Stochastic Variational Analysis

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Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample P^ν (random) empirical measure

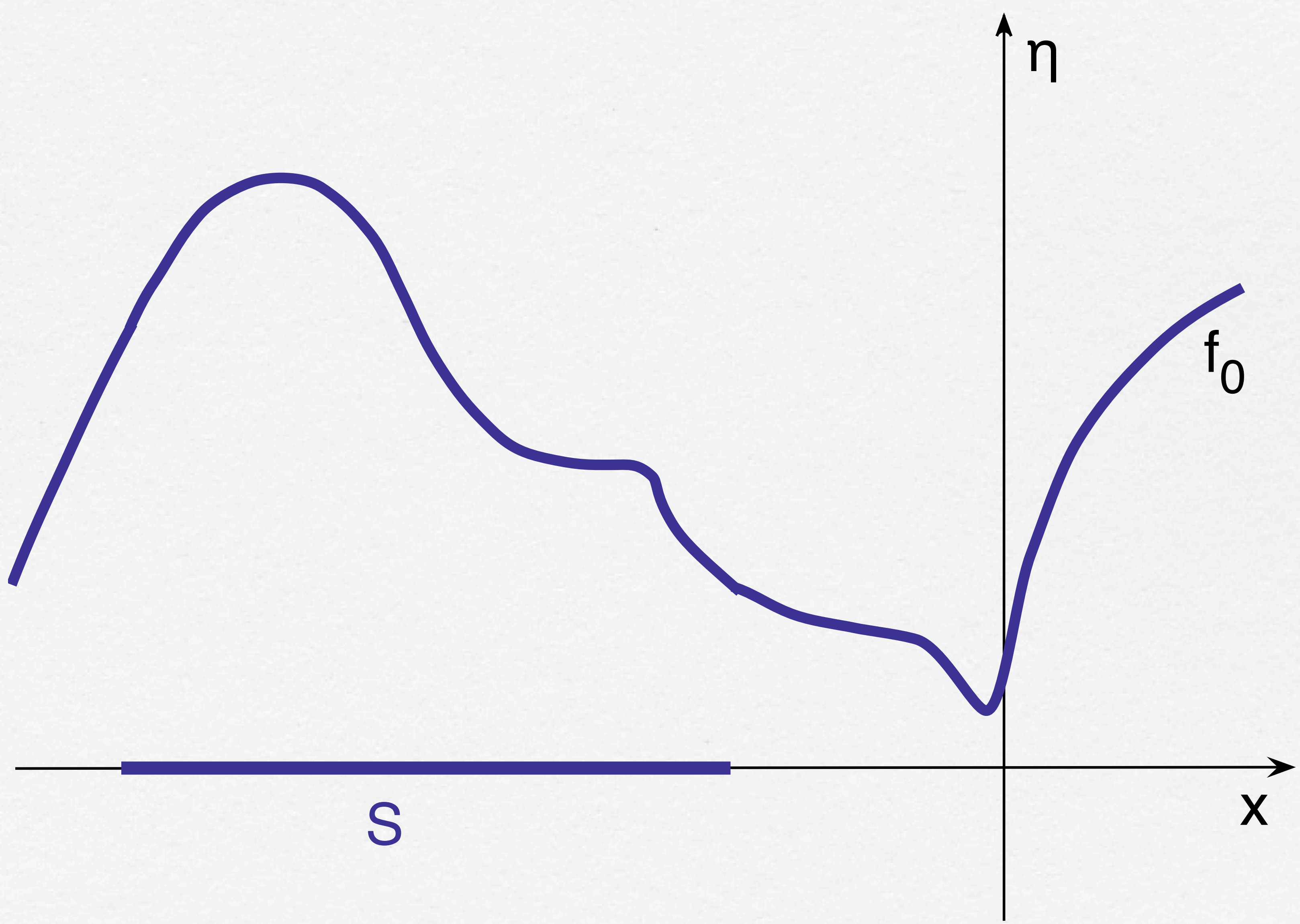
approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

Preliminaries (unavoidable)

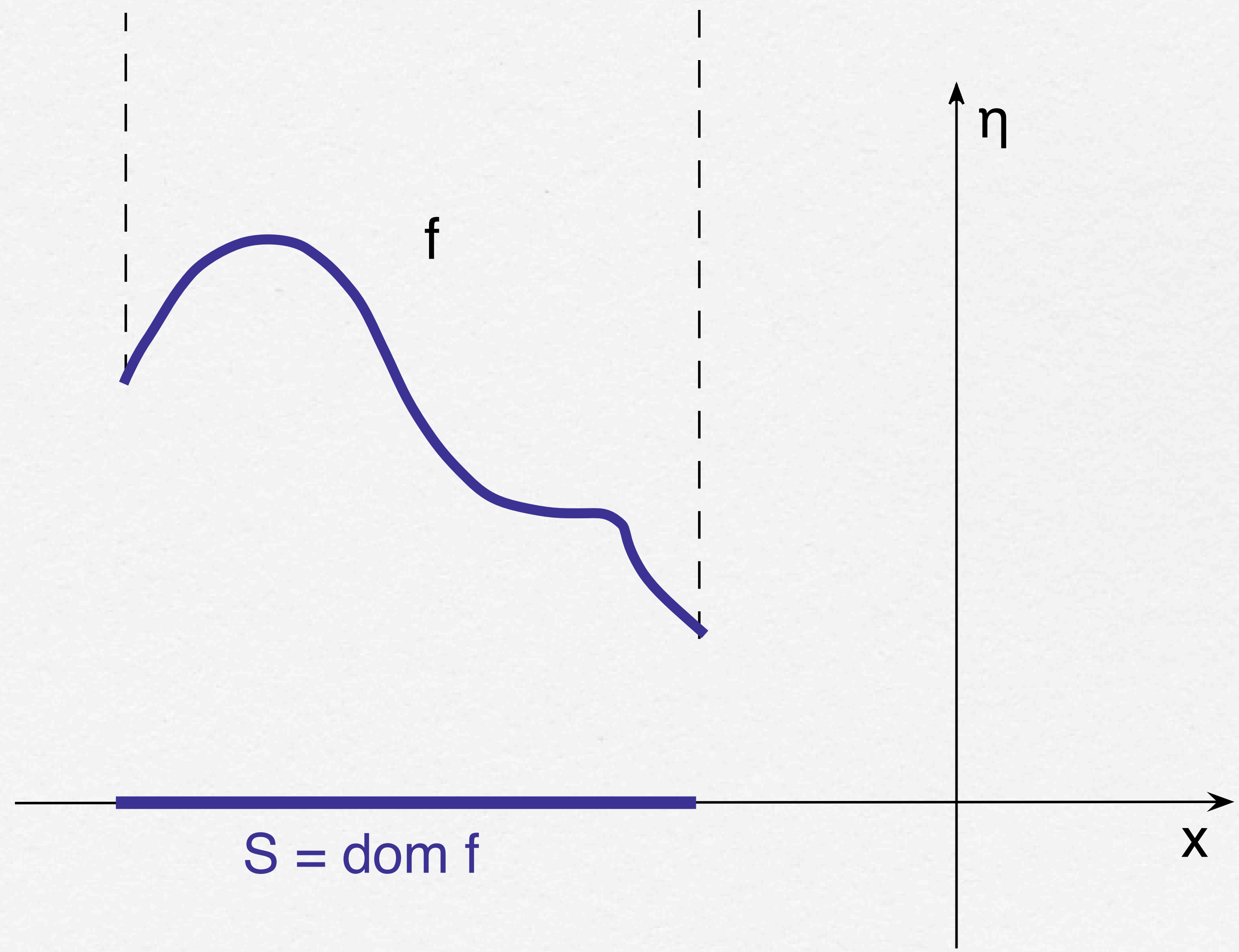
Optimization problem

$$\min f_0(x),\ x \in S,$$

$$S = \left\{ x \in \mathbb{R}^n \middle| f_i(x) \le 0,\ i = 1 \to s,\ f_i(x) = 0,\ i = s+1 \to m \right\}$$

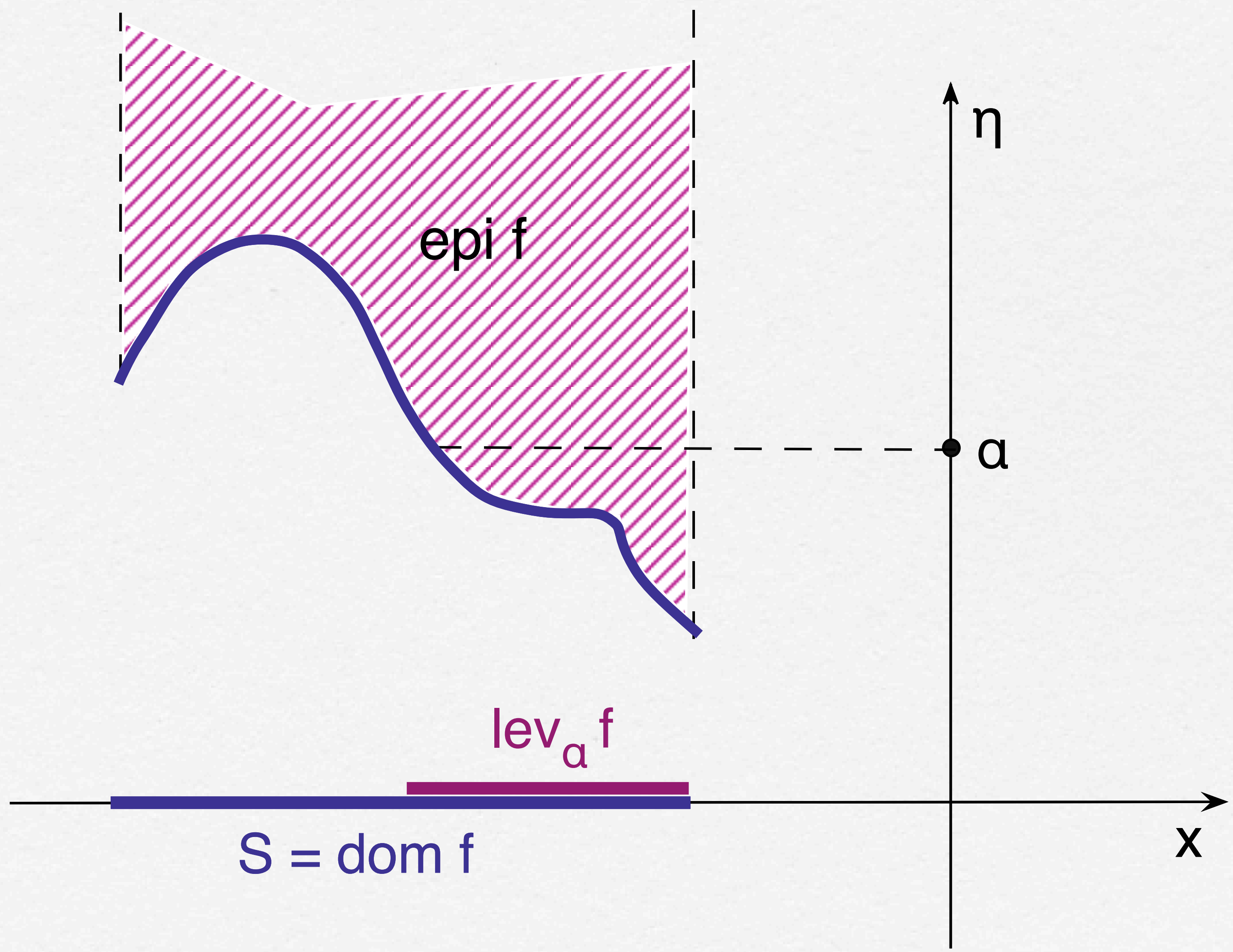


$$\min f \text{ on } E, \quad f = f_0 + \iota_S(x), \quad \iota_S \text{ indicator function of } S$$



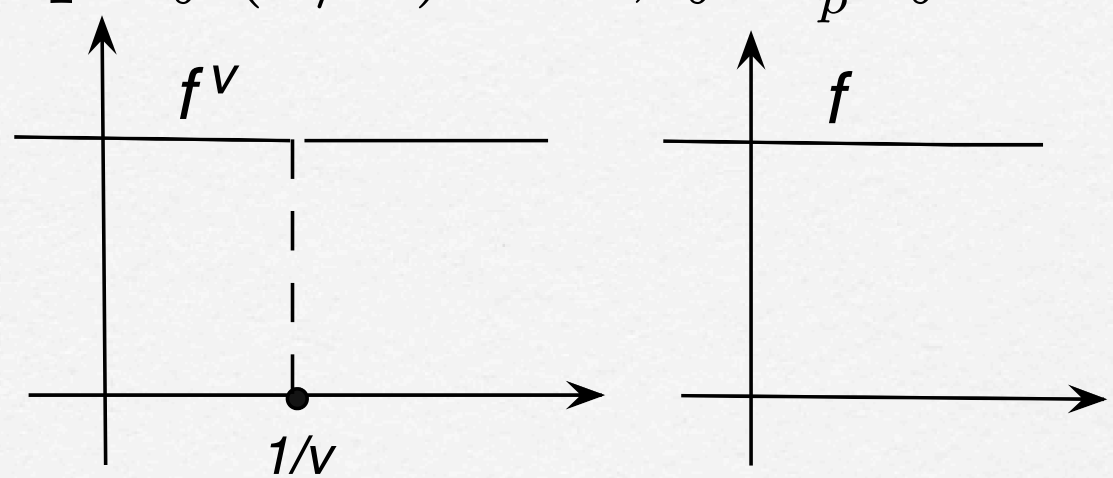
$$\min f \text{ on } E, \quad f = f_0 + \iota_S(x), \quad \iota_S \text{ indicator function of } S$$

$$\text{epi}\, f = \{(x,\alpha) \in E \times R \,|\, f(x) \le \alpha\}, \quad \text{lev}_\alpha f = \{x \in E \,|\, f(x) \le \alpha\}$$



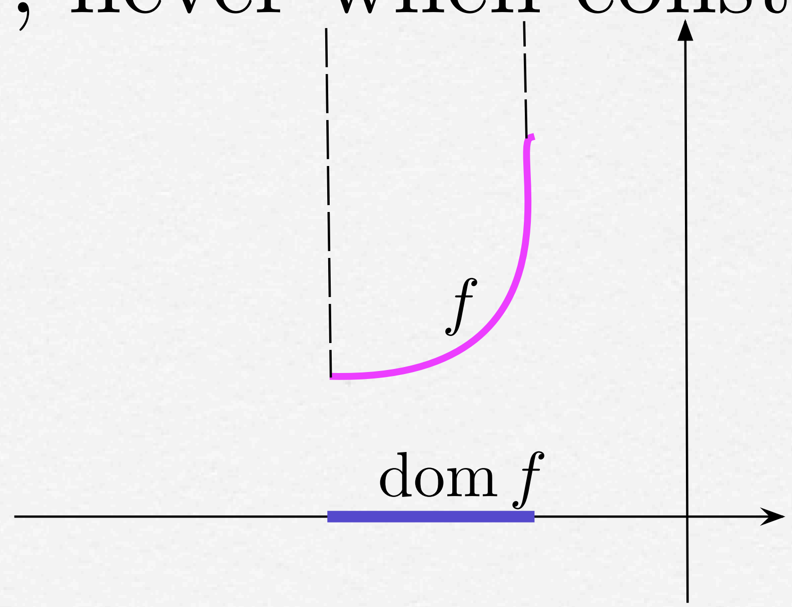
1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

   $f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$



2. uniform convergence implies convergence of minimizers

   but applies rarely, never when constraints depend on ν



variational
epi-
convergence

Epi-Convergence

$f^\nu \xrightarrow{e} f$ if for all $x \in E$,

1. $\forall\, x^\nu \to x,\ \liminf_\nu f^\nu(x^\nu) \geq f(x)$
2. $\exists\, x^\nu \to x,\ \limsup_\nu f^\nu(x^\nu) \leq f(x)$

"Geometrically": $\operatorname{epi} f^\nu \to \operatorname{epi} f$ (later)

Pointwise:

$$\liminf_\nu f^\nu(x) \geq f(x), \quad \limsup_\nu f^\nu(x) \leq f(x)$$

Continuous: $\forall x^\nu \to x$,

$$\liminf_\nu f^\nu(x^\nu) \geq f(x), \quad \limsup_\nu f^\nu(x^\nu) \leq f(x)$$

Epi-Convergence $\rightrightarrows$

$A^\nu = \arg\min f^\nu$, ε-A^ν : $\varepsilon > 0$ approximate minimizers,

$A = \arg\min f$ of limit problem, ε-A approx. minimizers

A^ν v-converges to A, written $A^\nu \rightrightarrows_v A$, if

a) $\bar{x} \in$ cluster-points$\{x^\nu \in A^\nu\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, x^\nu \in \varepsilon_\nu$-$A^\nu \to \bar{x}$

$f^\nu \xrightarrow{e} f$ implies ε-$A^\nu \rightrightarrows_v \varepsilon$-$A$, $\forall\, \varepsilon \geq 0$
A unique minimizer, ε^ν-$A^\nu \rightrightarrows A$ as $\varepsilon^\nu \searrow 0$.

($\inf f > -\infty$)

Stochastic Optimization

1. **Stochastic Programming** (recourse model)

$$f(\xi,x) = \begin{cases} f_{01}(x) + Q(\xi,x) & \text{if } x \in C_1 \\ \infty & \text{otherwise} \end{cases}$$

$$Q(\xi,x) = \inf_y \left\{ f_{02}(\xi,y) \,\middle|\, y \in C_2(\xi,x) \right\}$$

$$\min Ef(x) = \mathbb{E}\{f(\xi,x)\},$$

SAA-problem: $\min f^\nu(\vec{\xi}^{\,\nu}, x) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x)$

2. **Statistical Estimation** (fusion of hard & soft information)

$$L(\xi,h) = \begin{cases} -\ln h(\xi) & \text{if } h \geq 0, \int h = 1, h \in A^{\text{soft}} \subset E \\ \infty & \text{otherwise} \end{cases}$$

$EL(h) = \mathbb{E}\{L(\xi,h)\}$, $h^{\text{true}} = \operatorname{argmin}_E \mathbb{E}\{L(\boldsymbol{\xi},h)\}$

estimate: $h^\nu \in \operatorname{argmin}_E \mathbb{E}^\nu\{L(\boldsymbol{\xi},h)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} L(\xi^l,h)$

A^{soft} : constraints on support, moments, shape, smoothness, ...

Pricing financial instruments

3. A contingent claim: environment process: $\left\{\boldsymbol{\xi}^t \in \mathbb{R}^d\right\}_{t=0}^T$

history: $\vec{\xi}^{\,t}$, $\xi = \xi^T$, price process: $S^t(\vec{\xi}^{\,t}) \in \mathbb{R}^n$; numéraire (risk-free): $S_1^t \equiv 1$

claims: $\left\{G^t(\vec{\xi}^{\,t})\right\}_{t=1}^T$; i-strategy: $\left\{X^t(\vec{\xi}^{\,t})\right\}_{t=0}^T$; value @ t : $\langle S^t(\vec{\xi}^{\,t}), X^t(\vec{\xi}^{\,t})\rangle$

Instruments: T-bonds, options, swaps, insurace contracts, mortgages, ...

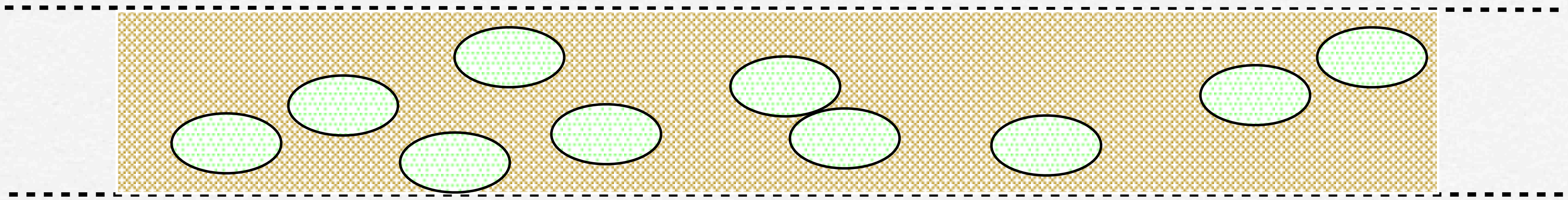
$$\max \mathbb{E}\left\{\langle S^T, X^T\rangle\right\} \text{ such that } \langle S^t, X^t\rangle \le G^t + \langle S^t, X^{t-1}\rangle,\ t = 1 \to T$$

$$\langle S^0, X^0\rangle \le G^0,\ \langle S^T, X^T\rangle \le G^T \text{ a.s.}$$

feasible if $G^0 + \cdots + G^T \ge 0 \ \ \forall \xi$; arbitrage $\Rightarrow$ unbounded

prob[$\boldsymbol{\xi} = \xi$] $= p_\xi$ (finite sample?): $\max \sum_{\xi \in \Xi} p_\xi \langle S^T(\xi), X^T(\xi)\rangle \ldots$

4.Stochastic homogenization, ...



$$-\nabla\cdot\big(a(\boldsymbol{\xi},x)\nabla u(\boldsymbol{\xi},x)\big) = h(x) \text{ for } x \in \Omega, \qquad u(\boldsymbol{\xi},x) = 0 \text{ on bdry } \Omega$$

Variational formulation: $\forall\, \xi, \quad g(\xi,u) := \frac{1}{2}\int_\Omega a(\xi,x)|\nabla u|^2 dx - \langle h, u\rangle$

find $u(\xi,x) \in \operatorname{argmin}_{u\in H_0^1(\Omega)} g(\xi,u), \quad g(\xi,\cdot) : L^2 \to (-\infty,\infty]$. convex

$\mathbb{E}\{u(\boldsymbol{\xi},x)\} \in \operatorname{argmin}_{u\in H_0^1(\Omega)} G(u)$ where $\operatorname{epi} G = \mathbb{E}\{\operatorname{epi} g(\boldsymbol{\xi},\cdot)\}$

$G(u) = \inf_z \big\{\mathbb{E}\{g\big(\boldsymbol{\xi}, z(\boldsymbol{\xi})\big)\,\big|\, \mathbb{E}\{z(\boldsymbol{\xi})\} = u\big\}$

$G^* = \mathbb{E}\{g^*(\boldsymbol{\xi},\cdot)\}, \quad g^*(\xi,v) = \sup_u \big\{\langle v,u\rangle - g(\xi,u)\big\}$, conjugate fcn

$\xi^1, \xi^2, \ldots$ stationary, use Ergodic Theorem for random lsc functions

$G = g^{\text{hom}} = \big(\operatorname{epi}_{\text{w}}\text{-}\lim_\nu \frac{1}{\nu}\sum_{l=1}^{\nu} g^*(\boldsymbol{\xi}^l,\cdot)\big)^* \quad \Longrightarrow \quad$ values of $a^{\text{hom}}(x)$

Expectation Functionals

$$Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot)\}$$

$f : \Xi \times E \to \overline{\mathbb{R}}$, random lsc function, $f(\xi,x) = f_0(\xi,x)$ when $x \in C(\xi)$

$$E \subset \mathcal{M}(\Xi,\mathcal{A};\mathbb{R}^n) : \mathcal{L}^p(\Xi,\mathcal{A},P;\mathbb{R}^n), \ldots$$

others: $C\big((\Xi,\tau);\mathbb{R}^n\big)$, Orlicz, Sobolev, lsc-fcns(E)

$$Ef(x) = \int_\Xi f(\xi, x(\xi))\, P(d\xi) = \mathbb{E}\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\}$$

$$= \infty \text{ whenever } \int_\Xi f_+(\xi, x(\xi))\, P(d\xi) = \infty$$

$Ef : E \to \overline{\mathbb{R}}$ always defined

Regression: (E is not a linear space)

$$\min\left\{ \int_{y\in\mathbb{R}} \int_{x\in[0,1]^n} \phi(y - h(x))\, P(dx,dy) \,\middle|\, h \in \text{ lsc-fcns}(\mathbb{R}^n) \cap \mathcal{H} \right\}$$

$\mathcal{H}$ shape restrictions (convex, unimodal, ...)

Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in x, $(\forall \xi \in \Xi)$

(b) (ξ,x)-measurable $\quad (\mathcal{A} \times B_E)$-measurable

recall: $f(\xi,x) = f_0(\xi,x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi,x) = \begin{cases} \frac{1}{\nu}\sum_{l=1}^{\nu}\left(f(\xi^l,x) \text{ if } \mathrm{x} \in C(\xi^l)\right) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi,\cdot)$ epi-converge to $\mathbb{E}\{f(\xi,h)\}$ P-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi,x)\}$ P-a.s.?

$$E^\nu f \xrightarrow{e} Ef \ a.s., \quad E^\nu f(x) = \tfrac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, x)$$

Random lsc functions

(via inf-projections)

D countable dense subset of E

$f : E \to \overline{\mathbb{R}}$, lsc fcn completely identified by

$\left\{ o_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$, countable

or $\left\{ c_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$

$f \quad x$ $\quad {}_{V \in \mathcal{N}(\bar{x})} \left[\quad {}_{x \in V} f \quad x \right], \quad f \quad , f \quad x \quad {}_{x \to \bar{x}} \quad f \quad x$

${}_{V \in \mathcal{Q}(\bar{x})} \left[\quad {}_{x \in V} f \quad x \right], \quad E \quad o$

$\mathcal{Q} \quad x \quad \left\{ \mathbb{B}^o \quad x, \delta \,\middle|\, x \in D, \delta \in \mathbb{Q}_+, x \in \mathbb{B}^o \quad x, \delta \right\}$

${}_{\delta \in \mathbb{Q}_+} \quad {}_{\{x \,|\, \mathbb{B}^o(x\delta) \in \mathcal{Q}(\bar{x})\}} \, o_{x,\delta}$

$\{c_{x,\delta}\}$

Epi-convergence

(via inf-projections)

$$f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^\nu \xrightarrow{e} f,\ f \text{ lsc}, \iff \forall \delta \in \mathbb{Q}_+,\ x \in D$$

$$\text{limsup}_\nu\, c^\nu_{x\delta} \le c_{x\delta}, \quad \text{liminf}_\nu\, o^\nu_{x\delta} \ge o_{x\delta}$$

for $x \in D, \delta \in \mathbb{Q}_+$: $o^\nu_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f^\nu$, $c^\nu_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f^\nu$

(fundamental) **Theorem.** $f^\nu : E \to \overline{\mathbb{R}}$ & f lsc (necessarily)

1. $\text{e-lim}\inf_\nu f^\nu \iff \liminf_\nu(\inf_B f^\nu) \ge \inf_B f$ for all compact B
2. $\text{e-lim}\sup_\nu f^\nu \iff \limsup_\nu(\inf_O f^\nu) \le \inf_O f$ for all open O

□ Hit-and-miss topology on the space of epigraphs, (later?). □

Scalarization of random lsc fcns

$f : \Xi \times E \to \overline{\mathbb{R}}$, random lsc fcn, completely identified by

$$\left\{ o_{x\delta}(\xi) = \inf_{\mathbb{B}^o(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}, \text{ countable}$$

$$\text{or } \left\{ c_{x\delta}(\xi) = \inf_{\mathbb{B}(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}, \quad \forall \xi \in \Xi$$

COUNTABLE

$\forall\, x \in \mathbb{R}^n, \delta > 0$

$\xi \mapsto o_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,

$o_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable

$\xi \mapsto c_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,

$c_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable.

- f random lsc fcn $\Rightarrow f + \iota_{\mathbb{B}(x,\delta)}$ random lsc fcn
- f random lsc fcn $\Rightarrow \xi \mapsto \alpha(\xi) = \inf_x f(x,\xi)$ measurable

Probabilistic properties

f random lsc fcn: $\{f(\boldsymbol{\xi}^\nu,\cdot)\}_{\nu\in\mathbb{N}}$ iid whenever $\{\boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid

$$\text{Effös field on lsc-fcns}(E) = \sigma\text{-}\left\{f \in \text{lsc-fcns}(E) \,\middle|\, \inf_O < \alpha\right\},\ O \text{ open},\ \alpha\in\mathbb{R}$$
$$= \mathcal{B}\big(\text{lsc-fcns}(E)\big),\ E \text{ Polish}$$

1. $\{f(\boldsymbol{\xi}^\nu,\cdot)\}_{\nu\in\mathbb{N}}$ "i" $\iff$ $\{o_{x\delta}(\boldsymbol{\xi}^\nu),\ \nu\in\mathbb{N}\}$ "i", $\forall x\in\mathbb{Q}^n, \delta\in\mathbb{Q}_+$
2. $f(\boldsymbol{\xi}^1,\cdot), f(\boldsymbol{\xi}^2,\cdot)$ "id" $\iff$ $o_{x\delta}(\boldsymbol{\xi}^1), o_{x\delta}(\boldsymbol{\xi}^2)$ "id", $\forall x\in\mathbb{Q}^n, \delta\in\mathbb{Q}_+$

the same holds for $\{c_{x\delta}(\cdot)\}$

Summary

$f : \Xi \times E \to \overline{\mathbb{R}}$ random lsc fcn,
$\boldsymbol{\xi}, \{\boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid
$\Rightarrow f(\boldsymbol{\xi}, \cdot), \{f(\boldsymbol{\xi}^\nu, \cdot)\}_{\nu\in\mathbb{N}}$ iid

$\Rightarrow \{o_{x\delta}(\boldsymbol{\xi}^\nu) : \Xi \to \overline{\mathbb{R}}\}_{\nu\in\mathbb{N}}$, iid,
countable, identify $f(\boldsymbol{\xi}^\nu \cdot)$

$\Rightarrow \{c_{x\delta}(\boldsymbol{\xi}^\nu) : \Xi \to \overline{\mathbb{R}}\}_{\nu\in\mathbb{N}}$, iid,
countable, identify $f(\boldsymbol{\xi}^\nu \cdot)$

Countable ⇒ a.s.

Lemma. $f, g : E \to \overline{\mathbb{R}}$, lsc. $D = \mathrm{prj}_E$ countable dense subset of $\mathrm{epi}\, f$.

$$f \le g \text{ on } D \implies f \le g \text{ on } E.$$

Proof. $f \le g$ on D only if $\{(x, \alpha) \mid \alpha \ge g(x), x \in R\} \subset \mathrm{epi}\, f$.
Taking closure on both sides $\implies$ $\mathrm{epi}\, g \subset \mathrm{epi}\, f$. $\square$

Implication. To check $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ $a.s.$ on E only needs $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ $a.s.$ on D a countable dense subset of E.
Restrict $\boldsymbol{\xi}$ to a set of P-measure 1, say Ξ itself (from now on), and $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on $D \implies f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on E.

LLN: random lsc functions?

$\forall\, x \in D,\ \delta \in \mathbb{Q}_+$

1. $\frac{1}{\nu}\sum_{l=1}^{\nu} o_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{o_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$

2. $\frac{1}{\nu}\sum_{l=1}^{\nu} c_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{c_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$

$\not\Longrightarrow \sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\,)$ because

$$\min\left\{\mathbb{E}\{f(\boldsymbol{\xi}, z)\} \,\middle|\, z \in \mathbb{B}(x,\delta)\right\} \neq \mathbb{E}\left\{ \min\{f(\boldsymbol{\xi}, z)\} \,\middle|\, z \in \mathbb{B}(x,\delta)\right\}$$

in general

Law of Large Numbers: Random lsc functions

LLN: Proof

1. $\exists\, x^\nu \to x : \limsup_\nu E^\nu f \le Ef$
   for any $x \in E$ and any sample $\xi^\infty = (\xi^1, \xi^2, \ldots)$
   $\lim_\nu \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x) \;\sim \lim_\nu \mathbb{E}^\nu\{f(\boldsymbol{\xi}^\infty, x)\} = Ef(x).$

2. $\forall x^\nu \to x, \liminf_\nu E^\nu f \ge Ef$
   for any $x \in E$ and any $\xi^\infty = (\xi^1, \xi^2, \ldots) \in \Xi^\infty$

$$\text{e-}\liminf_{\nu\to\infty} f^\nu(\xi^\infty, x) = \sup_{\delta \searrow 0} \liminf_{\nu\to\infty} \inf_{\mathbb{B}^o(x,\delta)} E^\nu f \ge \sup_{\delta^l \searrow 0} \liminf_{\nu\to\infty} \frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l\delta^l}(\xi^l)$$

where $x^l \in D \to x$, $\delta^l \in \mathbb{Q}_+ \searrow 0$: $\quad x \in \mathbb{B}^o(x^l, \delta^l)$ & $\{\mathbb{B}^o(x^l, \delta^l)\} \searrow$
$\frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l\delta^l}(\xi^l) \to \mathbb{E}\{o^l_{x^l\delta^l}(\boldsymbol{\xi})\}$ & $\mathbb{E}\{o^l_{x^l\delta^l}(\boldsymbol{\xi})\} \nearrow Ef(x)$

$\implies \quad \text{e-lim inf}_{\nu\to\infty} E^\nu f(x) \ge Ef(x) \quad \square$

Theorem

$f : \Xi \times E \to \overline{\mathbb{R}}$, locally inf-integrable random lsc function
$\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ are iid Ξ-valued random variables. Then,

$$E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \tfrac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\,\}$$

which means ε-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin Ef, $\forall\, \varepsilon \geq 0$

Ef unique minimizer, ε^ν-argmin $E^\nu f \rightrightarrows$ argmin Ef as $\varepsilon^\nu \searrow 0$.

SAA-applies without 'any' restrictions

loc.inf-integrable: $\int \inf\{f(\xi, \cdot) \,|\, \mathbb{B}(x, \delta)\} > \infty$ for some $\delta > 0$,
irrelevant in applications

Ergodic Theorem

(E, d) Polish, $(\Xi, \mathcal{A}, P)$ & $\mathcal{A}$ P-complete
$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, locally inf-integrable
$\varphi : \Xi \to \Xi$ ergodic measure preserving transformation. Then,

$$\frac{1}{\nu} \sum_{l=1}^{\nu} f(\varphi^l(\boldsymbol{\xi}, \cdot\,) \xrightarrow{e} Ef \quad a.s.$$

allows for stationary rather than iid.

Application: "samples" coming from dynamic systems,
time series, SDE, etc.

Random Sets

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$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$
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$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$
$\xi^1, \dots \xi^\nu$ sample P^ν (random) empirical measure
approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

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$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu} \sum_{l=1}^\nu G(\xi^l, x) \ni 0$

SAA-applies without 'any' restrictions

f on $\Xi \times E$, random lsc fcn (loc. inf-$\int$), $\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ iid

Then $E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \frac{1}{\nu} \sum_{l=1}^\nu f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$

ε-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin Ef, $\forall\, \varepsilon \geq 0$

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Stochastic Programming (with recourse)

$f(\xi, x) = f_{01}(x) + Q(\xi, x), \;\; Q(\xi, x) = \inf_y \left\{ f_{02}(\xi, y) \,\middle|\, y \in C_2(\xi, x) \right\}$

SAA-problem: $\min \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x) \xrightarrow{e} Ef(x) = \mathbb{E}\{f(\xi, x)\}$

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

Statistical Estimation (fusion of hard & soft information)

$L(\xi, h) = -\ln h(\xi)$ if $h \geq 0, \int h = 1, h \in A^{\text{soft}} \subset E$

Then, estimate $h^\nu \in \operatorname{argmin}_E \mathbb{E}^\nu\{L(\boldsymbol{\xi}, h)\} \to h^{\text{true}} = \operatorname{argmin} \mathbb{E}\{L(\boldsymbol{\xi}, h)\}$

example: Normal density

mean = (0,0) ... data samples correlated

covariance: MDM^T, $D =$ diag(4,1), $M = \begin{pmatrix} \cos(\pi/6) & \cos(2\pi/3) \\ \sin(\pi/6) & \sin(2\pi/3) \end{pmatrix}$

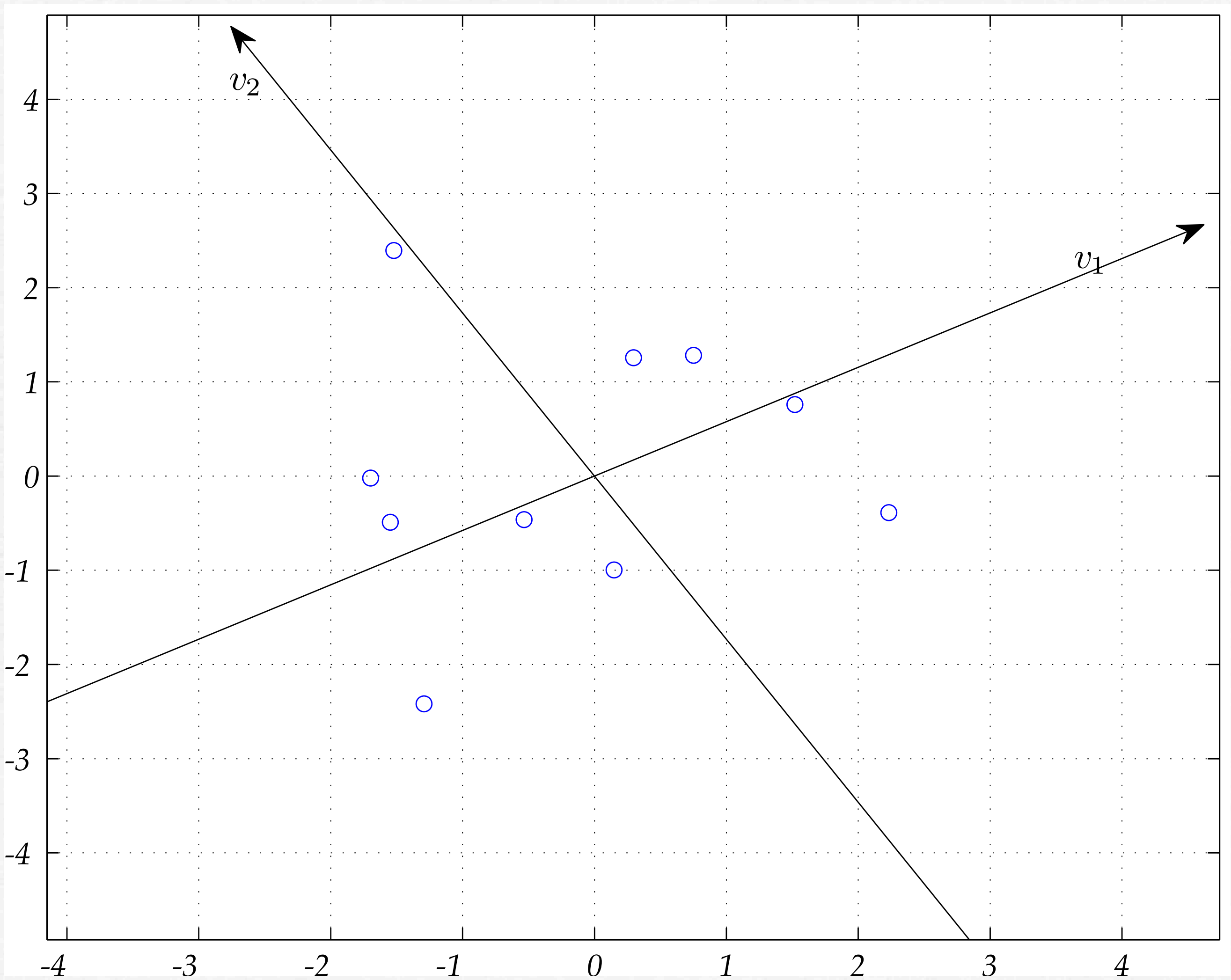
samples: $\nu = 10$,

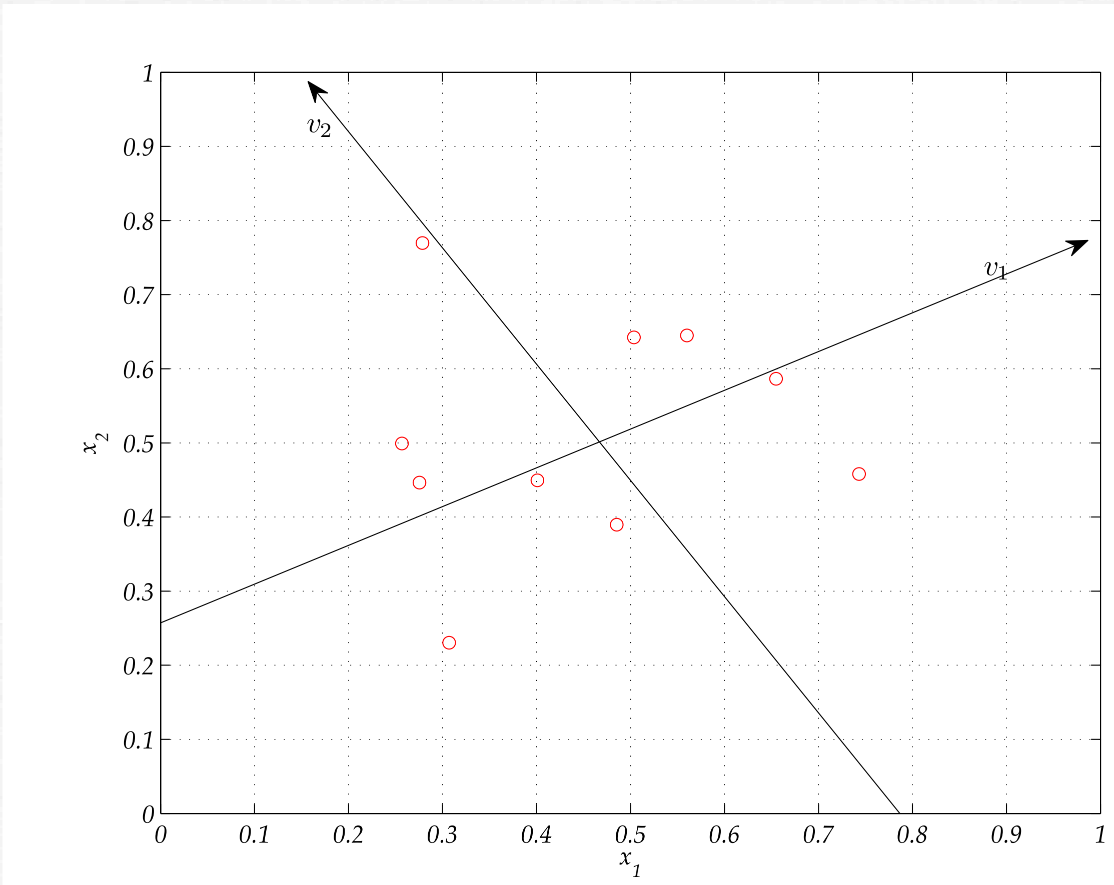
"soft" information: h unimodal

Results:

$$\left\| h^{true} - h^{est} \right\|_2^2 = 0.028, \quad \left\| h^{true} - h^{est} \right\|_\infty = 0.006$$

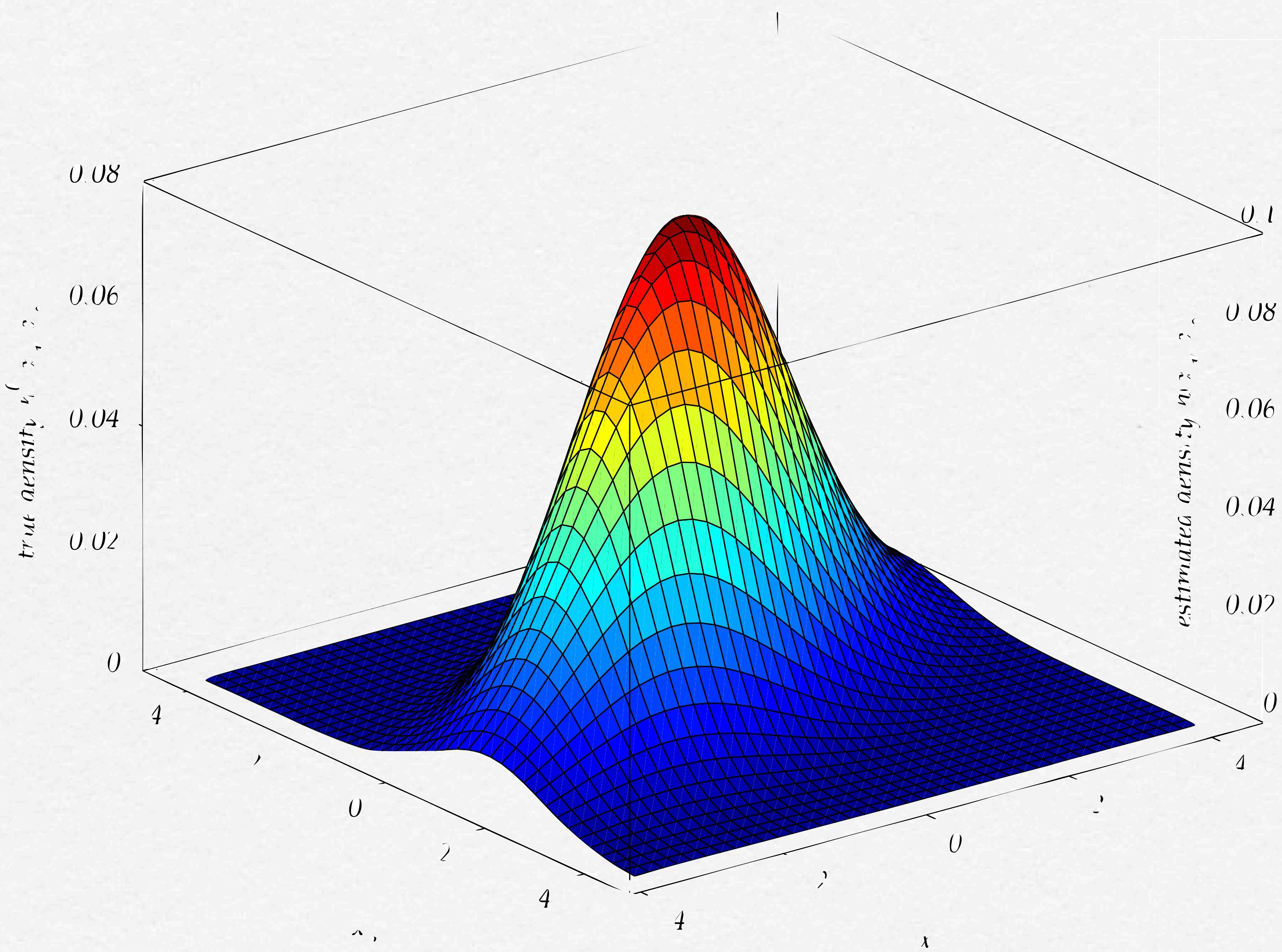
Sampled data

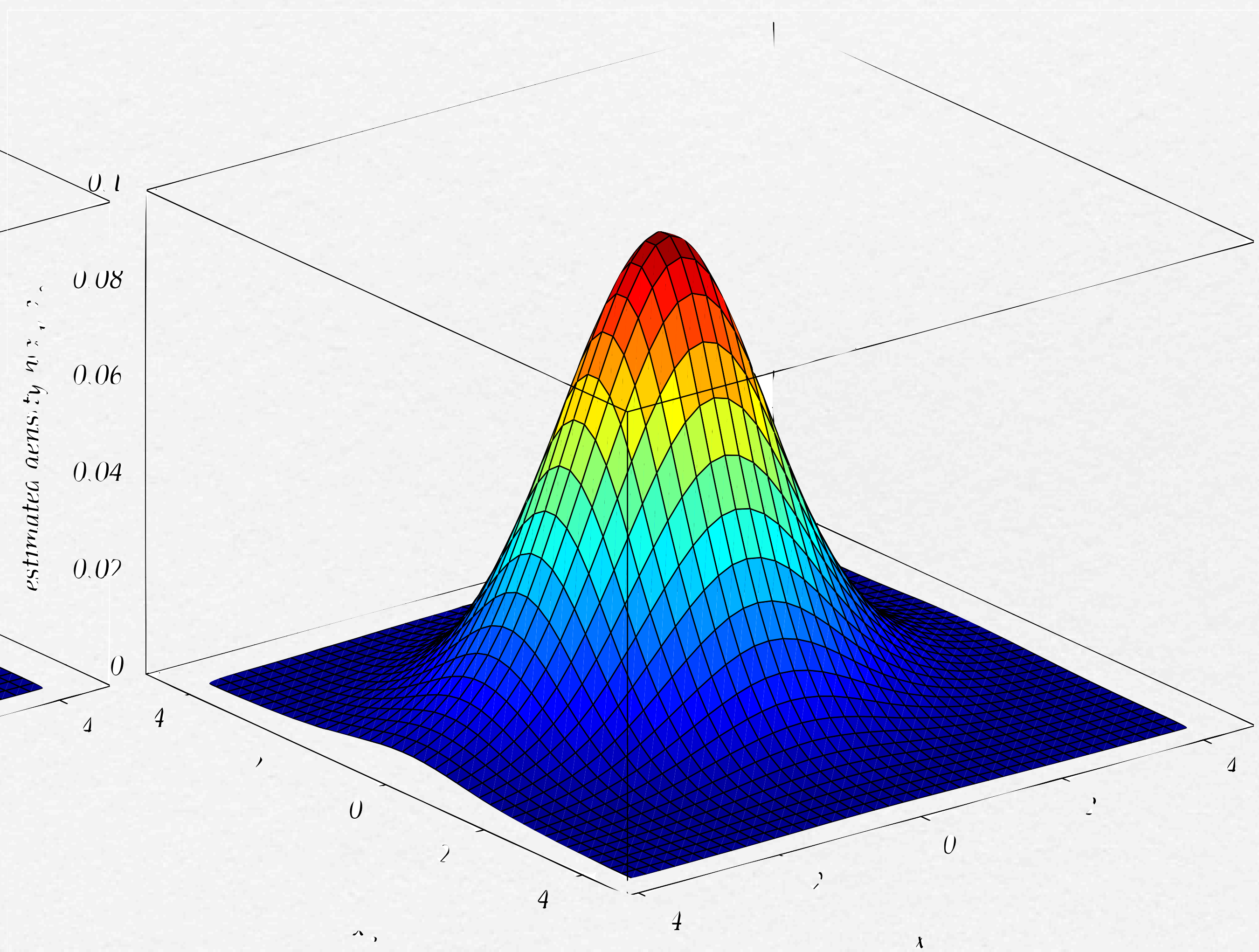




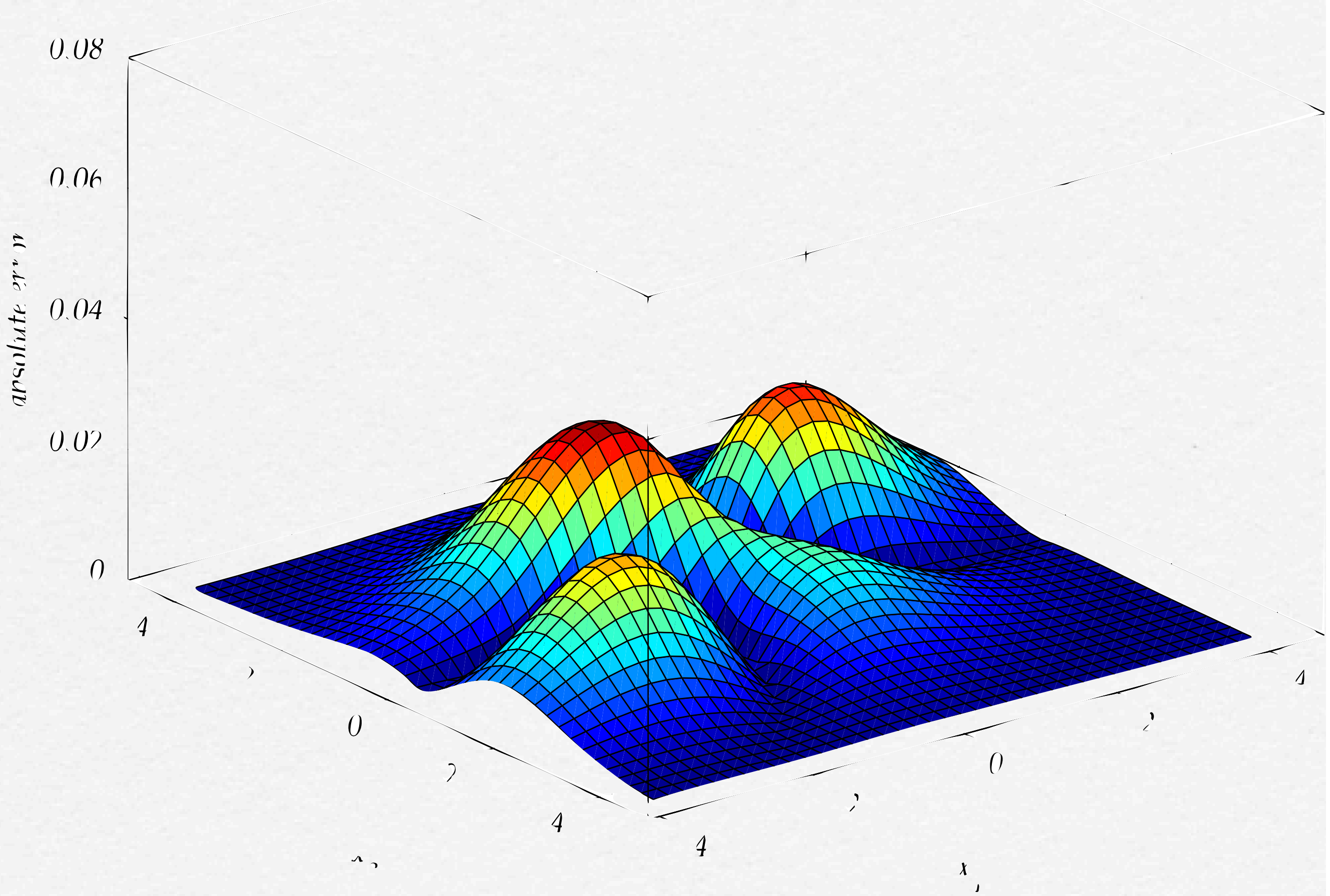
normalized

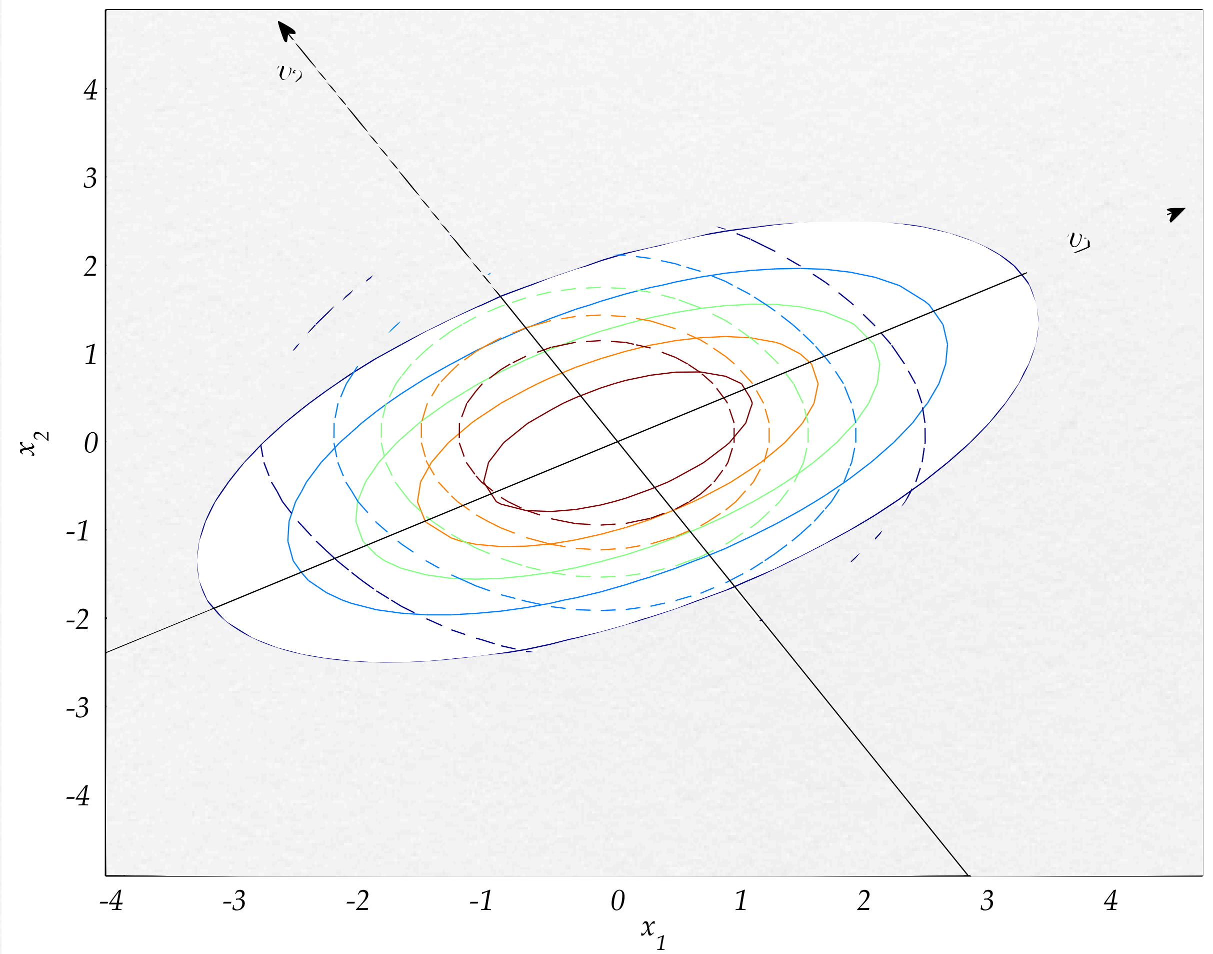
True & Estimated density





Measurement Errors





Absolute Error

Level curves: true & estimate

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \ldots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

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$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

SAA-applies without 'any' restrictions

f on $\Xi \times E$, random lsc fcn (loc. inf-$\int$), $\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ iid

Then $E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$

$\varepsilon\text{-argmin}\, E^\nu f \Rightarrow_v \varepsilon\text{-argmin}\, Ef, \ \forall\, \varepsilon \geq 0$

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \ldots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

Pricing contingent claims

claims $\left\{G^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}$, instrum. prices $\left\{S^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}_t$, invest. $\left\{X^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}$

$\max \mathbb{E}\{\langle S^T, X^T\rangle\}$ s.t. $\langle S^t, X^{t-1}\rangle \le G^t + \langle S^t, X^{t-1}\rangle$ + end conditions.

Use 'improved estimation' & sampling: $\max \sum p_\xi \langle S^T(\xi), X^T(\xi)\rangle$

Correct pricing = well regulated market??

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

Stochastic homogenization: Variational formulation

given $u(\xi, x) \in \operatorname{argmin}_{H_0^1(\Omega)} g(\xi, u) = \frac{1}{2} \int_\Omega a(\xi, x) |\nabla u|^2 \, dx - \langle h, u \rangle$

find g^{hom} such that $\mathbb{E}\{u(\boldsymbol{\xi}, \cdot\,)\} \in \operatorname{argmin} g^{\text{hom}}$

via Ergodic Thm: $g^{\text{hom}} = \Big(\text{epi}_w \text{ - lim}) \nu \frac{1}{\nu} \sum l = 1^\nu g^*(\boldsymbol{\xi}^l, \cdot\,) \Big)^*$

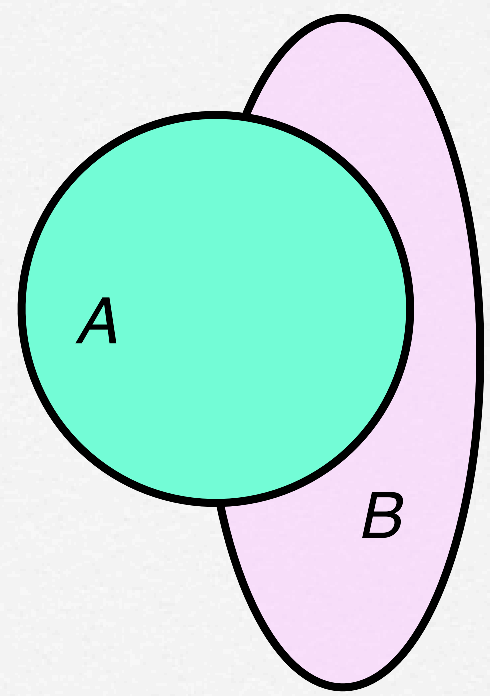
Topology of Hyperspaces

Painlevé, Pompeiu, Zoretti
Zarankiewicz, Hausdorff, Lubben, Moore
Choquet, Vietoris, Fell, Attouch-Wets, Beer, ...

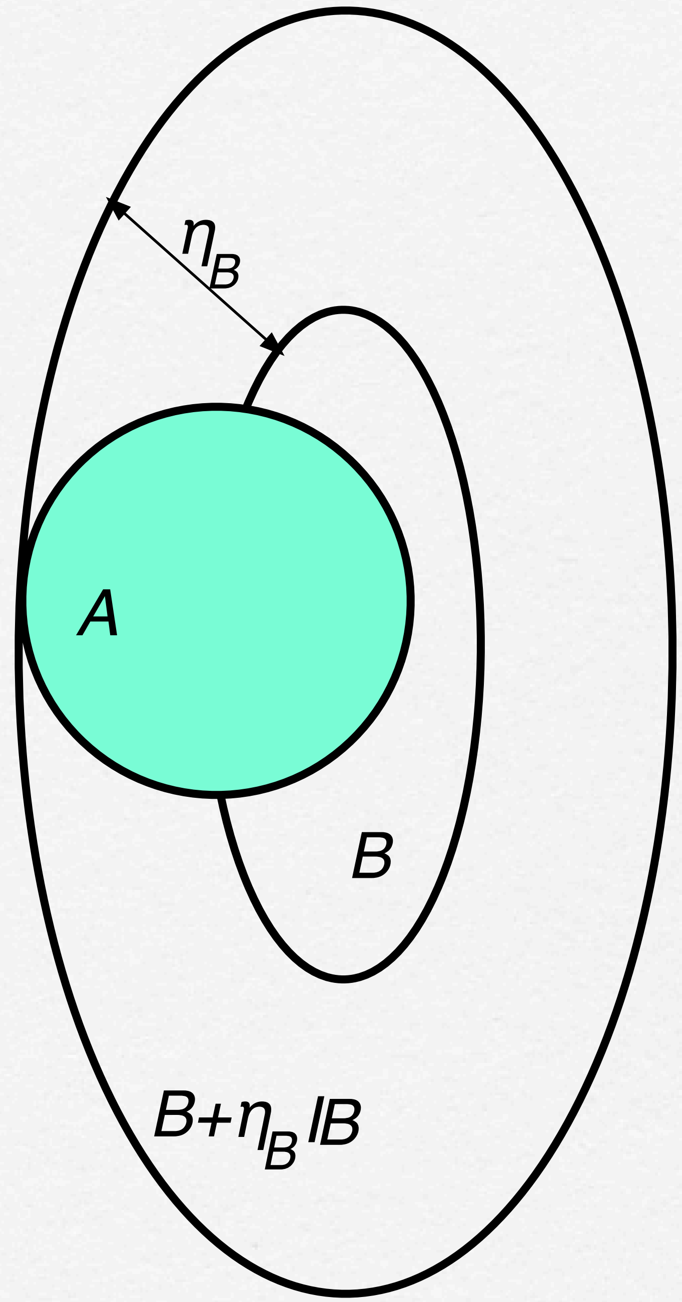
Hyperspace: sets(E)

- (E,d) always a Polish space
- $C \subset E,\ d(x,C) = \inf\{d(z,x) \mid z \in C\}, \quad d(x,\emptyset) = \infty$
- $\text{cl-sets}(E) = \{\text{all closed subsets of } E\}, \quad \emptyset, E \in \text{cl-sets}(E)$
- $dl(A,B) =$ distance between A & B, metric(?) on cl-sets(E)
- $(\text{cl-sets}(E), dl)$ Polish space = complete separable metric ??
- $dl(C^\nu, C) \to 0$ means $C^\nu \to C$ (set-convergence)

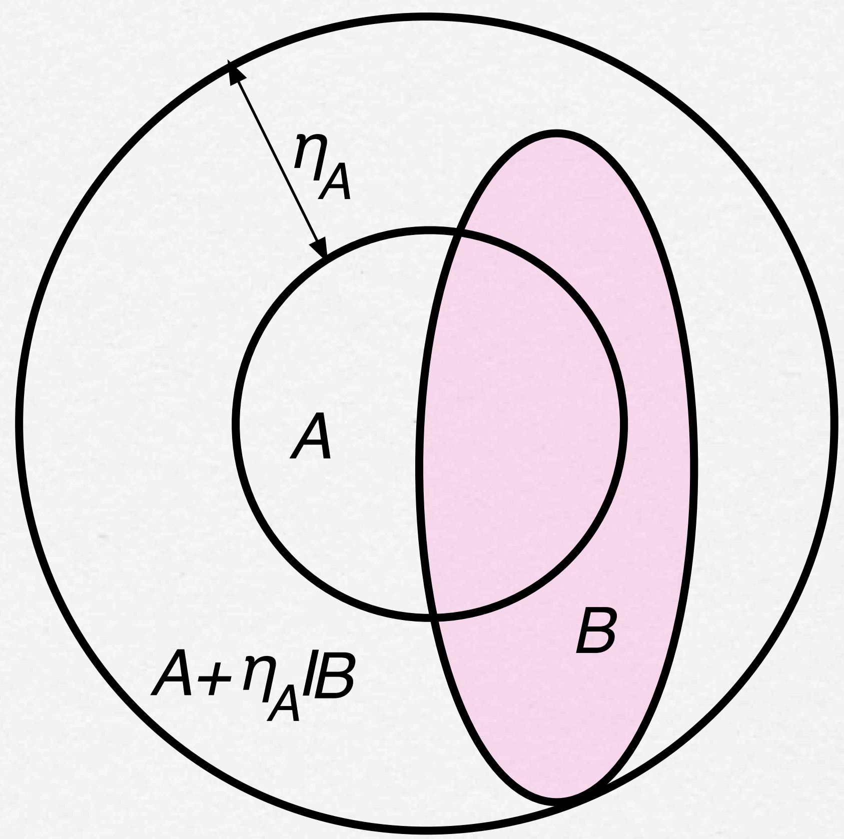
Pompeiu-Hausdorff distance



Pompeiu-Hausdorff distance

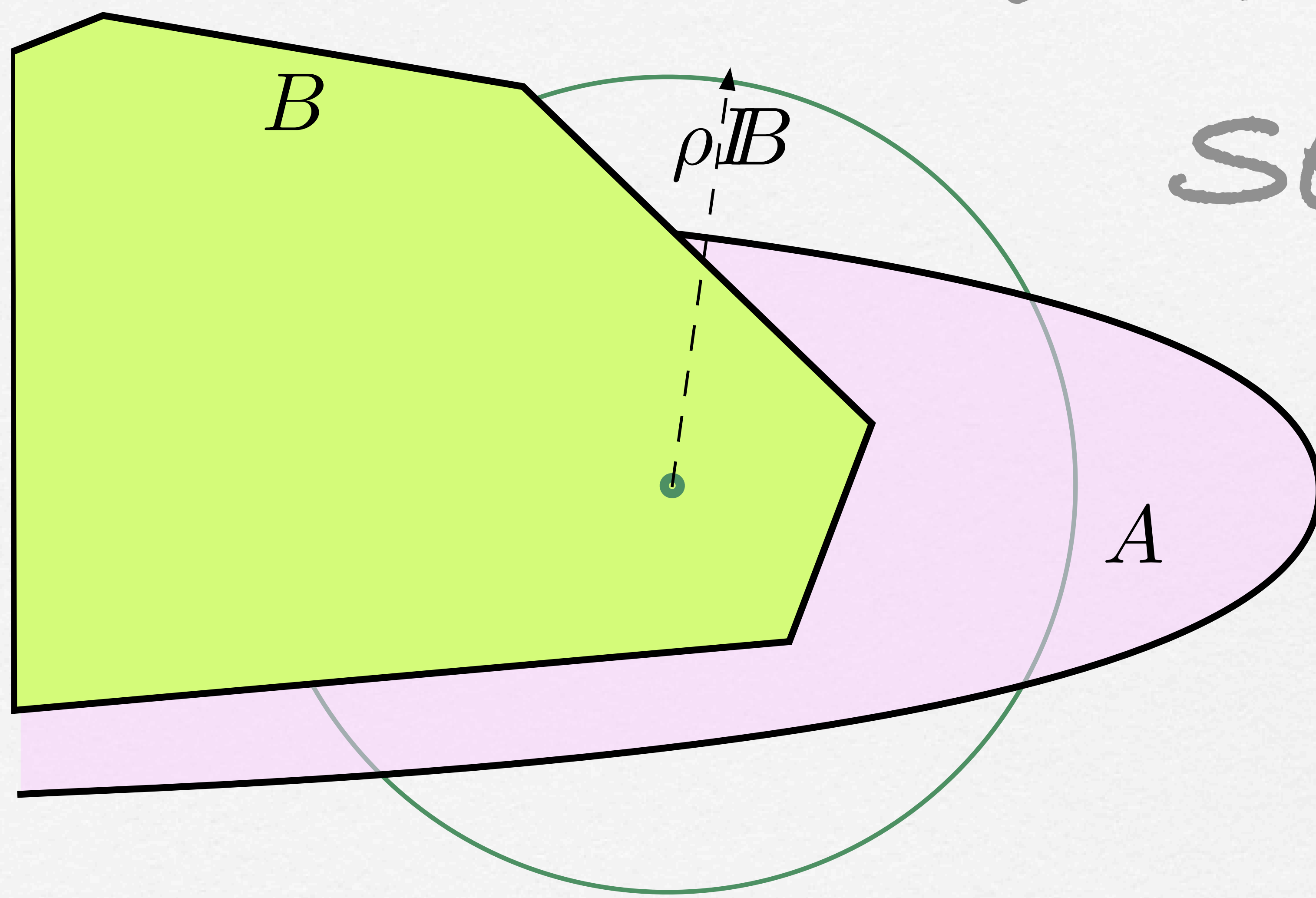


Pompeiu-Hausdorff distance

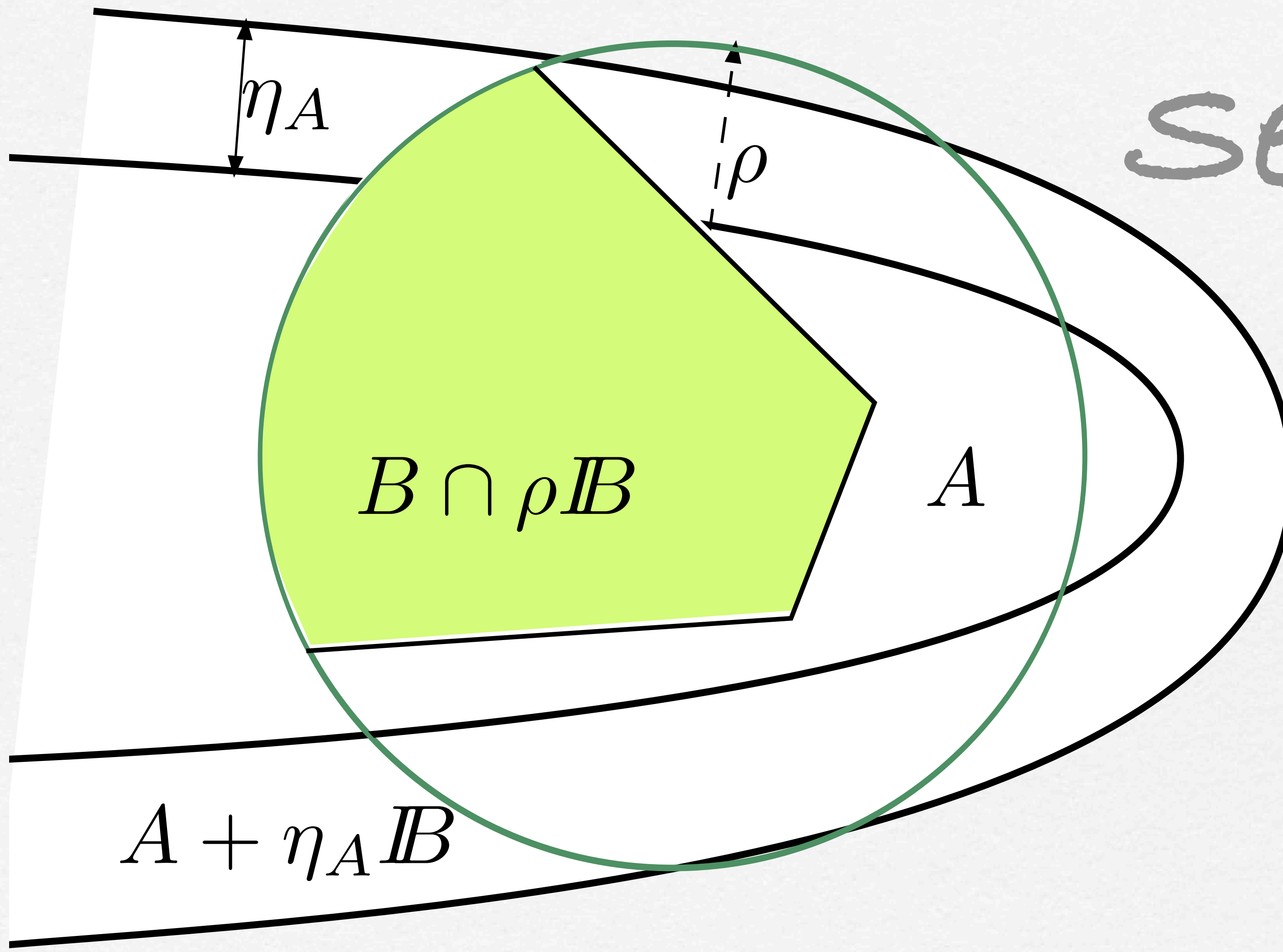


$$\hat{d\!l}(A,B) = \max[\,\eta_A, \eta_B\,]$$
$$= d\!l_\infty(A,B)$$

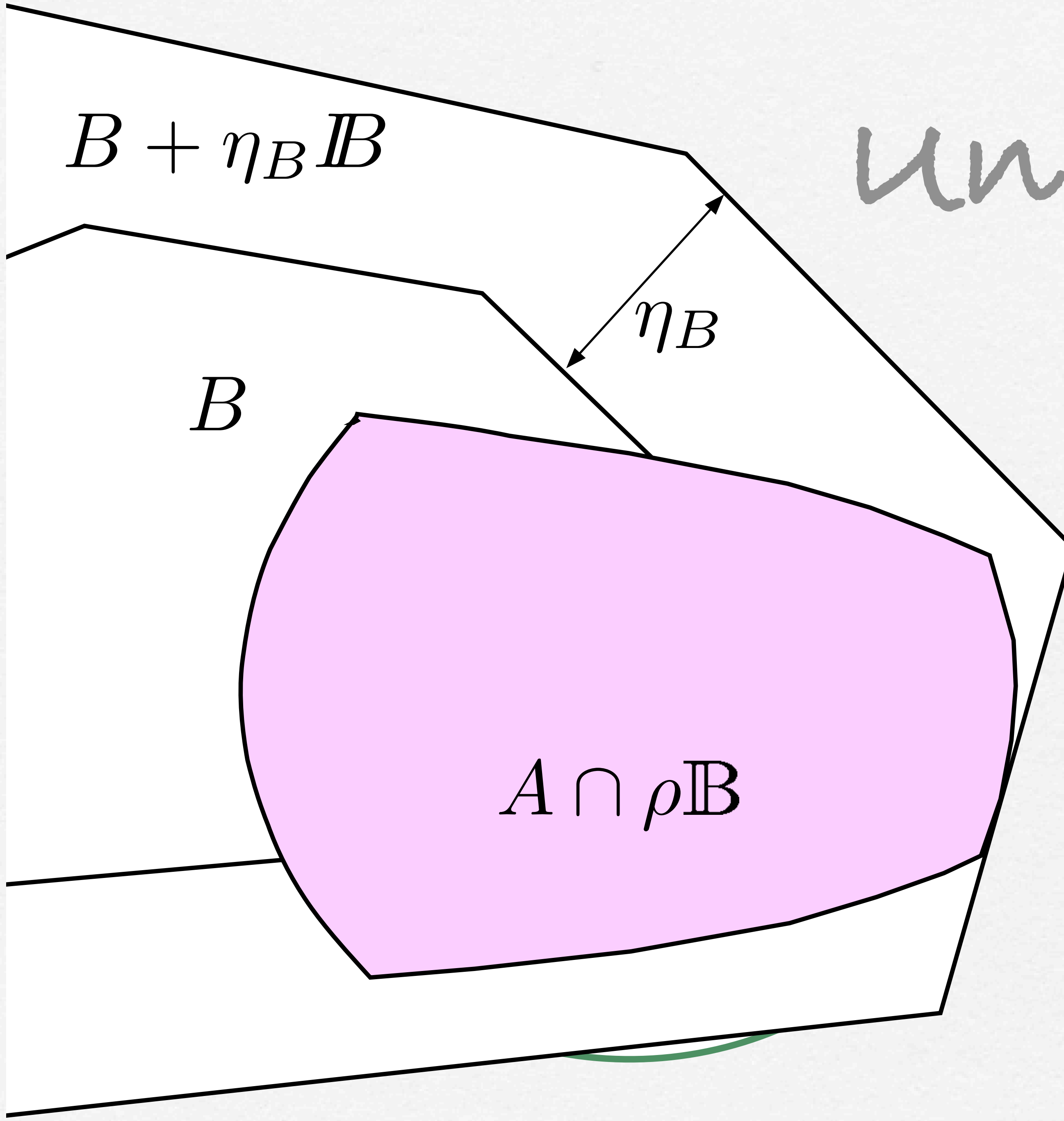
Unbounded Sets



Unbounded Sets



Unbounded Sets



$$\hat{d\!\!l}_\rho(A, B) = \max\left[\, \eta_A, \eta_B \,\right]$$

set distance (~Attouch-Wets)

τ_{aw} topology

- $\hat{d\!l}_\rho(A,B) \geq 0,\ \hat{d\!l}(A,A) = 0, \triangle$ inequality
- but $\hat{d\!l}_\rho(A,B) = 0$ possibly when $A \neq B$
- $d\!l_\rho(A,B) = \sup_{x\in\rho\boldsymbol{B}} \big[\, d(x,A),\, d(x,B) \big]$
- for all $\rho \geq 0,\ d\!l_\rho$ is a pseudo-metric
- $d\!l(A,B) = \int_{\rho\geq 0} d\!l_\rho(A,B)e^{-\rho}\, d\rho,$ set-metric
- $\hat{d\!l}_\rho(A,B) \leq d\!l_\rho(A,B) \leq \hat{d\!l}_{\rho'}(A,B) \quad \rho' \geq 2\rho + d_0$

Properties of the set-distance

$$C^\nu \to C \text{ if } dl(C^\nu, C)) \to 0 \iff \text{for any } \bar\rho \geq 0,$$

$$\begin{cases} dl_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \\ \hat{dl}_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \end{cases}$$

$$(E, d) \text{ Polish} \implies (\text{cl-sets}(E), dl) \text{ complete, metric space}$$

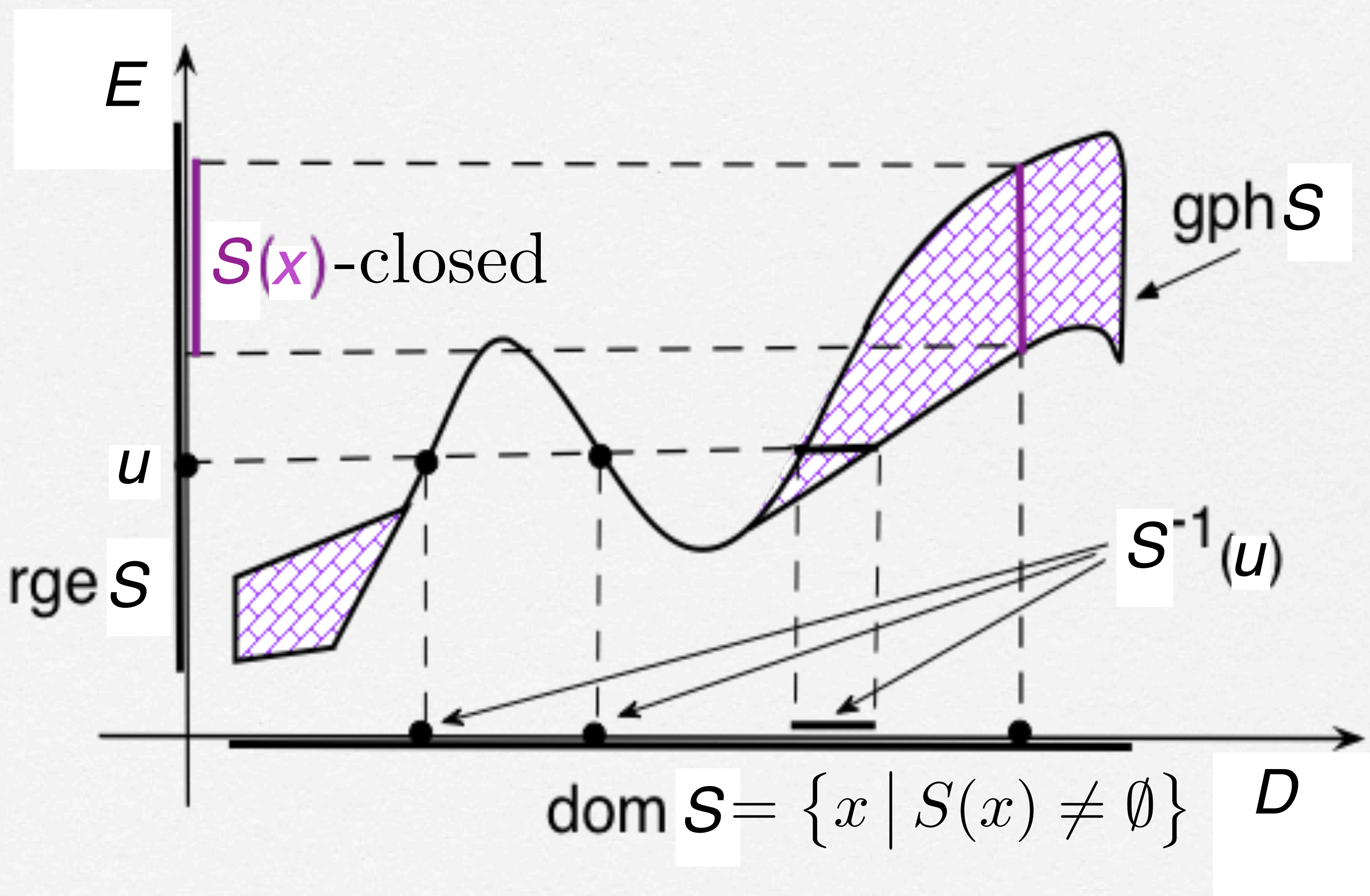
$$(\text{cl-sets}(E), dl) \text{ Polish} \iff E = \mathbb{R}^n$$

space of *osc-mappings*

outer semicontinuous

$$S : D \rightrightarrows E \text{ osc} \iff \text{gph}\, S \subset D \times E \text{ closed}$$

$$\text{gph}\, S = \{(x,u) \,|\, u \in S(x), x \in E\}$$



space of osc-mappings

outer semicontinuous

$$\mathbb{B} = \mathbb{B}_D \times \mathbb{B}_E \text{ (or } \mathbb{B}_{E\times D})$$

$$d\!l(R,S) = d\!l(\text{gph}\,R, \text{gph}\,S), \quad d\!l_\rho,\ \hat{d\!l}_\rho$$

$(\text{osc-maps}(D,E), d\!l)$ complete metric, Polish: $D = \mathbb{R}^n, E = \mathbb{R}^m$

$S : D \to E$ (single-valued) continuous $\Longrightarrow$ osc, ...

$$d\!l(f^\nu, f) \to 0 \Longrightarrow \text{argmin } f^\nu \rightrightarrows_v \text{argmin } f$$

$S^{-1}(0) =$ sol'ns of $S(x) \ni 0$

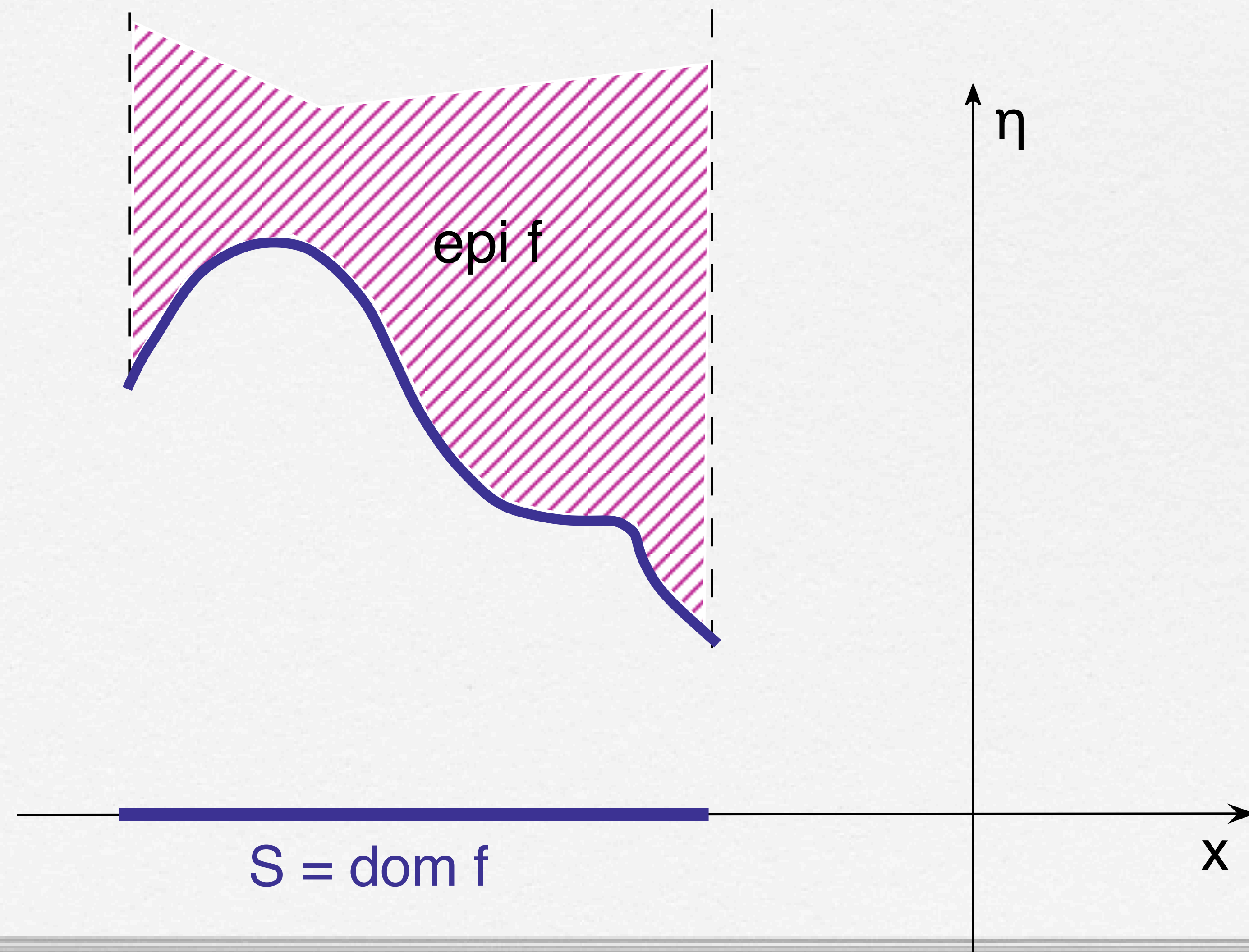
$S^\nu \to S$ uniformly $\Rightarrow d\!l(S^\nu, S) \to 0$

space of lsc-fcns(E)

lower semicontinuous

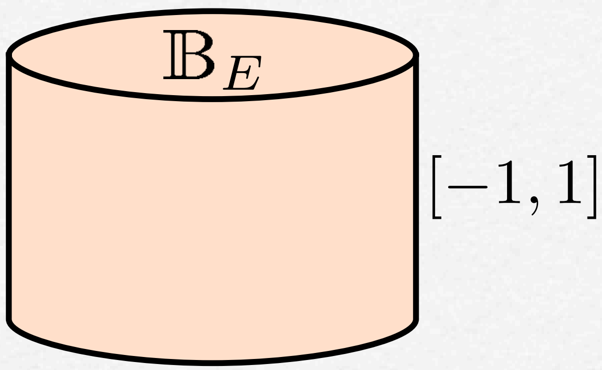
$$f : E \to \overline{\mathbb{R}} \text{ lsc} \iff \text{epi } f \subset E \times \mathbb{R} \text{ closed}$$

$$\text{epi } f = \left\{ (x, \eta) \,\middle|\, \eta \geq f(x) \right\}$$



space of lsc-fcns(E)

lower semicontinuous



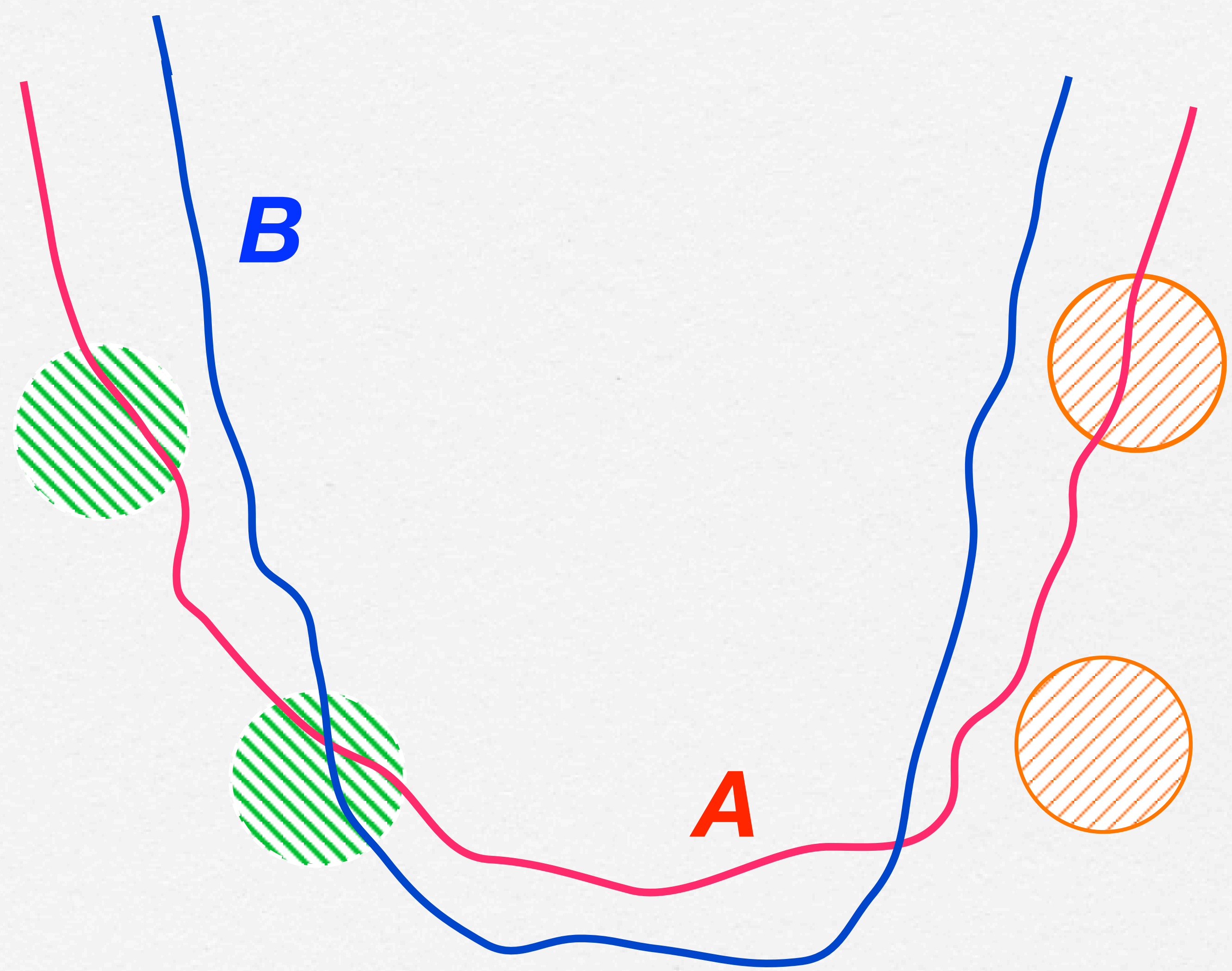
unit ball $\mathbb{B} = \mathbb{B}_E \times [-1, 1]$

$d\!l(f, g) = d\!l(\text{epi}\, f, \text{epi}\, g) \qquad d\!l_\rho,\ \hat{d\!l}_\rho$

$(\text{lsc-fcns}(E), d\!l)$ complete metric, Polish $E = \mathbb{R}^n$

$d\!l(f^\nu, f) \to 0 \Longrightarrow \text{argmin}\, f^\nu \rightrightarrows_v \text{argmin}\, f$

Hit-Open & Miss-Compact Sets



$\mathbb{R}^n$: Set-convergence ($\tau_{aw} = \tau_f$) topology

$\mathcal{F}$ = cl-sets($\mathbb{R}^n$), all closed subsets of $\mathbb{R}^n$

$\mathcal{F}^D$ = subsets $\mathbb{R}^n$ that *miss* $D = \{F \cap D = \emptyset\}$

$\mathcal{F}_D$ = subsets $\mathbb{R}^n$ that *hit* $D = \{F \cap D \neq \emptyset\}$

Hit-and-miss topology (= τ_f Fell topology)

subbase: $\{\mathcal{F}^K \mid K \text{ compact}\}$ & $\{\mathcal{F}_O \mid O \text{ open}\}$

$\mathbb{B}(x,\rho)$ closed ball, center x radius ρ, $\mathbb{B}^o(x,\rho)$ open

a subbase $\left\{\mathcal{F}^{\mathbb{B}(x,\rho)}, \mathcal{F}_{\mathbb{B}^o(x,\rho)} \;\middle|\; x \in \mathbb{Q}^d, \rho \in \mathbb{Q}_{++}\right\}$

countable base: $\left\{\mathcal{F}^{\mathbb{B}(x^1,\rho_1)\cup\ldots\cup\mathbb{B}(x^r,\rho_r)} \bigcap \mathcal{F}_{\mathbb{B}^o(x^1,\rho_1)\cup\ldots\cup\mathbb{B}^o(x^s,\rho_s)}\right\}$

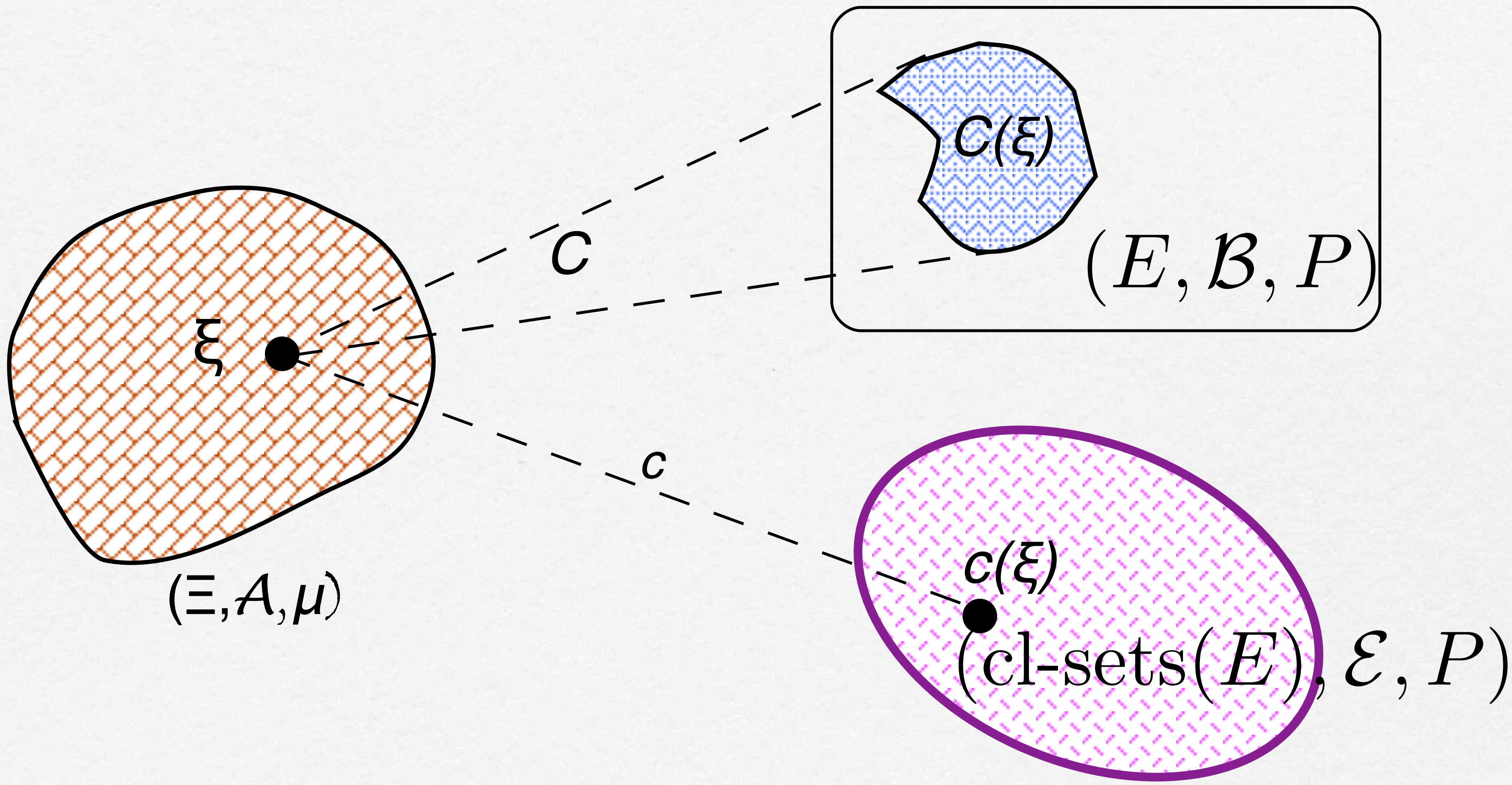
(cl-sets($\mathbb{R}^n$), τ_{aw}) Polish space (separable, complete metric)

Thursday, December 20, 12

Random Sets

Mattheron, Choquet
Salinetti-Wets, Castaing, Valadier, Hess, Stoyan, ...

Random sets



Random Closed Sets

$(\Xi, \mathcal{A}, P)$, $\quad \Xi \subset \mathbb{R}^N$ & E Polish, for example $\mathbb{R}^n$

$C : \Xi \rightrightarrows E$, $C(\xi) \subset E$ closed set for all $\xi \in \Xi$

& $C^{-1}(O) = \{\xi \,|\, C(\xi) \cap O \neq \emptyset\} \in \mathcal{A}$, $\forall O \subset E$, open

$\Rightarrow$ dom $C = C^{-1}(E) \in \mathcal{A}$, measurability ~ hit open sets

==

$c : \Xi \to$ cl-sets(E), $c(\xi) \sim C(\xi)$, $\mathcal{F}_O = \{F \subset E \text{ closed} \,|\, F \cap O \neq \emptyset\}$

$(\text{sets}(E), \mathcal{E})$, $\mathcal{E}$ Effrös field $= \sigma\text{-}\{\mathcal{F}_O \in \text{ sets}(\mathbb{R}^n), O \text{ open}\}$,

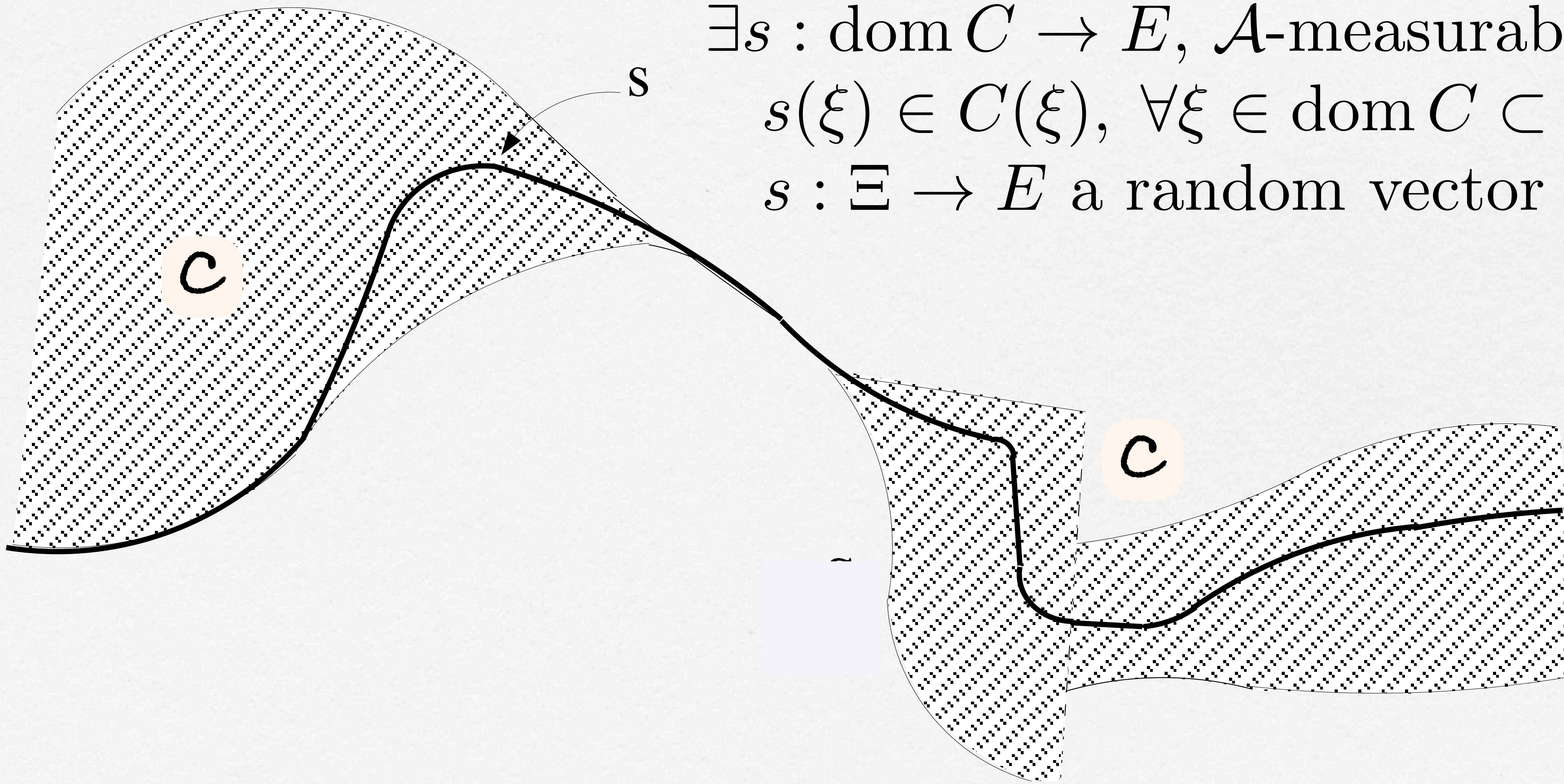
C measurable $\Leftrightarrow c$ measurable $[c^{-1}(\mathcal{F}_O) \in \mathcal{A}]$

$\mathcal{E} = \mathcal{B}$ Borel field when E Polish (complete separable metric space)

Measurable selection

- a random closed set C always admits a measurable selection!

$$\exists s : \operatorname{dom} C \to E,\ \mathcal{A}\text{-measurable},$$
$$s(\xi) \in C(\xi),\ \forall \xi \in \operatorname{dom} C \subset \Xi$$
$$s : \Xi \to E \text{ a random vector}$$



Castaing Representation

- C is a random closed set (& dom C measurable) $\Leftrightarrow$ it admits a Castaing representation: $\exists$ a <u>countable</u> family

$$\left\{ s^{\nu} : \text{ dom } C \to E, \text{ meas.-selections} \right\}$$

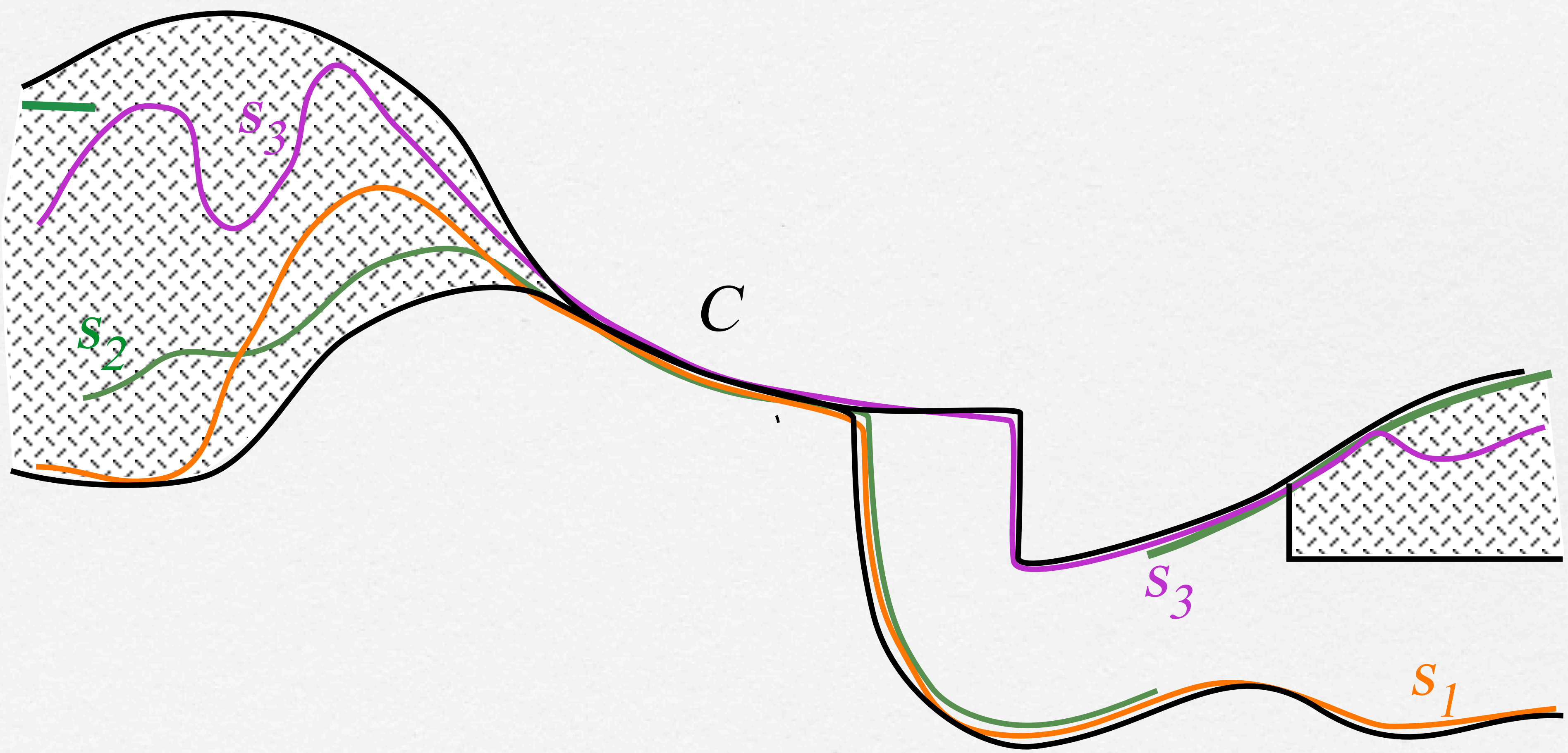
$$\text{cl} \bigcup_{\nu \in \mathbb{N}} s^{\nu}(\xi) = C(\xi), \forall \xi \in \text{ dom } C \subset \Xi$$

- Graph measurability

$(\Xi, \mathcal{A})$ P-complete for some P, (negligible sets are P-measurable)

C random set $\Leftrightarrow$ gph C $\mathcal{A} \otimes \mathcal{B}_n$-measurable

Castaing Representation



Random Elements: Convergence *(review)*

$$\boldsymbol{\xi} : (\Omega, \mathcal{F}, \mu) \to (\Xi, \mathcal{A}, P), \quad \boldsymbol{\xi}^\nu \xrightarrow{\star} \boldsymbol{\xi}$$

- a.s. (almost sure) convergence:

$$P\big\{\xi \,\big|\, \lim_\nu \boldsymbol{\xi}^\nu(\omega) = \xi \neq \boldsymbol{\xi}(\omega),\, \omega \in \Omega\big\} = 0$$

- convergence in probability:

$$P(|\boldsymbol{\xi}^\nu - \boldsymbol{\xi}| > \varepsilon) \to 0 \text{ for all } \varepsilon > 0$$

- convergence in distribution: $P^\nu \xrightarrow{\mathcal{D}} P$

a.s.-Convergence

* $\left\{C^\nu : \Xi \rightrightarrows \mathbb{R}^d, \nu \in \mathbb{N}\right\}$ random closed sets

* a.s. convergence: $dl\left(C^\nu(\xi), C(\xi)\right) \to 0$ for P-almost all $\xi \in \Xi$

  $C^\nu \to C$ a.s. $\Rightarrow C$ random closed set on $\Xi_0, \mu(\Xi_0) = 1$

* $C^\nu \to C$ P-a.s. and dom $C^\nu =$ dom C. Then,

  $\exists$ Castaing representations of $C^\nu \to$ a Castaing representation of C

  If $s : \Xi \to E$ is a measurable selection of C, then

  $\exists\, s^\nu : \Xi \to E$ selections of C^ν converging P-a.s. to s

* ('Egorov's Theorem': $C^\nu \to C$ μ-a.s. $\Leftrightarrow C^\nu \to C$ almost uniformly)

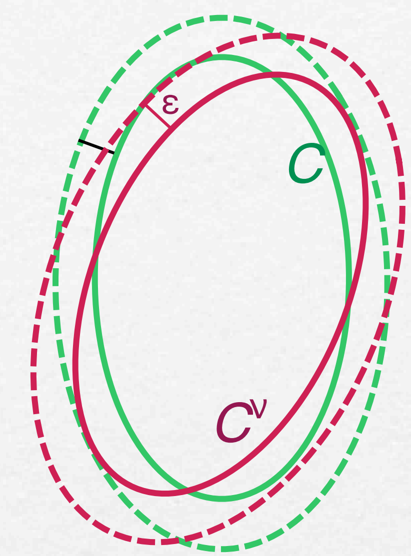
Convergence in probability

Let $\varepsilon^o C = \left\{ x \in \mathbb{R}^m \,\middle|\, d(x,C) < \varepsilon \right\}$, C^ν, C random sets

$$\Delta_{\varepsilon,\nu} = \left(C^\nu \setminus \varepsilon^o C \right) \cup \left(C \setminus \varepsilon^o C^\nu \right)$$

μ-a.s. convergence: $\mu\left\{ \xi \,\middle|\, C^\nu(\xi) \to C(\xi) \right\} = 1$

in probability: $P\left[\Delta^{-1}_{\varepsilon,\nu}(K) \right] \to 0, \forall \varepsilon > 0, K \in \mathcal{K} =$ cpct-sets



C^ν converges to C in probability

$\Leftrightarrow P(dl(C^\nu, C) > \varepsilon) \to 0$ for all $\varepsilon > 0$

$\Leftrightarrow$ every subsequence of $\{C^\nu\}_{\nu \in \mathbb{N}}$

contains a sub-subsequence converging μ-a.s to C

i.e., in probability $\Rightarrow$ in distribution $\left[\int h(\xi)\, dl(C^\nu(\xi), C(\xi))\, P(d\xi) \to 0 \right]$

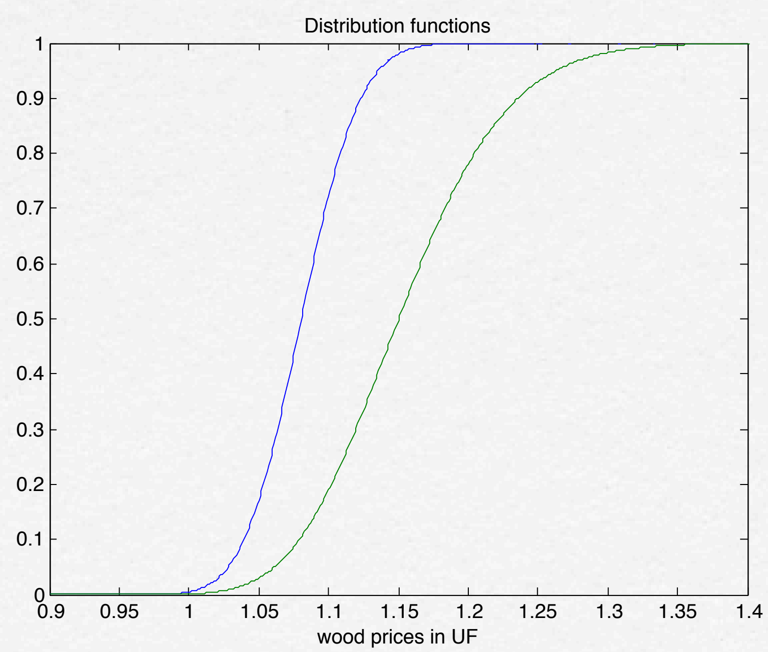
$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

P^ν, P defined on $(\mathbb{R}, \mathcal{B})$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\ \forall h$ continuous

$F^\nu(z) = P^\nu\big((-\infty, z)\big), \quad F(z) = P\big((-\infty, z)\big)$, cumulative distributions

$P^\nu \xrightarrow{\mathcal{D}} P \iff F^\nu \xrightarrow{p} F$ on cont $F = \{$ all continuity points of $F\}$



$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

$(F^\nu \xrightarrow{h} F,\ F$ usc $= -$lsc $)$

$\xrightarrow{h}$: hypo-convergence

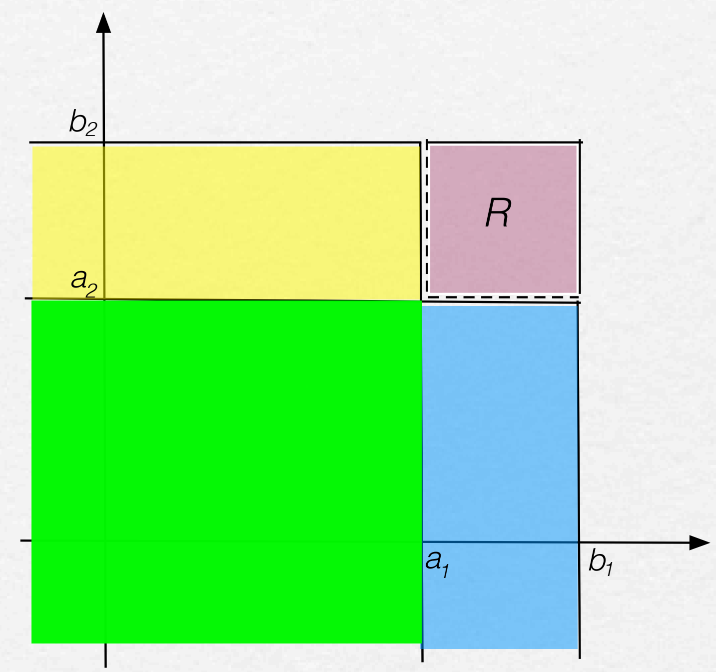
$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

P^ν, P defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\;\; \forall h$ continuous

$F^\nu(z) = P^\nu(\xi_i \le z_i, i = 1, \dots, n), \quad F(z) = P(\xi_i \le z_i, i = 1, \dots, n)$

1. $z \le \tilde{z} \implies F(z) \le F(\tilde{z})$ "increasing"
2. $\lim_{z \to \infty} F(z) = 1, \;\; \lim_{z_j \to -\infty} F(z) \to 0,$
3. F is usc (upper sc) $\limsup_{z' \to z} F(z') \le F(z),$
4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of R
   $\forall R \subset \mathbb{R}^n, \; P(\boldsymbol{\xi} \in R) = \sum_{v \in V} \operatorname{sgn}(v) F(v), \quad \operatorname{sgn}(v \in V) = (-1)^{\#a \text{ in } v}$



$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

Distribution of a random set

Borel σ-field: $\mathcal{B} = \sigma\text{-}\{\mathcal{F}^K \mid K \text{ compact}\}$ or $\sigma\text{-}\{\mathcal{F}_O \mid O \text{ open}\}\ldots$

Distribution $(P, \mathcal{B})$ regular, $\mathcal{K}$ compact subsets E

determined by values on $\{\mathcal{F}^K \mid K \in \mathcal{K}\}$ or $\{\mathcal{F}_K \mid K \in \mathcal{K}\}$

Distribution function (Choquet capacity):

$T : \mathcal{K} \to [0,1]$, (2) $T(\varnothing) = 0$ and $\forall \{K^\nu, \nu \in \{0\} \cup \mathbb{N}\} \subset \mathcal{K}$: (1,3)

a) $T(K^\nu) \searrow T(K)$ when $K^\nu \searrow K$ (~ usc on $\mathbb{R}^n$)

b) $\{D_\nu : \mathcal{K} \to [0,1]\}_{\nu \in \mathbb{N}}$ where $D_0(K^0) = 1 - T(K^0)$

(4) $D_1(K^0; K^1) = D_0(K^0) - D_0(K^0 \cup K^1)$ and for $\nu = 2, \ldots$

$D_\nu(K^0; K^1, \ldots, K^\nu) = D_{\nu-1}(K^0; K^1, \ldots, K^{\nu-1}) - D_{\nu-1}(K^0 \cup K^\nu; K^1, \ldots, K^{\nu-1})$

(~ rectangle condition on $\mathbb{R}^n$)

Thursday, December 20, 12

Existence-Uniqueness T

P on $\mathcal{B}$ determines a unique distribution function T on $\mathcal{K}$

$$T(K) = P(\mathcal{F}_K)$$

$$D_\nu(K^0; K^1, \ldots, K^\nu) = P(\mathcal{F}^{K^0} \cap \mathcal{F}_{K^1} \cap \cdots \cap \mathcal{F}_{K^\nu})$$

T on $\mathcal{K}$ determines a unique probability measure P.

Proof. via Choquet Capacity Theorem ***(Matheron)***

(refined) via probabilistic arguments ***(Salinetti-Wets)***

$C : \Xi \rightrightarrows \mathbb{R}^d$ a random closed set

$(P, \mathcal{B})$ induced probability measure:

$$P(\mathcal{F}_G) = P\left[C^{-1}(G)\right] \quad \forall G \in \mathcal{B}, \quad T(K) = P\left[C^{-1}(K)\right] \quad \forall K \in \mathcal{K}$$

Convergence in Distribution

random sets C^ν converge in distribution to C when

induced P^ν narrow-converge to $P : P^\nu \to_n P \quad = P^\nu \xrightarrow{\mathcal{D}} P$

$\Leftrightarrow T^\nu \to_p T$ on $\mathcal{K}_{T\text{-cont}}$ (convergence of distribution functions)

$\mathcal{K}_{T\text{-cont}}$?

a) $\forall C^\nu, \nu \in N, \exists$ converging subsequence (pre-compact)

b) $K^\nu \nearrow K = \text{ cl} \bigcup_\nu K^\nu$ regularly if int $K \subset \bigcup_\nu K^\nu$

c) distribution (fcn) continuity: $\lim_\nu T(K^\nu) = T(\text{cl} \bigcup_\nu K^\nu)$

d) convergence $T^\nu \to_p T$ on $\mathcal{C}_T$ continuity set $\Rightarrow P^\nu \to_n P$

e) $P^\nu \to_n P \Leftrightarrow T^\nu \to_p T$ on $\mathcal{C}_T^{ub} = \mathcal{C}_T \cap \mathcal{K}^{ub}$

$\mathcal{K}^{ub} =$ finite union of rational ball, positive radius

f) $\varepsilon \mapsto T(K + \varepsilon \mathbb{B})$: countable number of discontinuities

a detour about rates

$T^\nu \to_p T$ on $C_T \Leftrightarrow P^\nu \to_n P$ (Polish space: E, d)

P^ν, P defined on $\mathcal{B}$

probability sc-measures on cl-sets(E): λ

(i) $\lambda \geq 0$, (ii) $\lambda \nearrow$ $\lambda(C^1) \leq \lambda(C^2)$ if $C^1 \subset C^2$

(iii) λ is τ_f-usc on cl-sets(E), (iv) $\lambda(\emptyset) = 0, \lambda(E) = 1$

(v) λ modular: $\lambda(C^1) + \lambda(C^2) = \lambda(C^1 \cup C^2) + \lambda(C^1 \cap C^2)$

P and $\lambda = P_{\text{cl-sets}}$ define each other uniquely (E complete $\Rightarrow$ tight)

$\{P^\nu, \nu \in \mathbb{N}\}$ tight: $P^\nu \to_n P \Leftrightarrow \lambda^\nu \to_h \lambda$ $(\sim -\lambda^\nu \to_e -\lambda)$ on cl-sets(E)

tightness ~ equi-usc of $\{\lambda^\nu\}_{\nu \in \mathbb{N}}$ at $\emptyset$

rates: $dl(\lambda^\nu, \lambda) \to 0$ (for $\mathbb{R}$-valued r.v., "~" Skorohod distance)

Random Sets
Convergence & Expectation

Artstein-Vitale-Hart-Wets,
Cressis, Hiai, Weyl, ...

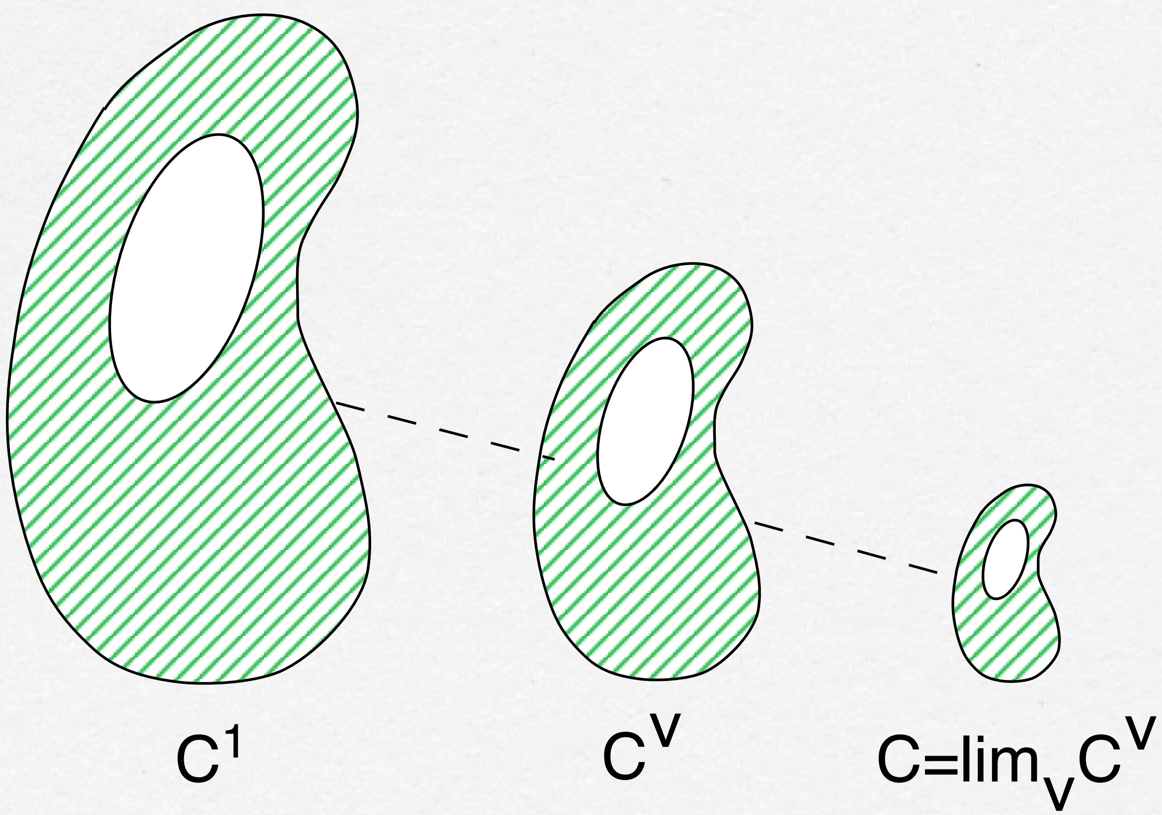
Outer/Inner Limits

outer limit: $\mathrm{Lo}_\nu C^\nu = \left\{ x \in \text{ cluster-points}\{x^\nu\},\ x^\nu \in C^\nu \right\} = \mathrm{Ls}_\nu C^\nu$

inner limit: $\mathrm{Li}_\nu C^\nu = \left\{ x = \lim_\nu x^\nu, x^\nu \in C^\nu \subset \mathbb{R}^n \right\} \subset \mathrm{Lo}_\nu C^\nu$

limit: $C^\nu \to C$ if $C = \mathrm{Li}_\nu C^\nu = \mathrm{Lo}_\nu C^\nu$ (Painlevé - Kuratowski)

All limit sets are closed



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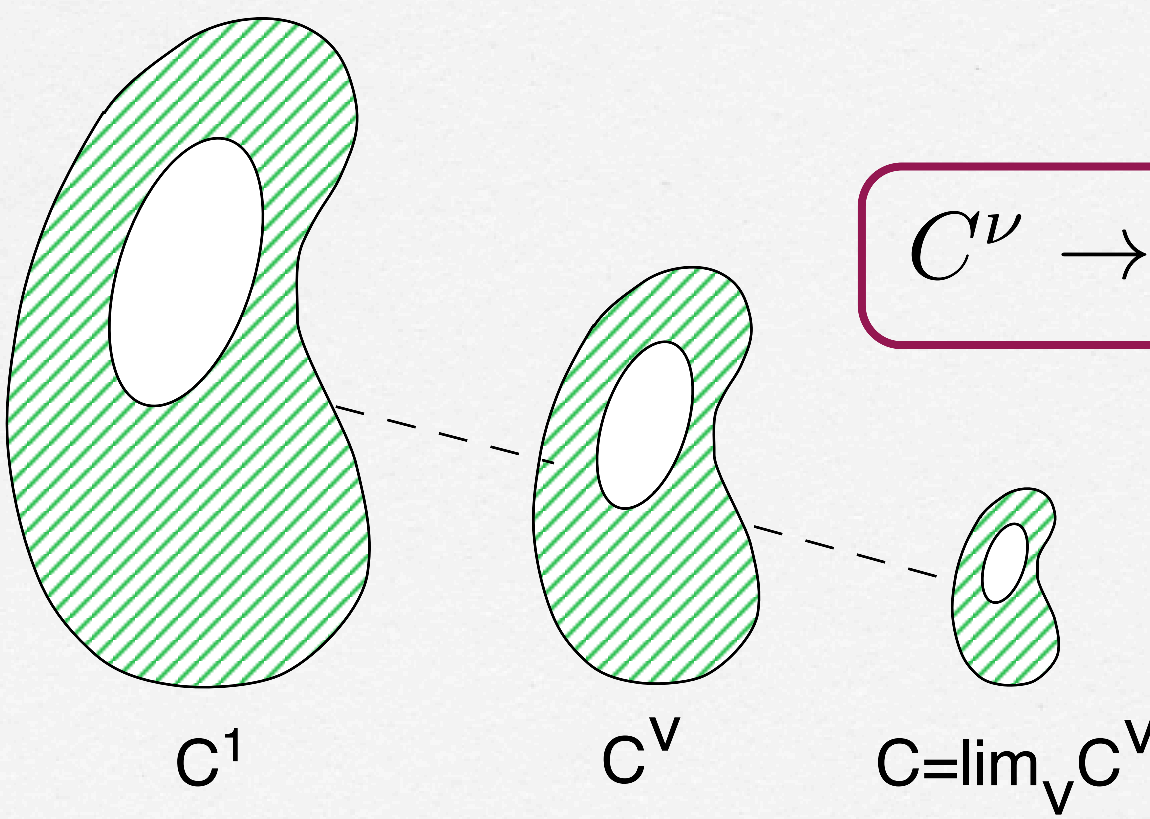

$$C^\nu \to C \iff d\!l(C^\nu, C) \to 0$$

Characterizing a.s. convergence

$\Big\{C; C^\nu : \Xi \rightrightarrows \mathbb{R}^n,\, \nu \in \mathbb{N}\Big\}$ random closed sets. Then,

1. $C^\nu \to C$ *a.s.*, $d\!l(C^\nu, C) \to 0$ *a.s.*, $\mathrm{Lo}_\nu(C^\nu) \subset C \subset \mathrm{Li}_\nu(C^\nu)$ *a.s.*,

2. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$, $d(x, C^\nu(\xi)) \to d(x, C(\xi))$,

3. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$,

$$\lim_{\rho \nearrow \infty} \mathrm{Lo}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x,\rho)\big) \subset C(\xi) \subset \lim_{\rho \nearrow \infty} \mathrm{Li}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x,\rho)\big).$$

"Proof 1. ⇔ 2."

$$C^\nu \to C \iff \forall x \in \mathbb{R}^n,\ d(x, C^\nu) \to d(x, C) \text{ provided } E = \mathbb{R}^n.$$

$C^\nu \to C$ if and only if the hit-miss criterion is satisfied

C hits $\mathbb{B}^o(x,\rho)$ then C^ν hits $\mathbb{B}^o(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \subset \mathrm{Li}_\nu C^\nu \iff d(x,C) \geq \mathrm{limsup}_\nu\, d(x, C^\nu), \forall x$

C misses $\mathbb{B}(x,\rho)$ then C^ν misses $\mathbb{B}(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \supset \mathrm{Lo}_\nu C^\nu \iff d(x,C) \geq \mathrm{liminf}_\nu\, d(x, C^\nu), \forall x$

Building Castaing representations

$C : \Xi \rightrightarrows \mathbb{R}^n$, a random closed set. Let

$$A = \left\{ a_k = (a_k^1, \dots, a_k^n, a_k^{n+1}) \,\middle|\, a_k^i \in \mathbb{Q}^n \text{ \& aff. independent } \right\}$$

for $\emptyset \neq D = D^0$ closed, define $\operatorname{prj}_D a_k = \operatorname{prj}_{D^n} a_k^{n+1}$
where $D^l = \operatorname{prj}_{D^{l-1}} a_k^l$ for $l = 1, \dots, n$

$\operatorname{prj}_D a_k$ is a singleton: intersection of n+1 "aff. independent" spheres.
Moreover, $\{ \operatorname{prj}_D a_k, a_k \in A \}$ also dense in D

$s_k : \Xi \to \mathbb{R}^n$ with $s_k(\xi) = \operatorname{prj}_{C(\xi)} a_k$ is a measurable selection of C

$\square$ When D is a random closed set, so is $\xi \mapsto \operatorname{prj}_{D(\xi)} a$, $a \in \mathbb{R}^n$
repeat the argument $n+1$ times to obtain s_k measurable. $\square$

Converging Castaing representations

$C^\nu : \Xi \rightrightarrows \mathbb{R}^n$ random closed sets converging P-$a.s.$ to C, $\operatorname{dom} C^\nu = \operatorname{dom} C$.
Then, $\exists \{ s_k^\nu, k \in \mathbb{N} \}$ Castaing representations of C^ν converging for each k to a Castaing representation $\{ s_k, k \in \mathbb{N} \}$ of C.

$\square$ All Castaing representations are built via our earlier "projections".
Then, $\forall \xi \in \Xi_1, s_k^\nu(\xi) \to s_k(\xi)$, $P(\Xi_1) = 1$ the set of $a.s.$-convergence.
Since, P-$a.s.$ convergence of $C^\nu \to C \Longrightarrow$ (rely on 2. earlier)

$$d(a_k^1, s_k^\nu(\xi)) = d(a_k^1, C^\nu(\xi)) \to d(a_k^1, C(\xi)) = d(a_k^1, s_k(\xi)), \forall \xi \in \Xi_1. \quad \square$$

(a) Convergence of Castaing representations $\not\Rightarrow$ convergence of random sets!
(b) v meas-selection of $C \Rightarrow \exists v^\nu$ meas-selection of C^ν converging $a.s.$ to v.

“Simple” random sets

$C : \Xi \rightrightarrows \mathbb{R}^n$ is a *simple* random set if $\operatorname{rge} C$ is finite.
C is a closed random set $\iff$ $C = P\text{-}a.s.$ limit of simple random sets.

$\square$ $\Leftarrow$: the limit of a sequence of random sets is a random set
$\Rightarrow$: let $C^\nu = C \cap \nu\mathbb{B}$, unif. bounded closed random set, $C = \operatorname{Lm}_\nu C^\nu$
build (via "prj") Castaing representations $\{r_k^\nu\}_{k\in\mathbb{N}}$ of the C^ν
let $\{s_k^\nu\}_{k\in\mathbb{N}'} = \bigcup_{\upsilon\leq\nu} \{r_k^\upsilon\}_{\upsilon\in\mathbb{N}}$, also Castaing for C^ν
$D_k^\nu = \bigcup_{j\leq k} s_j^\nu$ $d\!l$-converge uniformly to C^ν as $k \to \infty$
since each $s_k^\nu = \lim_{l\to\infty} s_{kl}^\nu$ uniformly, s_{kl}^ν simple random variables
$\Delta_{kl}^\nu = \bigcup_{j\leq k} s_{jl}^\nu$ is a simple random set, $C(\xi) = \operatorname{Lm}_\nu \operatorname{Lm}_k \operatorname{Lm}_l \Delta_{kl}^\nu(\xi)$

$\Delta_{kl}^\nu \overset{u}{\to} D_k^\nu \overset{u}{\to} C^\nu$ allows diagonalization to find $\Delta_{k^\nu l^\nu}^\nu \to C$. $\square$

Sierpiński-Lyapunov Theorems

$(\Xi, \mathcal{A})$ a measure space

Sierpiński (1922). Suppose P is an atomless probability measure.
Given $A_0, A_1 \in \mathcal{A}$ with $0 \leq P(A_0) \leq P(A_1) \leq 1$, then
$\forall \lambda \in [0,1], \exists\ A_\lambda \in \mathcal{A}$ such that $P(A_\lambda) = (1-\lambda)P(A_0) + \lambda P(A_1)$.
In particular, it implies $\forall\, \lambda \in [0,1], \exists\ A \in \mathcal{A}$ such that $P(A) = \lambda$;
choose $A_0 = \emptyset$ and $A_1 = \Xi$.

Lyapunov (1940) $\mu : \mathcal{A} \to \mathbb{R}^n$ atomless, σ-additive measure.
For $A \in \mathcal{A}$, define $\operatorname{rge} \mu(A) = \left\{ \mu(B) \,\middle|\, B \subset A \cap \mathcal{A} \right\}$. Then,
$\operatorname{rge} \mu(\Xi) \subset \mathbb{R}^n$ is convex and if μ is also bounded, it's compact.

Expectation: simple random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a simple random set, i.e., $\operatorname{rge} C = \left\{ z^k \in \mathbb{R}^n \,\middle|\, k \in K, |K| \text{ finite } \right\}$
Given $\bar{r}, \bar{s} \in EC = \mathbb{E}\{C(\boldsymbol{\xi})\} \implies$

$\exists$ simple selections $r, s : \Xi \to \mathbb{R}^n$ with $\mathbb{E}\{r(\boldsymbol{\xi})\} = \bar{r}, \mathbb{E}\{s(\boldsymbol{\xi})\} = \bar{s}$.

Let $\lambda \in [0,1]$. Define $v : \Xi \to \mathbb{R}^n$ as follows:

1. partition Ξ into subsets $A_=$ and $\mathcal{A}_{\neq}$
2. $A_= = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = s(\xi) \right\} \in \mathcal{A}$
3. $A = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = z_k, s(\xi) = z_l, k \neq l \right\} \in \mathcal{A}_{\neq}$, a finite collection
4. split each $A \in \mathcal{A}_{\neq}$, $P(A_r) = \lambda P(A)$ & $A_s = A \setminus A_r$ (Sierpiński)

$$\text{set } v(\xi) = \begin{cases} r(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_r \;\cup\; A_= \\ s(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_s \end{cases}$$

then $\bar{v} = \mathbb{E}\{v(\xi)\} = \lambda \bar{r} + (1-\lambda)\bar{s} \implies EC$ convex.
Clearly EC is bounded and it's easy to show it's also closed $\implies$ compact.

Expectation of random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a closed random set

$\Longrightarrow C = P\text{-}a.s.$ limit of simple random sets,

say $C^\nu \underset{a.s.}{\to} C$ with $C^\nu \nearrow$ w.l.o.g

$EC^\nu = \mathbb{E}\{C^\nu(\boldsymbol{\xi})\} \nearrow$ are convex, compact $\Rightarrow$

$EC = \mathbb{E}\{C(\boldsymbol{\xi})\} = \bigcup_\nu EC^\nu$

$\Longrightarrow EC$ convex

$\Longrightarrow EC$ closed if C is integrably bounded

$\Longrightarrow$ compact if $\operatorname{rge} C$ is bounded

Random Mappings

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$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0 \;\; \text{“approximated” by } G^\nu(x) = 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample}, \;\; G^\nu(x) = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x)$$

$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
$$\xi^1, \dots, \xi^\nu \;\; \text{sample, approximation } \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$$

An appendix: more about solution bounds

$$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\}, \; x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \textstyle\int_\Xi f(\xi, x)\, P(d\xi)$$
$$\xi^1, \dots \xi^\nu \;\; \text{sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\xi^l, x), \; x \in C$$

ε-Solutions Estimates

$f, g : E \to \overline{\mathbb{R}}$ lsc, convex & $\operatorname{argmin} f \cap \bar\rho\mathbb{B} \neq \emptyset \neq \operatorname{argmin} g \cap \bar\rho\mathbb{B}$

$\min f \geq -\bar\rho, \quad \min g \geq -\bar\rho$

with $\rho > \bar\rho$, $\varepsilon > 0$, $\quad \bar\eta = \hat{d\!l}_\rho(f, g)$:

$$\begin{aligned} \hat{d\!l}_\rho\big(\varepsilon\text{- argmin} f,\ \varepsilon\text{- argmin} g\big) \ &\leq\ \bar\eta\Big(1 + \frac{2\rho}{\bar\eta + \varepsilon/2}\Big) \\ &\leq\ \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(f, g) \end{aligned}$$

Epi-distance alternative

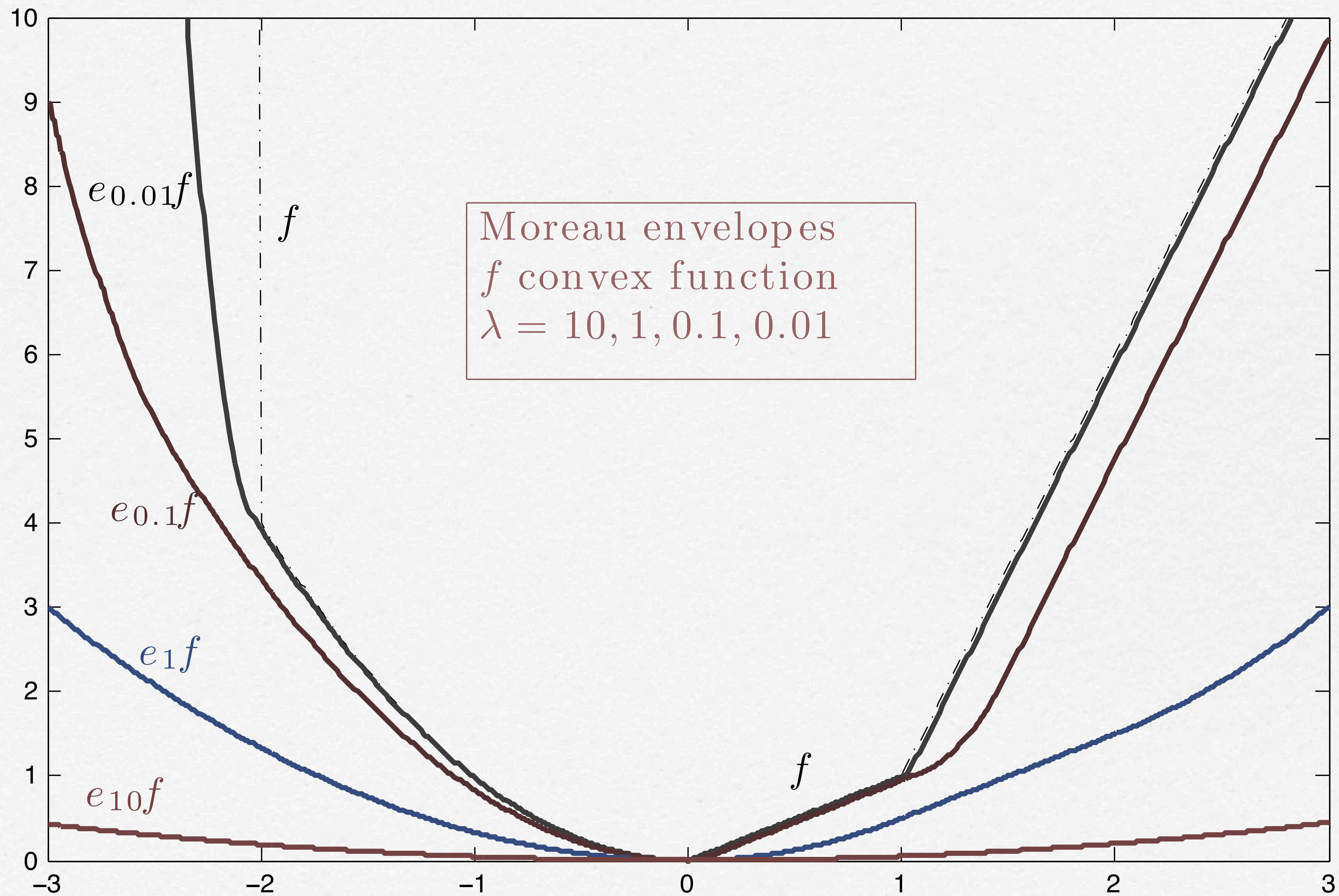
$$\check{d\!l}_{\lambda,\rho}(f,g)$$

same topology: τ_{aw}

Moreau envelopes

epi-sums ~ sum of epigraphs

$$(f \# g)(x) = \inf_u \{ f(u) + g(u-x) \}, \quad e_\lambda f(x) \text{ with } g = \frac{1}{2\lambda} | \cdot |^2$$



Alternative epi-distance

$$\check{d\!l}_{\lambda,\rho}(f,g) = \sup\left\{|f_\lambda(x) - g_\lambda(x)| \,\middle|\, x \in \rho\mathbb{B}\right\}$$

$$f, g \quad \text{majorizing} \quad -\alpha_1|\cdot|^p - \alpha_0$$

1. $\forall\, \lambda \geq 0,\ \check{d\!l}_{\lambda,\rho}(f,g) \leq \beta(\lambda,\rho)\hat{d\!l}_{\gamma(\lambda,\rho)}(f,g)$

2. $\hat{d\!l}_\rho(f_\lambda, g_\lambda) \leq \check{d\!l}_{\lambda,\rho}(f,g)$
   $\hat{d\!l}_\rho(f,g) \leq \check{d\!l}_{\lambda,9\rho}(f,g) + \kappa(\lambda,\alpha_1,\alpha_0,p)$

“Quantitative” LLN-a.s.

E separable Banach space, f random lsc function, $\{\boldsymbol{\xi}, \boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid

1. $\{f(\xi,\,\cdot\,), \xi \in \Xi\}$ separable subspace (lsc-fcns$(E), \tau_{aw})$
2. P-$a.s.$, $\forall\, \theta > 0, \rho \geq 0, \nu$:
   $\check{d\!l}_{\theta,\lambda}\left(\frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l,\,\cdot\,), \frac{1}{\nu}\sum_{l=1}^{\nu} f_\lambda(\boldsymbol{\xi}^l,\,\cdot\,)\right) \leq \varepsilon_{\theta,\rho}(\lambda)$
   with $\varepsilon_{\theta,\rho}(\lambda) \to 0$ as $\lambda \searrow 0$
3. $\forall\, \theta > 0, \rho \geq 0, \check{d\!l}_{\theta,\rho}\big(Ef_\lambda, Ef\big) \searrow 0$ as $\lambda \searrow 0$.

Then, $d\!l(E^\nu f, Ef) \to 0 \quad P^\infty$-$a.s.$

convex

$x \mapsto f(\xi, x)$ convex $\Longrightarrow$ conditions 2 & 3.

$\hat{d\!l}_\rho\big(\varepsilon\text{- argmin}\, E^\nu f,\ \varepsilon\text{- argmin}\, Ef\big) \leq \ \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(E^\nu f, Ef)$

E reflexive, $E^\nu f \xrightarrow[s,w]{} Ef \implies d\!l(E^\nu f, Ef) \to 0\ \ a.s.$

Approximating Mappings

Why?

$G: E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) \ni 0$, approximations?

$$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0 \ \text{ “approximated” by } G^\nu(x) = 0$$
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$$\xi^1, \ldots, \xi^\nu \ \text{ sample, approximation } \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$$

$$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\}, \ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \textstyle\int_\Xi f(\xi, x)\, P(d\xi)$$
$$\xi^1, \ldots \xi^\nu \ \text{ sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\xi^l, x), \ x \in C$$

Examples:

$\min f = f_0 + \iota_C$, optimality: "$0 \in \partial f(\bar{x}) = S(x)$" $\sim 0 = \nabla f(\bar{x}))$

generally, $\partial\big(f + g\big) \neq \partial f + \partial g$

$\mathbb{C}.\mathbb{Q}.$ (Constraint Qualification): $-N_C(\bar{x}) \cap \partial^\infty f_0(\bar{x}) = \{0\}$

$v \in \partial^\infty f_0(\bar{x}) =$ horizon subgradient if

$\exists\, \mathrm{x}^\nu \to \bar{x}$ with $f(x^\nu) \to f(\bar{x}), v^\nu \in \hat{\partial} f(x^\nu), \lambda_\nu \searrow 0 \;\&\; \lambda_\nu v^\nu \to v$

with $\mathbb{C}.\mathbb{Q}.$ $\bar{x}$ locally optimal $\Rightarrow \partial f_0(\bar{x}) + N_C(\bar{x}) = S(\bar{x}) \ni 0$

f convex ($\Rightarrow$ regular), $\partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0$

$\Rightarrow$ globally optimal (without $\mathbb{C}.\mathbb{Q}$)

When f_0, C are convex: $-\partial f_0(\bar{x}) \in N_C(\bar{x})$,

a functional variational inequality

“Variational” Approximations

(E, d) Polish, in paricular $E = \mathbb{R}^n$

$(\text{cl-sets}(E), d\!\!l)$ complete metric space; Polish if $E = \mathbb{R}^n$

$d\!\!l(C^\nu, C) \to 0 \iff C^\nu \to C$

osc-mappings = closed graph

$(\text{osc-maps}(S), d\!\!l)$ complete, metric space;

Polish if $\text{dom} \subset \mathbb{R}^n,\ \text{rge} \subset \mathbb{R}^m$

Convergence:

$S^\nu \xrightarrow{g} S$ if $d\!\!l(\text{gph}\, S^\nu, \text{gph}\, S) \to 0 \implies (S^\nu)^{-1}(0) \rightrightarrows_v S^-1(0)$

Thursday, December 20, 12

Why?

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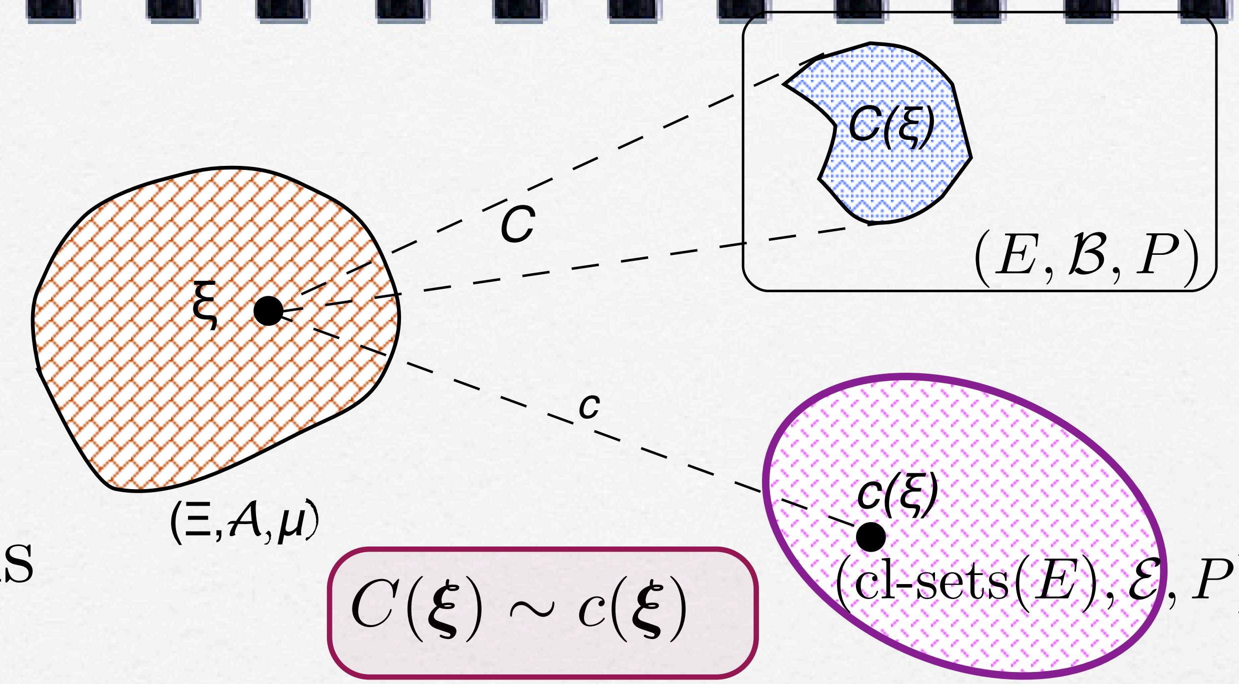
$$\xi^1, \dots \xi^\nu \ \text{ sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$$

Random sets



C 'covered' by countable selections

Castaing represnetation

a.s convergence: $P\{\xi \,|\, d\!\!l(C^\nu(\xi), C(\xi)) \to 0\} = 0$

$\Rightarrow$ in probability: $\forall \varepsilon > 0,\ P\{\xi \,|\, d\!\!l(C^\nu(\xi), C(\xi)) > \varepsilon\} \to 0$

$\Rightarrow$ in distribution T : cpct-sets$(E) \to [0,1],\ T(\emptyset) = 0,$

(a) $T(K^\nu) \searrow T(K)$ for $K^\nu) \searrow K$, (b) 'rectangle cond'n'

$P^\nu \overset{\mathcal{D}}{\to} P \iff T^\nu \to T$ on cpct-sets$(\mathbb{R}^n)$

or, even, on finite union of closed rational balls.

Random Sets: Expectation

Random set: Expectation

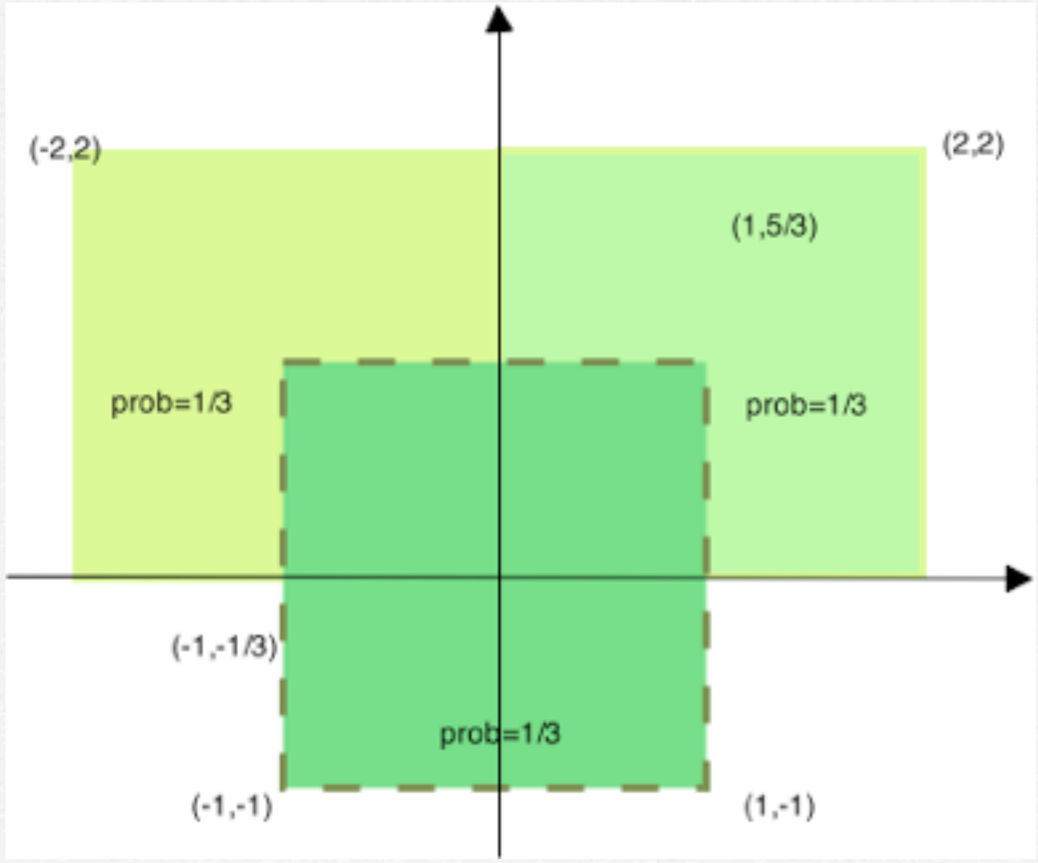
$$EC = \mathbb{E}\{C(\xi)\} = \left\{ \int_{\Xi} s(\xi)P(d\xi) \,\middle\|\, s(\bullet)\ P\text{-summable selection} \right\}$$

..not necessarily closed even when C is closed-valued

Convexity:

C P-atom convex $\Rightarrow EC$ is convex

(certainly when P is atomless).



Random set: Expectation

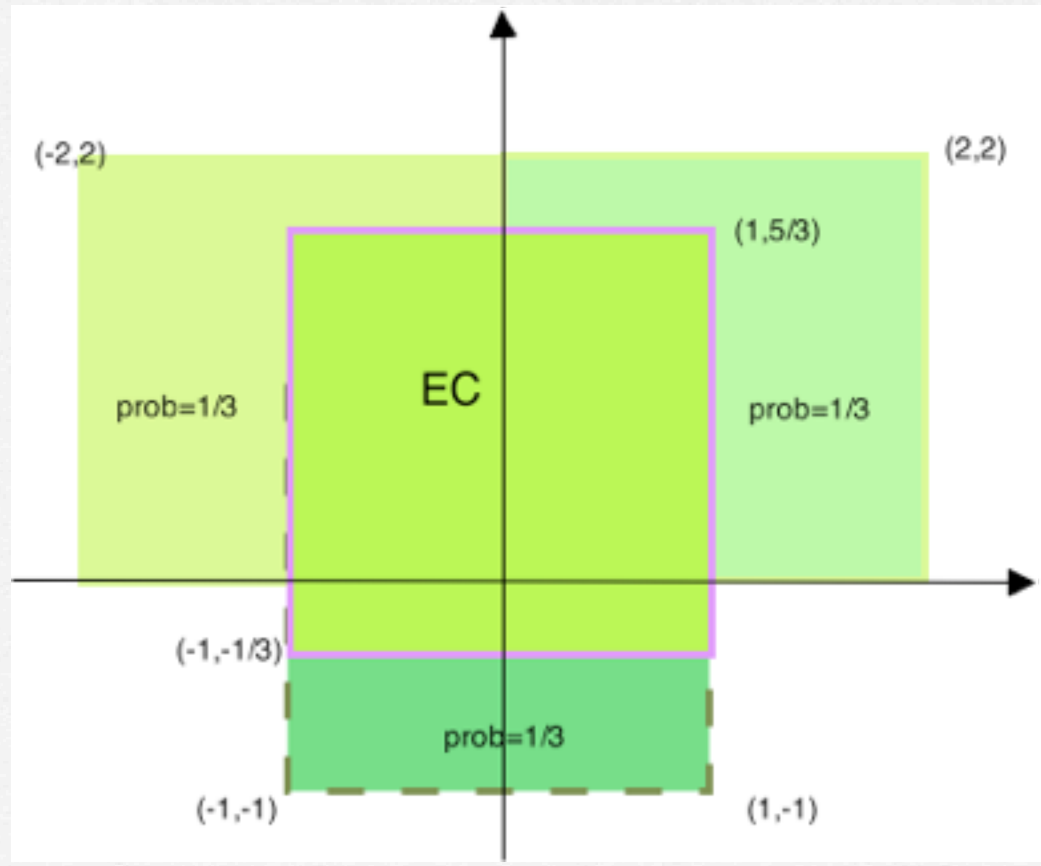
$$EC = \mathbb{E}\{C(\xi)\} = \left\{ \int_\Xi s(\xi)\,P(d\xi) \,\Big\|\, s(\bullet)\ P\text{-summable selection} \right\}$$

..not necessarily closed even when C is closed-valued

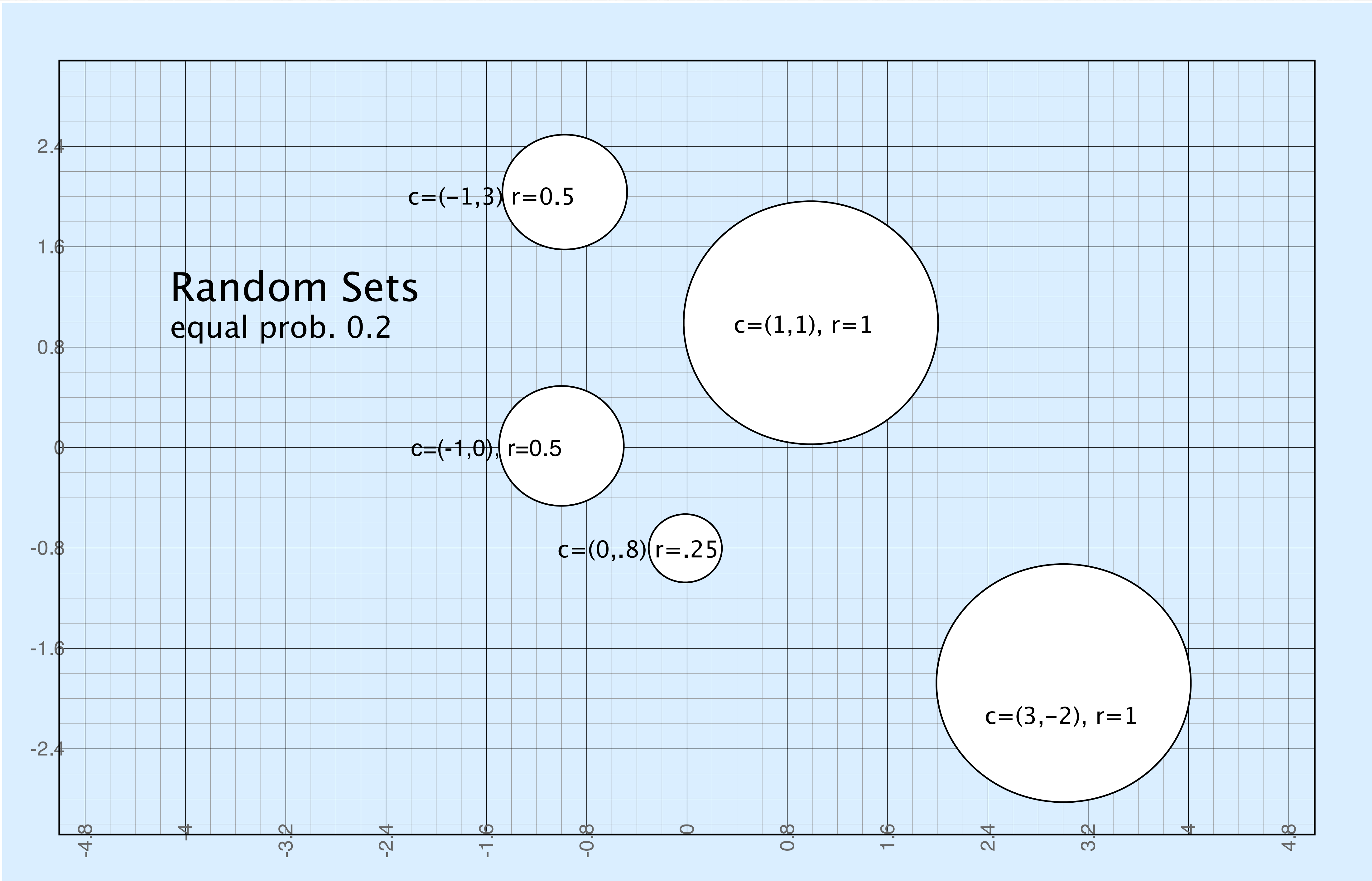
Convexity:

C P-atom convex $\Rightarrow EC$ is convex

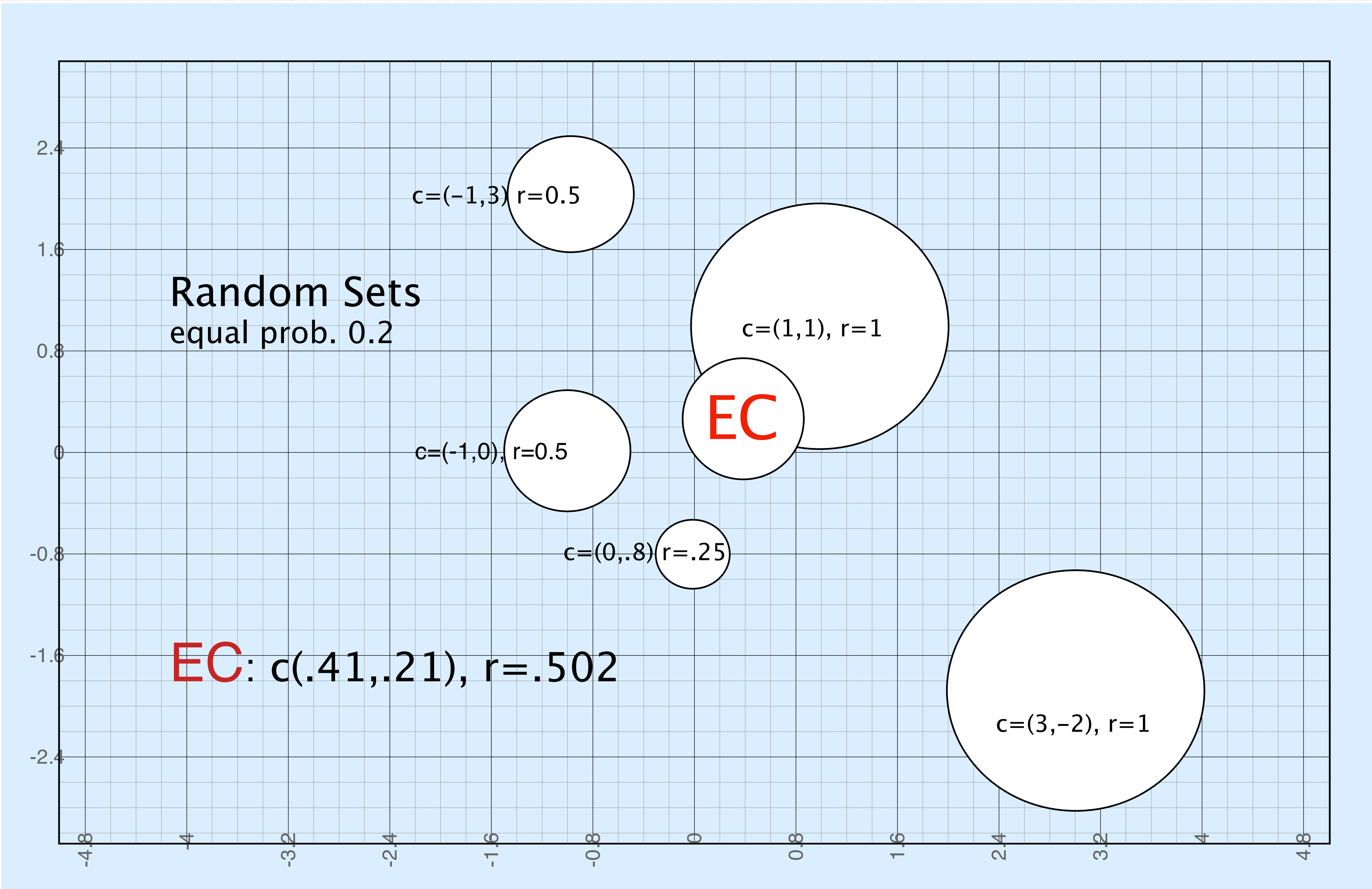
(certainly when P is atomless).



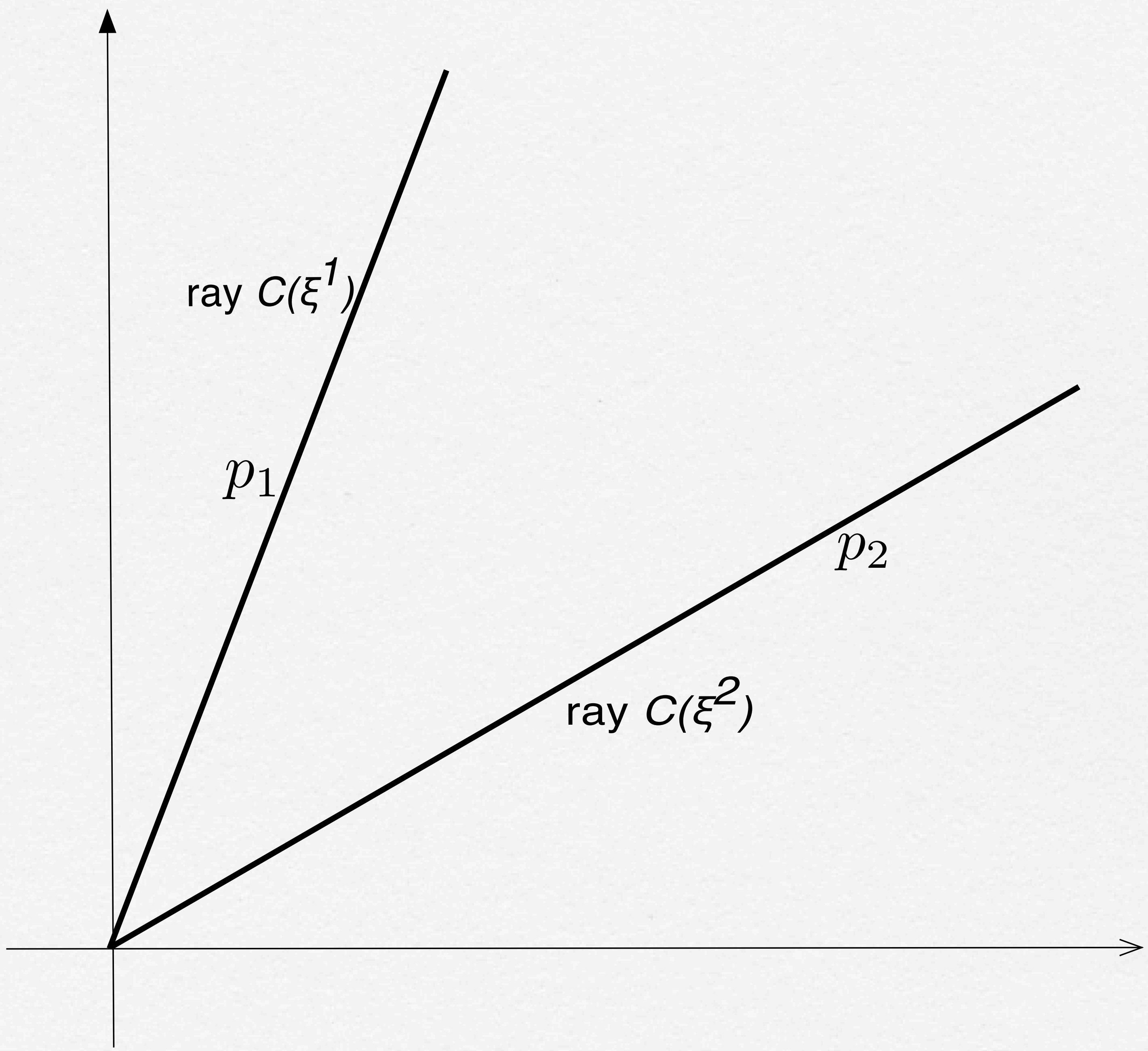
Bounded random set



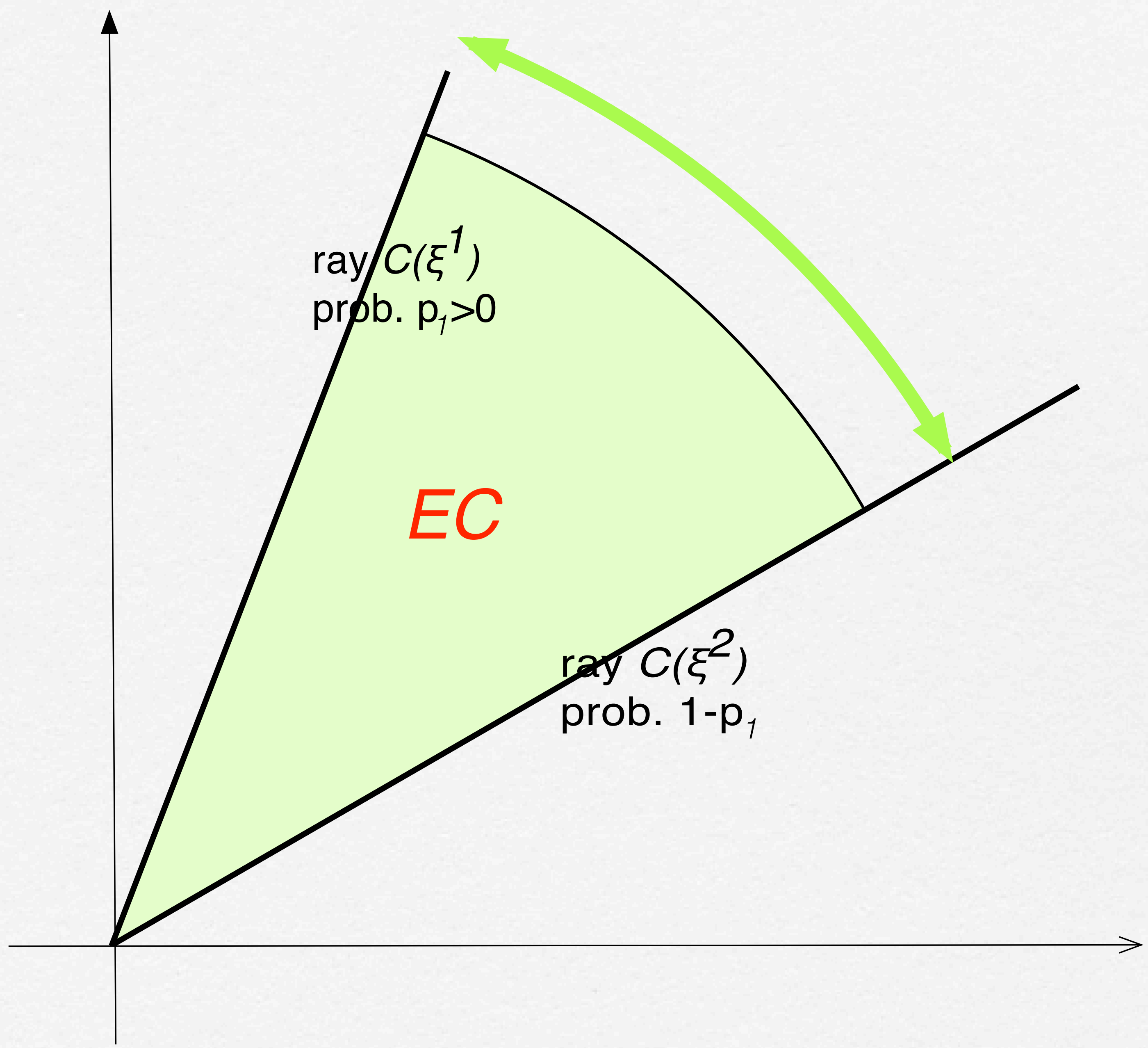
Expectation: Bounded r. set



Expectation: Unbounded r. sets



Expectation: Unbounded r. sets



Some properties: $\mathbb{E}\{C(\boldsymbol{\xi})\}$

- measure P atomless, then $EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ is convex (Richter, Lyapounov,...)
- P is P-atom convex $\Longrightarrow$ EC is convex; [an atom contains no (measurable) subset of positive probability]
- C a random set, $\emptyset \neq EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ contains no line, then

$$\operatorname{con} EC = \mathbb{E}\{\operatorname{con} X(\boldsymbol{\xi})\}$$

  this essentially requires that $C(\xi) \subset$ a pointed cone
- in general, the expectation of a (closed-valued) random set is *not* closed
- if $|C| = \mathbb{E}\big\{ \sup \left[\, |s(\xi)| \,\big|\, s(\xi) \in C(\xi) \,\right] \big\} < \infty$ then EC is closed; C is then *integrably bounded.*

Strong law of large numbers
for random sets *(Artstein-Hart)*

$C : \Xi \rightrightarrows E$ measurable, $\{\xi^\nu, \nu \in \mathbb{N}\}$ iid Ξ-valued random variables

$C(\xi^\nu)$ iid random sets (i.e. induced P^ν independent and identical)

$EC = \mathbb{E}\{C(\bullet)\} = \left\{ \int_\Xi s(\xi) P(d\xi) \,\middle|\, s : P\text{-summable } C(\xi)\text{-selection} \right\}$

independence $\Rightarrow$ all (measurable) selections are independent

$\{C(\xi^\nu) : \Xi \rightrightarrows \mathbb{R}^m \; \nu \in \mathbb{N}\}$ iid with $EC \neq \emptyset$. Then, with

$$C^\nu(\xi^\infty) = \nu^{-1}\left(\sum_{k=1}^{\nu} C(\xi^k)\right) \to \bar{C} = \text{cl con } EC \;\; P^\infty\text{-a.s.}$$

$\text{Lo}_\nu C^\nu(\xi^\infty) \subset \bar{C} \Leftrightarrow \limsup_\nu \sigma_{C^\nu} \leq \sigma_{\bar{C}}$ support functions

$\text{Li}_\nu C^\nu(\xi^\infty) \supset \bar{C}$ relies on LLN for (vector-valued) selections

Proof: time allowing

Random mappings

$S : \Xi \times E \rightrightarrows \mathbb{R}^m, \quad E \subset \mathbb{R}^n$

$\mathcal{A} \otimes \mathcal{B}^n$-jointly measurable: $S^{-1}(O) \in \mathcal{A} \otimes \mathcal{B}^n$, O open

$\Rightarrow \forall\, x : \xi \mapsto S(\xi, x)$ a random set

random closed set when S is closed-valued

$ES : E \rightrightarrows \mathbb{R}^m$ with $ES(x) = \mathbb{E}\{S(\xi, x)\}$ expected mapping

ES convex-valued when $\xi \mapsto S(\xi, \cdot)$ P-atom convex

Law of Large Numbers for random sets

applies pointwise

Thursday, December 20, 12

Sample Average Approximation (SAA)

stochastic variational problem: $\bar{S}(x) = \mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

$S : \Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ random set-valued mapping

$\boldsymbol{\xi}$ random vector with values $\xi \in \Xi \subset \mathbb{R}^N$

solution (a 'stationary point') $\quad \bar{x} \in \bar{S}^{-1}(0)$

—o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o

sample $\overset{\to\nu}{\xi} = (\xi^1, \dots, \xi^\nu)$ of $\boldsymbol{\xi}$

$\frac{1}{\nu}\left(\sum_{k=1}^{\nu} S(\xi^k, x)\right) = S^\nu(\overset{\to\nu}{\xi}, x) \ni 0$, approximating system?

i.e., $\left(S^\nu\right)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$ a.s. ???

Thursday, December 20, 12

So far ...

➪ *generalized equations*

$S : \Xi \times D \rightrightarrows E$, set-valued $S(\xi, x) \subset E$, inclusion $\mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

iid-sample $\vec{\xi}^{\,\nu} = \xi^1, \dots, \xi^\nu$ and $x \mapsto S(\xi, x)$ osc

SAA-mapping $S^\nu : \Xi^\infty \times D \rightrightarrows E$, random osc mappings

$S^\nu(\xi, x) = \frac{1}{\nu} \sum_{k=1}^{\nu} S(\xi^k, x) \approx S^\nu(\vec{\xi}^{\,\nu}, x)$, $\forall \xi \in \Xi^\infty$

$\forall x \in D$, $S(\cdot, x)$, closed random set,

let $\bar{S} = \operatorname{cl} \operatorname{con} ES$, $\quad ES(x) = \mathbb{E}\{S(x, \boldsymbol{\xi})\}$

Artstein-Hart LLN applies: $S^\nu \xrightarrow{p} \bar{S}$ $a.s.$ when $E = \mathbb{R}^m$

but $\xrightarrow{p}$ $\not\Rightarrow$ $(S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

recall: $\bar{S}(x) = \operatorname{cl} ES(x)$ when P-atom convex, $ES(x)$ closed if $\xi \mapsto S(\xi, x)$ is integrably bounded and compact if $\operatorname{rge} S(\cdot, x)$ is bounded.

Consistent approximations?

$$S^\nu(\xi,\cdot) \to_p \bar{S} \;\; P^\infty\text{-a.s.} \Rightarrow ? \;\; S^\nu(\xi,\cdot)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$$

sometimes!

graphical rather than pointwise convergence is required

$$S^\nu(\xi,\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S} \;\; P^\infty\text{-a.s. is needed}$$

relationship between graphical and pointwise convergence?

Some Examples

Stochastic VI, Variational Inequality

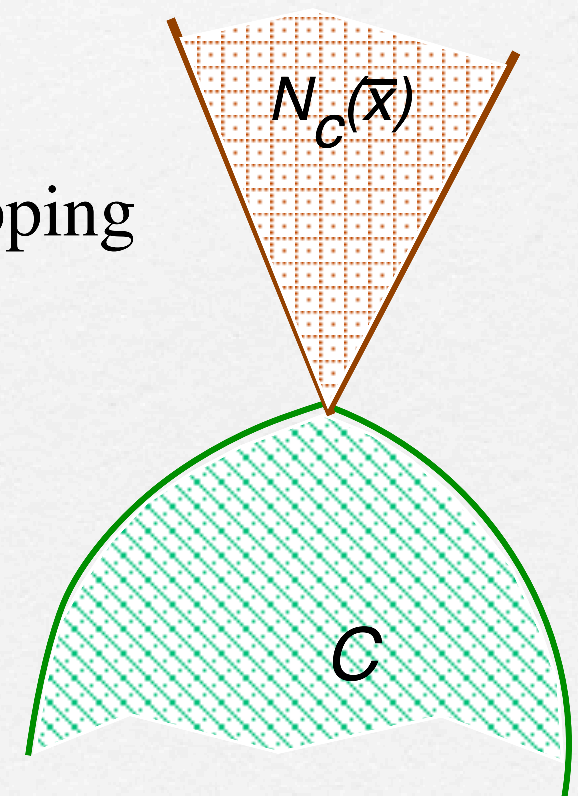
Network flow equilibrium with stochastic demand and link capacities
Economic equilibrium in a stochastic environment

$\xi = (\xi^1, \xi^2, \ldots)$, $G^\nu(\cdot, x)$ σ-$(\xi^1, \ldots \xi^\nu)$ measurable

$-G^\nu(\xi, x) \in N_C(x)$, C compact, convex

$N_{C(x)} + G^\nu(\xi, x) = S^\nu(x) \ni 0$, S^ν closed set-valued mapping

$$G^\nu(\xi, \cdot) \to^? G(\xi, \cdot)$$



$x^\nu(\xi)$ solution of $-G^\nu(\xi, x) \in N_C(x)$ for sample $\xi \approx \overset{\to\nu}{\xi}$

does $x^\nu(\xi) \to$ a solution of $-G(\xi, x) \in N_C(x)$? a.s.

what if C depends on (ξ, ν): sequence of random sets $C^\nu(\xi)$?

“static” Walras Equilibrium

agent's problem: $a \in \mathcal{A}$, $|\mathcal{A}|$ finite, possibly "large"

$\bar{x}_a \in \arg\max u_a(x_a)$ so that $\langle p, x_a \rangle \le \langle p, e_a \rangle$, $x_a \in X_a$

e_a: endowment of agent a, $e_a \in \operatorname{int} X_a$

u_a: utility of agent a, concave, usc

$u_a : X_a \to \mathbb{R}$, $X_a \subset \mathbb{R}^n$ (survival set) convex

market clearing: $s(p) = \sum_{a \in \mathcal{A}} (e_a - \bar{x}_a)$ excess supply

equilibirum price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \ge 0$, Δ unit simplex

Walras: a Variational Inequality

$$c_a = \arg\max_x u_a(x) \text{ so that } \langle p,x\rangle \le \langle p,e\rangle,\ x \in C_a$$

$$\sum_a (e_a - c_a) = s(p) \ge 0.$$

$$\Updownarrow$$

$$N_D(\bar{z}) = \left\{ v \middle| \langle v, z - \bar{z}\rangle \le 0, \forall z \in D \right\}$$

$$G(p,(x_a),(\lambda_a)) = \left[\sum_a (e_a - x_a);(\lambda_a p - \nabla u_a(x_a));\langle p, e_a - x_a\rangle\right]$$

$$D = \Delta \times \left(\prod_a C_a\right) \times \left(\prod_a \mathbb{R}_+\right)$$

$$-G(\bar{p},(\bar{x}_a),(\bar{\lambda}_a)) \in N_D(\bar{p},(\bar{x}_a),(\bar{\lambda}_a))$$

$$D \text{ unbounded} \to \hat{D} \text{ bounded}$$

Equilibrium: stochastic environment

$$(c_a^1, y_a, c_{a,\xi}^2) = \arg\max_{x^1, y \in \mathbb{R}^L, x_\bullet^2 \in \mathcal{M}} u_a^1(x^1) + \mathbb{E}^a \left\{ u_a^2(\boldsymbol{\xi}, x^2(\boldsymbol{\xi})) \right\}$$

$$\text{such that } \left\langle p^1, x_a^1 + T_a^1 y \right\rangle \le \left\langle p^1, e_a^1 \right\rangle$$

$$\left\langle p_\xi^2, x_{a,\xi}^2 \right\rangle \le \left\langle p_\xi^2, e_{a,\xi}^2 + T_{a,\xi}^2 y \right\rangle, \ \forall \xi \in \Xi$$

$$x_a^1 \in X_a^1, \ x_{a,\xi}^2 \in X_{a,\xi}^2, \quad \forall \xi \in \Xi$$

$\mathbb{E}^a \left\{ \bullet \right\}$ expecttion with respect to a-beliefs, Ξ finite support

2-stage stochastic programs with recourse

solution procedures & approximation theory "well-estblished"

$T_a^1, T_{a,\xi}^2$: input-output matrices (production, investments)

$e_a^1 \in \operatorname{int} X_a^1, \ e_{a,\xi}^2 \in \operatorname{int} X_{a,\xi}^2$ for all ξ

Market Clearing ~Equilibrium

excess supply: agent-a: $\left(c_a^1, y_a^1, \left\{ c_{a,\xi}^2 \right\}_{\xi \in \Xi} \right)$

$$\sum\nolimits_{a \in \mathcal{A}} \left(e_a^1 - (c_a^1 + T_a^1 y_a) \right) = s^1 \left(p^1, \{p_\xi^2\}_{\xi \in \Xi} \right) \geq 0$$

$$\forall \xi, \ \sum\nolimits_{a \in \mathcal{A}} \left((e_{a,\xi}^2 + T_{a,\xi}^2) - c_{a,\xi}^2 \right) = s_\xi^2 \left(p^1, \{p_\xi^2\}_{\xi \in \Xi} \right) \geq 0$$

Variational inequality: $-G(p,(x_a),(\lambda_a)) \in N_D(p,(x_a),(\lambda_a))$,

$$p = \left(p^1, \{p_\xi^2\}_{\xi \in \Xi} \right), x = \left(x^1, \{x_\xi^2\}_{\xi \in \Xi} \right), \lambda = \left(\lambda^1, \{\lambda_\xi^2\}_{\xi \in \Xi} \right)$$

$$S(\xi,(p,x,\lambda)) = G(\xi,(x,p,\lambda)) + N_{D(\xi)}(p,x,\lambda)),$$

$$\mathbb{E}\left\{ S(\xi,(p,x,\lambda)) \right\} \ni 0$$

a.s. Congergence of SAA-mappings

Graphical vs Pointwise convergence

$S, S^\nu : X \rightrightarrows \mathbb{R}^m$. Then, $S^\nu \underset{\text{point}}{\longrightarrow} S$ and $S^\nu \underset{\text{gph}}{\longrightarrow} S$ (at x)

$\Leftrightarrow \{C^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically) (at x)

~ Arzela-Ascoli Theorem for set-valued mappings

S random mapping, P^∞-a.s., $S^\nu(\xi, \cdot) \underset{\text{point}}{\longrightarrow} \operatorname{cl} \operatorname{con} ES = \bar{S}$

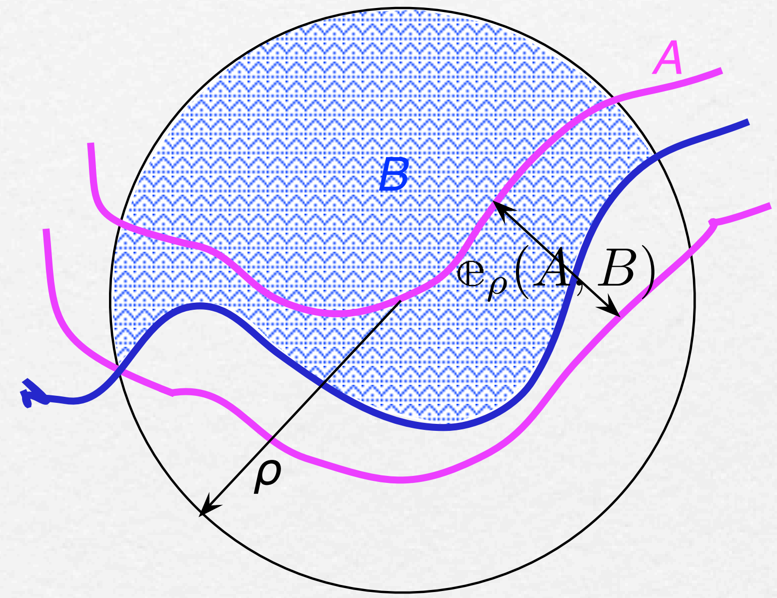
then $S^\nu \underset{\text{gph}}{\longrightarrow} \bar{S} \Leftrightarrow \{S^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically)

Semicontinuity: osc/isc

$$S : D \rightrightarrows \mathbb{R}^m \text{ continuous at } \bar{x} \text{ if } \lim_{x^\nu \to \bar{x}} d\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$$

$$d\!l(S(x^\nu), S(\bar{x})) \to 0 \iff d\!l_\rho(S(x^\nu), S(\bar{x})) \to 0$$

$$\iff \hat{d\!l}_\rho(S(x^\nu), S(\bar{x})) \to 0. \forall\, \rho > \bar{\rho} \geq 0$$



$$\hat{d\!l}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\big[\, e_\rho\big(S(x^\nu), S(\bar{x})\big), e_\rho\big(S(\bar{x}), S(x^\nu)\big)\,\big]$$

S is osc (outer semicontinuous) at $\bar{x}$ if $e_\rho\big(S(x^\nu), S(\bar{x})\big) \to 0$ as $x^\nu \to \bar{x}$

S is isc (inner semicontinuous) at $\bar{x}$ if $e_\rho\big(S(\bar{x}), S(x^\nu)\big) \to 0$ as $x^\nu \to \bar{x}$

Equi-osc mappings

$S : D \rightrightarrows \mathbb{R}^m,\ D \subset \mathbb{R}^n$ is osc if gph S is closed

osc at $\bar{x}$: given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) : d\!l_\rho\big(S(x), S(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$\left\{S^\nu : D \rightrightarrows \mathbb{R}^m\right\}$ are equi-osc at $\bar{x}$

given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) :\ d\!l_\rho\big(S^\nu(x), S^\nu(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$V = V(\rho, \epsilon)$ doesn't depend on ν.

G-convergence of SAA-mappings

$S:\ \Xi \times X \rightrightarrows \mathbb{R}^m$ random mapping, $(\Xi, \mathcal{A}, P)$

P^∞-a.s.: $S^\nu(\xi,\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S}$ at $\bar{x} \Leftrightarrow$ SAA-mappings $\{S^\nu(\xi,\cdot)\}$ equi-osc at $\bar{x}$

$\Rightarrow$ sol'ns of $S^\nu(\xi,\cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: P^∞-a.s.

$S(\xi,\cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\xi,\cdot)\}$ equi-osc

<u>Law of large Numbers for Random Mappings</u>

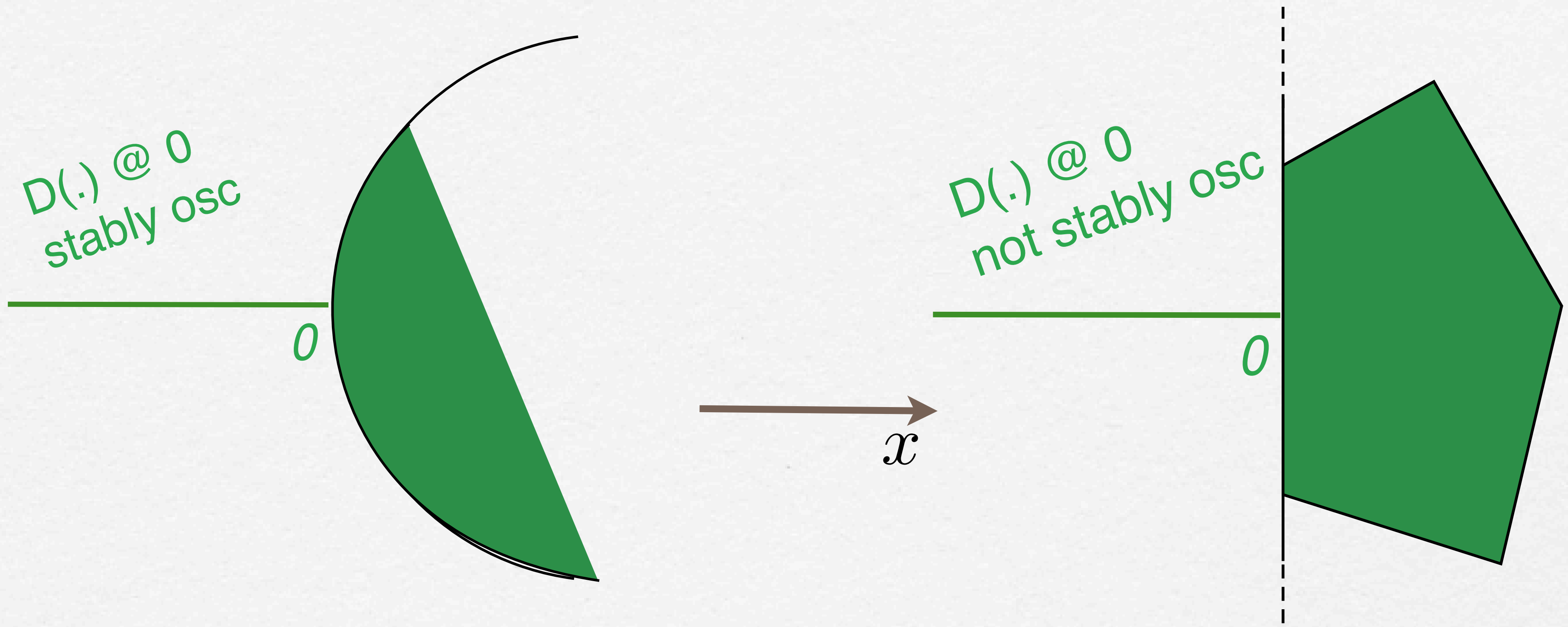
S random osc mapping: $\Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

stably osc & steady under averaging

$\xi^1, \xi^2, \ldots,$ iid random variables (values in Ξ), distribution P

Then, $\nu^{-1}\sum_{k=1}^{\nu} S(\xi^k,\cdot) \to_{\text{gph}} \bar{S} = \text{cl}\,\text{con}\, E\{S(\xi^0,\cdot)\}$ P^∞-a.s.

Stably osc mapping

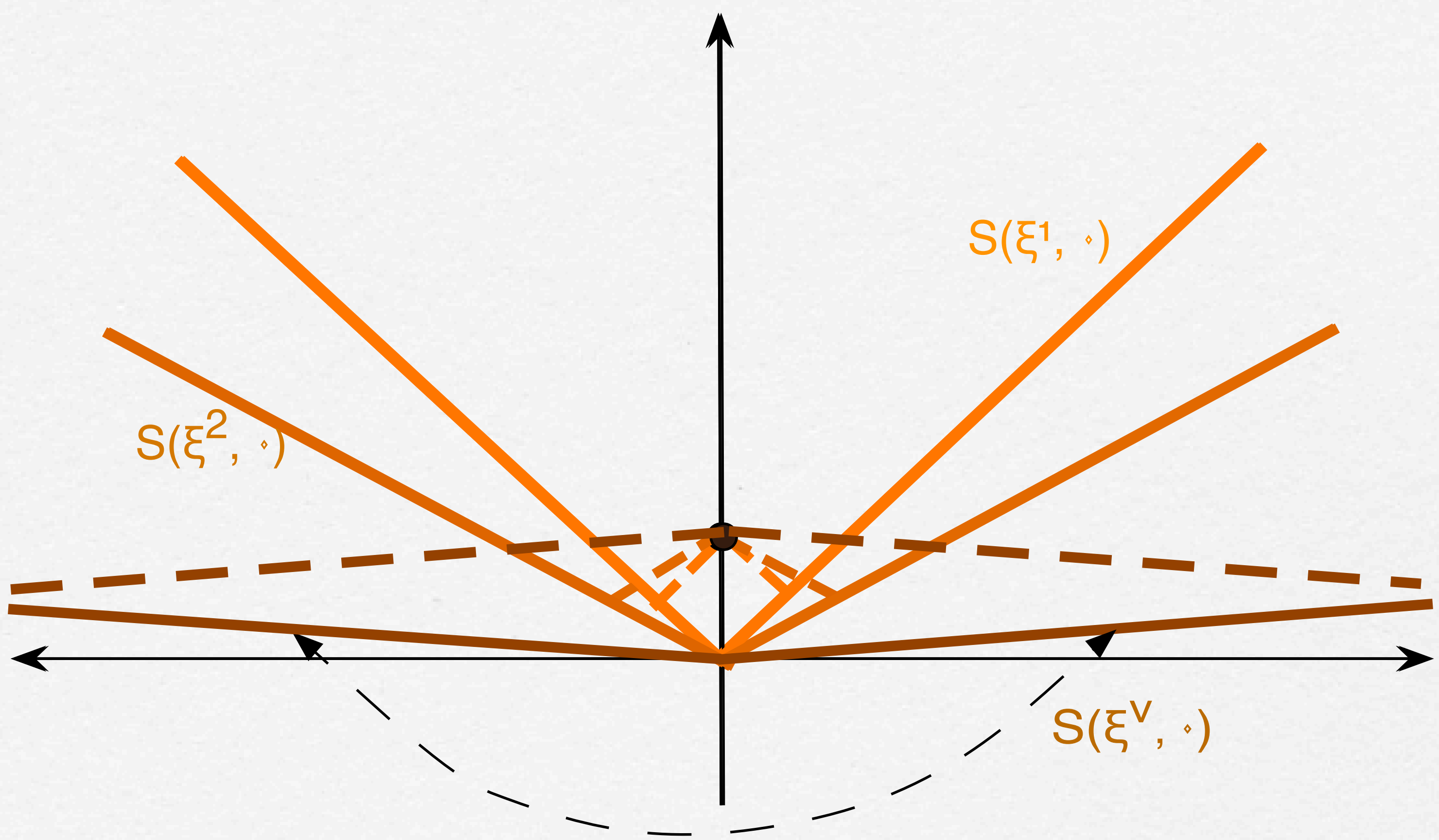


S stably osc near $\bar{x}$ if P-a.s.,

$$\forall \rho > 0, \varepsilon > 0,\ \exists W \in \mathcal{N}(\bar{x})\ \&\ \eta\mathbb{B}\ (\eta > 0):$$

$$d\!\!l_\rho\big(S(\xi, x'), S(\xi, x)\big) < \varepsilon,\ \forall\, x' \in x + \eta\mathbb{B},\ x \in W$$

Steady under averaging



$$u \in S^\nu(\vec{\xi}^{\,\nu}, x) \cap \rho\mathbb{B} \Rightarrow \exists\, \hat{\rho} \geq \rho,\ u^k \in S(\xi^k, x) \cap \hat{\rho}\mathbb{B} \text{ such that}$$

$$u = \nu^{-1}(u^1 + \cdots + u^\nu);\ S^\nu(\vec{\xi}^{\,\nu}, x) \cap \rho\mathbb{B} \subset \frac{1}{\nu}\left[\sum_{k=1}^{\nu} S(\xi^k, x) \cap \hat{\rho}\mathbb{B}\right]$$

Steady u. averaging & stably osc

rge $S \subset B$ bounded $\Rightarrow$ steady under averaging

S cone-valued and rge $S \subset$ pointed cone K. Then,

$\bar{S} = ES$ and $\Rightarrow$ steady under averaging.

S, R steady under averaging $\Rightarrow$ so is $S + R$

$R(\xi, x) = R(x) \Rightarrow R$ steady under averaging

rge S bounded + R constant $\Rightarrow$ steady under averaging

$G(\xi, x) + N_C(x) \Rightarrow$ steady under averaging (V.I.)

provided $G : \Xi \times X \to \mathbb{R}^n$ is bounded

S, R stably osc $\Rightarrow S + R$ stably osc

although D^1, D^2 osc $\not\Rightarrow D^1 + D^2$ osc

$\mathbb{B}$ closed, convex $\quad x \mapsto N_{\mathbb{B}}(x)$ osc

but not stably osc ($x^\nu \in$ int $\mathbb{B} \to \bar{x} \in$ bdry $\mathbb{B}$)

Implementing SAA ** locally

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi},x)\} \in R(x)$

(V.I.: $S = N_C$, applied to option pricing, ...)

$G^\nu(\vec{\xi}^\nu,\cdot) = \nu^{-1}\sum_{k=1}^{\nu} G(\xi^k,x)$. Assume $G^\nu(\vec{\xi}^\nu,\cdot)$, $EG \in C^1(\mathbb{R}^n;\mathbb{R}^n)$,

$\bar{x}$ strongly regular solution [Robinson] of $EG(x) \in R(x)$,

$\exists V \in \mathcal{N}(\bar{x}), \rho > 0$ such that $\forall z \in \rho\mathbb{B}$:

$$z + EG(\bar{x}) + \nabla EG(\bar{x})\big(x - \bar{x}\big) \in S(x)$$

has a unique solution $\bar{x}(z) \in V$, Lipschitz continuous on $\rho\mathbb{B}$, and

$\left\| G^\nu(\vec{\xi}^\nu,\cdot) - EG \right\| \to 0$ P-a.s. Then, for ν sufficiently large

on a neighborhood of $\bar{x}$, $G^\nu(\vec{\xi}^\nu,\cdot) \in R(x)$ has a unique solution

$$\bar{x}(\vec{\xi}^\nu) \to \bar{x} \quad P\text{-a.s.}$$