Random sets

$C$ ‘covered’ by countable selections

Castaing representation

\[ a.s \text{ convergence: } P \{ \xi \mid \| C(\nu)(\xi) - C(\xi) \| \to 0 \} = 0 \]

\[ \Rightarrow \text{ in probability: } \forall \varepsilon > 0, P \{ \xi \mid \| C(\nu)(\xi) - C(\xi) \| > \varepsilon \} \to 0 \]

\[ \Rightarrow \text{ in distribution } T : \text{cptc-sets}(\mathbb{R}^n) \to [0, 1], T(\emptyset) = 0, \]

\[ (a) \; T(K^{\nu}) \downarrow T(K) \text{ for } K^{\nu} \downarrow K, \; (b) \; \text{‘rectangle cond’n’} \]

\[ P^{\nu} \rightharpoonup P \iff T^{\nu} \to T \text{ on cptc-sets}(\mathbb{R}^n) \]

or, even, on finite union of closed rational balls.
Random set: Expectation

\[ EC = \mathbb{E}\left\{ C(\xi) \right\} = \left\{ \int_{\Xi} s(\xi) \mu(d\xi) \right\} \text{s.t. } s(\cdot) \text{ P-summable selection} \]

..not necessarily closed even when \( C \) is closed-valued

Convexity:

\( C \) P-atom convex \( \Rightarrow EC \) is convex

(certainly when \( P \) is atomless).

\[ (1,1) \quad \text{prob}=1/3 \]

\[ (-1,-1) \quad \text{prob}=1/3 \]

\[ (2,2) \quad \text{prob}=1/3 \]

\[ (-2,2) \quad \text{prob}=1/3 \]
Expectation: Bounded random set

Random Sets
equal prob. 0.2

EC: $c(0.41, 0.21), r = 0.502$

EC: $c(-1.3, r = 0.5)$

EC: $c(1.1, r = 1)$

EC: $c(0.8, r = 0.25)$

EC: $c(3, -2), r = 1$
Expectation: Unbounded random sets

ray $C(\xi^1)$
prob. $p_1 > 0$

ray $C(\xi^2)$
prob. $1 - p_1$

$EC$
Some properties: $\mathbb{E}\{C(\xi)\}$

• measure $P$ atomless, then $EC = \mathbb{E}\{C(\xi)\}$ is convex (Richter, Lyapounov,...)

• $P$ is $P$-atom convex $\implies EC$ is convex; [an atom contains no (measurable) subset of positive probability]

• $C$ a random set, $\emptyset \neq EC = \mathbb{E}\{C(\xi)\}$ contains no line, then

$$\text{con } EC = \mathbb{E}\{\text{con } X(\xi)\}$$

this essentially requires that $C(\xi) \subset$ a pointed cone

• in general, the expectation of a (closed-valued) random set is not closed

• if $|C| = \mathbb{E}\{\sup |s(\xi)||s(\xi) \in C(\xi)\} < \infty$ then $EC$ is closed; $C$ is then integrably bounded.
Strong law of large numbers
for random sets  (Artstein-Hart)

\[ C : \Xi \Rightarrow \mathbb{R}^m \text{ measurable, } \{\xi^v, v \in \mathbb{N}\} \text{ iid } \Xi\text{-valued random variables} \]

\[ C(\xi^v) \text{ iid random sets (i.e. induced } P^v \text{ independent and identical)} \]

\[ EC = \mathbb{E}\{C(\cdot)\} = \left\{ \int_{\Xi} s(\xi) P(d\xi) \mid s : P\text{-summable } C(\xi)\text{-selection} \right\} \]

independence  \Rightarrow  \text{ all (measurable) selections are independent}

\[ \left\{ C(\xi^v) : \Xi \Rightarrow \mathbb{R}^m, v \in \mathbb{N} \right\} \text{ iid with } EC \neq \emptyset. \text{ Then, with} \]

\[ C^v(\xi^\infty) = v^{-1}\left( \sum_{k=1}^{v} C(\xi^k) \right) \rightarrow \bar{C} = \text{cl con } EC \text{ } P^\infty\text{-a.s.} \]

Lo$_v C^v(\xi^\infty) \subset \bar{C} \iff \limsup_v \sigma_{C^v} \leq \sigma_{\bar{C}}$ support functions

Li$_v C^v(\xi^\infty) \supset \bar{C}$ relies on LLN for (vector-valued) selections  

\[ \text{Proof: time allowing} \]
Resources allocations ⇒ Average of epi-sums

\[ w \in \mathbb{R}^n_+, \ w \text{ central resources to allocate to } \nu \text{ firms} \]
suppose \( \nu \) is relatively large ⇒ expected optimal allocation?
\[ r_i \text{ production functions, } r_i = r_i(\xi, x) \text{ with } \xi \in \Xi, \]
\[ \forall \xi : \ z_\nu(\xi, q) = \min -\nu^{-1} \sum_{i=1}^{\nu} r_i(\xi, x^i) \] s.t. \[ \nu^{-1} \sum_{i=1}^{k} x^i \leq w \]
\((\Xi, A, \mu), -r_i : \text{lsc in } x, \text{jointly measurable } A \otimes B\)
"Limit" Problem (as \( \nu \to \infty \))
\[ z(q) = \max -\int r(\xi, x(\xi)) dP \] s.t. \[ \int x(\xi) dP \leq w \]
Suppose \( \{-r_i(\xi, \cdot) \in \text{lsc-fcns}(\mathbb{R}^n)\} \) are iid ⇔ epi \( -r_i \) iid
Then, \( z_\nu(\xi, q) \to z(q) \) \( P \)-a.s. where \( r = r_i \) if \( P \) atomless
or \( -r \) is \( P \)-atom convex

Use LLN on epigraphs : limit of sums of 'finite' # of epigraphs
Random mappings

\( S : \mathbb{E} \times E \Rightarrow \mathbb{R}^m, \; E \subset \mathbb{R}^n \)

\( A \otimes \mathcal{B}^n \)-jointly measurable: \( S^{-1}(O) \in A \otimes \mathcal{B}^n \), \( O \) open

\[ \Rightarrow \forall \; x : \xi \mapsto S(\xi, x) \; \text{a random set} \]

random closed set when \( S \) is closed-valued

\( ES : E \Rightarrow \mathbb{R}^m \) with \( ES(x) = \mathbb{E}\{S(\xi, x)\} \) expected mapping

\( ES \) convex-valued when \( \xi \mapsto S(\xi, \cdot) \) \( P \)-atom convex

Law of Large Numbers for random sets applies
Sample Average Approximation (SAA)

stochastic variational problem: \( \bar{S}(x) = \mathbb{E}\{S(\xi, x)\} \geq 0 \)

\( S : \Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m \) random set-valued mapping

\( \xi \) random vector with values \( \xi \in \Xi \subset \mathbb{R}^N \)

solution (a 'stationary point') \( \bar{x} \in \bar{S}^{-1}(0) \)

sample \( \xi^\nu = (\xi^1, \ldots, \xi^\nu) \) of \( \xi \)

\[
\frac{1}{\nu} \left( \sum_{k=1}^{\nu} S(\xi^k, x) \right) = S^\nu(\bar{\xi}, x) \geq 0, \text{ approximating system?}
\]

i.e., \( (S^\nu)^{-1}(0) \rightarrow \bar{S}^{-1}(0) \) a.s.
$w \in T_C(x)$, tangent to $C$ at $x \in C$, if $\left| x^\nu - x \right| / \tau_v \rightarrow w$ for $x^\nu \rightarrow x, \tau_v \searrow 0$
Variational Geometry

Normal Cone

\[ \nu \in \hat{N}_C(x), \text{ regular normal at } x \in C, \text{ if } \langle \nu, x - \bar{x} \rangle \leq o(l x - \bar{x} l), \forall x \in C \]

\[ \nu \in N_C(\bar{x}), \text{ normal at } \bar{x} \in C, \text{ if } \exists x^\nu \rightarrow x \text{ and } \nu^\nu \rightarrow \nu \text{ with } \nu^\nu \in \hat{N}_C(x^\nu) \]

normal cones: closed cones, \( \hat{N}_C(\bar{x}) \) convex
Clarke regularity

C Clarke regular at \( \bar{x} \) if \( C \) locally closed & \( N_C(x) = \hat{N}_C(\bar{x}) \)

which implies \( N_C(\bar{x}) \) is convex if \( C \) regular at \( \bar{x} \)

In general, \( N_C(\bar{x}) = \text{Lo}_{x \to \bar{x}} \hat{N}_C(x) \supset \hat{N}_C(\bar{x}) \)

Smooth manifolds and closed convex set are regular
Subgradients

\[ N_E(x, f(x)) \]

\[ E = \text{epi } f \]

\[ (\bar{x}, f(\bar{x})) \]

\[ (v, -1) : v \in \partial f(\bar{x}) \]

\[ (v, -0) : v \in \partial_\infty f(x) \]

\[ (v, -1) : v \in \partial f(x) \]
SUBGRADIENTS

\[ \nabla f(x) = \left\{ v \mid f(x) = f(x) + \langle v, x - \bar{x} \rangle + o(\| x - \bar{x} \|) \right\} \quad \text{(singleton)} \]

\[ v \in \hat{\partial} f(\bar{x}) \text{ regular subgradient if } f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(\| x - \bar{x} \|) \]

\[ \hat{\partial} f(\bar{x}) = \left\{ v \mid (v, -1) \in \hat{N}_{\text{epi}\ f}(\bar{x}, f(\bar{x})) \right\}, \text{ closed and convex} \]

\[ \partial f(\bar{x}) = \left\{ v \mid (v, -1) \in N_{\text{epi}\ f}(\bar{x}, f(\bar{x})) \right\}, \text{ closed} \]

\[ x \mapsto \partial f(x) \text{ osc } f \text{-attentive convergence: } \Rightarrow \text{Lo}_{x \mapsto f, \bar{x}} \partial f(x) \subset \partial f(\bar{x}) \]

\[ f \text{ differentiable at } \bar{x} : \hat{\partial} f(\bar{x}) = \nabla f(\bar{x}) = \partial f(\bar{x}) \]

\[ f \text{ regular at } \bar{x} : f \text{ locally lsc with } \partial f(\bar{x}) = \hat{\partial} f(\bar{x}) \text{ (} f \text{ locally convex, e.g.)} \]

\[ \partial t_c(x) = N_c(x) \text{ when } C \text{ is convex} \]
Optimality

\[ \min f = f_0 + t_C, \text{ optimality: } 0 \in \partial f(\bar{x}) \]

\[ 0 = \nabla f(\bar{x}) \]

Generally, \( \partial (f + g) \neq \partial f + \partial g \)

\( \mathbb{C.Q.} \) (Constraint Qualification): \( -N_C(\bar{x}) \cap \partial^{\infty} f_0(\bar{x}) = \{0\} \)

\[ \nu \in \partial^{\infty} f_0(\bar{x}) = \text{ horizon subgradient if} \]

\[ \exists x^\nu \to \bar{x} \text{ with } f(x^\nu) \to f(\bar{x}), \nu^\nu \in \hat{\partial} f(x^\nu), \lambda_\nu \searrow 0 \& \lambda_\nu \nu^\nu \to \nu \]

with \( \mathbb{C.Q.} \). \( \bar{x} \) locally optimal \( \Rightarrow \) \( \partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0 \)

\( f \) convex \( (\Rightarrow \text{ regular}) \), \( \partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0 \)

\( \Rightarrow \) globally optimal (without \( \mathbb{C.Q.} \))

When \( f_0, C \) are convex: \( -\partial f_0(\bar{x}) \in N_C(\bar{x}), \]

functional variational inequality
Stochastic Optimization

\( \min Ef(x) = \mathbb{E}\{f(\xi, x)\} \quad \text{--stationary point--} \quad \partial Ef(x) \ni 0 \)

assuming \( \mathbb{E}\{\partial f(\xi, x)\} = \partial Ef(x) \) (by no means clear cut)

could \( \partial Ef(x) \ni 0 \) get replaced (?) by

\[
\nu^{-1}\left( \sum_{k=1}^{\nu} \partial f(\xi^k, x) \right) \ni 0 \text{ from sample } \xi^\nu
\]

\textit{footnote}

\( \text{dom } Ef \approx \bigcap_{\xi \in \Xi} \text{dom } f(\xi, :) \)

unless \( \xi \mapsto \text{dom } f(\xi, :) \) constant, constraints don't depend on \( \xi \)

interchanging \( \mathbb{E} \) & \( \partial \) is only exceptionally valid
Stochastic Homogenization

(Varadhan, Bensoussan, Lions, Papanicolaou, Fannjiang, Licht-Michaille ...)

conductor: $\Omega \subset \mathbb{R}^3$, composite $\geq 2$ materials,

$0 \leq a(\xi, x) \leq \kappa_{\text{bdd}}$, stationary process w.r.t. location

heat $u$: PDE with rapidly varying stochastic coefficients

$\nabla \cdot (a(\xi, x) \nabla u(\xi, x)) = h(x), \ x \in \Omega \ & \ \text{bdry conditions}$

homogenized equation with effective coefficient $a$

$\nabla \cdot (a(x) \nabla u(x)) = h(x), \ x \in \Omega \ & \ \text{bdry cond.}$

such that  $u(x) = \mathbb{E} \{u(\xi, x)\}.$ \hspace{1cm} a(x) \neq \mathbb{E} \{a(\xi, x)\}!

$S(\xi, u) = \nabla \cdot (a(\xi, \cdot) \nabla u(\xi, \cdot)) - h(\cdot)$ on $\Omega$

for $P$-almost all $\xi$,  $u(\xi, \cdot) \in (S(\xi, \cdot))^{-1}(0)$

$\mathbb{E}\{u(\xi, \cdot)\} \in \mathbb{E}\left\{ (S(\xi, \cdot))^{-1} \right\} \ \ \ ?=? \ (S^{\text{hom}})^{-1}(0),$

$S^{\text{hom}}(u) = \nabla \cdot (a^{\text{hom}}(x) \nabla u(x)) - h(x)$

$\text{gph } S^{\text{hom}} = \mathbb{E} \{\text{gph } S(\xi, \cdot)\}$