i(u_2(i,\omega_m)) + \sum_{j\in\mathcal{T}} v_2(j,\omega_m) - \sum_{c\in\mathcal{C}} d_2(c,\omega_m) \perp \pi_2(\omega_m) \geq 0, & \\ & & m = 1,\ldots,M.\end{array}$$

We proceed to relate this to the optimization problem of a social planner.

A social planner optimizes total welfare using the following stochastic optimization problem.

$$\begin{array}{lll}\mathrm{SP}(\mu): & \min & \sum_{j\in\mathcal{T}} C_j(v_1(j)) - \sum_{c\in\mathcal{C}} D_c(d_1(c)) \\ & & + \sum_{m=1}^{M} \mu_m [\sum_{j\in\mathcal{T}} C_j(v_2(j,\omega_m)) - \sum_{c\in\mathcal{C}} D_c(d_2(c,\omega_m)) - \sum_{i\in\mathcal{H}} V_i(x_2(i,\omega_m))] \\ & \text{s.t.} & \sum_{i\in\mathcal{H}} U_i(u_1(i)) + \sum_{j\in\mathcal{T}} v_1(j) - \sum_{c\in\mathcal{C}} d_1(c) \geq 0, \\ & & \sum_{i\in\mathcal{H}} U_i(u_2(i,\omega_m)) + \sum_{j\in\mathcal{T}} v_2(j,\omega_m) - \sum_{c\in\mathcal{C}} d_2(c,\omega_m) \geq 0, \quad m = 1,2,\ldots,M, \\ & & (u_1(i), u_2(i,\omega_m), x_1(i), x_2(i,\omega_m)) \in \mathrm{HP}(i), \quad i \in \mathcal{H}, \\ & & (v_1(j), v_2(j,\omega_m)) \in \mathrm{TP}(j), \quad j \in \mathcal{T}, \\ & & (d_1(c), d_2(c,\omega_m)) \in \mathrm{CP}(c), \quad c \in \mathcal{C}.\end{array}$$

Let $\pi_1 \geq 0$ and $\mu_m \pi_2(\omega_m) \geq 0$ be the Lagrange multipliers corresponding to constraints

$$\sum_{i\in\mathcal{H}} U_i(u_1(i)) + \sum_{j\in\mathcal{T}} v_1(j) - \sum_{c\in\mathcal{C}} d_1(c) \geq 0 \tag{3}$$

and

$$\sum_{i\in\mathcal{H}} U_i\left(u_2(i,\omega_m)\right) + \sum_{j\in\mathcal{T}} v_2(j,\omega_m) - \sum_{c\in\mathcal{C}} d_2(c,\omega_m) \geq 0, \qquad m = 1,2,\ldots,M. \tag{4}$$

These give a Lagrangian defined by

$$\begin{aligned}
L(u,v,d,x,\pi,\mu) \;=\;& \sum_{j\in\mathcal{T}} C_j(v_1(j)) - \sum_{c\in\mathcal{C}} D_c(d_1(c)) \\
&+ \sum_{m=1}^{M} \mu_m[\sum_{j\in\mathcal{T}} C_j(v_2(j,\omega_m)) - \sum_{c\in\mathcal{C}} D_c(d_2(c,\omega_m)) - \sum_{i\in\mathcal{H}} V_i(x_2(i,\omega_m))] \\
&+ \pi_1[-\sum_{i\in\mathcal{H}} U_i\left(u_1(i)\right) - \sum_{j\in\mathcal{T}} v_1(j) + \sum_{c\in\mathcal{C}} d_1(c)] \\
&+ \sum_{m=1}^{M} \mu_m \pi_2(\omega_m)[-\sum_{i\in\mathcal{H}} U_i\left(u_2(i,\omega_m)\right) - \sum_{j\in\mathcal{T}} v_2(j,\omega_m) + \sum_{c\in\mathcal{C}} d_2(c,\omega_m)].
\end{aligned}$$

For fixed multipliers $\pi_1 \geq 0$ and $\mu_m \pi_2(\omega_m) \geq 0$, minimizing the Lagrangian decouples into problems HP(i,μ), TP(j,μ), and CP(c,μ). By Lagrangian duality, there exists a set of prices $\pi_1 \geq 0$ and $\pi_2(\omega_m) \geq 0$ such that the solutions to HP(i,μ), TP(j,μ) and CP(c,μ) will satisfy (3) and (4), and therefore solve SP(μ). Moreover the multipliers satisfy

$$0 \;\leq\; \sum_{i\in\mathcal{H}} U_i\left(u_1(i)\right) + \sum_{j\in\mathcal{T}} v_1(j) - \sum_{c\in\mathcal{C}} d_1(c) \perp \pi_1 \geq 0, \tag{5}$$

$$0 \;\leq\; \sum_{i\in\mathcal{H}} U_i\left(u_2(i,\omega_m)\right) + \sum_{j\in\mathcal{T}} v_2(j,\omega_m) - \sum_{c\in\mathcal{C}} d_2(c,\omega_m) \perp \pi_2(\omega_m) \geq 0, \tag{6}$$

and so the solutions solve CE(μ). Conversely any solution to CE(μ) will give multipliers $\pi_1 \geq 0$ and $\pi_2(\omega_m) \geq 0$ such that the solutions to HP(i,μ), TP(j,μ) and CP(c,μ) will satisfy (3) and (4) and therefore solve SP(μ). Thus the solutions to SP(μ) and CE(μ) coincide. We collect these results into the following propositions.

Proposition 2 *Suppose every agent is risk neutral and has knowledge of all deterministic data, as well as sharing the same probability distribution μ for inflows. Then the solution to SP(μ) is the same as the solution to CE(μ).*

Proposition 3 *Suppose for given $\pi_1 \geq 0$ and $\pi_2(\omega_m) \geq 0$ that (u,x,v,d) satisfies*

$$\begin{aligned}
(u_1(i), u_2(i,\omega_m), x_1(i), x_2(i,\omega_m)) &\in \arg\max HP(i,\mu), \\
(v_1(j), v_2(j,\omega_m)) &\in \arg\max TP(j,\mu), \\
(d_1(c), d_2(c,\omega_m)) &\in \arg\max CP(c,\mu).
\end{aligned}$$

Moreover if (5) and (6) hold then (u,x,v,d) solves SP(μ).

Proposition 4 *Suppose that (u,x,v,d) solves SP(μ) and $\pi_1 \geq 0$ and $\mu_m \pi_2(\omega_m) \geq 0$ are the Lagrange multipliers of (3) and (4). Then (u,x,v,d,π) solves CE(μ).*

3.1 Example

Throughout this paper we will illustrate the concepts using a hydro-thermal system with one reservoir, one thermal plant, and one consumer. We let thermal cost be $C(v) = v^2$, and define

$$
\begin{aligned}
U(u) &= 1.5u - 0.015u^2 \\
V(x) &= 10\log(0.1x + 0.01) \\
D(d) &= 40d - 2d^2
\end{aligned}
$$

We assume inflow 4 in period 1, and inflows of $1, 2, \ldots, 10$ with equal probability in each scenario in period 2. With an initial storage level of 10 units this gives the competitive equilibrium shown in Table 1.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		1.336	7.590	6.410	0.668				
1	1	2.539	2.865	5.725	1.269	2.057	20.417	362.283	384.758
1	2	2.053	3.590	6.000	1.027	1.500	19.418	366.863	387.781
1	3	1.696	4.387	6.203	0.848	1.165	18.809	370.264	390.238
1	4	1.431	5.236	6.355	0.716	0.958	18.514	372.809	392.281
1	5	1.231	6.121	6.470	0.616	0.825	18.445	374.746	394.016
1	6	1.076	7.031	6.559	0.538	0.735	18.529	376.252	395.516
1	7	0.953	7.961	6.629	0.477	0.673	18.716	377.446	396.835
1	8	0.855	8.904	6.686	0.427	0.629	18.969	378.411	398.008
1	9	0.774	9.857	6.733	0.387	0.596	19.264	379.204	399.064
1	10	0.706	10.818	6.772	0.353	0.571	19.585	379.866	400.022

Table 1: Competitive equilibrium (solution to CE) with initial storage of 10.

The social planner's problem that maximizes expected welfare (by minimizing expected generation and future cost) is shown in Table 2. One can observe that the two solutions are identical, as predicted by Proposition 2.

4 Risk aversion

We now turn attention to a setting in which the electricity generators are risk averse. To measure risk aversion we use a risk measure ρ defined by

$$\rho(Z) = (1-\lambda)\mathbb{E}[Z] + \lambda \mathsf{AVaR}_{1-\alpha}[Z].$$

Here Z represents the random negative profit of an agent (or random system cost of the social planner), and $\mathsf{AVaR}_{1-\alpha}[Z]$ denotes *average value at risk* at level $1-\alpha$ that can be expressed as the well-known formula [16]

$$\mathsf{AVaR}_{1-\alpha}[Z] = \inf_t \{t + \alpha^{-1}\mathbb{E}[(Z-t)_+]\}.$$

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	cost (total)
0		1.336	7.590	6.410	0.668				
1	1	2.539	2.865	5.725	1.269	2.057	20.417	362.283	-384.758
1	2	2.053	3.590	6.000	1.027	1.500	19.418	366.863	-387.781
1	3	1.696	4.387	6.203	0.848	1.165	18.809	370.264	-390.238
1	4	1.431	5.236	6.355	0.716	0.958	18.514	372.809	-392.281
1	5	1.231	6.121	6.470	0.616	0.825	18.445	374.746	-394.016
1	6	1.076	7.031	6.559	0.538	0.735	18.529	376.252	-395.516
1	7	0.953	7.961	6.629	0.477	0.673	18.716	377.446	-396.835
1	8	0.855	8.904	6.686	0.427	0.629	18.969	378.411	-398.008
1	9	0.774	9.857	6.733	0.387	0.596	19.264	379.204	-399.064
1	10	0.706	10.818	6.772	0.353	0.571	19.585	379.866	-400.022

Table 2: Social planning solution (solution to SP) with initial storage of 10.

The value $\lambda \in [0,1]$ is a measure of risk aversion, where $\lambda = 0$ corresponds to a risk-neutral agent (or system planner) and $\lambda = 1$ is the most risk averse setting in which all the weight in the objective is placed on $\mathsf{AVaR}_{1-\alpha}[Z]$. Observe that we assume all decision makers are minimizing $\rho(Z)$.

The risk measure we adopt is an example of a *coherent risk measure* as defined by [1]. Any coherent risk measure $\rho(Z)$ has a *dual representation* expressing it as

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_\mu[Z]$$

where $\mathcal{D}$ is a convex subset of probability measures (see e.g. [1],[10]). $\mathcal{D}$ is called the *risk set* of the coherent risk measure. The dual representation using a risk set plays an important role in the analysis we carry out in this paper.

The duality theorem for coherent risk measures means that given any set of random payoffs $Z(\omega_m)$ defined over a finite set of outcomes $m = 1, 2, \ldots, M$, and a polyhedral risk set

$$\mathcal{D}=\text{conv}\{(p_1^1, p_2^1, \ldots, p_M^1), (p_1^2, p_2^2, \ldots, p_M^2), \ldots, (p_1^K, p_2^K, \ldots, p_M^K)\}$$

we may write

$$\rho(Z) = \max_{\mu \in \mathcal{D}} \sum_{m=1}^{M} \mu_m Z(\omega_m) = \max_{k=1,..,K} \sum_{m=1}^{M} p_m^k Z(\omega_m),$$

since the maximum of a linear function over $\mathcal{D}$ is attained at an extreme point. By a standard dualization, this gives

$$\rho(Z) = \begin{cases} \min & \theta \\ \text{s.t.} & \theta \geq \sum_{m=1}^{M} p_m^k Z(\omega_m), \quad k = 1, 2, \ldots, K. \end{cases} \tag{7}$$

We can use the optimization problem (7) to measure the risk of any set of random costs when the risk measure is defined by a polyhedral risk set.

Example 5 *Suppose there are* 10 *scenarios with probability* $p_m = 0.1$, $m = 1, 2, \ldots, 10$, *and*

$$\rho(Z) = (1 - \lambda)\mathbb{E}[Z] + \lambda \mathsf{AVaR}_{0.9}[Z].$$

Then we set $K = M = 10$, *and* $\mathcal{D}$ *is the convex hull of 10 extreme points. The* k*th extreme point has probability* $\frac{9\lambda+1}{10}$ *for scenario* k *and* $\frac{1-\lambda}{10}$ *for the others. Thus*

$$\begin{aligned} \rho(Z) &= \max_{\mu \in \mathcal{D}} \sum_{m=1}^{M} \mu_m Z(\omega_m) \\ &= \min\{\theta : \theta \geq \sum_{m=1}^{M} p_m^k Z(\omega_m), \quad k = 1, 2, \ldots, M\} \end{aligned}$$

where

$$p_m^k = \begin{cases} \frac{9\lambda+1}{10}, & m = k \\ \frac{1-\lambda}{10}, & \textit{otherwise.} \end{cases}$$

We now return to the hydro-thermal scheduling problem. We first study each agent's risk-averse optimization problem with given prices $(\pi_1, \pi_2(\omega_m))$ that in the second stage depend on the scenario ω_m. Here each agent's optimization problem can be augmented with a risk term that models their aversion to risk. Because of translation equivariance we can write the objective in terms of the risk measure applied to each agent's total disbenefit, not only its uncertain part. In our example, the thermal problem for agent $j \in \mathcal{T}$ becomes:

$$\begin{array}{lll} \text{RTP}(j): & \min & \rho_j(-\pi_1 v_1 + C_j v_1(j) - \pi_2(\omega) v_2(j, \omega_m) + C_j\,(v_2(j, \omega_m)) \\ & \text{s.t.} & (v_1(j), v_2(j, \cdot)) \in \text{TP}(j), \quad j \in \mathcal{T}, \end{array}$$

each hydro agent $i \in \mathcal{H}$ solves

$$\begin{array}{lll} \text{RHP}(i): & \min & \rho_i(-\pi_1 U_i(u_1) - \pi_2(\omega_m) U_i(u_2(i, \omega_m)) - V_i(x_2(\omega_m))) \\ & \text{s.t.} & (u_1(i), u_2(i, \cdot), x_1(i), x_2(i, \cdot)) \in \text{HP}(i), \quad i \in \mathcal{H}, \end{array}$$

and each consumer group solves

$$\begin{array}{lll} \text{RCP}(c): & \min & \rho_c(-D_c(d(c)) + \pi_1 d_1(c) - D_c(d_2(c, \omega)) + \pi_2(\omega_m) d_2(c, \omega_m)) \\ & \text{s.t.} & (d_1(c), d_2(c, \omega_m)) \in \text{CP}(c), \quad c \in \mathcal{C}. \end{array}$$

In a similar way to the risk-neutral case, we can define a risk-averse competitive equilibrium to be a solution to the variational problem:

$$\begin{array}{lll} \text{RCE:} & (u_1(i), u_2(i, \omega_m)) \in \arg\min \text{RHP}(i), & i \in \mathcal{H}, \\ & (v_1(j), v_2(j, \omega_m)) \in \arg\min \text{RTP}(j), & j \in \mathcal{T}, \\ & (d_1(c), d_2(c, \omega_m)) \in \arg\min \text{RCP}(c), & c \in \mathcal{C}, \\ & 0 \leq \sum_{i \in \mathcal{H}} U_i\,(u_1(i)) + \sum_{j \in \mathcal{T}} v_1(j) - \sum_{c \in \mathcal{C}} d_1(c) \perp \pi_1 \geq 0, & \\ & 0 \leq \sum_{i \in \mathcal{H}} U_i\,(u_2(i, \omega_m)) + \sum_{j \in \mathcal{T}} v_2(j, \omega_m) - \sum_{c \in \mathcal{C}} d_2(c, \omega_m) & \\ & \qquad \perp \pi_2(\omega_m) \geq 0, & m = 1, \ldots, M. \end{array}$$

Finally, given a system risk measure ρ_0, a risk-averse social planner solves

$$\begin{aligned}
\text{RSP:}\quad \min \quad & \textstyle\sum_{j\in\mathcal{T}} C_j(v_1(j)) - \sum_{c\in\mathcal{C}} D_c(d_1(c)) \\
& +\rho_0\left(\textstyle\sum_{j\in\mathcal{T}} C_j(v_2(j,\omega_m)) - \sum_{c\in\mathcal{C}} D_c(d_2(c,\omega_m)) - \sum_{i\in\mathcal{H}} V_i(x_2(i,\omega_m))\right) \\
\text{s.t.}\quad & \textstyle\sum_{i\in\mathcal{H}} U_i\left(u_1(i)\right) + \sum_{j\in\mathcal{T}} v_1(j) - \sum_{c\in\mathcal{C}} d_1(c) \geq 0, \\
& \textstyle\sum_{i\in\mathcal{H}} U_i\left(u_2(i,\omega_m)\right) + \sum_{j\in\mathcal{T}} v_2(j,\omega_m) - \sum_{c\in\mathcal{C}} d_2(c,\omega_m) \geq 0, \quad m = 1,2,\ldots,M, \\
& (u_1(i), u_2(i,\omega_m), x_1(i), x_2(i,\omega_m)) \in \text{HP}(i,\mu), \quad i \in \mathcal{H}, \\
& (v_1(j), v_2(j,\omega_m)) \in \text{TP}(j,\mu), \quad j \in \mathcal{T}, \\
& (d_1(c), d_2(c,\omega_m)) \in \text{CP}(c,\mu), \quad c \in \mathcal{C}.
\end{aligned}$$

4.1 Example

We now examine solutions to RCE and RSP in the example problem when all decision makers use the risk measure

$$\rho(Z) = 0.8\mathbb{E}[Z] + 0.2\mathsf{AVaR}_{0.9}[Z].$$

High initial storage We assume an initial storage of 15 units, inflow 4 in period 1, and inflows of $1, 2, \ldots, 10$ with equal probability in each scenario in period 2. The solution to RSP is shown in Table 3 and the competitive equilibrium under risk is shown in Table 4.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	cost (total)
0		0.816	12.291	6.709	0.408				
1	1	1.118	6.757	6.534	0.559	0.479	14.131	380.898	-395.508
1	2	0.987	7.681	6.610	0.494	0.410	14.291	382.175	-396.876
1	3	0.882	8.621	6.671	0.441	0.361	14.527	383.202	-398.090
1	4	0.796	9.571	6.720	0.398	0.325	14.811	384.042	-399.178
1	5	0.725	10.530	6.761	0.363	0.298	15.127	384.740	-400.164
1	6	0.665	11.495	6.796	0.333	0.277	15.460	385.328	-401.065
1	7	0.614	12.466	6.825	0.307	0.261	15.803	385.829	-401.893
1	8	0.571	13.440	6.851	0.285	0.248	16.150	386.262	-402.659
1	9	0.532	14.418	6.873	0.266	0.237	16.497	386.638	-403.372
1	10	0.499	15.399	6.892	0.249	0.229	16.842	386.968	-404.039

Table 3: Risk averse social planning solution with initial storage of 15

The solutions as before are identical. Observe that scenario 1 is the worst-case outcome in this example. It leads to the highest system cost, as well as to the lowest profit for the hydro generator and worst welfare for the consumer. The thermal generator has highest profit in scenario 1, but, as shown above it is indifferent to risk in this model as it solves a wait-and-see model.

The risk set of the social planner (with $\lambda = 0.2$) is

$$\mathcal{D}{=}\text{conv}\{(\bar{p}, p, \ldots, p), (p, \bar{p}, \ldots, p), \ldots, (p, p, \ldots, \bar{p})\}.$$

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		0.816	12.291	6.709	0.408				
1	1	1.118	6.757	6.534	0.559	0.479	14.131	380.898	395.508
1	2	0.987	7.681	6.610	0.494	0.410	14.291	382.175	396.876
1	3	0.882	8.621	6.671	0.441	0.361	14.527	383.202	398.090
1	4	0.796	9.571	6.720	0.398	0.325	14.811	384.042	399.178
1	5	0.725	10.530	6.761	0.363	0.298	15.127	384.740	400.164
1	6	0.665	11.495	6.796	0.333	0.277	15.460	385.328	401.065
1	7	0.614	12.466	6.825	0.307	0.261	15.803	385.829	401.893
1	8	0.571	13.440	6.851	0.285	0.248	16.150	386.262	402.659
1	9	0.532	14.418	6.873	0.266	0.237	16.497	386.638	403.372
1	10	0.499	15.399	6.892	0.249	0.229	16.842	386.968	404.039

Table 4: Risk averse competitive equilibrium with initial storage of 15

where $\bar{p} = \frac{9\lambda+1}{10} = 0.28$, $p = \frac{1-\lambda}{10} = 0.02$. Since scenario 1 is the worst case, the risk-averse social planning solution is therefore the same as a risk-neutral social planning solution with adjusted probabilities

$$(\mu_1, \mu_2, \ldots, \mu_{10}) = (0.28, 0.08, \ldots, 0.08).$$

This corresponds to a risk-neutral competitive equilibrium in which all players maximize expected profit assuming these probabilities. The consumer will solve RCP, the hydro agent will solve RHP and the thermal agent will solve RTP. Therefore by Proposition 2 we get the same solutions in the social planning solution as we do in the competitive equilibrium, as confirmed by Table 3 and Table 4.

Low initial storage We now assume initial storage of 10 units, inflow 4 in period 1, and inflows of $1, 2, \ldots, 10$ with equal probability in each scenario in period 2. The risk-neutral results in equilibrium are the same as the social planning solution as predicted by Proposition 2 and demonstrated in Table 1 and Table 2. When we include risk aversion, we obtain the results shown in Table 5 and Table 6. Table 5 shows a solution to RSP when the social planner uses a risk measure ρ_0 with $\lambda = 0.2$, and Table 6 shows the solution to RCE when all agents use this measure.

Low initial storage - elastic demand The solutions above assume a consumer welfare measured by $D(d) = 40d - 2d^2$ which corresponds to a linear inverse demand function

$$P(d) = 40 - 4d.$$

We can see the effect of a more elastic inverse demand function by solving RSP and RCE using $D(d) = 20d - 0.5d^2$, still with initial storage of 10. This gives the results shown in Table 7 and Table 8. Table 7 shows a solution to RSP when the social planner uses a risk measure ρ_0 with $\lambda = 0.2$, and Table 8 shows the solution to RCE when all agents use this measure.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	cost (total)
0		1.545	7.710	6.290	0.773				
1	1	2.472	2.948	5.763	1.236	2.125	21.918	360.888	-384.931
1	2	2.004	3.682	6.028	1.002	1.601	20.968	365.307	-387.876
1	3	1.660	4.486	6.224	0.830	1.286	20.401	368.589	-390.276
1	4	1.404	5.340	6.370	0.702	1.090	20.138	371.050	-392.277
1	5	1.210	6.229	6.482	0.605	0.963	20.090	372.927	-393.980
1	6	1.060	7.142	6.568	0.530	0.878	20.189	374.390	-395.457
1	7	0.940	8.073	6.637	0.470	0.818	20.385	375.553	-396.756
1	8	0.844	9.018	6.692	0.422	0.775	20.644	376.495	-397.914
1	9	0.765	9.972	6.738	0.382	0.743	20.944	377.270	-398.957
1	10	0.699	10.934	6.776	0.349	0.719	21.267	377.919	-399.905

Table 5: Risk averse social planning solution with initial storage of 10

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		1.317	7.580	6.420	0.658				
1	1	2.545	2.858	5.722	1.272	2.053	20.280	362.407	384.740
1	2	2.057	3.582	5.998	1.029	1.492	19.277	367.002	387.771
1	3	1.700	4.378	6.202	0.850	1.156	18.664	370.413	390.233
1	4	1.434	5.226	6.353	0.717	0.948	18.366	372.965	392.279
1	5	1.233	6.111	6.469	0.616	0.814	18.295	374.908	394.017
1	6	1.077	7.022	6.558	0.539	0.724	18.378	376.418	395.520
1	7	0.955	7.951	6.629	0.477	0.661	18.564	377.615	396.840
1	8	0.856	8.894	6.686	0.428	0.617	18.816	378.582	398.015
1	9	0.775	9.847	6.733	0.387	0.584	19.111	379.377	399.071
1	10	0.707	10.808	6.772	0.353	0.559	19.432	380.040	400.031

Table 6: Risk averse competitive equilibrium with initial storage of 10

In these examples the worst case profits for the hydro and thermal producers both occur in scenario 10 when water is plentiful. By releasing large amounts of water the (elastic) price decreases to levels that erode their profits. The consumer welfare is maximized in this scenario. The worst-case overall welfare occurs in scenario 1.

In both examples with low storage one can see that the risk-averse social planning solution and the risk-averse competitive equilibrium are different. The social planning solution has highest system cost in scenario 1. In contrast the lowest hydro profit in the risk-averse competitive equilibrium is in scenario 5 in the inelastic case and scenario 10 in the elastic case. Since the hydro generator and the system do not agree on a worst-case outcome, the probability distributions that correspond to an equivalent risk neutral decision will not be common. This means that the competitive equilibrium differs from the plan maximizing total risk-adjusted welfare. We can attempt to construct some agreement on what would be the worst-case outcome by trading risk. This the subject of the next section.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	cost (total)
0		5.637	5.598	8.402	2.819				
1	1	7.917	0.851	5.746	3.959	23.614	105.874	176.144	-305.633
1	2	7.149	0.975	6.623	3.575	20.721	109.091	185.720	-315.532
1	3	6.416	1.122	7.476	3.208	18.236	110.626	195.407	-324.269
1	4	5.722	1.297	8.301	2.861	16.129	110.722	205.082	-331.932
1	5	5.069	1.507	9.090	2.535	14.369	109.636	214.610	-338.614
1	6	4.463	1.760	9.838	2.232	12.924	107.638	223.844	-344.406
1	7	3.907	2.062	10.536	1.954	11.761	105.006	232.633	-349.400
1	8	3.406	2.421	11.177	1.703	10.844	102.014	240.831	-353.689
1	9	2.961	2.844	11.754	1.480	10.136	98.914	248.314	-357.363
1	10	2.573	3.333	12.265	1.286	9.599	95.915	254.997	-360.511

Table 7: Risk averse social planning solution with elastic demand and initial storage of 10

5 Risk trading with polyhedral risk sets

We now turn our attention to the situation where each agent $a \in \mathcal{H} \cup \mathcal{T} \cup \mathcal{C}$ is endowed with a polyhedral risk set denoted

$$\mathcal{D}_a = \text{conv}\{(p_1^{a1}, p_2^{a1}, \ldots, p_M^{a1}), (p_1^{a2}, p_2^{a2}, \ldots, p_M^{a2}), \ldots, (p_1^{aK_a}, p_2^{aK_a}, \ldots, p_M^{aK_a})\}.$$

We also assume that the agents can trade risk with each other using instruments that pay a certain amount contingent on a given scenario occurring. Trading risk involves each agent a making a payment in the first stage of the amount $\sum_{m=1}^{M} \mu_m W_a(\omega_m)$ to receive contingent payments $W_a(\omega_m)$ in each outcome ω_m, $m = 1, 2, \ldots, M$. (By an abuse of notation we have denoted the price μ_m. In fact μ turns out to be a probability measure, so this notation is consistent with the previous section.) We assume that the number (M) of traded instruments equals the number of uncertain outcomes. In other words we assume that the market for risk is *complete* [15].

The contingent payments can be used to offset high disbenefits $Z(\omega_m)$ for some outcomes ω_m. The net disbenefit in outcome ω_m is then $Z(\omega_m) - W(\omega_m)$. Using (7) the risk of the net disbenefit is

$$\begin{aligned} \rho(Z - W) &= \max_{k=1,..,K} \sum_{m=1}^{M} p_m^k (Z(\omega_m) - W(\omega_m)) \\ &= \begin{cases} \min & \theta \\ \text{s.t.} & \theta \geq \sum_{m=1}^{M} p_m^k (Z(\omega_m) - W(\omega_m)), \quad k = 1, 2, \ldots, K. \end{cases} \end{aligned}$$

Given prices $(\pi_1, \pi_2(\omega_m), \mu)$ agent i solves the following optimization problem.

$$\begin{aligned}
\text{R}(i,\pi,\mu)\text{:}\ \min\quad & \theta_i + \textstyle\sum_{m=1}^M \mu_m W_i(\omega_m) \\
\text{s.t.}\quad & \theta_i + \textstyle\sum_{m=1}^M p_m^{ik} W_i(\omega_m) \geq \sum_{m=1}^M p_m^{ik} Z_i(\omega_m), \quad k = 1,\ldots,K_i, \\
& Z_i(\omega_m) = -(\pi_1 U_i(u_1(i)) + \pi_2(\omega_m) U_i(u_2(i,\omega_m)) - V_i(x_2(i,\omega_m))) \\
& (u_1(i), u_2(i,\cdot), x_1(i), x_2(i,\cdot)) \in \text{HP}(i).
\end{aligned}$$

Since $\text{R}(i,\pi,\mu)$ is a convex optimization problem, there exist multipliers λ_{ik} so that $\text{R}(i,\pi,\mu)$ is equivalent to

$$\begin{aligned}
\text{L}(i,\pi,\mu)\text{:}\ \min\quad & \theta_i + \textstyle\sum_{m=1}^M \mu_m W_i(\omega_m) \\
& + \textstyle\sum_{k=1}^{K_i} \lambda_{ik} \left(\sum_{m=1}^M p_m^{ik} Z_i(\omega_m) - \theta_i - \sum_{m=1}^M p_m^{ik} W_i(\omega_m) \right) \\
\text{s.t.}\quad & Z_i(\omega_m) = -(\pi_1 U_i(u_1(i)) + \pi_2(\omega_m) U_i(u_2(i,\omega_m)) - V_i(x_2(i,\omega_m))) \\
& (u_1(i), u_2(i,\cdot), x_1(i), x_2(i,\cdot)) \in \text{HP}(i).
\end{aligned}$$

The boundedness of the value of $\text{R}(i,\pi,\mu)$ implies that $\sum_{k=1}^{K_i} \lambda_{ik} = 1$, $\lambda_{ik} \geq 0$, and $\mu_m = \sum_{k=1}^{K_i} \lambda_{ik} p_m^{ik}$, so $\text{R}(i,\pi,\mu)$ is bounded only when $\mu \in \mathcal{D}_i$.

The prices μ in the market for risk must be the same for each agent. So thermal generator j can pay $\sum_{m=1}^M \mu_m W_j(\omega_m)$ to receive contingent payments $W_j(\omega_m)$ in each outcome ω_m. Given prices $(\pi_1, \pi_2(\omega_m), \mu)$ agent j has the following risk-averse optimization problem.

$$\begin{aligned}
\text{R}(j,\pi,\mu)\text{:}\ \min\quad & \theta_j + \textstyle\sum_{m=1}^M \mu_m W_i(\omega_m) \\
\text{s.t.}\quad & \theta_j + \textstyle\sum_{m=1}^M p_m^{jk} W_j(\omega_m) \geq \sum_{m=1}^M p_m^{jk} Z_j(\omega_m), \quad k = 1,\ldots,K_j, \\
& Z_j(\omega_m) = -\pi_1 v_1(j) + C_j(v_1(j)) - \pi_2(\omega_m) v_2(j,\omega_m) + C_j(v_2(j,\omega_m)) \\
& (v_1(j), v_2(j,\cdot)) \in \text{TP}(j).
\end{aligned}$$

Similarly to the above, we can show that $\text{R}(j,\pi,\mu)$ is bounded only if $\mu \in \mathcal{D}_j$. Finally suppose consumer c can pay $\sum_{m=1}^M \mu_m W_c(\omega_m)$ to receive contingent payments $W_c(\omega_m)$ in each outcome ω_m. Their risk-averse problem given $(\pi_1, \pi_2(\omega_m), \mu)$ is the following.

$$\begin{aligned}
\text{R}(c,\pi,\mu)\text{:}\ \min\quad & \theta_c + \textstyle\sum_{m=1}^M \mu_m W_c(\omega_m) \\
\text{s.t.}\quad & \theta_c + \textstyle\sum_{m=1}^M p_m^{ck} W_c(\omega_m) \geq \sum_{m=1}^M p_m^{ck} Z_c(\omega_m), \quad k = 1,\ldots,K_j, \\
& Z_c(\omega_m) = \pi_1 d_1(c) - D_c(d_1(c)) + \pi_2(\omega) d_2(c) - D_c(d_2(c)) \\
& (d_1(c), d_2(c,\cdot)) \in \text{CP}(c).
\end{aligned}$$

As before $\mathrm{R}(c, \pi, \mu)$ is bounded only if $\mu \in \mathcal{D}_c$. The payoffs from risk trading cannot be net positive in each outcome, so

$$-\sum_{i \in \mathcal{H}} W_i(\omega_m) - \sum_{j \in \mathcal{T}} W_j(\omega_m) - \sum_{c \in \mathcal{C}} W_c(\omega_m) \geq 0. \tag{8}$$

In other words the agents cannot swap positions in such a way to create extra benefit in any scenario. If there is no opportunity at the margin for decreasing risk from the trades W then the price μ_m will be zero. A risk-averse competitive equilibrium with risk trading then solves the variational problem:

$$\begin{array}{lll}
\text{RCET:} & (\theta_i, u_1(i), u_2(i,\omega_m), x_2(i,\omega_m), Z_i(\omega_m), W_i(\omega_m)) \in \arg\min \mathrm{R}(i,\pi,\mu), & i \in \mathcal{H}, \\
& (\theta_j, v_1(j), v_2(j,\omega_m), Z_j(\omega_m), W_j(\omega_m)) \in \arg\min \mathrm{R}(j,\pi,\mu), & j \in \mathcal{T}, \\
& (\theta_c, d_1(c), d_2(c,\omega_m), Z_c(\omega_m), W_c(\omega_m)) \in \arg\min \mathrm{R}(c,\pi,\mu), & c \in \mathcal{C}, \\
& & \\
& 0 \leq -\sum_{i \in \mathcal{H}} W_i(\omega_m) - \sum_{j \in \mathcal{T}} W_j(\omega_m) - \sum_{c \in \mathcal{C}} W_c(\omega_m) \perp \mu_m \geq 0, & m = 1, \ldots, M, \\
& & \\
& 0 \leq \sum_{i \in \mathcal{H}} U_i\left(u_1(i)\right) + \sum_{j \in \mathcal{T}} v_1(j) - \sum_{c \in \mathcal{C}} d_1(c) \perp \pi_1 \geq 0, & \\
& & \\
& 0 \leq \sum_{i \in \mathcal{H}} U_i\left(u_2(i,\omega_m)\right) & \\
& \quad + \sum_{j \in \mathcal{T}} v_2(j,\omega_m) - \sum_{c \in \mathcal{C}} d_2(c,\omega_m) \perp \pi_2(\omega_m) \geq 0, & m = 1, \ldots, M.
\end{array}$$

Clearly, any solution to RCET requires that $\mu \in \mathcal{D}_0 = (\cap_{i \in \mathcal{H}} \mathcal{D}_i) \cap (\cap_{j \in \mathcal{T}} \mathcal{D}_j) \cap (\cap_{c \in \mathcal{C}} \mathcal{D}_c)$.

Now, given the set $\mathcal{D}_0$, consider solving the risk averse social planning problem

$$\begin{array}{lll}
\text{RSP:} & \min & \sum_{j \in \mathcal{T}} C_j(v_1(j)) - \sum_{c \in \mathcal{C}} D_c(d_1(c)) \\
& & + \max_{\mu \in \mathcal{D}_0} \sum_{m=1}^{M} \mu_m [\sum_{j \in \mathcal{T}} C_j(v_2(j,\omega_m)) - \sum_{c \in \mathcal{C}} D_c(d_2(c,\omega_m)) - \sum_{i \in \mathcal{H}} V_i(x_2(i,\omega_m))] \\
& & \\
& \text{s.t.} & \sum_{i \in \mathcal{H}} U_i\left(u_1(i)\right) + \sum_{j \in \mathcal{T}} v_1(j) - \sum_{c \in \mathcal{C}} d_1(c) \geq 0, \\
& & \sum_{i \in \mathcal{H}} U_i\left(u_2(i,\omega_m)\right) + \sum_{j \in \mathcal{T}} v_2(j,\omega_m) - \sum_{c \in \mathcal{C}} d_2(c,\omega_m) \geq 0, \quad m = 1, 2, \ldots, M, \\
& & (u_1(i), u_2(i,\omega_m), x_1(i), x_2(i,\omega_m)) \in \mathrm{HP}(i,\mu), \quad i \in \mathcal{H}, \\
& & (v_1(j), v_2(j,\omega_m)) \in \mathrm{TP}(j,\mu), \quad j \in \mathcal{T}, \\
& & (d_1(c), d_2(c,\omega_m)) \in \mathrm{CP}(c,\mu), \quad c \in \mathcal{C},
\end{array}$$

This seeks a plan of generation that maximizes the total welfare of the agents by minimizing total fuel cost and value of unserved load. Let the optimal solution to RSP be $(u^*, v^*, x^*, d^*, \mu^*)$, and let the shadow prices of the market clearing constraints be π_1^*, and $\mu_m^* \pi_2^*(\omega_m)$. We know that (u^*, v^*, x^*, d^*) solves $\mathrm{SP}(\mu^*)$, and so $(u^*, v^*, x^*, d^*, \pi_1^*, \pi_2^*(\omega_m))$ is a solution to $\mathrm{CE}(\mu^*)$. We proceed to show that this defines a solution to RCET.

Consider fixing the solution $(u^*, v^*, x^*, d^*, \pi^*)$. This defines disbenefits for each agent in each scenario by

$$Z_i^*(\omega_m) = -(\pi_1^* U_i(u_1^*(i)) + \pi_2^*(\omega_m) U_i(u_2^*(i,\omega_m)) + V_i(x_2^*(i,\omega_m))) \tag{9}$$

$$Z_j^*(\omega_m) = -\pi_1^* v_1^*(i) - \pi_2^*(\omega_m) v_2^*(i,\omega_m) + C_j(v_1^*(j)) + C_j(v_2^*(j,\omega_m)) \tag{10}$$

$$Z_c^*(\omega_m) = \pi_1^* d_1^*(c) + \pi_2^*(\omega_m) d_2^*(c,\omega_m) - D_c(d_1^*(c)) - D_c(d_2^*(c,\omega_m)). \tag{11}$$

The total disbenefit of this solution in scenario ω_m is

$$\begin{aligned} & \sum_{i\in\mathcal{H}} Z_i^*(\omega_m) + \sum_{j\in\mathcal{T}} Z_j^*(\omega_m) + \sum_{c\in\mathcal{C}} Z_c^*(\omega_m) \\ = \;& C_j(v_1^*(j)) + C_j(v_2^*(j,\omega_m)) - D_c(d_1^*(c)) - D_c(d_2^*(c,\omega_m)) - V_i(x_2^*(i,\omega_m)), \end{aligned}$$

because π^* satisfies (5) and (6). The optimal value of RSP is

$$\begin{aligned} & \mathbb{E}_{\mu^*}\left[C_j(v_1^*(j)) + C_j(v_2^*(j,\omega_m)) - D_c(d_1^*(c)) - D_c(d_2^*(c,\omega_m)) - V_i(x_2^*(i,\omega_m))\right] \\ = \;& \max_{\mu\in\mathcal{D}_0} \mathbb{E}_{\mu}\left[C_j(v_1^*(j)) + C_j(v_2^*(j,\omega_m)) - D_c(d_1^*(c)) - D_c(d_2^*(c,\omega_m)) - V_i(x_2^*(i,\omega_m))\right] \\ = \;& \max_{\mu\in\mathcal{D}_0} \sum_{m=1}^{M} \mu_m\Big(\sum_{i\in\mathcal{H}} Z_i^*(\omega_m) + \sum_{j\in\mathcal{T}} Z_j^*(\omega_m) + \sum_{c\in\mathcal{C}} Z_c^*(\omega_m)\Big). \end{aligned}$$

For fixed $(u^*, v^*, x^*, d^*, \pi^*)$ (and hence $Z_i^*(\omega_m), Z_j^*(\omega_m), Z_c^*(\omega_m)$) let us define the problem

$$\text{DTP:}\quad \max_{\mu\in\mathcal{D}_0} \quad \textstyle\sum_{m=1}^{M} \mu_m\big(\sum_{i\in\mathcal{H}} Z_i^*(\omega_m) + \sum_{j\in\mathcal{T}} Z_j^*(\omega_m) + \sum_{c\in\mathcal{C}} Z_c^*(\omega_m)\big),$$

that we have just shown has the same optimal solution and value as RSP. Now consider the trading problem:

$$\begin{aligned} \text{RTP:}\quad & \min \quad \textstyle\sum_{i\in\mathcal{H}} \theta_i + \sum_{j\in\mathcal{T}} \theta_j + \sum_{c\in\mathcal{C}} \theta_c \\ \text{s.t.}\quad & -\textstyle\sum_{i\in\mathcal{H}} W_i(\omega_m) - \sum_{j\in\mathcal{T}} W_j(\omega_m) - \sum_{c\in\mathcal{C}} W_c(\omega_m) \geq 0, \qquad [\mu_m] \\ & \theta_i + \textstyle\sum_{m=1}^{M} p_m^{ik} W_i(\omega_m) \geq \sum_{m=1}^{M} p_m^{ik} Z_i^*(\omega_m), \quad k \leq K_i,\ i\in\mathcal{H}, \\ & \theta_j + \textstyle\sum_{m=1}^{M} p_m^{jk} W_j(\omega_m) \geq \sum_{m=1}^{M} p_m^{jk} Z_j^*(\omega_m), \quad k \leq K_j,\ j\in\mathcal{T}, \\ & \theta_c + \textstyle\sum_{m=1}^{M} p_m^{ck} W_c(\omega_m) \geq \sum_{m=1}^{M} p_m^{ck} Z_c^*(\omega_m), \quad k \leq K_c,\ c\in\mathcal{C}. \end{aligned}$$

The linear programming dual of RTP is

$$\begin{aligned} \text{RTP}^*:\quad & \max \quad \textstyle\sum_{m=1}^{M} \sum_{i\in\mathcal{H}} \left(\sum_{k=1}^{K_i} \lambda_{ik} p_m^{ik}\right) Z_i^*(\omega_m) \\ & + \textstyle\sum_{m=1}^{M} \sum_{j\in\mathcal{T}} \left(\sum_{k=1}^{K_j} \sigma_{jk} p_m^{jk}\right) Z_j^*(\omega_m) \\ & + \textstyle\sum_{m=1}^{M} \sum_{j\in\mathcal{C}} \left(\sum_{k=1}^{K_c} \eta_{ck} p_m^{jk}\right) Z_c^*(\omega_m) \\ \text{s.t.}\quad & \textstyle\sum_{k=1}^{K_i} \lambda_{ik} p_m^{ik} - \mu_m = 0, \\ & \textstyle\sum_{k=1}^{K_j} \sigma_{jk} p_m^{jk} - \mu_m = 0, \\ & \textstyle\sum_{k=1}^{K_c} \eta_{ck} p_m^{ck} - \mu_m = 0, \\ & \textstyle\sum_{k=1}^{K_i} \lambda_{ik} = 1, \quad \lambda_{ik} \geq 0 \qquad i\in\mathcal{H} \\ & \textstyle\sum_{k=1}^{K_j} \sigma_{jk} = 1, \quad \sigma_{jk} \geq 0 \qquad j\in\mathcal{T} \\ & \textstyle\sum_{k=1}^{K_c} \eta_{ck} = 1, \quad \eta_{ck} \geq 0 \qquad c\in\mathcal{C} \end{aligned}$$

which is easily seen to be equivalent to DTP, when μ_m is substituted for $\sum_{k=1}^{K_i} \lambda_{ik} p_m^{ik}$, $\sum_{k=1}^{K_j} \sigma_{jk} p_m^{jk}$, and $\sum_{k=1}^{K_c} \eta_{ck} p_m^{ck}$.

By the duality theorem of linear programming this means that there exists an optimal solution to RTP with the same value as DTP. Thus RTP has a solution $(\theta_i, \theta_j, \theta_c, W_i, W_j, W_c)$ that clears the market for risk trading, so for each $m = 1, 2, \ldots, M$,

$$0 \leq -\sum_{i \in \mathcal{H}} W_i(\omega_m) - \sum_{j \in \mathcal{T}} W_j(\omega_m) - \sum_{c \in \mathcal{C}} W_c(\omega_m) \perp \mu_m \geq 0,$$

and has the same value as DTP (and therefore RSP). With prices μ_m, such a solution to RTP minimizes a risk-trading problem for each agent, namely (θ, W) solves

$$\begin{aligned}
\text{R}(i,\mu):\ \min\quad & \theta_i + \textstyle\sum_{m=1}^{M} \mu_m W_i(\omega_m) \\
\text{s.t.}\quad & \theta_i + \textstyle\sum_{m=1}^{M} p_m^{ik} W_i(\omega_m) \geq \sum_{m=1}^{M} p_m^{ik} Z_i^*(\omega_m), \quad k = 1, \ldots, K_i, \\
\text{R}(j,\mu):\ \min\quad & \theta_j + \textstyle\sum_{m=1}^{M} \mu_m W_j(\omega_m) \\
\text{s.t.}\quad & \theta_j + \textstyle\sum_{m=1}^{M} p_m^{jk} W_j(\omega_m) \geq \sum_{m=1}^{M} p_m^{jk} Z_j^*(\omega_m), \quad k = 1, \ldots, K_j, \\
\text{R}(c,\mu):\ \min\quad & \theta_c + \textstyle\sum_{m=1}^{M} \mu_m W_c(\omega_m) \\
\text{s.t.}\quad & \theta_c + \textstyle\sum_{m=1}^{M} p_m^{ck} W_c(\omega_m) \geq \sum_{m=1}^{M} p_m^{ck} Z_c^*(\omega_m), \quad k = 1, \ldots, K_j.
\end{aligned}$$

This gives the following result.

Proposition 6 *Suppose the market for risk trading is complete, and $\mathcal{D}_0 = (\cap_{i \in \mathcal{H}} \mathcal{D}_i) \cap (\cap_{j \in \mathcal{T}} \mathcal{D}_j) \cap (\cap_{c \in \mathcal{C}} \mathcal{D}_c)$ is nonempty. Let the optimal solution to RSP be $(u^*, v^*, x^*, d^*, \mu^*)$, and let the shadow prices of the market clearing constraints be π_1^*, and $\mu_m^* \pi_2^*(\omega_m)$. Finally let Z^* denote the vector with components*

$$\begin{aligned}
Z_i^*(\omega_m) &= -(\pi_1^* U_i(u_1^*(i)) + \pi_2^*(\omega_m) U_i(u_2^*(i, \omega_m)) - V_i(x_2^*(i, \omega_m))) \\
Z_j^*(\omega_m) &= -\pi_1^* v_1^*(i) - \pi_2^*(\omega_m) v_2^*(i, \omega_m) + C_j(v_1^*(j)) + C_j(v_2^*(j, \omega_m)) \\
Z_c^*(\omega_m) &= \pi_1^* d_1^*(c) + \pi_2^*(\omega_m) d_2^*(c, \omega_m) - D_c(d_1^*(c)) - D_c(d_2^*(c, \omega_m)).
\end{aligned}$$

Then there exists $W^* = (W_i^*, W_j^*, W_c^*)$ so that $(u^*, v^*, x^*, d^*, \mu^*, Z^*, W^*, \pi^*)$ solves RCET.

The proposition shows that under an assumption of a complete market for risk, we may construct a competitive equilibrium with risk trading from a social planning solution. This entails identifying the set $\mathcal{D}_0$ defined by the intersection of the agents' risk sets. The trading in risk to give this equilbrium is not unique, since if $(\theta_i, \bar{W}_i(\omega_m))$ is feasible for $\text{R}(i, \mu^*)$, then so is $(\theta_i + 1, \bar{W}_i(\omega_m) - 1)$ with the same objective. In other words we can add a constant a to every payout from the risk contract, and improve the risked position θ, as long as we pay a back in stage 1 with the contract payment $\sum_{m=1}^{M} \mu_m a = a$. Thus there is a linear manifold of possible risk trades that will solve RCET.

5.1 Examples

The risk-trading analysis of the previous section can be applied to our example problem. Suppose we first assume that the thermal generator is risk-neutral (i.e. $\lambda = 0$). This seems reasonable without risk trading since given electricity prices he will act the same irrespective of his choice of λ. (The same is true of the consumer, although we need only one risk-neutral agent for the following example.) Any risk neutral agent has risk set

$$\mathcal{D}= \{(0.1, 0.1, \ldots, 0.1)\}.$$

It follows that the intersection of $\mathcal{D}$ and the other agent's risk sets (which contain $(0.1, 0.1, \ldots, 0.1)$) is $\mathcal{D}_0 = \mathcal{D}$. A complete market for trading risk will then result in an equilibrium that has risk measure

$$\sup_{\mu \in \mathcal{D}_0} \mathbb{E}_\mu[\sum_{i=1}^{N} Z_i] = \mathbb{E}[\sum_{i=1}^{N} Z_i].$$

Adding risk trading to the risked competitive equilibrium will then give the optimal risk-neutral social planning solution.

To illustrate this, suppose that the initial storage in the hydro reservoir is 10. Table 9 shows the competitive equilibrium when risk trading is allowed in a complete market. Since the thermal generator is risk neutral, the intersection of risk sets for the agents is the singleton $\{0.1, 0.1, \ldots, 0.1\}$. Agents in a competitive equilibrium with risk trading will optimize using the worst-case measure in this set. Thus the solution shown in Table 9 is the same as the risk neutral competitive equilibrium shown in Table 1 (which is reproduced for convenience as Table 10).

This result depends on the assumption that $\lambda = 0$ for the thermal generator. In section 3, where risk trading was not included, we argued that the thermal generator solves a wait-and-see optimization problem, and so he is indifferent to the choice of λ. We can then assume $\lambda = 0$ for the thermal generator with no loss in generality. However, once the thermal plant can trade risk the choice of λ makes a difference to his actions.

To see this consider the equilibrium with risk trading where both agents choose risk measures with $\lambda = 0.2$. The equilibrium solution is shown in Table 11. Observe that the risk-averse competitive equilibrium differs from the risk-neutral social planning solution in Table 10. The thermal agent, faced with the possibility of trading risk, is no longer indifferent to his choice of λ and now wishes to reduce his exposure to low profits. In the absence of risk trading he could not change his exposure by any actions at all, and so we could assume that he was risk neutral. Now we assume that he is endowed with the same risk measure as the hydro generator. The intersection of the risk sets of all three agents is non-empty, but is not a singleton: it is the risk set shared by each agent.

The risk trading that occurs is shown in Table 12. Risk trading produces the equilibrium shown in Table 11 that corresponds to a social planning solution that maximizes total expected profit with the worst-case probability distribution in this risk set. We can verify this by examining the total welfare of all agents in the risk-averse equilibrium. This is shown in the last column of Table 11. The smallest welfare occurs in scenario 1. This is the riskiest from the perspective of all agents objectives summed together, and they trade risk to give a minimum risk solution for the sum of their positions. The probability distribution in the intersection of risk sets that corresponds to this equilibrium assigns 0.28 to scenario 1 and 0.08 to the other scenarios.

The social planning solution that maximizes total social welfare with this risk set is the same as the the risked equilibrium with risk trading. The risk-neutral social planning solution computed with the probabilities defined by the risk price is shown in Table 13. One can observe that the generation levels are the same in each solution but the agents profits differ from those in the equilibrium with trading. Their total is however the same as the total welfare in the equilibrium with trading. The trades have enabled each agent to use the same risk measure, and agree on a worst-case probability distribution.

6 Conclusions

The comparison of competitive market equilibrium with social planning solutions is not straightforward when these markets involve uncertain inflows into hydro reservoirs. Even in the risk neutral setting we need to ensure that there are sufficiently many traded instruments (pricing water exchanges between reservoirs on the same river chain for example) to make a competitive equilibrium coincide with a social planning solution. We have presented a simple class of models for which this result is true in the risk-neutral case.

In a setting with risk-averse agents, further sets of traded instruments are needed to ensure that a social planning solution and a risked equilibrium coincide. Although our results are restricted to a two-stage model, they extend naturally to a multi-stage stochastic equilibrium. The conditional risk measures in this case need to be time-consistent, and so have a nested structure as described by [17]. When the inflow distributions have a finite number (M) of outcomes per stage then this enables a social planning solution to be computed using the scenario-tree methods outlined in [13] (which are really only tractable when the inflow distributions are stagewise independent). We assume that in each possible node of the scenario tree there are sufficient risk trading instruments to cover the M positions of each agent at that stage. A multi-stage competitive equilibrium with different risk trades in each node of the scenario tree will then correspond to a risk-averse social planning solution in this scenario tree.

Even allowing for stagewise independence, finding the appropriate social planning problem to solve is not straightforward since it requires that the social planner have knowledge of the risk sets of each agent (which is private information). However we show that given this information, the planner can in principle solve a risk-averse dynamic optimization problem with an appropriate coherent risk measure (using e.g. the methods discussed in [13]) to yield a stochastic process of energy prices that correspond to the outcomes of a competitive equilibrium with risk trading.

This result raises some interesting questions for regulators who are seeking competitive benchmarks with which to monitor the competiveness of electricity markets with hydro-electric reservoirs. A risk-neutral social planning solution is likely to incur some energy shortages and corresponding high prices, as these will happen occasionally to minimize the expected cost. A hydro-thermal market that avoids these shortages is preferable, but prices in periods without shortages will incur risk premiums that are often attributed to unilateral exercise of market power by hydro-generators. The models we develop in this paper are a first step towards estimating perfectly competitive risk premia for these markets, and will assist regulators to diagnose strategic behaviour by generators.

A further question raised by this work is the effect on hydro-firming investment. This requires high prices in dry periods to cover its long-run marginal cost of supply. This raises the possibility of devising an investment model that incorporates risk premia from a risk-averse competitive hydro-thermal model to cover these costs. This would provide an interesting comparison to observed investment in hydro firming plant.

Finally we remark that the RCET models we solve assume market completeness, which is unrealistic in practice. However in many circumstances, these models admit equilibrium solutions in incomplete markets as well (as we have demonstrated in solving RCE). This provides regulators and market analysts with a methodology to test the welfare gains that might be realized by introducing practical hedging instruments into markets in which these are absent.

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stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		5.004	4.829	9.171	2.502				
1	1	8.530	0.770	5.060	4.265	24.448	99.555	178.230	302.233
1	2	7.736	0.878	5.951	3.868	21.222	104.221	187.644	313.087
1	3	6.976	1.007	6.823	3.488	18.426	107.028	197.256	322.710
1	4	6.252	1.160	7.670	3.126	16.031	108.208	206.951	331.190
1	5	5.567	1.342	8.487	2.783	14.006	108.007	216.605	338.619
1	6	4.925	1.562	9.267	2.462	12.322	106.686	226.079	345.086
1	7	4.330	1.825	10.004	2.165	10.946	104.516	235.223	350.685
1	8	3.786	2.140	10.690	1.893	9.843	101.777	243.888	355.508
1	9	3.298	2.513	11.316	1.649	8.978	98.739	251.930	359.646
1	10	2.866	2.951	11.878	1.433	8.312	95.645	259.234	363.191

Table 8: Risk averse competitive equilibrium with elastic demand and initial storage of 10

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		1.336	7.590	6.410	0.668				
1	1	2.539	2.865	5.725	1.269	-8.123	19.067	373.814	384.758
1	2	2.053	3.590	6.000	1.027	-5.100	19.067	373.814	387.781
1	3	1.696	4.387	6.203	0.848	-2.643	19.067	373.814	390.238
1	4	1.431	5.236	6.355	0.716	-0.600	19.067	373.814	392.281
1	5	1.231	6.121	6.470	0.616	1.135	19.067	373.814	394.016
1	6	1.076	7.031	6.559	0.538	2.635	19.067	373.814	395.516
1	7	0.953	7.961	6.629	0.477	3.954	19.067	373.814	396.835
1	8	0.855	8.904	6.686	0.427	5.127	19.067	373.814	398.008
1	9	0.774	9.857	6.733	0.387	6.183	19.067	373.814	399.064
1	10	0.706	10.818	6.772	0.353	7.142	19.067	373.814	400.023

Table 9: Risk averse competitive equilibrium with initial storage of 10 and risk trading with $\lambda = 0$ for the thermal generator. The equilibrium price of risk is $P(\omega) = 0.1$.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		1.336	7.590	6.410	0.668				
1	1	2.539	2.865	5.725	1.269	2.057	20.417	362.283	384.758
1	2	2.053	3.590	6.000	1.027	1.500	19.418	366.863	387.781
1	3	1.696	4.387	6.203	0.848	1.165	18.809	370.264	390.238
1	4	1.431	5.236	6.355	0.716	0.958	18.514	372.809	392.281
1	5	1.231	6.121	6.470	0.616	0.825	18.445	374.746	394.016
1	6	1.076	7.031	6.559	0.538	0.735	18.529	376.252	395.516
1	7	0.953	7.961	6.629	0.477	0.673	18.716	377.446	396.835
1	8	0.855	8.904	6.686	0.427	0.629	18.969	378.411	398.008
1	9	0.774	9.857	6.733	0.387	0.596	19.264	379.204	399.064
1	10	0.706	10.818	6.772	0.353	0.571	19.585	379.866	400.022

Table 10: Risk neutral competitive equilibrium with initial storage of 10.

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	welfare (total)
0		1.545	7.710	6.290	0.773				
1	1	2.472	2.948	5.763	1.236	-1.232	18.320	367.842	384.931
1	2	2.004	3.682	6.028	1.002	-0.039	19.568	368.347	387.876
1	3	1.660	4.486	6.224	0.830	0.700	20.309	369.267	390.276
1	4	1.404	5.340	6.370	0.702	1.405	21.045	369.826	392.277
1	5	1.210	6.229	6.482	0.605	1.999	21.663	370.319	393.980
1	6	1.060	7.142	6.568	0.530	2.510	22.189	370.758	395.457
1	7	0.940	8.073	6.637	0.470	2.956	22.647	371.153	396.756
1	8	0.844	9.018	6.692	0.422	3.353	23.050	371.511	397.914
1	9	0.765	9.972	6.738	0.382	3.708	23.410	371.838	398.957
1	10	0.699	10.934	6.776	0.349	4.031	23.735	372.139	399.905

Table 11: Risk-averse competitive equilibrium with risk trading.

stage	ω_m	price	trade (T)	trade (H)	trade (C)
0					
1	1	0.280	1.658	0.768	-2.426
1	2	0.080	3.375	2.966	-6.341
1	3	0.080	4.429	4.274	-8.703
1	4	0.080	5.330	5.274	-10.604
1	5	0.080	6.051	5.938	-11.989
1	6	0.080	6.647	6.366	-13.013
1	7	0.080	7.153	6.627	-13.781
1	8	0.080	7.593	6.772	-14.364
1	9	0.080	7.980	6.832	-14.813
1	10	0.000	8.327	6.834	-15.161

Table 12: Risk trading between three agents in equilibrium

stage	ω_m	price	storage	release	thermal	profit (T)	profit (H)	welfare (C)	cost (total)
0		1.545	7.710	6.290	0.773				
1	1	2.472	2.948	5.763	1.236	2.125	21.918	360.888	-384.931
1	2	2.004	3.682	6.028	1.002	1.601	20.968	365.307	-387.876
1	3	1.660	4.486	6.224	0.830	1.286	20.401	368.589	-390.276
1	4	1.404	5.340	6.370	0.702	1.090	20.138	371.050	-392.277
1	5	1.210	6.229	6.482	0.605	0.963	20.090	372.927	-393.980
1	6	1.060	7.142	6.568	0.530	0.878	20.189	374.390	-395.457
1	7	0.940	8.073	6.637	0.470	0.818	20.385	375.553	-396.756
1	8	0.844	9.018	6.692	0.422	0.775	20.644	376.495	-397.914
1	9	0.765	9.972	6.738	0.382	0.743	20.944	377.270	-398.957
1	10	0.699	10.934	6.776	0.349	0.719	21.267	377.919	-399.905

Table 13: Risk-neutral social planning solution with probabilities defined by risk prices