Stochastic Variational Problems

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“Optimization & Equilibrium”

“Nothing at all takes place in the universe in which some rule of maximum or minimum does not appear” L. Euler 1744

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“Il n’est pas certain que tout soit incertain” B. Pascal   ±1645

Prelude: 14th-Century ~ 1950

~ 1350 Oresme's rule: $f:\mathbb{R}\to\mathbb{R}$

$$x^* \in \arg\min f \Rightarrow df(x^*;w) = \lim_{\tau\to 0}\frac{f(x^*+\tau w)-f(x^*)}{\tau} = 0,\ \forall w\in\mathbb{R}$$

~ 1650 Fermat's rule: $f:\mathbb{R}\to\mathbb{R},\ f$ smooth

$$x^* \in \arg\min f \Rightarrow f'(x^*) = \nabla f(x^*) = 0$$

$$df(x;w) = \langle \nabla f(x0,w\rangle \Rightarrow \text{(Oresme rule} \Leftrightarrow \text{Fermat rule)}$$

(~ 1950 Dantzig simplex method for linear programming)

Oresme's rule: $f:\mathbb{R}^n\to\mathbb{R}$

$$x^* \in \arg\min f \Rightarrow df(x^*;w) = 0,\ \forall w\in\mathbb{R}^n$$

Fermat's rule: $f:\mathbb{R}^n\to\mathbb{R},\ f$ smooth

$$x^* \in \arg\min f \Rightarrow \nabla f(x^*) = 0, \qquad df(x;w) = \langle \nabla f(x), w\rangle$$

Curve Fitting

find $h:[0,1] \to \mathbb{R}$, given $h(z_1),\ldots,h(z_L)$

approximate by $p(x) = a_n x^n + \cdots + a_1 x + a_0$

$$\min_{a\in\mathbb{R}^{n+1}} \sum_{l=1}^{L} \Big(\sum_{j=0}^{n} a_j z_l^j - h(z_l)\Big)^2 = \min_{a\in\mathbb{R}^{n+1}} \langle Za - y, Za - y\rangle$$

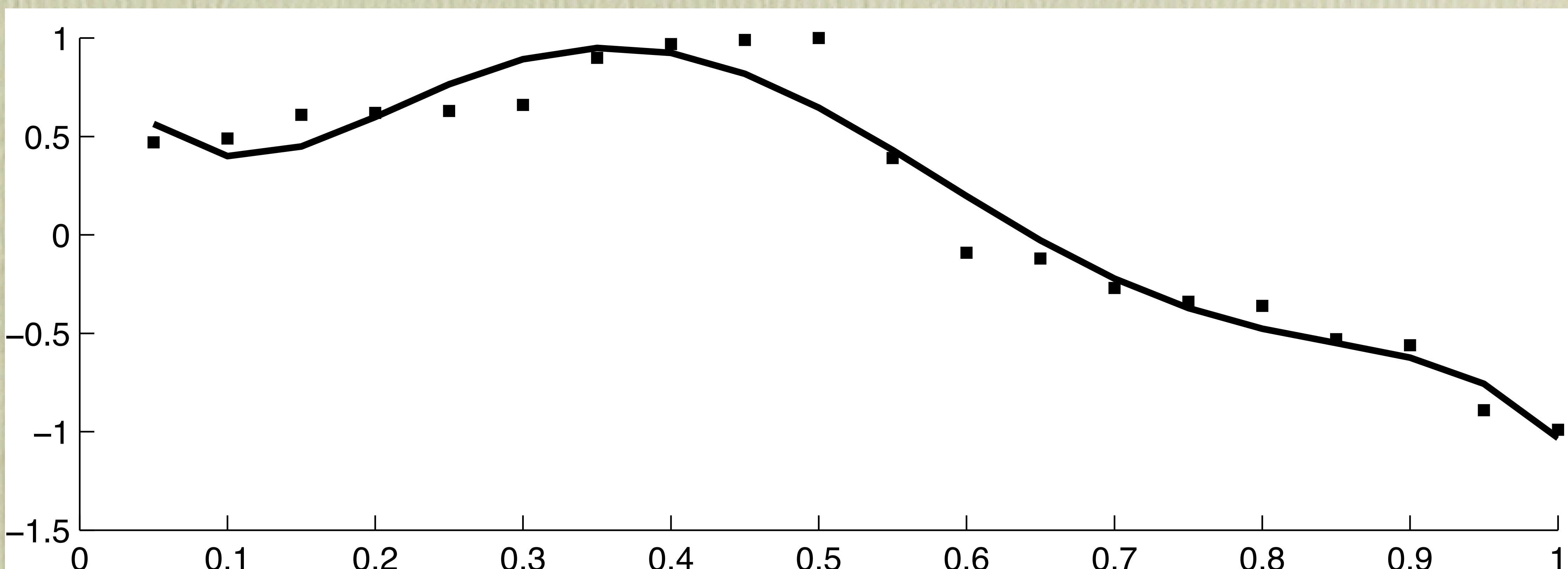
$$Z = \begin{bmatrix} z_1^n & \ldots & z_1 & 1 \\ z_2^n & \ldots & z_2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ z_L^n & \ldots & z_L & 1 \end{bmatrix}, \quad (y_1,\ldots,y_L) = \big(h(z_1),\ldots,h(z_l)\big)$$

Solution: applying Fermat's rule,

$a^* = \big(Z^\top Z)^{-1} Z^\top y$; assuming col. Z linearly independent

Curve fitting: polynomial-approx.

with best degree $n = 5$



Steepest descent - Newton Methods

direction of descent $\exists \tau_\star > 0, \forall\, \tau \in (0, \tau_\star): \quad f(\bar{x} + \tau d) < f(\bar{x})$

$df(\bar{x}; d) < 0$ smooth: $\langle \nabla f(\bar{x}), d \rangle \le 0 \Rightarrow d$ direction of descent

Steepest descent: x^0; $\nabla f(x^\nu) = 0$ stop;

$$d^\nu = -\nabla f(x^\nu), x^{\nu+1} = \arg\min \left\{ f(x) \middle| x \in [x^\nu, x^\nu + \lambda d^\nu), \lambda \ge 0 \right\}$$

Newton direction: $d^\nu = -\left(\nabla^2 f(x^\nu)\right)^{-1} \nabla f(x^\nu) \Rightarrow$ Newton Method

quadratic convergence locally: (f $C^2 \ldots$), with local sol'n @ $\bar{x}$

$\exists \rho > 0, \kappa \ge 0: \quad \| \nabla^2 f(x) - \nabla^2 f(x') \| \le \kappa \,|\, x - x' \,|, \ \forall x, x' \in \mathbb{B}(\bar{x}, \rho)$

Quasi-Newton Method(s)

0. $x^{\nu}, B^{0}(=I), \nu = 0$

1. $\nabla f(x^{\nu}) = 0$ stop or $B^{\nu} d^{\nu} = -\nabla f(x^{\nu})$

2. $x^{\nu+1} = \arg\min\left\{ f(x) \middle| x \in [x^{\nu}, x^{\nu} + \lambda d^{\nu}), \lambda \geq 0 \right\}$

$B^{\nu} \approx$ Hessian $\nabla^2 f(x^{\nu})$ in Newton (curvature)

$B^{\nu+1}(x^{\nu+1} - x^{\nu}) = (B^{\nu} + U)(x^{\nu+1} - x^{\nu}) = \nabla f(x^{\nu+1}) - \nabla f(x^{\nu})$, i.e.,

$U^{\nu} s^{\nu} = c^{\nu} - B^{\nu} s^{\nu}, s^{\nu} = x^{\nu+1} - x^{\nu}, c^{\nu} = \nabla f(x^{\nu+1}) - \nabla f(x^{\nu})$

Quasi-Newton condition

Integral Functionals (Calculus of Variations)

$$\min\left\{f(x)=\int_0^1 L(t,x(t),\dot{x}(t))\,dt \,\middle|\, x\in \text{ fcns}\big([0,1],\mathbb{R}\big),\; x(0)=\alpha, x(1)=\beta\right\}$$

Oresme rule : $\quad x^* \in \arg\min f \Rightarrow df(x^*;w)=0, \forall w\in W \subset X,$

W admissible variations: $=\left\{w\in \text{ fcns}\big([0,1],\mathbb{R}\big)\,\middle|\, w(0)=w(1)=0\right\}$

Bernoulli (Jacob, Johan), Newton (~ 1700) $\Rightarrow$ Euler equation

$$\boxed{L_x\big(t,x^*(t),\dot{x}^*(t)\big)=\frac{d}{dt}L_{\dot{x}}\big(t,x^*(t),\dot{x}^*(t)\big) \quad \text{for } t\in[0,1]}$$

Mathematical Shift (a paradigm change)

differentiability $\rightarrow$ non-smooth

typical example: $f(x) = \min\left\{ g(x,y) \middle| y \in S(x) \right\}$

dom f $\rightarrow$ open to closed

INEQUALITIES!

A formulation -- Product mix problem

A furniture manufacturer must choose $x_j \geq 0$, how many dressers of type $j = 1, \ldots, 4$ to manufacture so as to maximize profit

$$\sum_{j=1}^{4} c_j x_j = 12x_1 + 25x_2 + 21x_3 + 40x_4$$

The constraints:

$$t_{c1}x_1 + t_{c2}x_2 + t_{c3}x_3 + t_{c4}x_4 \leq d_c$$
$$t_{f1}x_1 + t_{f2}x_1 + t_{f3}x_1 + t_{f4}x_1 \leq d_f$$

t_{cj} (t_{fj}) carpentry (finishing) man-hours: dresser type j

d_c (d_f) = total time available for carpentry (finishing)

Product mix problem (2)

Solution via linear programming:

$$\max\langle c, x\rangle \text{ so that } Tx \le d,\ x \in I\!R^n_+.$$

With

$$T = \begin{bmatrix} t_{c1} & t_{c2} & t_{c3} & t_{c4} \\ t_{f1} & t_{f2} & t_{f3} & t_{f4} \end{bmatrix} = \begin{bmatrix} 4 & 9 & 7 & 10 \\ 1 & 1 & 3 & 40 \end{bmatrix}, \quad \begin{bmatrix} d_c \\ d_f \end{bmatrix} = \begin{bmatrix} 6000 \\ 4000 \end{bmatrix}$$

Optimal: $\boxed{x^d = (4000/3,\ 0,\ 0,\ 200/3)}$

Value: \$ 18,667.

Product mix problem (3)

But . . . "reality" can't be ignored!

$$\boldsymbol{t}_{cj} = t_{cj} + \boldsymbol{\eta}_{cj}, \quad \boldsymbol{t}_{fj} = t_{fj} + \boldsymbol{\eta}_{fj}$$

entry	possible values			
$d_c + \zeta_c$:	5,873	5,967	6,033	6,127
$d_f + \zeta_f$:	3,936	3,984	4,016	4,064

10 random variables, say, 4 possible values each

$$\boxed{L = 1,048,576 \text{ possible pairs } (T^l, d^l)}$$

Product mix problem (4)

What if $\sum_{j=1}^{4}(t_{cj}+\eta_{cj})x_j > d_c + \zeta_c$? $\implies$ overtime

With $\xi = (\eta_{\{\cdot,\cdot\}}, \zeta_{\{\cdot\}})$, **recourse**: $(y_c(\xi), y_f(\xi))$ @ cost (q_c, q_f).

$$
\begin{array}{lllllll}
\max & \langle c, x\rangle & -p_1\langle q, y^1\rangle & -p_2\langle q, y^2\rangle & \cdots - & p_L\langle q, y^L\rangle & & \\
\text{s.t.} & T^1 x & -y^1 & & & & \leq & d^1 \\
 & T^2 x & & -y^2 & & & \leq & d^2 \\
 & \vdots & & & \ddots & & & \vdots \\
 & T^L x & & & & -y^L & \leq & d^L \\
 & x \geq 0, & y^1 \geq 0, & y^2 \geq 0, & \cdots & y^L \geq 0. & &
\end{array}
$$

Structured large scale l.p. ($L \approx 10^6$)

Product mix problem (5)

Define $\Xi = \{\xi = (\eta, \zeta)\}$, $p_\xi =$ prob $[\boldsymbol{\xi} = \xi]$

$$Q(\xi, x) = \max\left\{\langle -q, y\rangle \,\middle|\, T_\xi x - y \geq d_\xi,\ y \geq 0\right\}$$

$$EQ(x) = E\{Q(\boldsymbol{\xi}, x)\} = \sum_{\xi \in \Xi} p_\xi Q(\xi, x)$$

the equivalent deterministic program (DEP):

$$\max\langle c, x\rangle + EQ(x) \ \text{ so that } \ x \in I\!R^n_+.$$

a *non-smooth* convex optimization problem: EQ concave.

Product mix problem (6)

Solution of DEP, or large scale l.p.,:

Optimal: $\boxed{x^* = (257,\ 0,\ 665.2,\ 33.8)}$

expected Profit: \$ 18,051

The solution x^* is *robust* : it considered all $\approx 10^6$ possibilities.

$$\text{Recall: } x^d = (1{,}333.33,\ 0,\ 0,\ 66.67)$$

$$\text{expected "profit" relying on } x^d = \$\,16{,}942$$

- x^d is not close to optimal
- x^d isn't pointing in the right direction

Curve Fitting (2)

find $h:[0,1] \to \mathbb{R}$, given $h(t_1),\ldots,h(t_L)$ & h smooth

approximate by z, C^{1+}-curve

mesh $=\{0,\delta,2\delta,\ldots,N\delta=1\}$, on $((k-1)\delta,k\delta]$ $\quad z''(t)=a_k$

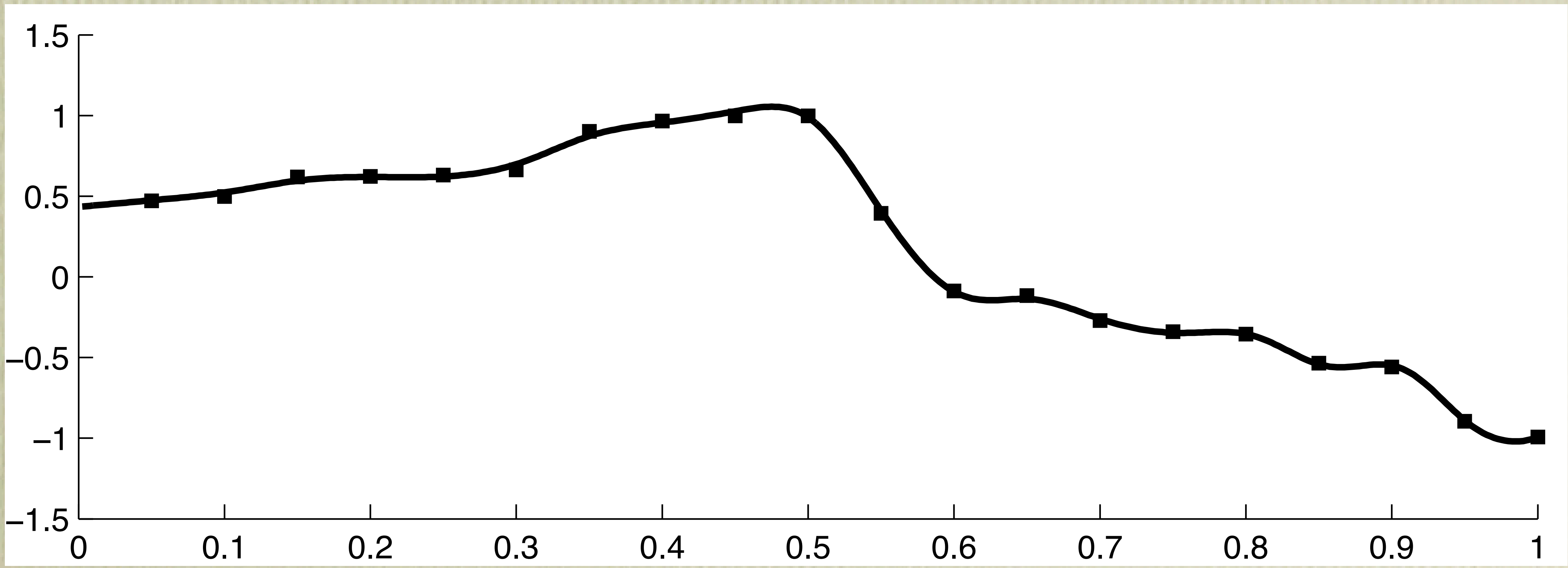
$$z(t)=z_0+v_0t+\delta\sum_{j=1}^{k-1}(t-t_j+\delta/2)a_j+\tfrac{1}{2}(t-t_{k-1})^2a_k,\quad t\in((k-1)\delta,k\delta]$$

$$\boxed{\begin{array}{l}\text{find } z_0,v_0,a_1,\ldots,a_N \quad \text{all } \in\mathbb{R}\\ \quad \min\left\|\big(z(t_l)-h(t_l)\big),l=1,\ldots,L\right\|_{\square} \quad -\kappa\le a_k\le\kappa,\ k=1,\ldots,N\end{array}}$$

Epi-spline fit

z is an epi-spline of order 2,

on each (open) sub-interval a polynomial of order 2.



Preliminaries -- Convexity
"minimization framework"

$$f : \mathbb{R}^n \to \bar{\mathbb{R}} = [-\infty, \infty], \ \ \text{dom} f = \{x \mid f(x) < \infty\}$$

$$\text{proper: dom} f \neq \emptyset, f > -\infty$$

$$f(x) = f_0(x) \text{ when } f_i(x) \leq 0, \ f_i(x) = 0, \ x \in X \subset \mathbb{R}^n$$

$$i = 1,..,s \qquad i = s+1,..,m$$

$$= \infty \ \ \text{otherwise}$$

$$\text{epi} f = \{(x,\alpha) \mid f(x) \geq \alpha\} \subset \mathbb{R}^{n+1}$$

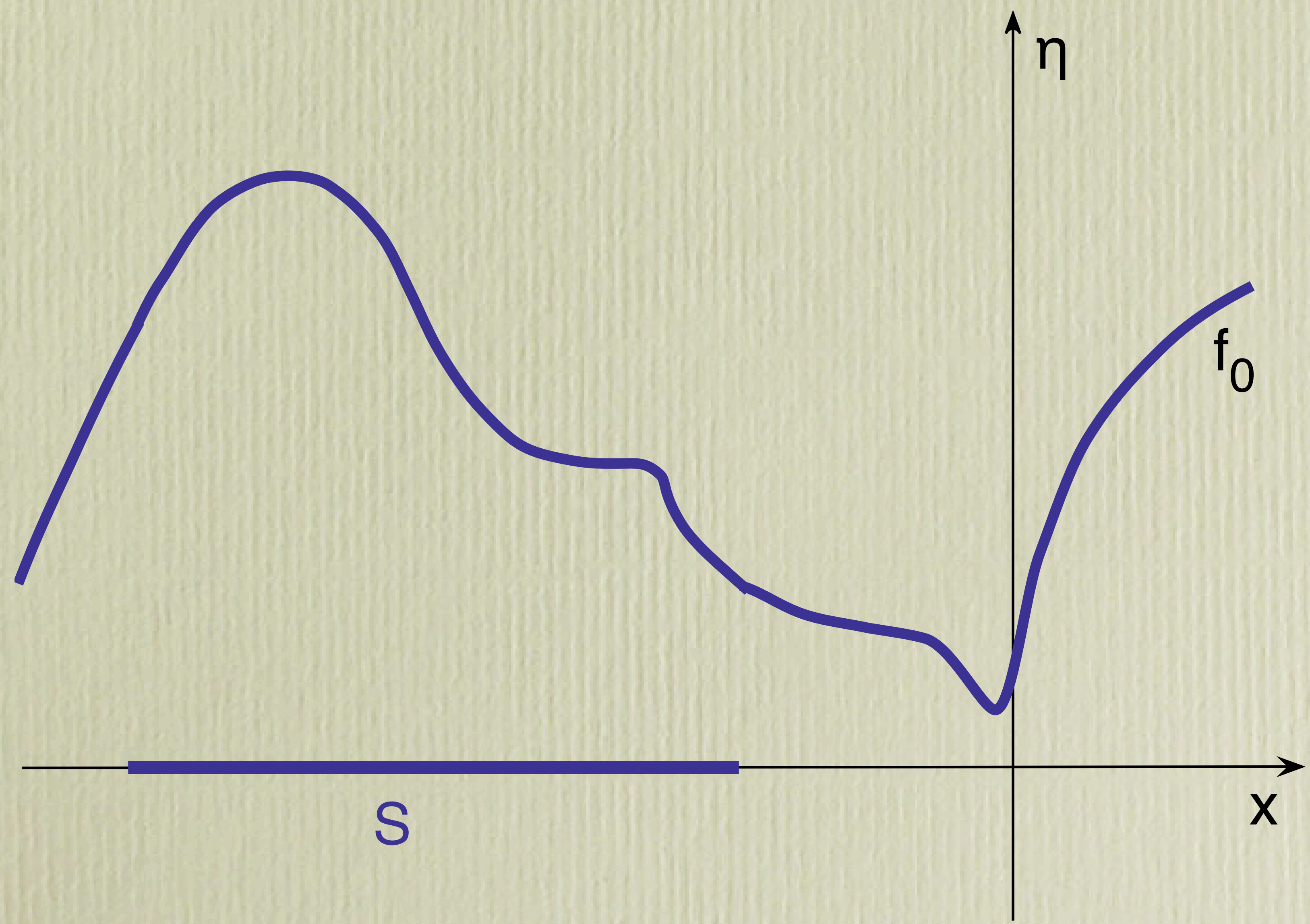
$$f \text{ lsc } \Leftrightarrow \text{epi} f \text{ closed (lower semicontinuous)}$$

$$f \text{ convex } \Leftrightarrow \text{epi} f \text{ convex}$$

Optimization problem

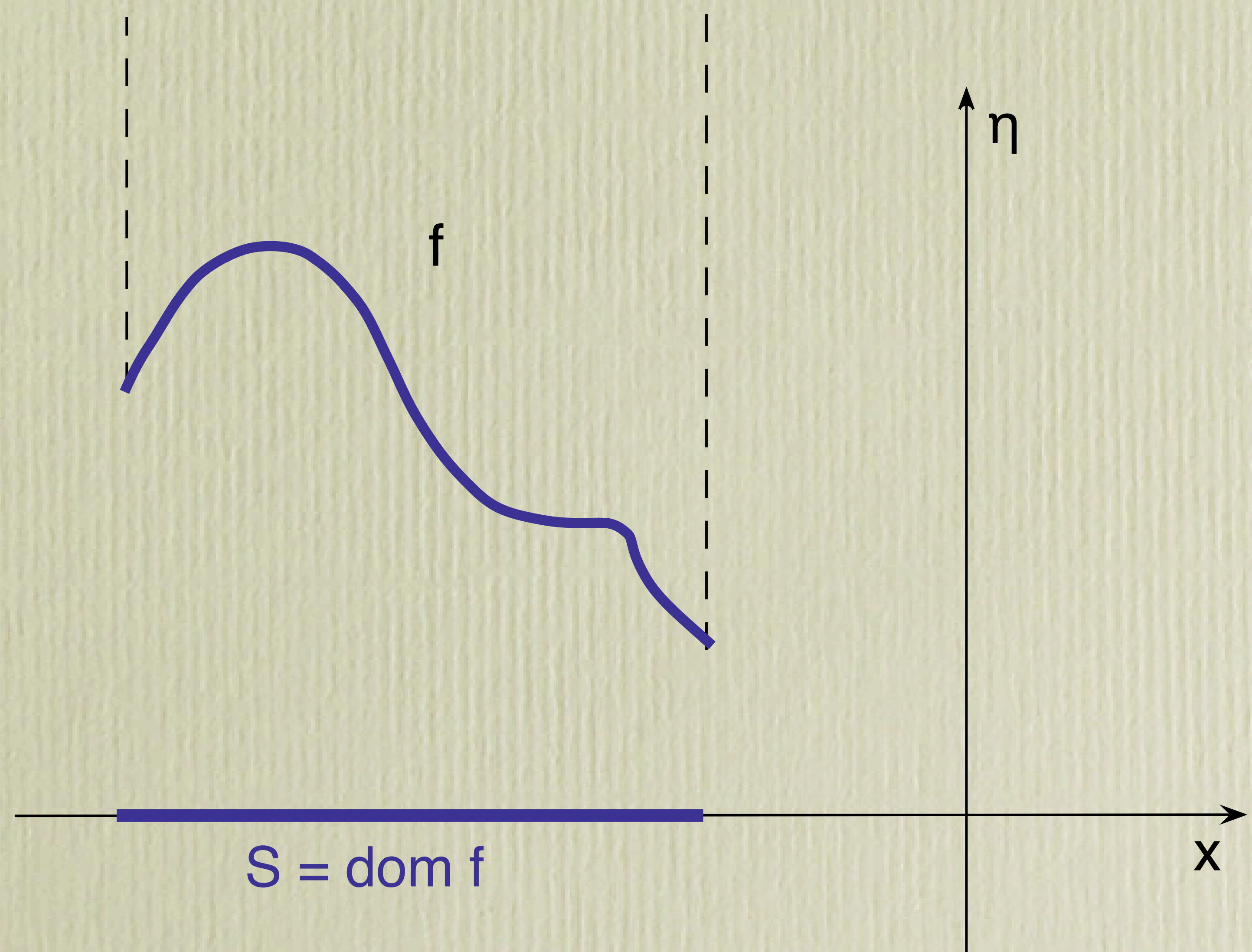
$$\min f(x),\ x \in S,$$

$$S = \left\{ x \in \mathbb{R}^n \middle| f_i(x) \le 0,\ i = 1 \to s,\ f_i(x) = 0,\ i = s+1 \to m \right\}$$



Extended-real valued fcn

$\min f$ on $\mathbb{R}^n$, $\quad f = f_0 + \iota_S(x)$, $\quad \iota_S$ indicator function of S

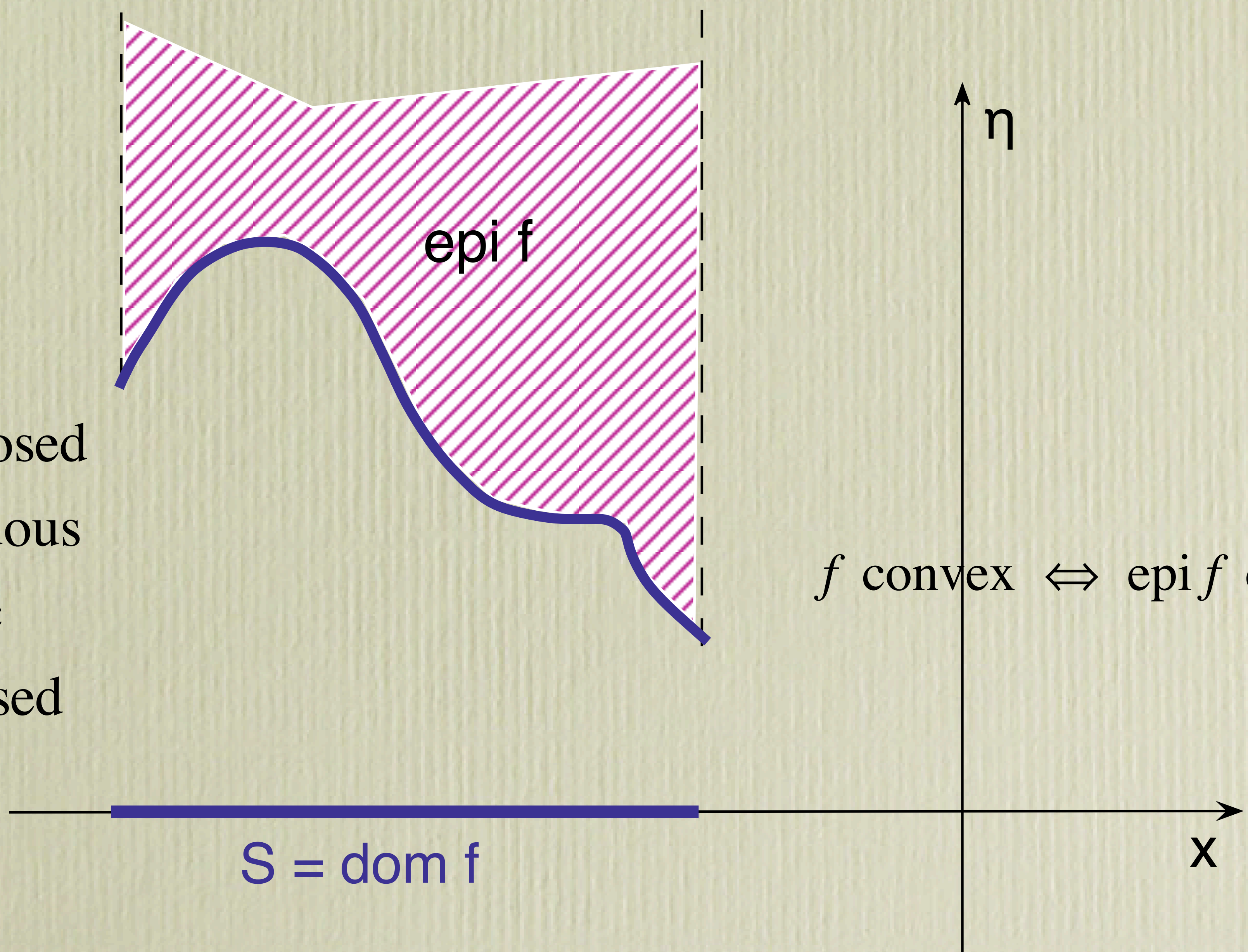


Functions & Epigraphs

$$\operatorname{epi} f = \left\{(x,\alpha) \middle| f(x) \le \alpha\right\}$$

f lsc at x : $\liminf_{x' \to x} f(x') \ge f(x)$, f usc at x : $\limsup_{x' \to x} f(x') \le f(x)$

f lsc $\Leftrightarrow$ epi f closed f usc $\Leftrightarrow$ hypo f closed



f lsc $\Leftrightarrow$ epi f closed

lower semicontinuous

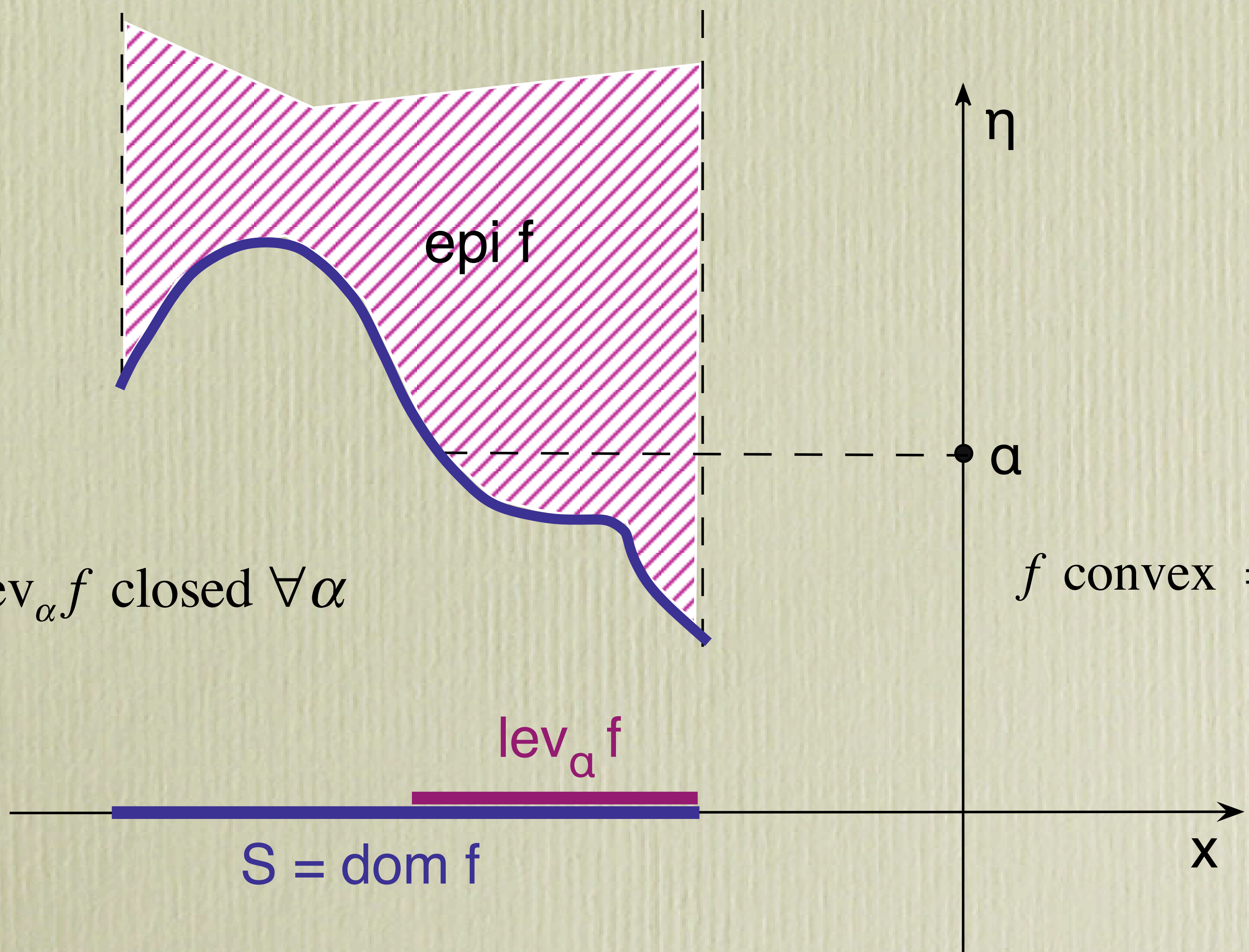
f usc $\Leftrightarrow$ $-f$ lsc

$\Leftrightarrow$ hypo f closed

f convex $\Leftrightarrow$ epi f convex

Level sets & Constraints

$$\text{lev}_\alpha = \left\{ x \in \mathbb{R}^n \,\middle|\, f(x) \leq \alpha \right\}$$

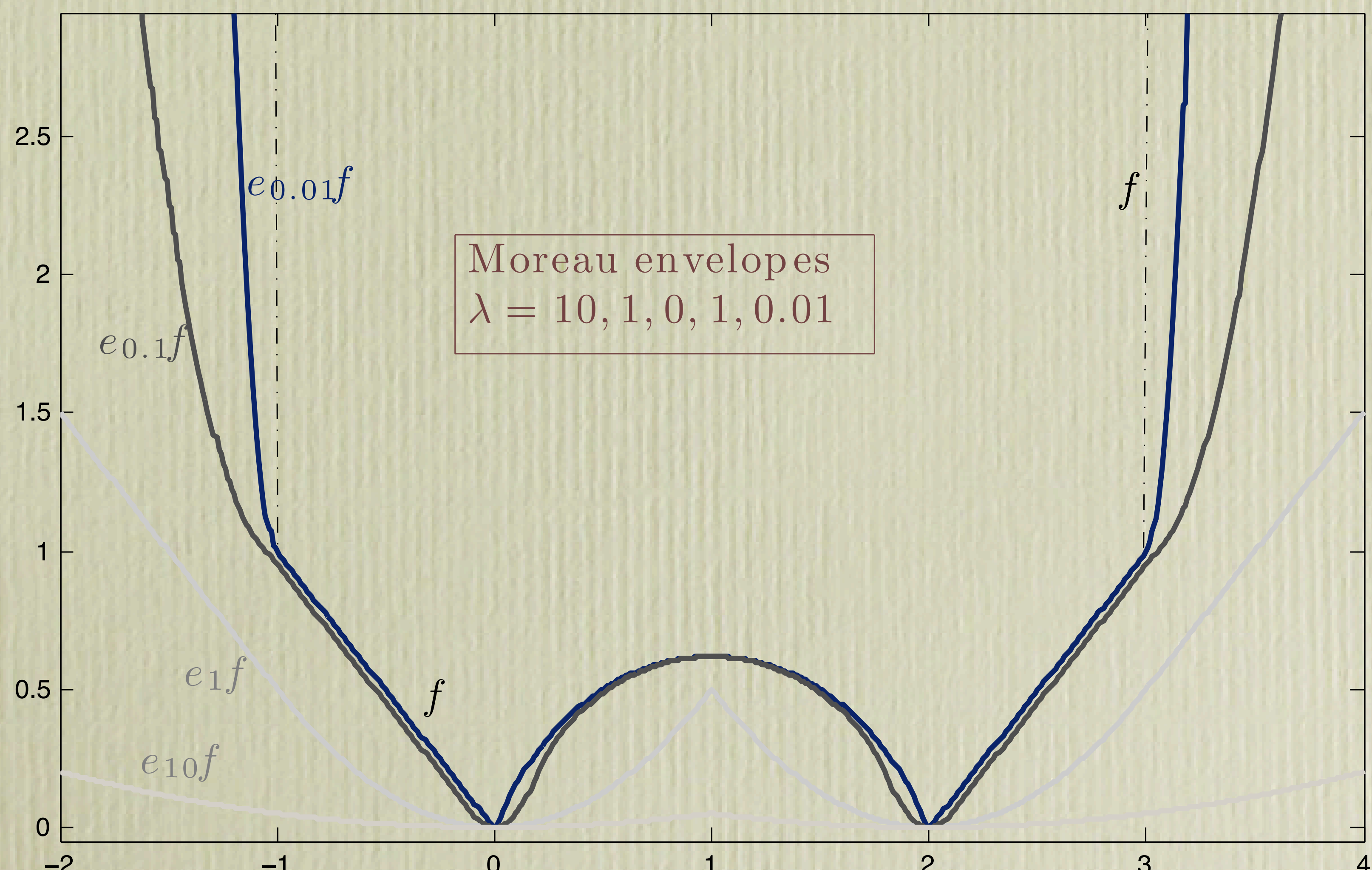


f lsc $\Leftrightarrow$ $\text{lev}_\alpha f$ closed $\forall \alpha$

f convex $\Rightarrow$ $\text{lev}_\alpha f$ convex

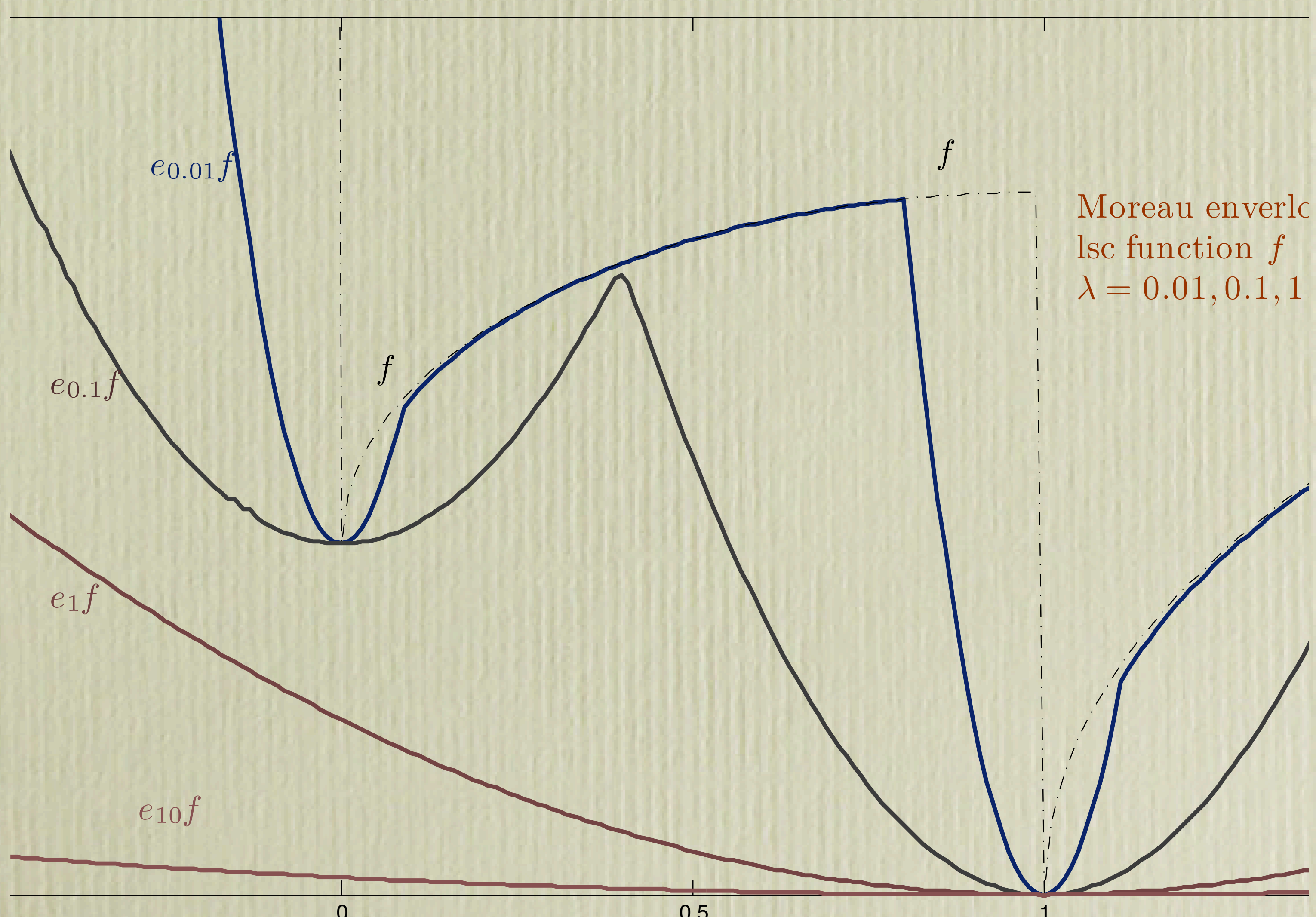
MOREAU ENVELOPES

$$e_\lambda f = \inf_w \left\{ f(w) + \frac{1}{2\lambda} | w - x |^2 \right\} \qquad \text{epi } e_\lambda f \approx \text{epi } f + \text{epi} \frac{1}{2\lambda} \left| \bullet \right|^2$$



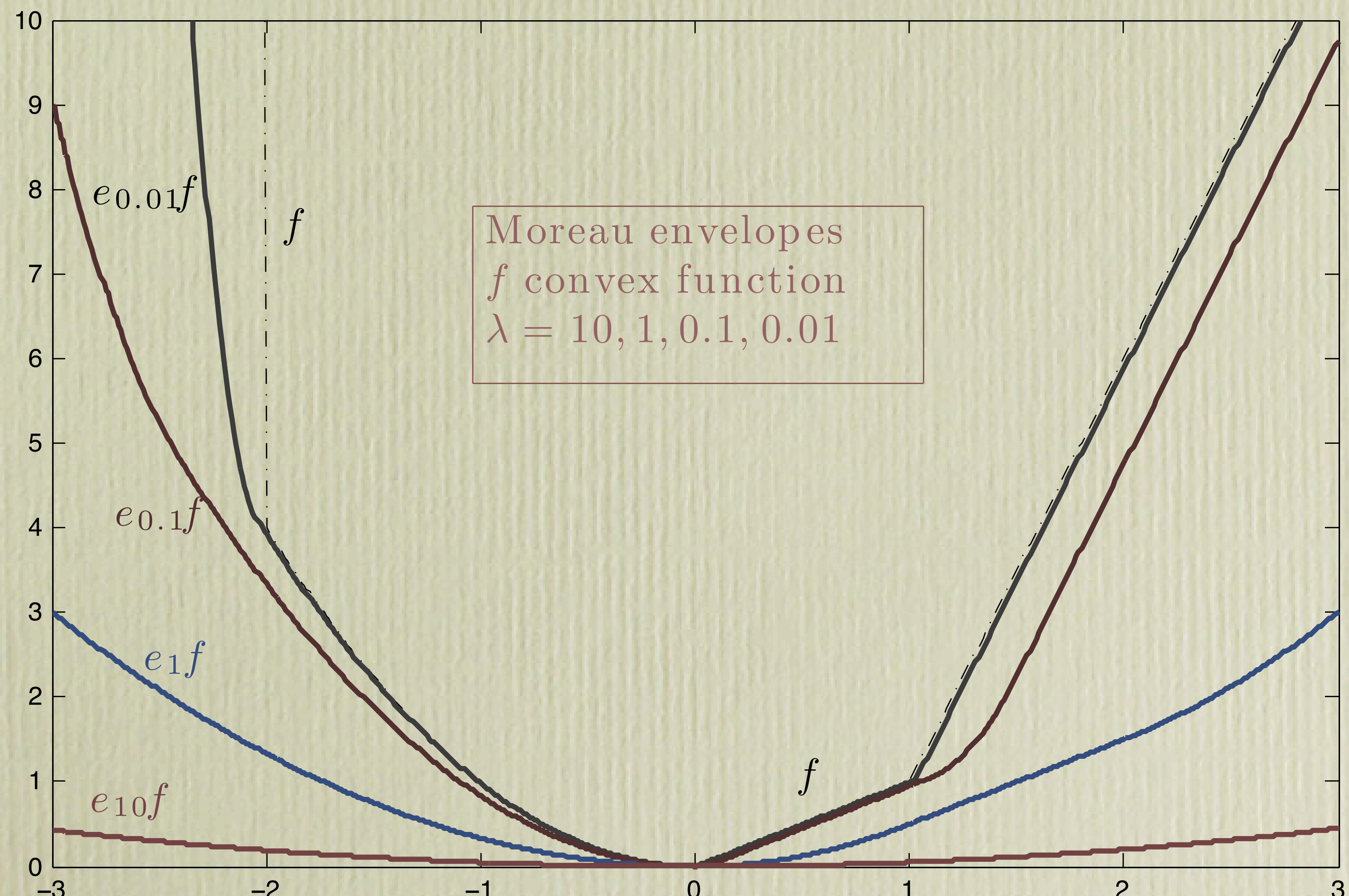
MOREAU ENVELOPE: APPROXIMATIONS

$$e_\lambda f = \inf_w \left\{ f(w) + \frac{1}{2\lambda} |w - x|^2 \right\} \qquad \text{epi } e_\lambda f \approx \text{epi } f + \text{epi} \frac{1}{2\lambda} |\bullet|^2$$



EPI-SUMS (inf-convolution)

$$\operatorname{epi} f \;\#\; \operatorname{epi} g = \inf_w \left\{ f(w) + g(w - x) \right\} \qquad e_\lambda f(x) \;\text{ with } g = \frac{1}{2\lambda}|\bullet|$$



Convex functions: Properties

- $f_1 + f_2,\ \mathbb{E}\{f(\xi,\cdot)\} = \int_\Xi f(\xi,\bullet)P(d\xi) = Ef$ convex $\quad (\Xi,\mathcal{A},P)$

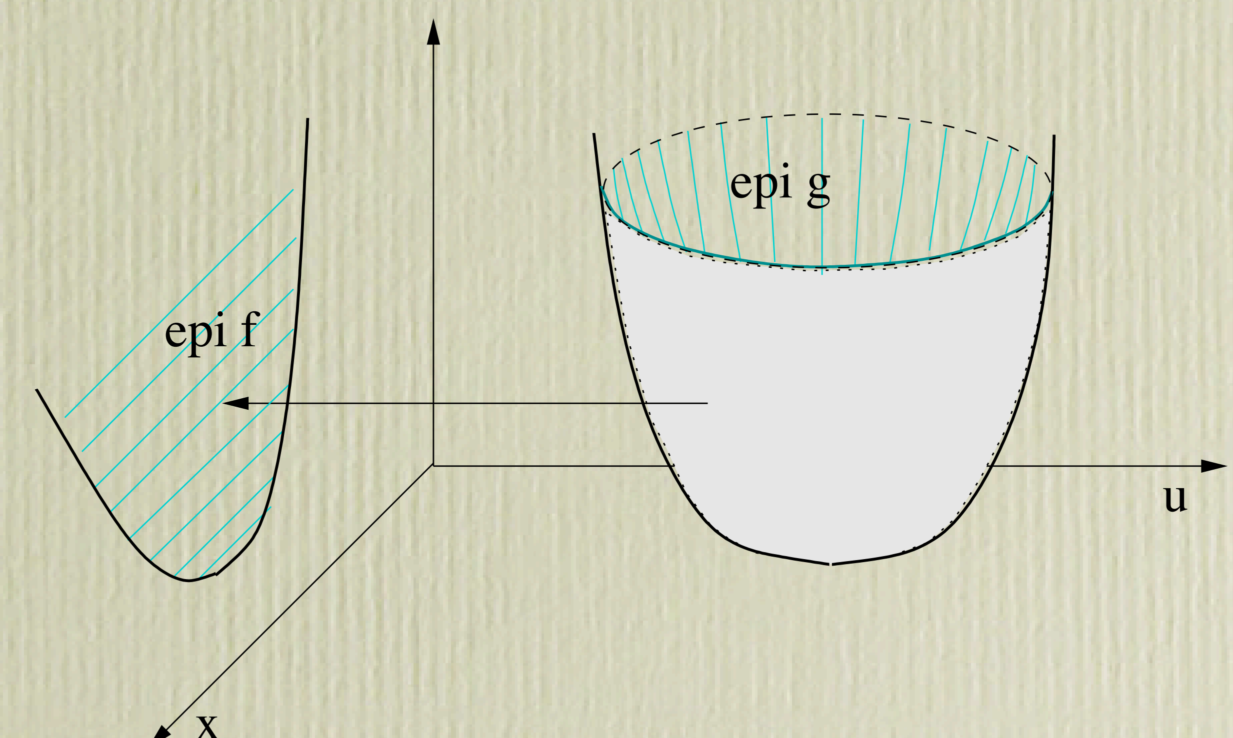
  $f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty,\infty]$ random lsc function (normal integrand)

  $x \mapsto f(\xi,x)$ lsc $\forall \xi \in \Xi$

  $(\xi,x) \mapsto f(\xi,x)$ (jointly) $\mathcal{A} \otimes \mathcal{B}^n$-measurable

- inf-projection: $f(x) = \inf_{u\in\mathbb{R}^m} g(u,x)$ convex when g is convex

Proof:



- f convex, $\boxed{\text{local minimum} \iff \text{global minimum}}$

Convex fcns: differentiability

- $f : O_{\text{open}} \subset \mathbb{R}^n$ differentiable, convex $\Leftrightarrow$
  - (a) $\langle x^1 - x^0, \nabla f(x^1) - \nabla f(x^0)\rangle \geq 0 \;\; \forall x^0, x^1$; montonicity
  - (b) $f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle, \forall\, x, y \in \mathbb{R}^n$; affine support
  - (c) $\nabla^2 f(x)$ positive semi-definite $\forall x$; f twice differentiable
- $f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle$ quadratic form, convex $\Leftrightarrow$ Q psd

Convex fcns: subdifferentiability

- $df(\bar{x};w) = \lim_{\tau \searrow 0} \dfrac{f(\bar{x}+\tau w) - f(\bar{x})}{\tau}$ subderivative (fcn) direction w
- $\partial f(\bar{x}) = \left\{ v \in \mathbb{R}^n \,\middle|\, f(x) \geq f(\bar{x}) + \langle v, x-\bar{x}\rangle,\ \forall\, x \in \mathbb{R}^n \right\}$ subgradients

  $= \left\{ v \in \mathbb{R}^n \,\middle|\, \langle v, w\rangle \leq df(\bar{x};w), \forall\, w \in \mathbb{R}^n \right\}$

  $= \nabla f(\bar{x}),\ f$ diferentiable
- $\forall\, w : df(\bar{x};w) = \max_v \left\{ \langle v,w\rangle \,\middle|\, v \in \partial f(\bar{x}) \right\}$

- $f(x) = \max_{i \in I} f_i(x)$ of convex fcns is convex, ∂f(x) = ??
- $f =$ inf-projection of $g(u,\bullet)$, ∂f(x) = ??

Generalized Oresme, Fermat Rules

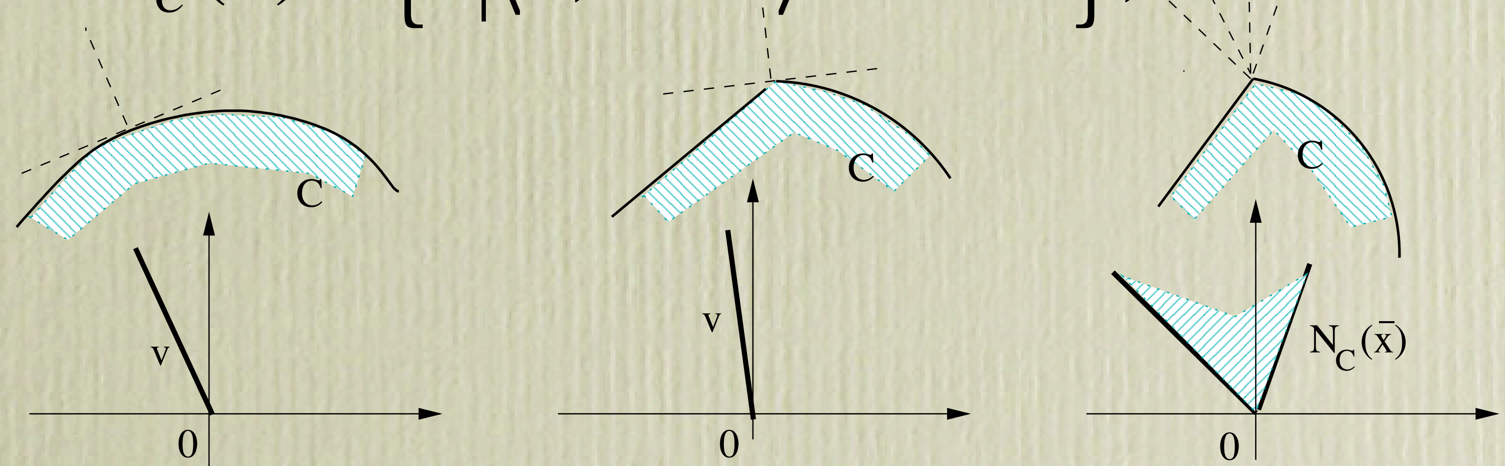
$f:\mathbb{R}^n \to \bar{\mathbb{R}}$, convex

- Oresme rule: $x^* \in \arg\min f \iff df(x^*;\cdot) \geq 0$
- Fermat rule: $x^* \in \arg\min f \iff 0 \in \partial f(x^*)$

$g = \iota_C$, C convex -- indicator function (of constraints, e.g.)

$f = f_0 + \iota_C$, $\quad 0 \in \partial f(x^\star) \iff ?\ 0 \in \partial \mathrm{f}_0(x^\star) + \partial \iota_C(x^\star)$

- $\partial \iota_C(\bar{x}) = N_C(x) = \left\{ v \middle| \langle v, x - \bar{x} \rangle \ \forall x \in C \right\}$, normal cone to C @ $\bar{x}$



Subdifferential Calculus

proper fcn: $\operatorname{dom} f \neq \emptyset,\ f > -\infty.\quad f, g$ proper, convex

$$df(x;\cdot) + dg(x;\cdot) = d(f+g)(x;\cdot),$$

$$\partial f(x) + \partial g(x) \subset \partial(f+g)(x)$$

when $\operatorname{int}\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$, then $\boxed{\partial f(x) + \partial g(x) = \partial(f+g)(x)}$

$$f = f_0 + \iota_S,\ f_0(x) \in \mathbb{R} \Rightarrow x^\star \in \arg\min f \Leftrightarrow 0 \in \partial f(x^\star) + N_C(x^\star)$$

convex programs:

$$\min f = f_0 + \iota_S,\ S = \left\{ x \in X \,\middle|\, f_i \le 0, i:1 \to s,\quad f_i = 0,\ i: s+1 \to m \right\}$$

X closed, convex   convex   affine

linearly constrained: (linear, quadratic, ... programs)

X polyhedral (box)   affine   affine

Convex, Linearly Constrained

$C = \{x | \langle A_i, x\rangle \le b_i, i = 1,\ldots,s,\ \langle A_i, x\rangle = b_i, i = s+1,\ldots,m\}$ polyhedral

with $\bar{x} \in C$,

$v \in N_C(\bar{x}) \Leftrightarrow \exists \lambda_1 \ge 0,\ldots,\lambda_s \ge 0,\ \lambda_{s+1}\ldots,\lambda_m \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{m} \lambda_i A_i \quad i = 1,\ldots,s, \quad \lambda_i(\langle A_i, \bar{x}\rangle - b_i) = 0 \qquad \left(\lambda \perp (A\bar{x} - b)\right)$$

$x^\star$ sol'n of linearly constrained convex program if and only if

one can find KKT-multipliers $y \in \mathbb{R}^m$ such that

(a) $\left\langle A_i, x^\star \right\rangle \ge b_i, i = 1,\ldots,s, \quad \left\langle A_i, x^\star \right\rangle = b_i, i = s+1,\ldots,m,$

(b) $i = 1,\ldots,m, \quad y_i \ge 0,\ y_i(\langle A_i, x^\star\rangle - b_i) = 0,$

(c) $x^\star \in \arg\min f_0(x) - \left\langle A^\top y, x\right\rangle, x \in X$ (box)

(~c) -$v \in N_X(x^\star)$ such that $\partial f_0(x^\star) \ni v + A^\top y$

Expectation Functionals

plain probability measures: for $A \in \mathcal{A}$, $P(A) = P_d(A) + P_c(A)$

$$\Rightarrow \mathbb{E}\{h(\xi)\} = \sum_{\xi \in \Xi_d} h(\xi) p_\xi + \int_{\xi \in \Xi_c \subset \mathbb{R}^N} h(\xi)\, p(\xi)\, d\xi)$$

density

$$\mathbb{E}\{S(\xi)\} = \{\mathbb{E}\{s(\xi)\} \,|\, s(\xi) \in_{a.s.} S(\xi), s \text{ summable}\} \subset \mathbb{R}^N$$

properties of E :

linearity, order preserving, dominated convergence

▵ f random convex lsc functions $\Rightarrow x \mapsto Ef(x)$ convex

when Ef finite-valued:

$$dEf(x;w) = \mathbb{E}\{df(\xi,x;w)\} \quad \& \quad \partial Ef(x) = \mathbb{E}\{\partial f(\xi,x)\}$$

▵ $x^* \in \arg\min Ef \Leftrightarrow \exists\, v \in \partial f(\bullet, x^*)$, $\mathbb{E}\{v(\xi)\} = 0$ &

$$\boxed{\forall \xi \in \Xi : \; x^* \in \arg\min_x \left[f(\xi,x) - \langle v(\xi), x \rangle \right]}$$

!! Remark: to solve $0 \in \partial Ef(x^*)$ it suffices to know $v(\xi)$ for just one ξ and solve one problem of the same size as the deterministic version

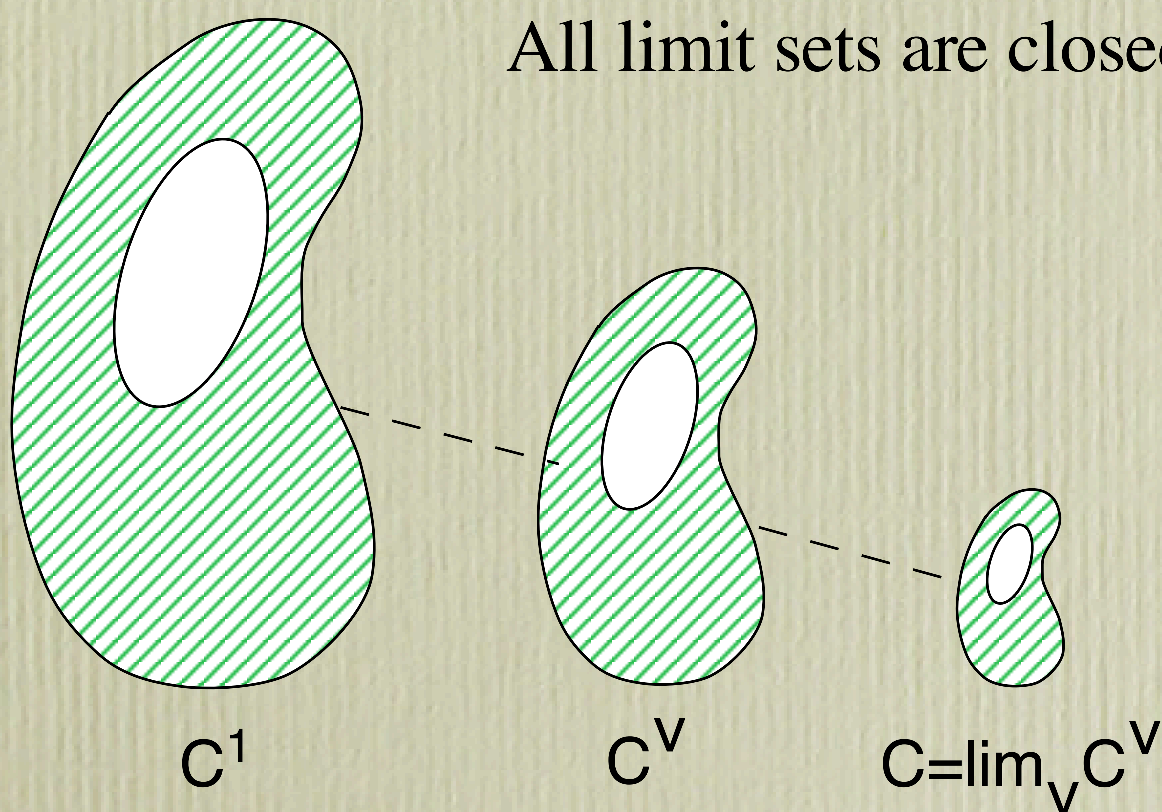
Approximation: Convergence

outer limit: $\mathrm{Ls}_v C^v = \left\{ x \in \text{ cluster-points}\{\mathrm{x}^v\}, x^v \in C^v \right\}$

inner limit: $\mathrm{Li}_v C^v = \left\{ x = \lim_v x^v, x^v \in C^v \subset \mathbb{R}^n \right\} \subset \mathrm{Ls}_v C^v$

limit: $C^v \to C$ if $C = \mathrm{Li}_v C^v = Ls_v C^v$ (Painlevé]

All limit sets are closed

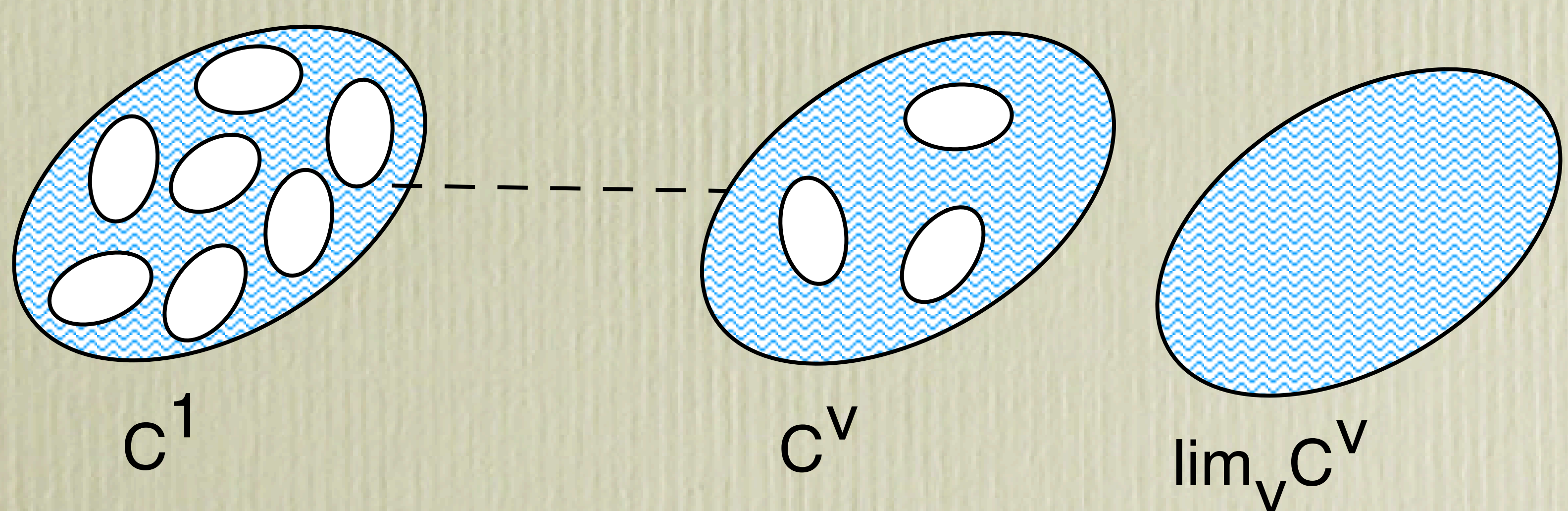


Convex limit sets

C^ν convex $\Rightarrow$ $\mathrm{Li}_\nu C^\nu$ convex $\Rightarrow$ $\mathrm{Lm}_\nu C^\nu$ convex (if it exists)

$\nRightarrow$ $\mathrm{Ls}_\nu C^\nu$ convex

but convexity can result from taking limits



EPI-Limits

$\left\{ f^\nu : \mathbb{R}^n \to \bar{\mathbb{R}}, \nu \in \mathbb{N} \right\}$

lower epi-limit: $\text{e-li}_\nu f^\nu$ such that $\text{epi}(\text{e-li}_\nu f^\nu) = \text{Ls}_\nu \, \text{epi} \, f^\nu$

upper epi-limit: $\text{e-ls}_\nu f^\nu$ such that $\text{epi}(\text{e-ls}_\nu f^\nu) = \text{Li}_\nu \, \text{epi} \, f^\nu$

epi-limit: $f^\nu \underset{e}{\to} f$ when $f = \text{e-li}_\nu f^\nu = \text{e-ls}_\nu f^\nu$, $f = \text{e-lm}_\nu f^\nu$

all epi-limits are lsc (closed epigraphs), $\text{e-li}_\nu f^\nu \le \text{e-ls}_\nu f^\nu$

f^ν convex $\Rightarrow$ $\text{e-ls}_\nu f^\nu$ is convex and so is $\text{e-lm}_\nu f^\nu$ (if it exists)

Convergence of level sets / constraint sets:

$f \le \text{e-li}_\nu f^\nu \Leftrightarrow \text{Ls}_\nu(\text{lev}_{\alpha_\nu} f^\nu) \subset \text{lev}_\alpha f \quad \forall \alpha_\nu \to \alpha$

$f \ge \text{e-ls}_\nu f^\nu \Leftrightarrow \text{Ls}_\nu(\text{lev}_{\alpha_\nu} f^\nu) \subset \text{lev}_\alpha f$ for some $\alpha_\nu \to \alpha$

Operations: sums, scalar multiplication, epi-sums

SV-Convergence
solutions, minimizers, ...

A^ν solutions of (generalized) equations

minimizers of a sequence of functions

saddle points or min-sup points of bifunctions

ε-$A^\nu : \varepsilon > 0$ approximate solutions, minimizers, ...

A solution set, minimizers, ... of corresponding limit

Definition: A^ν sv-converge to A, written $A^\nu \rightrightarrows_\nu A$, if

a) $\bar{x} \in$ cluster-points$\left\{x^\nu \in A^\nu\right\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, \mathrm{x}^\nu \in \varepsilon_\nu\text{-}A^\nu \to \bar{x}$

Convergence of Minimizers
Sv-convergence of Minimizers

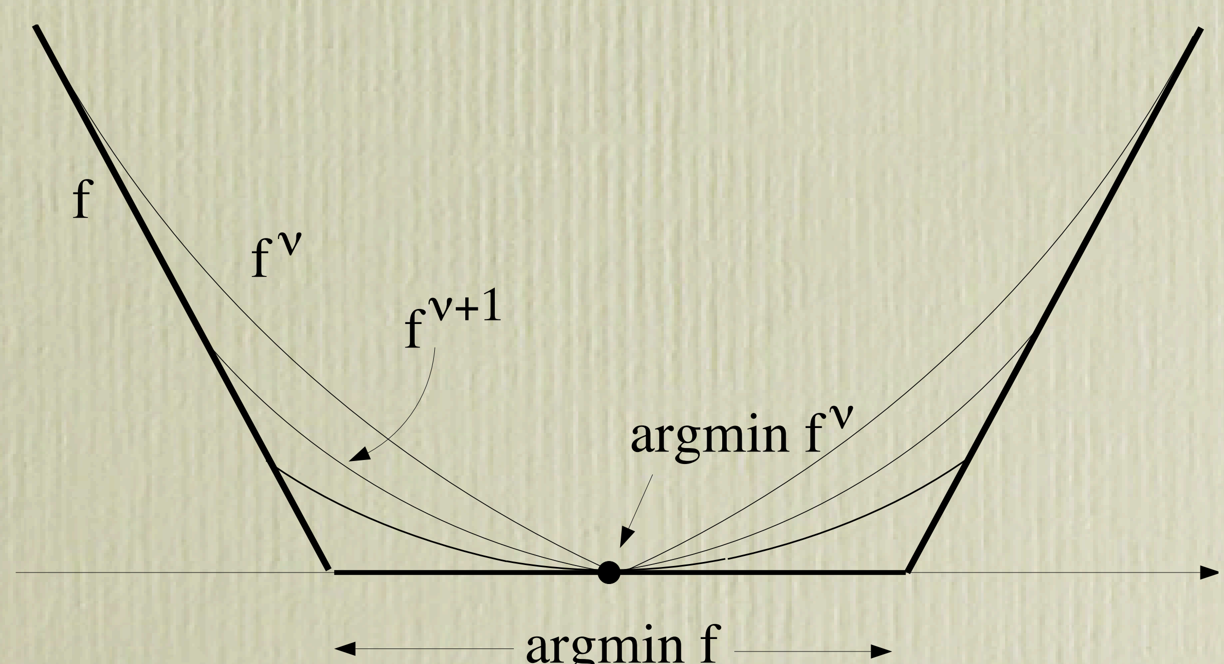
$$f^\nu \underset{e}{\to} f, x \in \text{ cluster}\left\{x^\nu \in \arg\min f^\nu\right\} \Rightarrow x \in \arg\min f$$

$$f^\nu \underset{e}{\to} f, \inf f \in \mathbb{R}, x \in \arg\min f \Rightarrow \exists \varepsilon_\nu \searrow 0, x^\nu \in \varepsilon_\nu\text{-}\arg\min f^\nu \to x$$

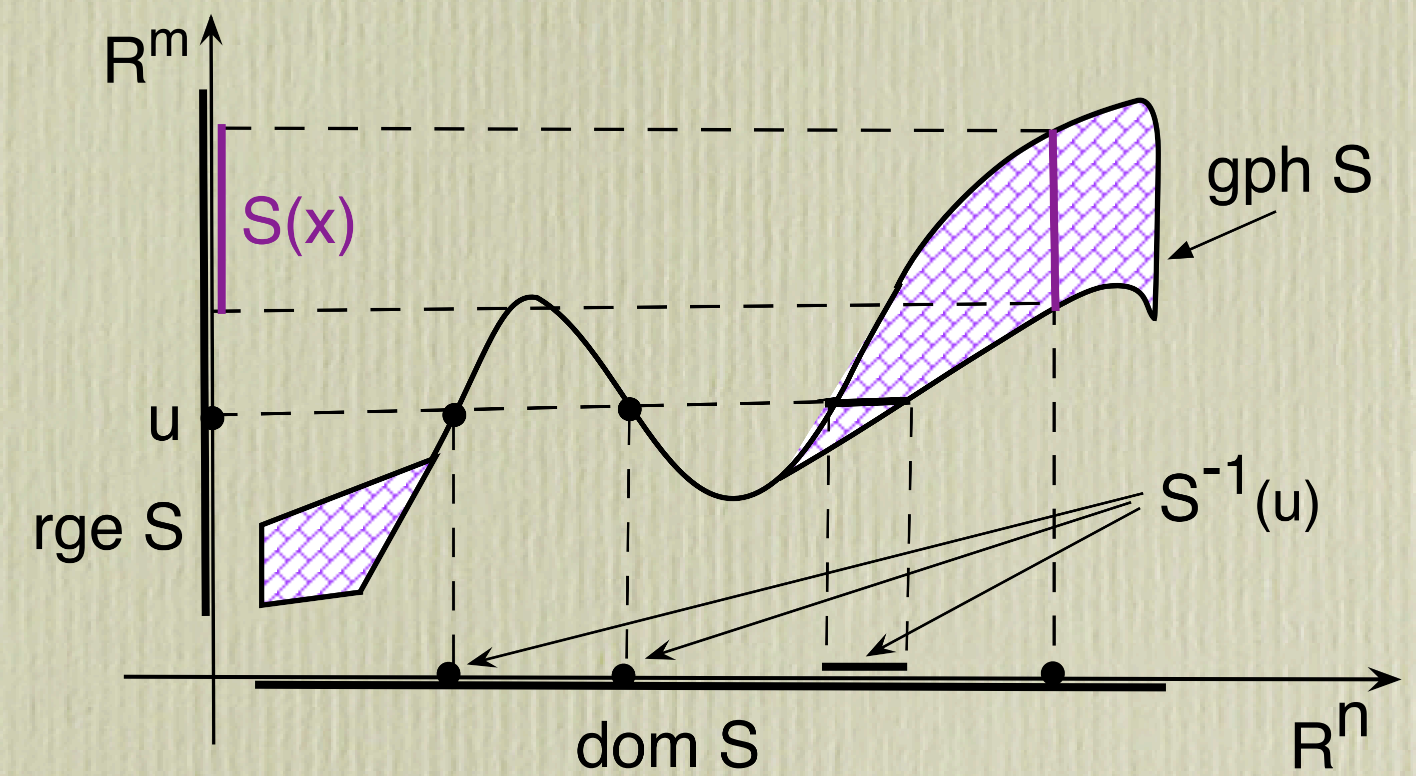
$$f^\nu \underset{e}{\to} f \not\Rightarrow \arg\min f^\nu \to \arg\min f$$

$$f^\nu \underset{e}{\to} f,\ \inf f^\nu \to \inf f \in \mathbb{R} \Leftrightarrow \left\{f^\nu\right\}_{\nu \in \mathbb{N}} \text{ epi-tight, i.e.}$$

$$\forall \varepsilon > 0, \exists B \text{ compact s.t. } \inf_B f^\nu \leq \inf f^\nu + \varepsilon,\ \forall \nu \geq \nu_\epsilon$$



Set-valued mappings



S osc (outer semicontinuous) at $\bar{x}$ if $\mathrm{Ls}_{x\to\bar{x}} S(x) \subset S(\bar{x})$

S osc $\Leftrightarrow$ gph S closed

S isc (inner semicontinuous) at $\bar{x}$ if $\mathrm{Li}_{x\to\bar{x}} S(x) \supset S(\bar{x})$

S continuous if it's isc and osc

GRAPHICAL CONVERGENCE

SV-convergence of solutions

$S^\nu \to_g S$ when gph $S^\nu \to$ gph S (as subsets of $\mathbb{R}^n \times \mathbb{R}^m$)

Generalized Equations ~ Inclusions

$S^\nu, S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $S^\nu(x) \ni u^\nu$, $S(x) \ni \bar{u}$ and $S^\nu \to_g S, u^\nu \to \bar{u}$. Then

$$\bar{x} \in \text{ cluster-pts}\left\{ x^\nu \,\middle|\, S^\nu(x^\nu) \ni u^\nu \right\} \Rightarrow S(\bar{x}) \ni \bar{u}$$

$$S(\bar{x}) \ni \bar{u} \Rightarrow \exists \hat{u}^\nu \to \bar{u} \text{ with } S^\nu(\hat{x}^\nu) \ni \hat{u}^\nu \text{ and } \hat{x}^\nu \to \bar{x}$$

$S^\nu \to_p S$ pointwise doesn't yield convergence of sol'ns

CONVERGENCE RATES

Excess distance function:

$$e_\rho(A,B) = \inf\left\{\eta \ge 0 \,\middle|\, A \cap \rho\mathbb{B} \subset B + \eta\mathbb{B}\right\}, \quad \rho > 0$$

Estimate of set distance:

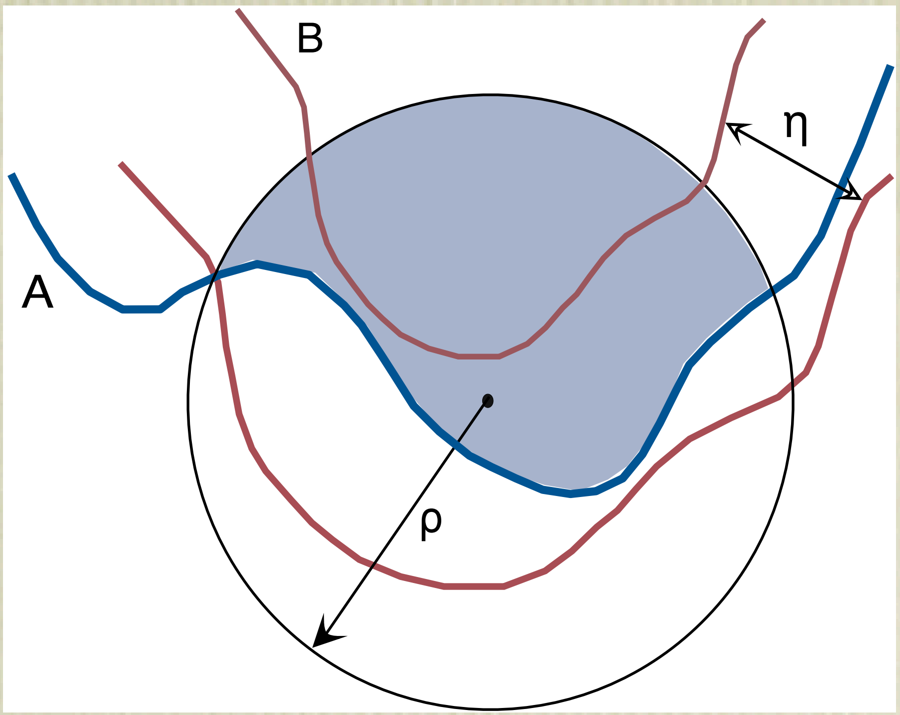
$$d\hat{l}_\rho(A,B) = \max[\,e_\rho(A,B), e_\rho(B,A)\,]$$

Set-distance:

$$dl_\rho(A,B) = \max_{x \in \rho\mathbb{B}} \left|d(x,A) - d(x,B)\right|,$$

$$d(x,C) = \inf_{y \in C} |y - x|$$

Pompeiu-Hausdroff distance: $\rho = \infty$



$$d\hat{l}_\rho(A,B) \le dl_\rho(A,B) \le d\hat{l}_{\rho'}(A,B),$$

$$\rho' \ge 2\rho + \max[\,d(0,A), d(0,B)\,]$$

$$C^\nu \to C \Leftrightarrow dl_\rho(C^\nu, C) \to 0 \Leftrightarrow d\hat{l}_\rho(C^\nu, C) \to 0 \quad \forall \rho \ge 0$$

EPI-DISTANCE

lsc-fcns($\mathbb{R}^n$) = space of all lsc functions from $\mathbb{R}^n \to \overline{\mathbb{R}} = [-\infty, \infty]$

$d\hat{l}_\rho(f,g) = d\hat{l}_\rho(\text{epi}\, f, \text{epi}\, g)$, $\quad dl_\rho(f,g) = dl_\rho(\text{epi}\, f, \text{epi}\, g)$, $\rho \geq 0$

$$\mathbb{B}^{n+1} = \mathbb{B}^n \times [-1,1]$$

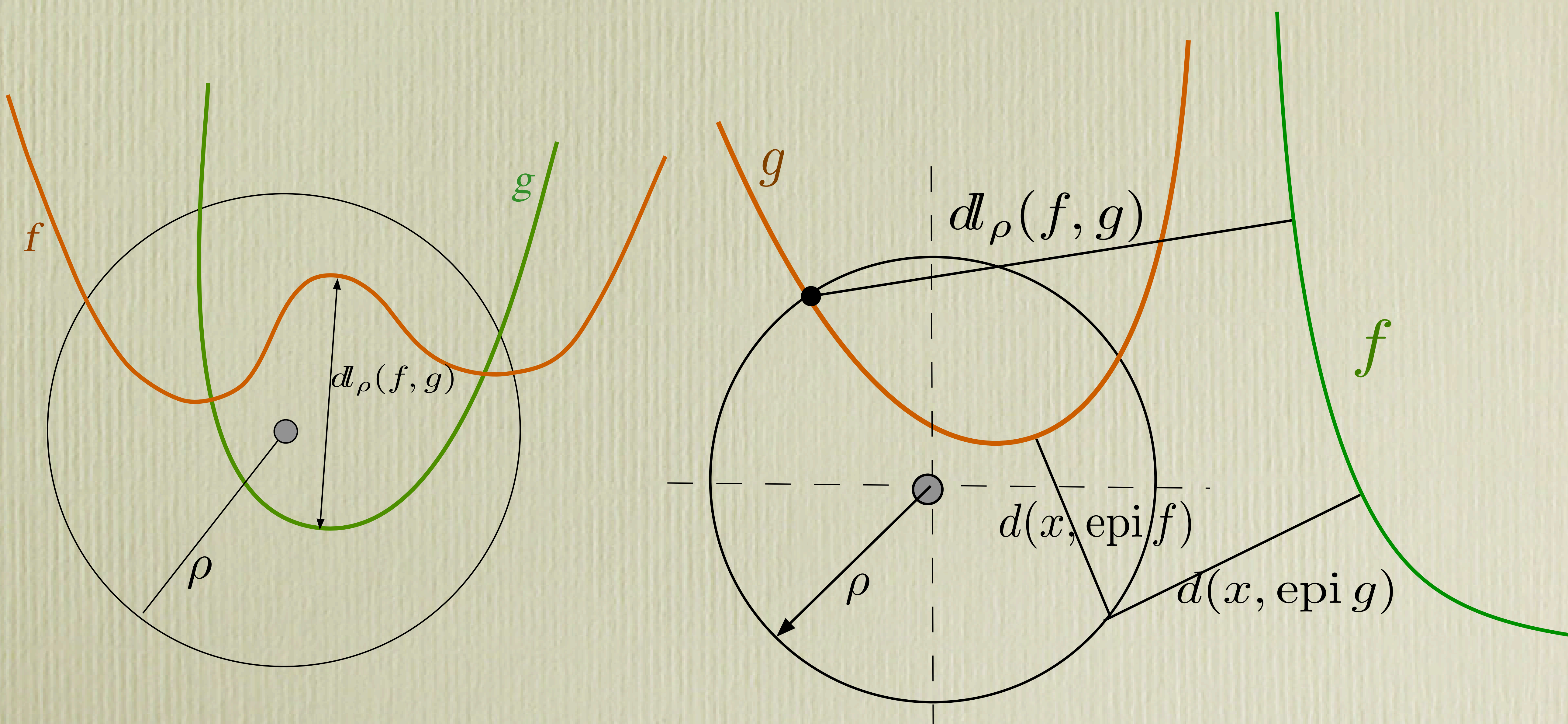
$dl(f,g) = \int_{\rho \geq 0} e^{-\rho} dl_\rho(f,g)\, d\rho$, epi-distance, Attouch-Wets topology

$f^\nu, f \in$ lsc-fcns($\mathbb{R}^n$), $f^\nu \to_e f \Leftrightarrow dl(f^\nu, f) \to 0$

also $dl_\rho(f^\nu, f) \to 0, \forall \rho \geq \bar{\rho} > 0, \ldots$

$\big($lsc-fcns($\mathbb{R}^n$) $\setminus \{f \equiv \infty\}, dl\big)$ complete metric space

Epi-distance

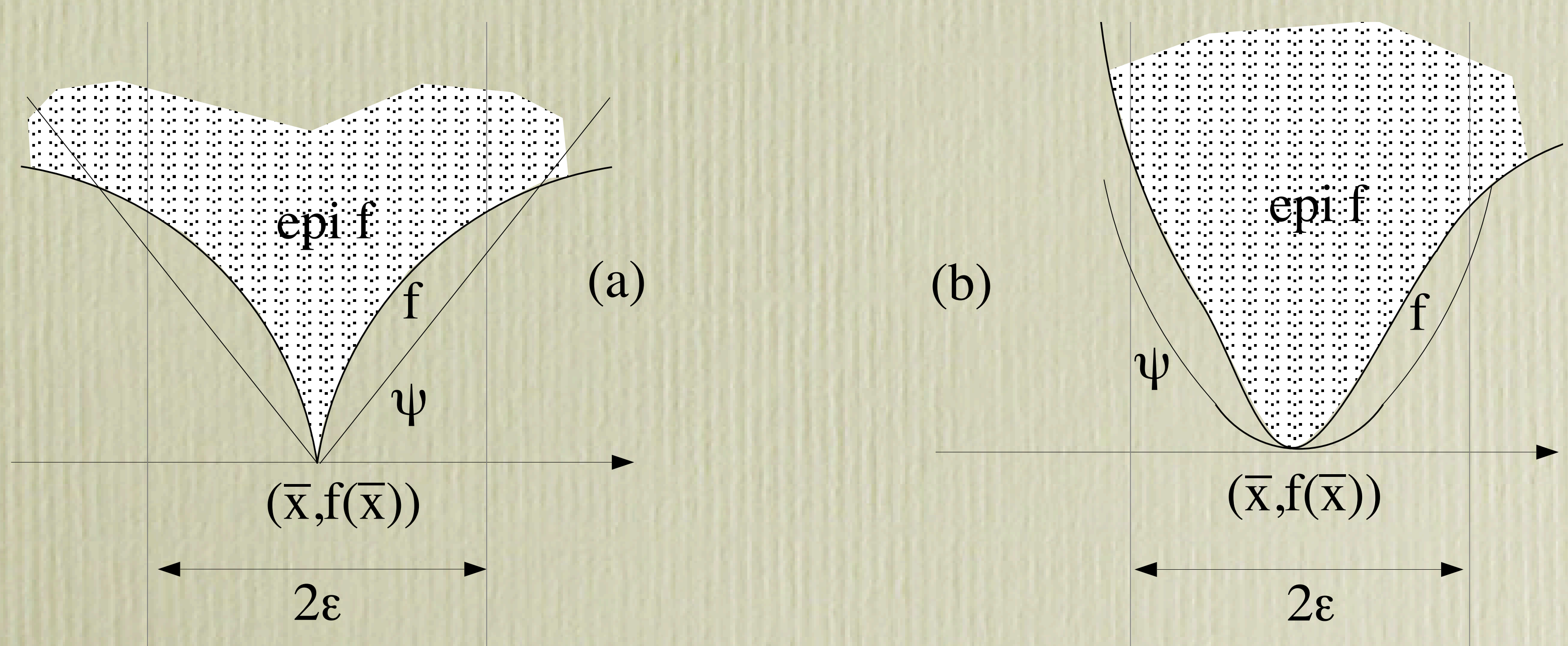


QUANTITATIVE ESTIMATES

under ψ-conditioning for f, $\quad f, g \in \text{lsc-fcns}(\mathbb{R}^n), \quad \inf f, \inf g \in \mathbb{R}$

$$\left|\min_{\rho\mathbb{B}} g - \min f\right| \le dl_\rho(f, g)$$

$$\arg\min_{\rho\mathbb{B}} g \subset \arg\min f + \psi(dl_\rho(f, g))\mathbb{B}$$



QUANTITATIVE ESTIMATE
convex functions

$f, g : \mathbb{R}^n \to \overline{\mathbb{R}}$, proper, lsc, convex functions

$\arg\min f, \arg\min g \neq \emptyset$

ρ_0 large enough so that $\rho_0 \mathbb{B}$ meets $\arg\min f$ & $\arg\min g$

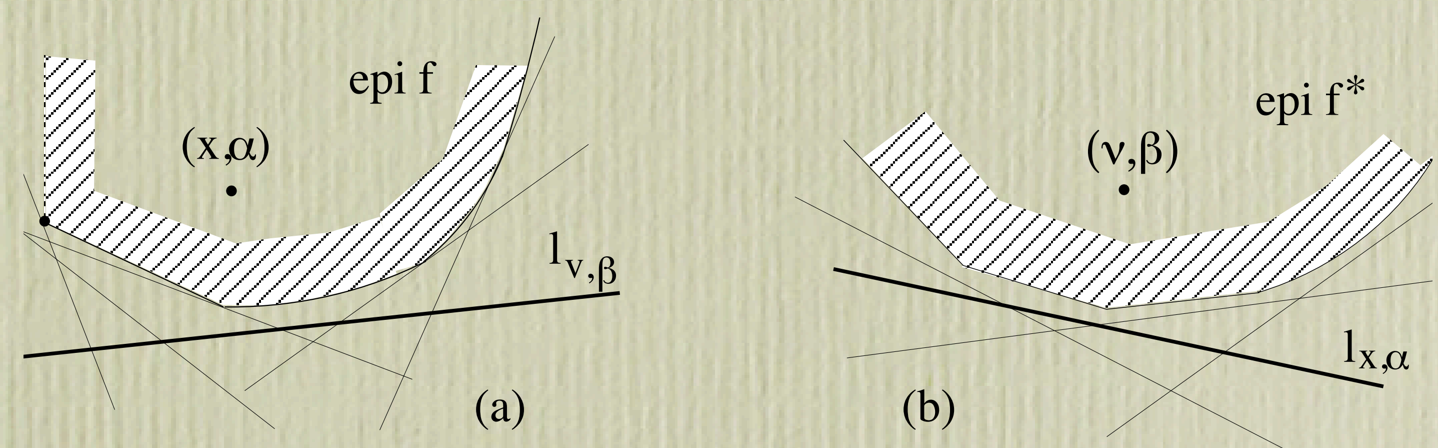
$\min f \geq -\rho_0, \ \min g \geq -\rho_0$

Then, with $\rho > \rho_0, \ \varepsilon > 0, \ \bar{\eta} = dl_\rho(f, g)$

$$\boxed{\begin{aligned} d\hat{l}_\rho(\varepsilon\text{-arg min} f, \varepsilon\text{-arg min g}) &\leq \bar{\eta}\left(1 + \frac{2\rho}{\bar{\eta} + \epsilon/2}\right) \\ &\leq (1 + 4\rho/\epsilon) d\hat{l}_\rho(f, g) \end{aligned}}$$

Convex functions

(Wijsman) $f^\nu \underset{e}{\rightarrow} f \Leftrightarrow (f^\nu)^* \underset{e}{\rightarrow} f^* = \sup_x \big(\langle v,x\rangle - f(x)\big),\quad f^\nu$ lsc, convex



conjugate functions

$f^\nu \underset{e}{\rightarrow} f \not\Rightarrow f^\nu \underset{p}{\rightarrow} f$ (pointwise) & $f^\nu \underset{p}{\rightarrow} f \not\Rightarrow f^\nu \underset{e}{\rightarrow} f$

$$f^\nu \underset{e}{\rightarrow} f \equiv f^\nu \underset{p}{\rightarrow} f \Leftrightarrow \left\{f^\nu\right\}_{\nu\in\mathbb{N}} \text{ is equi-lsc}$$

(Walkup-Wets) $dl_{csm}\big(f,g\big) = dl_{csm}\big(f^*,g^*\big) \quad \big[\approx dl(f,g) = dl(f^*,g^*)\big]$

Attouch's Theorem

(initial proof: via Moreau envelopes)

$f^\nu, f : \mathbb{R}^n \to \overline{\mathbb{R}}$, proper, convex, lsc and $\lambda > 0$

The following are equivalent:

a) $f^\nu \to_e f$

b) the mappings $\partial f^\nu \to_g \partial f$ and

$\exists v^\nu \in \partial f^\nu(x^\nu), \bar{v} \in \partial f(\bar{x}),\ \left(x^\nu, v^\nu\right) \to (\bar{x}, \bar{v}), f^\nu(x^\nu) \to f(\bar{x})$

(convergence of an integration constant)

c) $P_\lambda f^\nu \to_p P_\lambda f = \arg\min_w \left\{ f(w) + \frac{1}{2\lambda}|w - \bullet|^2 \right\}$ and

$\exists\, \bar{x}, x^\nu \to \bar{x}$ such that $e_\lambda f^\nu(x^\nu) \to e_\lambda f(\bar{x})$

in situation b): also $f^{\nu*}(v^\nu) \to f^*(\bar{v})$

Epi-Convergence

review

$f^\nu \xrightarrow{e} f$ if for all $x \in E$,

1. $\forall\, x^\nu \to x,\ \liminf_\nu f^\nu(x^\nu) \geq f(x)$

2. $\exists\, x^\nu \to x,\ \limsup_\nu f^\nu(x^\nu) \leq f(x)$

"Geometrically": $\mathrm{epi}\, f^\nu \to \mathrm{epi}\, f$ (later)

Pointwise:

$$\liminf_\nu f^\nu(x) \geq f(x), \qquad \limsup_\nu f^\nu(x) \leq f(x)$$

Continuous: $\forall x^\nu \to x$,

$$\liminf_\nu f^\nu(x^\nu) \geq f(x), \qquad \limsup_\nu f^\nu(x^\nu) \leq f(x)$$

Epi-Convergence $\Rightarrow$ convergence of minimizers

$A^\nu = \arg\min f^\nu$, ε-A^ν : $\varepsilon > 0$ approximate minimizers,

$A = \arg\min f$ of limit problem, ε-A approx. minimizers

A^ν v-converges to A, written $A^\nu \Rightarrow_v A$, if

a) $\bar{x} \in$ cluster-points$\{x^\nu \in A^\nu\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, x^\nu \in \varepsilon_\nu$-$A^\nu \to \bar{x}$

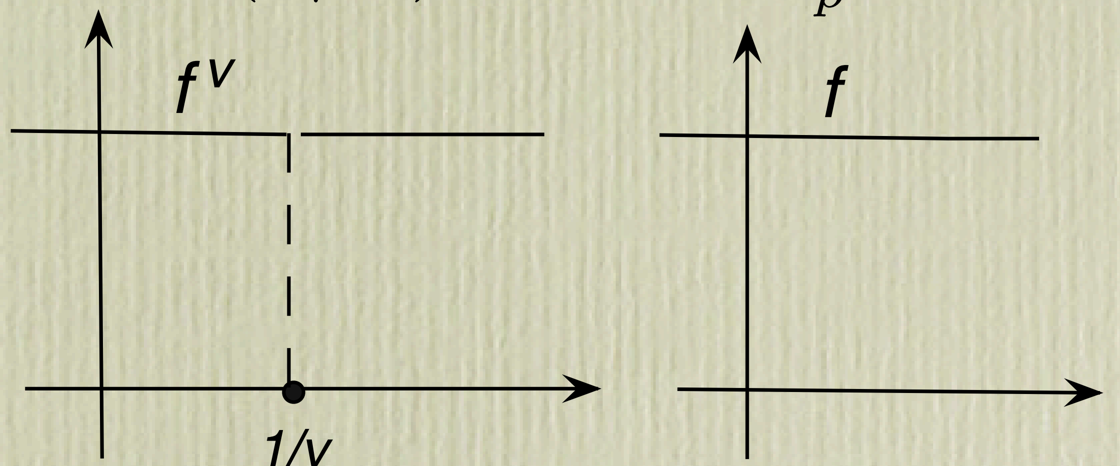
$f^\nu \xrightarrow{e} f$ implies ε-$A^\nu \Rightarrow_v \varepsilon$-$A$, $\forall\, \varepsilon \geq 0$
A unique minimizer, ε^ν-$A^\nu \rightrightarrows A$ as $\varepsilon^\nu \searrow 0$.

(inf f > -∞)

Why epi-convergence?

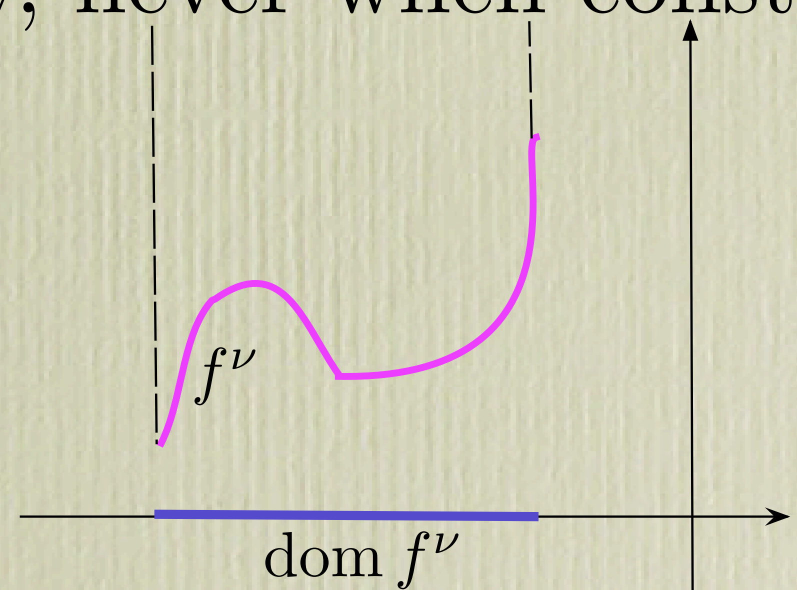
1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

   $f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$



2. uniform convergence implies convergence of minimizers

   but applies rarely, never when constraints depend on ν



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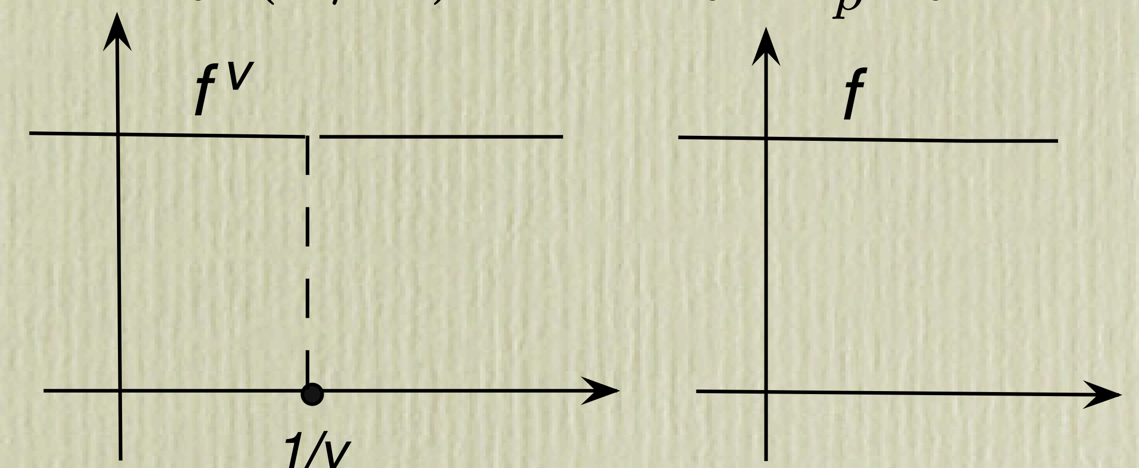

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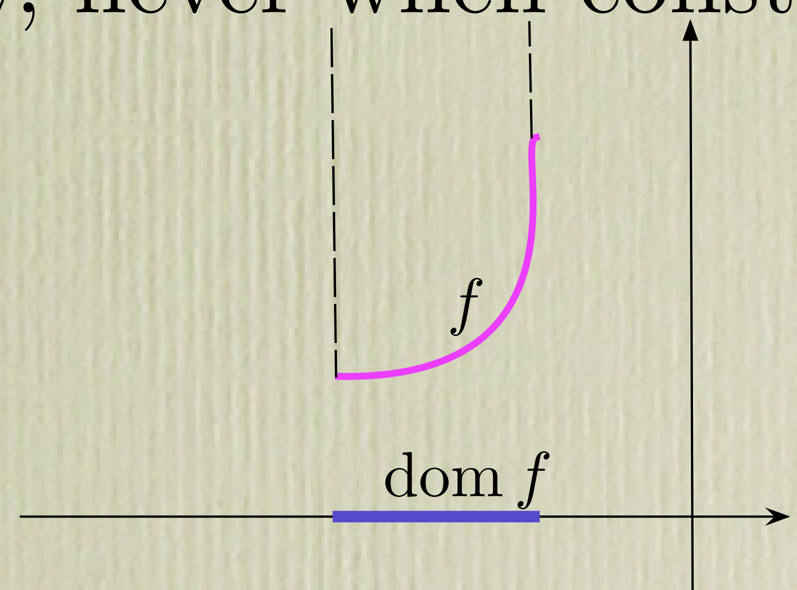
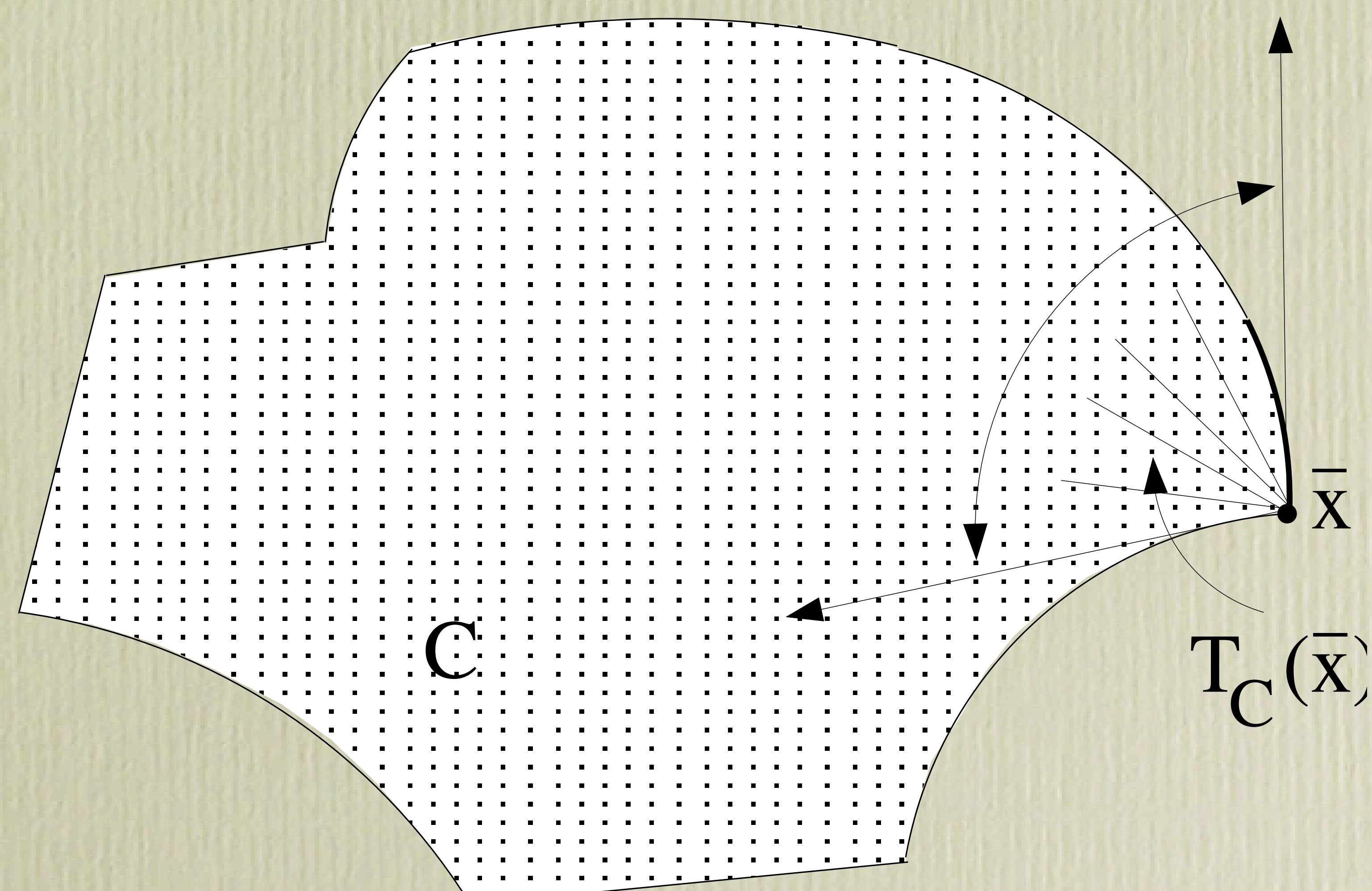


Variational epi-convergence

Variational geometry
Tangent Cone

$w \in T_C(x)$, tangent to C at $x \in C$, if $\left|x^\nu - x\right| / \tau_\nu \to w$ for $x^\nu \underset{C}{\to} x, \tau_\nu \searrow 0$

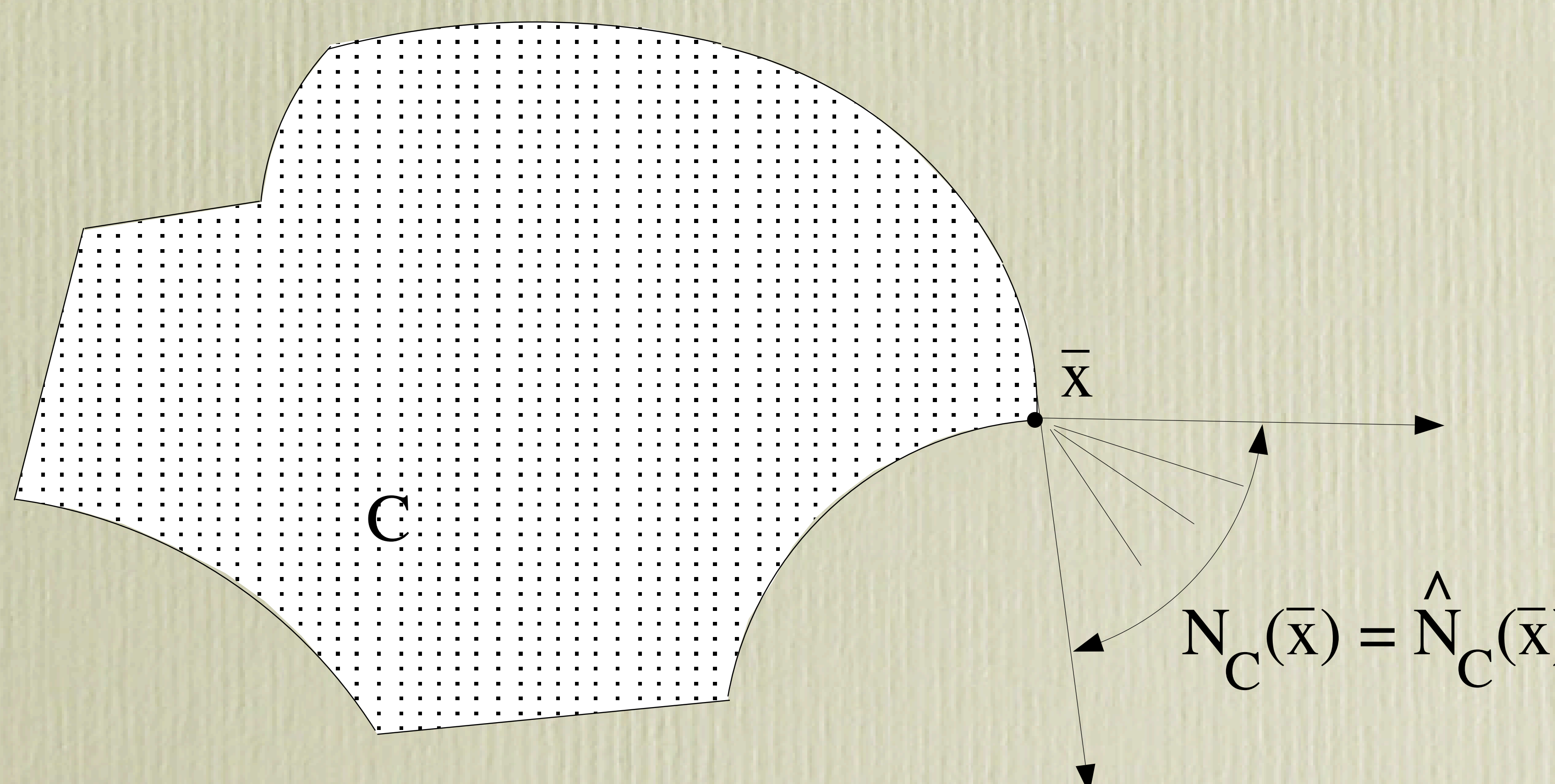


Variational Geometry
Normal Cone

$v \in \hat{N}_C(\bar{x})$, regular normal at $\bar{x} \in C$, if $\langle v, x - \bar{x} \rangle \leq o(|x - \bar{x}|), \forall x \in C$

$v \in N_C(\bar{x})$, normal at $\bar{x} \in C$, if $\exists x^\nu \underset{C}{\rightarrow} x$ and $v^\nu \rightarrow v$ with $v^\nu \in \hat{N}_C(x^\nu)$

normal cones: closed cones, $\hat{N}_C(\bar{x})$ convex



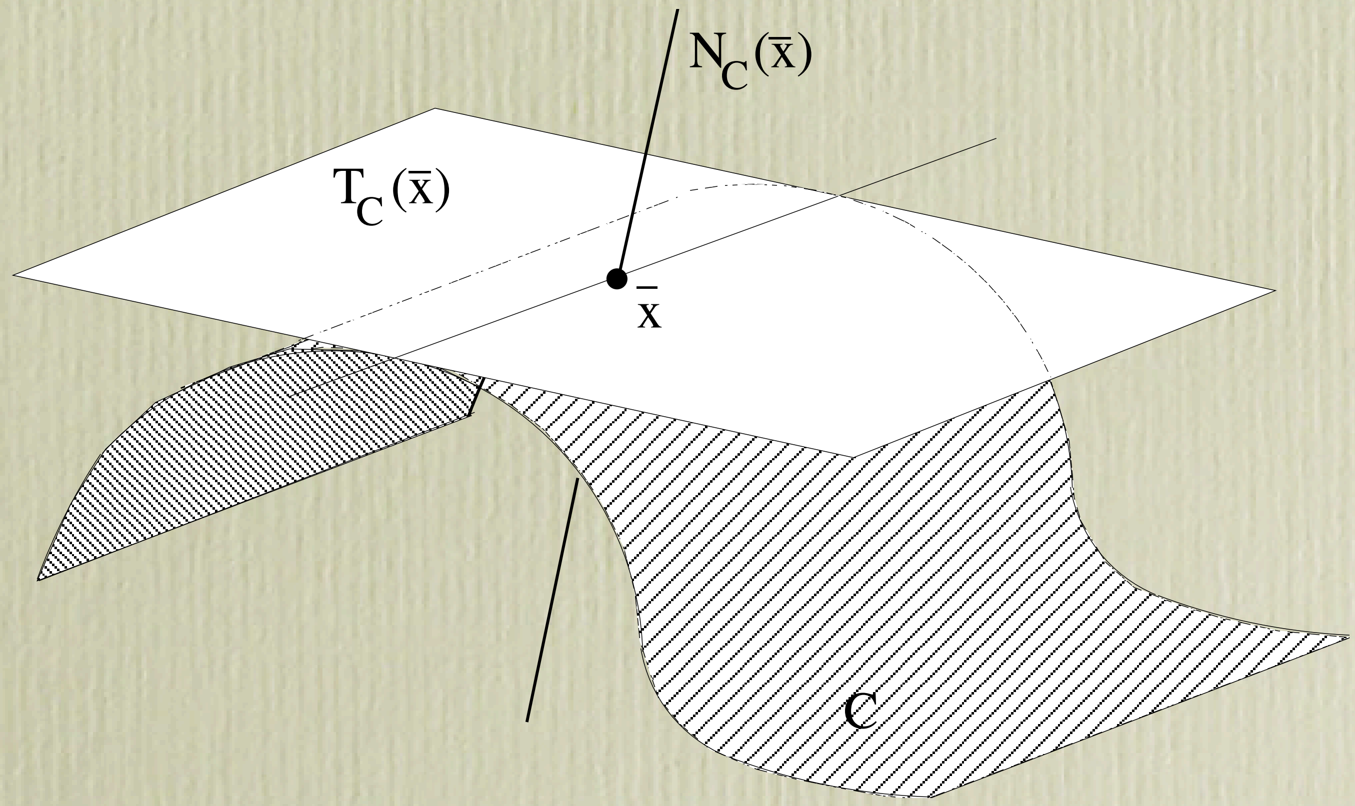
Clarke regularity

C Clarke regular at $\bar{x}$ if C locally closed & $N_C(x) = \hat{N}_C(\bar{x})$

which implies $N_C(\bar{x})$ is convex if C regular at $\bar{x}$

In general, $N_C(\bar{x}) = Ls_{x \to_C \bar{x}} N_C(x) \supset \hat{N}_C(\bar{x})$

Smooth manifolds and closed convex set are regular (also locally)



SUBGRADIENTS

(never supergradients)

$v \in \hat{\partial} f(\bar{x})$ regular subgradient if $f(x) \geq f(\bar{x}) + \langle v, x - \bar{x} \rangle + o(|x - \bar{x}|)$

$$\hat{\partial} f(\bar{x}) = \left\{ v \,\middle|\, (v,-1) \in \hat{N}_{\text{epi}\, f}(\bar{x}, f(\bar{x})) \right\}, \text{ closed and convex}$$

$v \in \partial f(\bar{x})$ subgradient if $\exists\, \mathrm{x}^\nu \to_f \bar{x}, v^\nu \in \hat{\partial} f(x^\nu)$ with $v^\nu \to v$

$$\partial f(\bar{x}) = \left\{ v \,\middle|\, (v,-1) \in N_{\text{epi}\, f}(\bar{x}, f(\bar{x})) \right\}, \text{ closed}$$

$\mathrm{x} \mapsto \partial f(x)$ osc f-attentive convergence: $\Rightarrow \mathrm{Ls}_{x \to_f \bar{x}} \partial f(x) \subset \partial f(\bar{x})$

f differentiable at $\bar{x}$: $\hat{\partial} f(\bar{x}) = \nabla f(\bar{x}) = \partial f(\bar{x})$

f regular at $\bar{x}$: f locally lsc with $\partial f(\bar{x}) = \hat{\partial} f(\bar{x})$ (f locally convex, e.g)

$\partial \iota_C(x) = N_C(x)$ when C is convex

OPTIMALITY

$\min f = f_0 + \iota_C$, optimality: ``$0 \in \partial f(\bar{x})$''

generally, $\partial\left(f+g\right) \neq \partial f + \partial g$

$\mathbb{C}.\mathbb{Q}.$ (Constraint Qualification): $-N_C(\bar{x}) \cap \partial^{\infty} f_0(\bar{x}) = \{0\}$

$v \in \partial^{\infty} f_0(\bar{x}) =$ horizon subgradient if

$$\exists\, \mathrm{x}^{\nu} \to_f \bar{x}, v^{\nu} \in \hat{\partial} f(x^{\nu}), \lambda_{\nu} \searrow 0 \ \&\ \lambda_{\nu} v^{\nu} \to v$$

Fermat's Rule (quite a bit generalized):

with $\mathbb{C}.\mathbb{Q}.$ $\bar{x}$ locally optimal $\Rightarrow \partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0$

f convex ($\Rightarrow$ regular), $\partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0 \ \Rightarrow$

globally optimal (no $\mathbb{C}.\mathbb{Q}.$ in this form)

Stochastic Variational Analysis

Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample P^ν (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

Some Examples:

1. **Stochastic Programming** (recourse model)

$$f(\xi,x) = \begin{cases} f_{01}(x) + Q(\xi,x) & \text{if } x \in C_1 \\ \infty & \text{otherwise} \end{cases}$$

$$Q(\xi,x) = \inf_y \left\{ f_{02}(\xi,y) \,\middle|\, y \in C_2(\xi,x) \right\}$$

$$\min Ef(x) = \mathbb{E}\{f(\xi,x)\},$$

SAA-problem: $\min f^\nu(\vec{\xi}^{\,\nu}, x) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x)$

2. **Statistical Estimation** (fusion of hard & soft information)

$$L(\xi,h) = \begin{cases} -\ln h(\xi) & \text{if } h \geq 0, \int h = 1, h \in A^{\text{soft}} \subset E \\ \infty & \text{otherwise} \end{cases}$$

$$EL(h) = \mathbb{E}\{L(\xi,h)\},\ h^{\text{true}} = \operatorname{argmin}_E \mathbb{E}\{L(\boldsymbol{\xi},h)\}$$

estimate: $h^\nu \in \operatorname{argmin}_E \mathbb{E}^\nu\{L(\boldsymbol{\xi},h)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} L(\xi^l, h)$

A^{soft} : constraints on support, moments, shape, smoothness, ...

Pricing financial instruments

3. A contingent claim: environment process: $\left\{\boldsymbol{\xi}^t \in \mathbb{R}^d\right\}_{t=0}^T$

history: $\vec{\xi}^{\,t}$, $\xi = \xi^T$, price process: $S^t(\vec{\xi}^{\,t}) \in \mathbb{R}^n$; numéraire (risk-free): $S_1^t \equiv 1$

claims: $\left\{G^t(\vec{\xi}^{\,t})\right\}_{t=1}^T$; i-strategy: $\left\{X^t(\vec{\xi}^{\,t})\right\}_{t=0}^T$; value @ t : $\langle S^t(\vec{\xi}^{\,t}), X^t(\vec{\xi}^{\,t})\rangle$

Instruments: T-bonds, options, swaps, insurace contracts, mortgages, ...

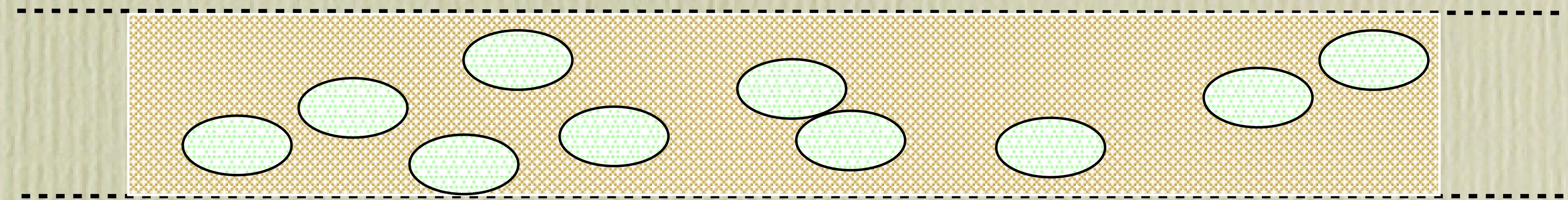
$$\max \mathbb{E}\left\{\langle S^T, X^T\rangle\right\} \text{ such that } \langle S^t, X^t\rangle \le G^t + \langle S^t, X^{t-1}\rangle,\ t = 1 \to T$$

$$\langle S^0, X^0\rangle \le G^0,\ \langle S^T, X^T\rangle \ge G^T \text{ a.s.}$$

feasible if $G^0 + \cdots + G^T \ge 0\ \forall \xi$; arbitrage $\Rightarrow$ unbounded

prob$[\boldsymbol{\xi} = \xi] = p_\xi$ (finite sample?): $\max \sum_{\xi \in \Xi} p_\xi \langle S^T(\xi), X^T(\xi)\rangle \ldots$

4. Stochastic homogenization, ...



$$-\nabla\cdot\big(a(\boldsymbol{\xi},x)\nabla u(\boldsymbol{\xi},x)\big) = h(x) \text{ for } x\in\Omega, \qquad u(\boldsymbol{\xi},x)=0 \text{ on bdry } \Omega$$

Variational formulation: $\forall\,\xi, \quad g(\xi,u) := \frac{1}{2}\int_\Omega a(\xi,x)|\nabla u|^2 dx - \langle h,u\rangle$
find $u(\xi,x)\in \operatorname{argmin}_{u\in H_0^1(\Omega)} g(\xi,u), \quad g(\xi,\cdot): L^2\to(-\infty,\infty]$. convex

$\mathbb{E}\{u(\boldsymbol{\xi},x)\}\in \operatorname{argmin}_{u\in H_0^1(\Omega)} G(u)$ where $\operatorname{epi} G = \mathbb{E}\{\operatorname{epi} g(\boldsymbol{\xi},\cdot)\}$
$G(u) = \inf_z \big\{\mathbb{E}\{g\big(\boldsymbol{\xi},z(\boldsymbol{\xi})\big)\,\big|\,\mathbb{E}\{z(\boldsymbol{\xi})\}=u\big\}$
$G^* = \mathbb{E}\{g^*(\boldsymbol{\xi},\cdot)\}, \quad g^*(\xi,v)=\sup_u\big\{\langle v,u\rangle - g(\xi,u)\big\}$, conjugate fcn
$\xi^1,\xi^2,\ldots$ stationary, use Ergodic Theorem for random lsc functions

$$G = g^{\text{hom}} = \big(\operatorname{epi}_{\text{w}}\text{-}\lim_\nu \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} g^*(\boldsymbol{\xi}^l,\cdot)\big)^* \quad\Longrightarrow\quad \text{values of } a^{\text{hom}}(x)$$

$Ef = \mathbb{E}\{f(\boldsymbol{\xi},\cdot)\}$

$f:\Xi\times E\to\overline{\mathbb{R}}$, random lsc function, $f(\xi,x)=f_0(\xi,x)$ when $x\in C(\xi)$

$E\subset\mathcal{M}(\Xi,\mathcal{A};\mathbb{R}^n):\mathcal{L}^p(\Xi,\mathcal{A},P;\mathbb{R}^n),\ldots$

others: $C\big((\Xi,\tau);\mathbb{R}^n\big)$,Orlicz, Sobolev, lsc-fcns(E)

$$Ef(x)=\int_\Xi f(\xi,x(\xi))\,P(d\xi)=\mathbb{E}\big\{f(\boldsymbol{\xi},x(\boldsymbol{\xi}))\big\}$$

$$=\infty \text{ whenever } \int_\Xi f_+(\xi,x(\xi))\,P(d\xi)=\infty$$

$Ef:E\to\overline{\mathbb{R}}$ always defined

Regression: (E is not a linear space)

$$\min\left\{\int_{y\in\mathbb{R}}\int_{x\in[0,1]^n}\phi(y-h(x))\,P(d(x,y))\,\middle|\, h\in \text{ lsc-fcns}(\mathbb{R}^n)\cap\mathcal{H}\right\}$$

$\mathcal{H}$ shape restrictions (convex, unimodal, ...)

Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in x, $(\forall \xi \in \Xi)$

(b) (ξ, x)-measurable $\quad$ $(\mathcal{A} \otimes B_E)$-measurable

recall: $f(\xi,x) = f_0(\xi,x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi,x) = \begin{cases} \frac{1}{\nu}\sum_{l=1}^{\nu}\Big(f(\xi^l,x) \text{ if } \mathrm{x} \in C(\xi^l)\Big) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi,\cdot)$ epi-converge to $\mathbb{E}\{f(\xi,h)\}$ P-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi,x)\}$ P-a.s.?

Law of Large Numbers for random lsc functions
$\sim$ LLN for Stochastic Optimization Problems.

Random lsc functions

(via inf-projections)

D countable dense subset of E

$f : E \to \overline{\mathbb{R}}$, lsc fcn completely identified by

$\left\{ o_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$, countable

or $\left\{ c_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$

$$f(\bar{x}) = \sup_{V \in \mathcal{N}(\bar{x})} \left[\inf_{x \in V} f(x) \right], \quad f \text{ lsc}, \; f(\bar{x}) = \liminf_{x \to \bar{x}} f(x)$$

$$= \sup_{V \in \mathcal{Q}(\bar{x})} \left[\inf_{x \in V} f(x) \right], \quad E \text{ separable} \quad \text{(Polish)}$$

$$\mathcal{Q}(\bar{x}) = \left\{ \mathbb{B}^o(x,\delta) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+, \bar{x} \in \mathbb{B}^o(x,\delta) \right\}$$

$$= \sup_{\delta \in \mathbb{Q}_+} \inf_{\{x \,|\, \mathbb{B}^o(x\delta) \in \mathcal{Q}(\bar{x})\}} o_{x,\delta}$$

$\{c_{x,\delta}\}$ same argument

Epi-convergence: Characterization

(via inf-projections)

$$f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^\nu \xrightarrow{e} f,\ f \text{ lsc}, \iff \forall \delta \in \mathbb{Q}_+,\ x \in D$$

$$\boxed{\operatorname{limsup}_\nu c^\nu_{x\delta} \le c_{x\delta}, \quad \operatorname{liminf}_\nu o^\nu_{x\delta} \ge o_{x\delta}}$$

for $x \in D, \delta \in \mathbb{Q}_+$: $o^\nu_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f^\nu$, $c^\nu_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f^\nu$

(fundamental) **Theorem.** $f^\nu : E \to \overline{\mathbb{R}}$ & f lsc (necessarily)

1. $\text{e-}\liminf_\nu f^\nu \iff \liminf_\nu (\inf_B f^\nu) \ge \inf_B f$ for all compact B
2. $\text{e-}\limsup_\nu f^\nu \iff \limsup_\nu (\inf_O f^\nu) \le \inf_O f$ for all open O

$\square$ Hit-and-miss topology on the space of epigraphs, (later?). $\square$

Scalarization of random lsc fcns

$f : \Xi \times E \to \overline{\mathbb{R}}$, random lsc fcn, completely identified by
$\left\{ o_{x\delta}(\xi) = \inf_{\mathbb{B}^o(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$, countable
or $\left\{ c_{x\delta}(\xi) = \inf_{\mathbb{B}(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$, $\forall \xi \in \Xi$

COUNTABLE

$\forall\, x \in \mathbb{R}^n, \delta > 0$
$\xi \mapsto o_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,
$o_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable
$\xi \mapsto c_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,
$c_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable.

- f random lsc fcn $\Rightarrow f + \iota_{\mathbb{B}(x,\delta)}$ random lsc fcn
- f random lsc fcn $\Rightarrow \xi \mapsto \alpha(\xi) = \inf_x f(x,\xi)$ measurable

Probabilistic properties
iid-properties (pairwise "i")

f random lsc fcn: $\{f(\boldsymbol{\xi}^\nu, \cdot\,)\}_{\nu\in\mathbb{N}}$ iid whenever $\{\boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid

Effös field on lsc-fcns$(E) = \sigma\text{-}\{f \in \text{lsc-fcns}(E) \,\big|\, \inf_O < \alpha\},\ O$ open, $\alpha \in \mathbb{R}$
$= \mathcal{B}\big(\text{lsc-fcns}(E)\big)$, E Polish

1. $\{f(\boldsymbol{\xi}^\nu, \cdot\,)\}_{\nu\in\mathbb{N}}$ "i" $\iff$ $\{o_{x\delta}(\boldsymbol{\xi}^\nu),\, \nu \in \mathbb{N}\}$ "i", $\forall x \in \mathbb{Q}^n, \delta \in \mathbb{Q}_+$
2. $f(\boldsymbol{\xi}^1, \cdot\,), f(\boldsymbol{\xi}^2, \cdot\,)$ "id" $\iff$ $o_{x\delta}(\boldsymbol{\xi}^1), o_{x\delta}(\boldsymbol{\xi}^2)$ "id", $\forall x \in \mathbb{Q}^n, \delta \in \mathbb{Q}_+$

the same holds for $\{c_{x\delta}(\,\cdot\,)\}$

Countable $\Rightarrow$ a.s.

Lemma. $f, g : E \to \overline{\mathbb{R}}$, lsc. $D = \mathrm{prj}_E$ countable dense subset of epi f.

$$f \le g \text{ on } D \implies f \le g \text{ on } E.$$

Proof. $f \le g$ on D only if $\{(x,\alpha) \,|\, \alpha \ge g(x), x \in R\} \subset \operatorname{epi} f$.
Taking closure on both sides $\implies$ $\operatorname{epi} g \subset \operatorname{epi} f$. $\square$

Implication. To check $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ *a.s.* on E only needs $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ *a.s.* on D a countable dense subset of E. Restrict $\boldsymbol{\xi}$ to a set of P-measure 1, say Ξ itself (from now on), and $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on $D \implies f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on E.

LLN: random lsc functions?

$$\forall\, x \in D,\ \delta \in \mathbb{Q}_+$$

1. $\frac{1}{\nu}\sum_{l=1}^{\nu} o_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{o_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$
2. $\frac{1}{\nu}\sum_{l=1}^{\nu} c_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{c_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$

$\not\Longrightarrow \sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\,)$ because

$$\min\big\{\mathbb{E}\{f(\boldsymbol{\xi}, z)\}\,\big|\, z \in \mathbb{B}(x,\delta)\big\} \neq \mathbb{E}\big\{\min\{f(\boldsymbol{\xi}, z)\}\,\big|\, z \in \mathbb{B}(x,\delta)\big\}$$

in general

Law of Large Numbers: Random lsc functions

LLN: Proof

1. $\exists\, x^\nu \to x : \limsup_\nu E^\nu f \leq Ef$
   for any $x \in E$ and any sample $\xi^\infty = (\xi^1, \xi^2, \dots)$
   $\lim_\nu \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x) \;\sim \lim_\nu \mathbb{E}^\nu\{f(\boldsymbol{\xi}^\infty, x)\} = Ef(x).$

2. $\forall x^\nu \to x, \liminf_\nu E^\nu f \geq Ef$
   for any $x \in E$ and any $\xi^\infty = (\xi^1, \xi^2, \dots) \in \Xi^\infty$

$$\text{e-}\liminf_{\nu\to\infty} f^\nu(\xi^\infty, x) = \sup_{\delta \searrow 0} \liminf_{\nu\to\infty} \inf_{\mathbb{B}^o(x,\delta)} E^\nu f \geq \sup_{\delta^l \searrow 0} \liminf_{\nu\to\infty} \frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l\delta^l}(\xi^l)$$

where $x^l \in D \to x$, $\delta^l \in \mathbb{Q}_+ \searrow 0$: $\quad x \in \mathbb{B}^o(x^l, \delta^l)$ & $\{\mathbb{B}^o(x^l, \delta^l)\} \searrow$
$\frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l\delta^l}(\xi^l) \to \mathbb{E}\{o^l_{x^l\delta^l}(\boldsymbol{\xi})\}$ & $\mathbb{E}\{o^l_{x^l\delta^l}(\boldsymbol{\xi})\} \nearrow Ef(x)$

$\implies \quad$ e-lim $\inf_{\nu\to\infty} E^\nu f(x) \geq Ef(x) \quad \square$

Law of Large Numbers
(random lsc fcns)

$f : \Xi \times E \to \overline{\mathbb{R}}$, locally inf-integrable random lsc function
$\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ are iid Ξ-valued random variables. Then,

$$E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\,\}$$

which means ε-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin Ef, $\forall\, \varepsilon \geq 0$

Ef unique minimizer, ε^ν-argmin $E^\nu f \rightrightarrows$ argmin Ef as $\varepsilon^\nu \searrow 0$.

SAA-applies without 'any' restrictions

loc.inf-integrable: $\int \inf\{f(\xi, \cdot) \,\big|\, \mathbb{B}(x, \delta)\} > \infty$ for some $\delta > 0$,
irrelevant in applications

Ergodic Theorem

(E, d) Polish, $(\Xi, \mathcal{A}, P)$ & $\mathcal{A}$ P-complete
$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, locally inf-integrable
$\varphi : \Xi \to \Xi$ ergodic measure preserving transformation. Then,

$$\frac{1}{\nu} \sum_{l=1}^{\nu} f(\varphi^l(\boldsymbol{\xi}, \cdot\,) \xrightarrow{e} Ef \quad a.s.$$

allows for stationary rather than iid.

Application: "samples" coming from dynamic systems,
time series, SDE, etc.

Approximation: Probability Measures

1. $\{P^\nu,\ v = 1, \ldots\} \underset{\mathcal{D}}{\to} P$, $f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ random lsc fcn function and for given x, $\xi \mapsto f(\xi, x)$ continuous on Ξ, $\forall\, \varepsilon > 0$,

(a) there exists a neighborhood V of x such that

$$|f(\xi, y) - f(\xi, x)| < \varepsilon, \quad \text{for all } y \in V.$$

(b) there exists a subset $\Xi_\varepsilon \in \mathcal{A}$ such that

$$\int_{\Xi \setminus \Xi_\varepsilon} |f(\xi, x)| P^\nu(d\xi) < \varepsilon, \quad \text{for all } \nu \in \mathbb{N}.$$

Then, $E^\nu f = \xrightarrow{c} Ef = \int f(\xi, \cdot)\, dP(\xi)$ at x, i.e.,

$$\forall\, \{x^\nu,\ v = 1, \ldots\} \to x, \quad Ef(x) = \lim_{\nu \to \infty} E^\nu f(x^\nu).$$

Approx.: Probability Measures 2

2. $\{P^\nu\}_{\nu\in\mathbb{N}} \underset{\mathcal{D}}{\to} P$

$f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ random lsc function

$\forall\, \varepsilon > 0, \exists\, \Xi_\varepsilon \in \mathcal{A}$ such that $\forall\, x \in \operatorname{dom} f$,

$$\int_{\Xi\setminus\Xi_\varepsilon} |f(\xi,x) P^\nu(d\xi) < \varepsilon, \quad \forall\, \nu \quad (f\text{-tight})$$

Then, $E^\nu f \xrightarrow{e} EF = \int_\Xi f(\xi,\cdot)\, P(d\xi)$.

Quantitative Approximation

3. f random convex lsc fcn, $\operatorname{dom} f(\xi,\cdot) = X$ (constant)

$$\mathcal{F} = \left\{ \xi \mapsto f(\xi,x) \,\middle|\, x \in X \right\}$$

$$\mathcal{P}_{\mathcal{F}} = \left\{ Q \in \mathcal{P}(\Xi) \,\middle|\, \begin{array}{c} \forall \rho > 0,\ \sup_{\rho\mathbb{B}} \int f(\xi,\cdot) Q(d\xi) < \infty \\ \int \inf_{\rho\mathbb{B}} f(\xi,\cdot) Q(d\xi) > -\infty \end{array} \right\}$$

$$d_{\mathcal{F},\rho}(P,Q) = \sup_{x \in \rho\mathbb{B}} \left| E^P f(x) - E^Q f(x) \right|, \text{ pseudo-metric on } \mathcal{P}_{\mathcal{F}}$$

$$P \in \mathcal{P}_{\mathcal{F}}, \emptyset \neq \operatorname{arg\,min} Ef \text{ bounded } \Rightarrow \exists \hat{\rho} > 0, \hat{\varepsilon} > 0:$$

$$\forall \varepsilon \in (0,\hat{\varepsilon}),\ Q \in \mathcal{P}_{\mathcal{F}} \text{ such that } d_{\mathcal{F},\hat{\rho}+\varepsilon}(P,Q) < \varepsilon:$$

$$\boxed{dl_\infty(\varepsilon\text{-}\operatorname{arg\,min} E^P f, \varepsilon\text{-}\operatorname{arg\,min} E^Q f) \leq \frac{4\hat{\rho}}{\varepsilon} d_{\mathcal{F},\hat{\rho}+\varepsilon}(P,Q)}$$

Proof: via epi distance between $E^P f$ and $E^Q f$

Solving Stochastic Programs

L-Shaped Strategy

Benders, Dantzig-Wolfe dual

taking advantage of structure

Two-Stage Recourse Model

With (arbitrary linear) recourse:

$\min_x \langle c, x\rangle + \mathbb{E}\big\{Q(\boldsymbol{\xi}, x)\big\},\ Ax = b,\ \ x \geq 0$

where $\boldsymbol{\xi} = (q(\boldsymbol{\xi}, \cdot\,), \boldsymbol{T}, \boldsymbol{d}, \boldsymbol{W})$

$Q(\xi, x) = \inf\big\{\langle q(\xi, y)\rangle \,\big|\, W_\xi y = d_\xi - T_\xi x,\ y \geq 0\big\},\ \ q(\xi, \cdot\,)$ convex

The *deterministic equivalent problem*, a convex program:

$\min_x \langle c, x\rangle + EQ(x)$ such that $Ax = b,\ x \geq 0$

with $EQ(x) = E\{Q(\boldsymbol{\xi}, x)\} = \int_\Xi Q(\xi, x)\, P(d\xi)$

but, generally, EQ is not finite valued.

Q finite-valued implies for all decision x,

for all events ξ a recourse is available

.

Multi-Stage: Deterministic Equivalent

$$\min_{x \in \mathcal{N}^a} \mathbb{E}\{f(\xi, x(\xi))\} = \mathbb{E}\Big\{\mathbb{E}\cdots\Big\{\mathbb{E}\Big\{f(\xi, x(\xi))\Big|\mathcal{A}_T\Big|\cdots\Big|\mathcal{A}_1\Big|\mathcal{A}_0\Big\}\Big\}\Big\}$$

"time-staged objective":

$$= f_1(x^1) + \mathbb{E}\Big\{f_2\left(\xi; x^1, x^2(\xi)\right) + \mathbb{E}\Big\{f_3\left(\xi; x^1, x^2(\xi), x^3(\xi)\right)\Big|\mathcal{A}_2\Big\}\Big|\mathcal{A}_1\Big\} \quad \cdots$$

$$= f_1(x^1) + \mathbb{E}\Big\{f_2\left(\xi; x^1, x^2(\xi)\right) + EQ_2\left(\xi; x^1, x^2(\xi)\right)\Big|\mathcal{A}_1\Big\}$$

$$EQ_2\left(\xi; x^1, x^2(\xi)\right) = \mathbb{E}\Big\{\inf_{x^3 \in \mathbb{R}^{n_3}} f_3(\xi; x^1, x^2(\xi), x^3)\Big|\mathcal{A}_2\Big\}$$

$$= f_1(x^1) + \mathbb{E}\Big\{EQ_1(\xi; x^1, x)\Big|\mathcal{A}_1\Big\}$$

$$EQ_1(\xi; x^1) = \mathbb{E}\Big\{\inf_{x^2 \in \mathbb{R}^{n_2}} f_2(\xi; x^1, x^2) + EQ_2(\xi; x^1, x^2)\Big|\mathcal{A}_1\Big\}$$

$$= f_1(x^1) + EQ_1(x^1)$$

Solution procedures

$$\min_{x \in \mathcal{N}^a} \mathbb{E}\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\} = \min_{x^1 \in \mathbb{R}^{n_1}} f_1(x^1) + EQ_1(x^1)$$

$$EQ_1(\xi; x^1) = \mathbb{E}\left\{\inf_{x^2 \in \mathbb{R}^{n_2}} f_2(\xi; x^1, x^2) + EQ_2(\xi; x^1, x^2) \middle| \mathcal{A}_1\right\}$$

$$EQ_2(\xi; x^1, x^2(\xi)) = \mathbb{E}\left\{\inf_{x^3 \in \mathbb{R}^{n_3}} f_3(\xi; x^1, x^2(\xi), x^3) \middle| \mathcal{A}_2\right\}$$

deterministic optimization! convex when f convex random lsc function

in theory: any algorithmic procedure!

hurdles: values, (sub)gradients, "Hessians" of $f_1(x^1) + EQ_1(x^1)$

are either not acessible or at best, prohibitively EXPENSIVE

Approaches: $P^\nu \sim P \Rightarrow$ approximating stochastic process $\{\boldsymbol{\xi}_t, t \le T\}$

sampling: a) same as approximation except P^ν random measure

b) SAA-strategy for $\partial\Big(\mathbb{E}\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\} + N_{\mathcal{N}^a}(x(\boldsymbol{\xi}))\Big)$

Sequential l.p. Strategy

$\min f_0(x),\quad x \in X \in \mathbb{R}^n,\quad f_0$ linear (not essential)

$\quad f_i(x) \le 0,\ \ i = 1,\ldots,s,\ \ f_i(s) = 0, i = s+1,\ldots,m$ (affine)

in the $s+1$ first constraints: $f_i(x) = \sup_{t\in T} f_{i,t}(x),\quad f_i \ge f_{i,t}$ affine

0. $\nu = 0$, pick polytope (box) $K^0 \ni x^{opt}$

1. $x^\nu \in \arg\min f_0$ on K^ν, set $\mathrm{i}_\nu : f_{i_\nu}(x^\nu) = \max\left\{ {}_{1\le i\le s} f_i(x^\nu),\ {}_{s+1<i<m} \left| f_i(x^\nu) \right| \right\}$

   if $f_{i_\nu}(x^\nu) \le 0, x^\nu$ optimal, otherwise go to 2.

2. return to 1. with $K^{\nu+1} = K^\nu \cap \left\{ \left\langle \nabla f_{i_\nu}(x^\nu), x - x^\nu \right\rangle + f_{i_\nu}(x^\nu) \le 0 \right\}$

when f_0 is not linear (but convex): $\min \theta$ such that $f_0(x) - \theta \le 0$

convergence: finite # of steps or iterates cluster to optimal sol'n

SLP for Stochastic Programs

$$\min f_1(x) + EQ_1(x) \text{ s.t. } Ax = b,\, x \geq 0 \quad (x = x^1)$$

$$EQ_1(x) = \sum_{l=1}^{L} p_l Q_1(\xi^l, x) \quad L \text{ large}$$

$$Q_1(\xi^l, x) = \inf_{x^2 \in X_2} \left\{ f_2(\xi^l; x, x^2) + (EQ_2(\cdots) \right\}$$

$$\text{dom } EQ_1 = \bigcap_{l=1}^{L} \text{dom } Q_1(\xi^l, \cdot) = \bigcap_{l=1}^{L} \left\{ x \,\middle|\, \exists x^2 \in X_2, f_2(\xi^l; x, x^2) < \infty \right\}$$

0. $\nu = r = s = 0$

1. $\nu = \nu + 1$, solve: $\min f_1(x) + \theta,\ Ax = b,\ x \geq 0$ such that

(feasibility cuts) $\quad \langle E_k, x \rangle \geq e_k,\ k = 1 \to r$

(optimality cuts) $\quad \langle F_k, x \rangle + \theta \geq f_k,\ k = 1 \to s$

2. generate feasibility cuts: check if $x \in \text{dom } EQ_1$.

   No: E_k separates x from dom EQ_1, go to 1. Yes, go to 3.

3. generate optimality cuts: $F_k \in \partial EQ_1(x^k)$, go to 1.

Cut Generation: Fixed Recourse

(l.p.)-solution: (x^ν, θ^ν)

x^ν feasible?

$\forall \xi \in \Xi : \; z_\xi = \operatorname{argmax}_z \left\{ \langle d_\xi - T_\xi x^\nu, z \rangle \,\middle|\, W^\top z \leq 0, \; -1 \leq z_j \leq 1 \right\}$

if $\eta_\xi = \langle d_\xi - T_\xi x^\nu, z \rangle = 0$, x^ν feasible

for some $\xi, \eta_\xi > 0$, then $E_{k+1} = (T_\xi)^\top z_\xi, \; e_{k+1} = \langle d_\xi, z_\xi \rangle$

.

x^ν optimal?

$\forall \xi \in \Xi : \; v_\xi = \operatorname{argmax}_v \left\{ \langle d_\xi - T_\xi x^\nu, v \rangle \,\middle|\, W^\top z \leq q_\xi \right\}$

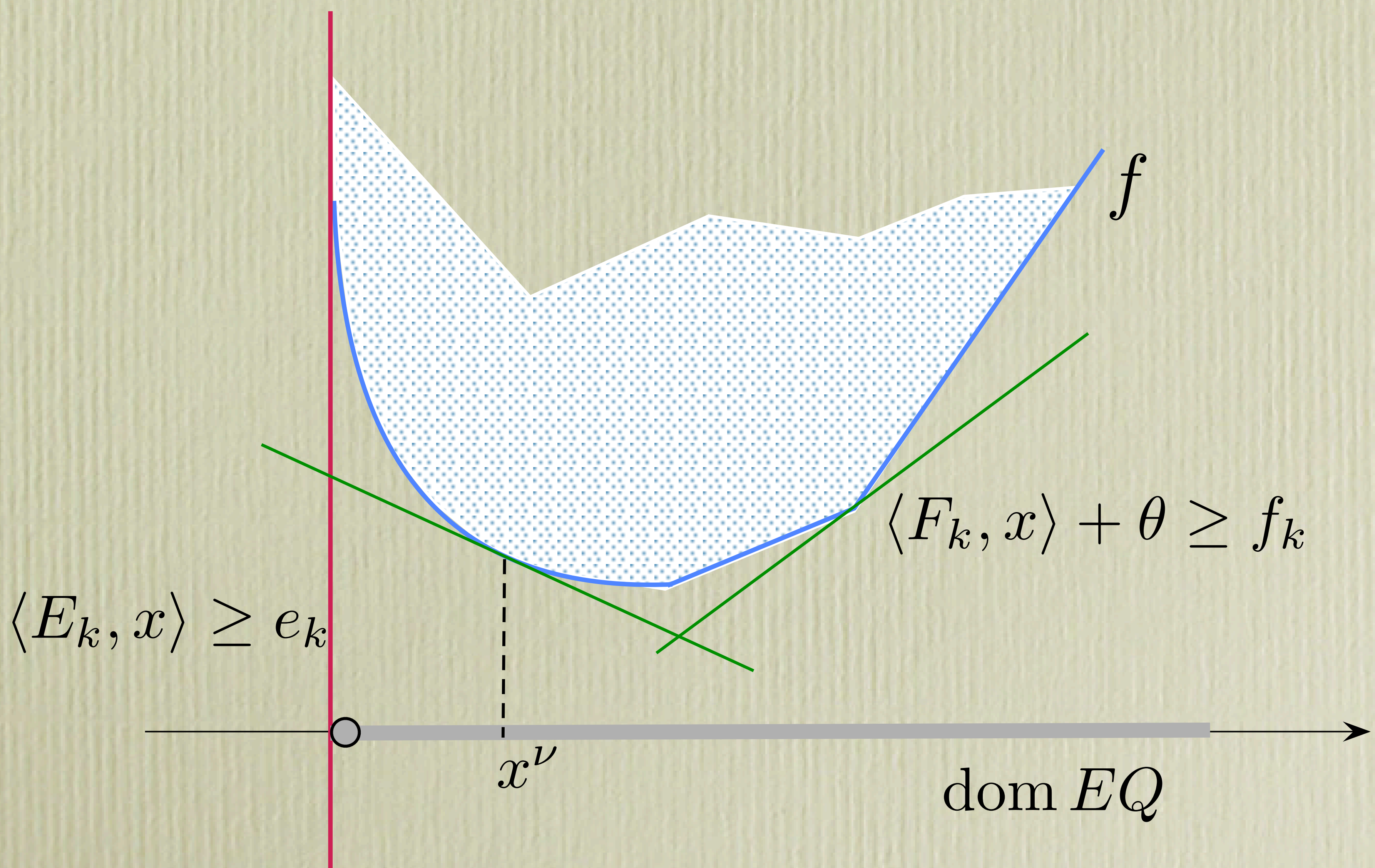
if infeasible for some $\xi \quad \Longrightarrow \quad$ unbounded problem

otherwise $F_{k+1} = \mathbb{E}\{\boldsymbol{T v}\}, \quad f_{k+1} = \mathbb{E}\{\langle \boldsymbol{d}, \boldsymbol{v} \rangle\}$

if $\theta^\nu \geq fk + 1 - \langle F_{k+1}, x^\nu \rangle \quad \Longrightarrow \quad x^\nu$ optimal

add optimality cut

Generating cutting hyperplanes



Aggregation Principle

in

Stochastic

Optimization

Interchanging: $\boxed{\mathbb{E} \;\&\; \min}$

Evident: with $E = \{x : \Xi \to \mathbb{R}^N \mid \text{ measurable, ... }\}$
$$\min \mathbb{E}\left\{ f\big(\boldsymbol{\xi}, x(\boldsymbol{\xi})\big) \,\middle|\, x \in E \right\} = \mathbb{E}\left\{ \min f(\boldsymbol{\xi}, x) \,\middle|\, x \in \mathbb{R}^N \right\}$$
when $\exists\, x(\,\cdot\,) \in E$ such that P-$a.s.$ $x(\xi) \in \operatorname{argmin} f(\xi, \cdot\,)$
x is measurable, ...

But our problem is: $\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\}$, equivalently,

$$\min Ef(x) = \mathbb{E}\{f\big(\boldsymbol{\xi}, x(\boldsymbol{\xi})\big)\}$$
$$\text{such that } \; x(\boldsymbol{\xi}) = \mathbb{E}\{x(\boldsymbol{\xi})\} \; P\text{–}a.s.$$

x can not depend on 'anticipated' (future) information

Dynamic Information Process

So far, x mostly restricted to $\{\emptyset, \Xi\}$-measurable, i.e., constant on Ξ

Generally, as $t \nearrow T$ (possibly ∞) additional information is acquired

$$\mathcal{A}_0 = \{\emptyset, \Xi\} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_T = \mathcal{A}, \text{ a filtration}$$

with x_t decision @ time t depend on available information, i.e. $\mathcal{A}_t$-measurable

<u>Reformulation</u>

Let $x(\xi) = \big(x_0(\xi), x_1(\xi), \dots, x_T(\xi)\big) : \Xi \to \mathbb{R}^N,\ N = \sum_{t=0}^T n_t$

$$\mathcal{N}_a = \left\{x \in E \,\middle|\, x_t\ \mathcal{A}_t\text{-measurable},\ t = 0, \dots T\right\}$$

$$\boxed{\text{find } x \in \mathcal{N}_a \text{ such that } Ef(x) = \mathbb{E}\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\} \text{ is minimized}}$$

Nonanticipativity <u>constraints</u>: $x \in \mathcal{N}_a$ (linear subspace)

Here-&-Now vs. Wait-&-See

- Basic Process: decision --> observation --> decision

$$x^1 \quad \to \quad \xi \quad \to \quad x^2_\xi$$

- Here-&-now problem! x^1

  not all contingencies available at time 0

  can't depend on ξ!

- Wait-&-see problem

  implicitly all contingencies available at time 0

  choose (x^1_ξ, x^2_ξ) after observing ξ

- incomplete information to anticipative information ?

Stochastic Optimization: Fundamental Theorem

A here-and-now problem can be “reduced” to a wait-and-see problem by introducing the

appropriate ‘information’ costs
(price of non-anticipativity)

Price of Nonanticipativity

Here-&-now

$$\min\ \mathbb{E}\left\{ f(\boldsymbol{\xi}, x^1, x^2_\xi) \right\}$$

$$x^1 \in C^1 \subset \mathbb{R}^n,$$

$$x^2_\xi \in C^2(\xi, x^1),\ \forall \xi.$$

Explicit non-anticipativity

$$\min \mathbb{E}\left\{ f(\xi, x^1_\xi, x^2_\xi) \right\}$$

$$x^1_\xi \in C^1 \subset \mathbb{R}^n,$$

$$x^2_\xi \in C^2(\xi, x^1_\xi),\ \forall \xi.$$

$$\boxed{x^1_\xi = \mathbb{E}\left\{ x^1_\xi \right\} \quad \forall \xi}$$

$w_\xi \perp$ subspace of constant fcns

multipliers

$$\Rightarrow \mathbb{E}\left\{ w_\xi \right\} = 0$$

$$\min\ \mathbb{E}\left\{ f(\boldsymbol{\xi}, x^1_\xi, x^2_\xi) - \langle w_\xi, x^1_\xi \rangle + \langle w_\xi, \mathbb{E}\{x^1_\xi\} \rangle \right\}$$

such that $x^1_\xi \in C_1, \quad x^2_\xi \in C_2(\boldsymbol{\xi}, x^1_\xi)$

Adjusted Here-&-Now

$\min \; \mathbb{E}\left\{ f(\boldsymbol{\xi}, x^1, x^2_\xi) \right\}$ such that $x^1 \in C^1 \subset \mathbb{R}^n, \; x^2_\xi \in C^2(\xi, x^1), \forall \xi$

x^1 must be $\mathcal{G}$-measurable, $\mathcal{G} = \sigma\text{-}\{\emptyset, \Xi\}$

x^2 is $\mathcal{A}$-measurable, $\mathcal{A} \supset \mathcal{G}$,

in general, interchange $\mathbb{E}$ & ∂ is not valid

required: $\forall \; \xi, x^1 \in C^1, C^2(\xi, x^1) \neq \emptyset$ $\;\mathcal{G}$-measurability of constraints

Now, suppose w_ξ are the (optimal) non-anticipativity multipliers (prices)

$\min \; \mathbb{E}\left\{ f(\boldsymbol{\xi}, x^1_\xi, x^2_\xi) - \langle w_\xi, x^1_\xi \rangle + \langle w_\xi, \mathbb{E}\{x^1_\xi\} \rangle \right\}$

such that $x^1_\xi \in C^1 \subset \mathbb{R}^n, \; x^2_\xi \in C^2(\xi, x^1_\xi), \forall \xi$

Interchange is now O.K. , $\mathbb{E}\left\{ \langle w_\xi, \mathbb{E}\{x^1_\xi\} \rangle \right\} = \langle \mathbb{E}\{w_\xi\}, \mathbb{E}\{x^1_\xi\} \rangle = 0$, yields

$\boxed{\forall \; \xi, \text{ solve: } \min \; f(\boldsymbol{\xi}, x^1, x^2) - \langle w_\xi, x^1 \rangle \text{ s.t. } \; x^1 \in C^1, \; x^2 \in C^2(\xi, x^1)}$

a collection of deterministic optimization problems in $\mathbb{R}^{n_1 + n_2}$

Progressive Hedging Algorithm

0. w^0_ξ such that $\mathbb{E}\{w^0_\xi\} = 0,\ \nu = 0.$ Pick $\rho > 0$

1. for all ξ :

$$(x^{1,\nu}_\xi, x^{2,\nu}_\xi) \in \arg\min f(\xi; x^1, x^2) - \langle w^\nu_\xi, x^1 \rangle$$

$$x^1 \in C^1 \subset \mathbb{R}^{n_1},\ x^2 \in C^2(\xi, x^1) \subset \mathbb{R}^{n_2}$$

2. $\bar{x}^{1,\nu} = \mathbb{E}\{x^{1,\nu}_\xi\}$. Stop if $\left| x^{1,\nu}_\xi - \bar{x}^{1,\nu} \right| = 0$ (approx.)

otherwise $w^{\nu+1}_\xi = w^\nu_\xi + \rho\left[x^{1,\nu}_\xi - \bar{x}^{1,\nu} \right]$, return to 1. with $\nu = \nu + 1$

Convergence: add a proximal term

$$f(\xi; x^1, x^2) - \langle w^\nu_\xi, x^1 \rangle - \frac{\rho}{2}\left| x^1 - \bar{x}^{1,\nu} \right|^2$$

linear rate in $(\mathrm{x}^{1,\nu}, w^\nu)$ ... eminently parallelizable

Nonanticipativity

Recall $\quad \min Ef(x) = \mathbb{E}\{f\big(\boldsymbol{\xi}, x(\boldsymbol{\xi})\big)\}$ such that $x(\xi) = \mathbb{E}\{x(\boldsymbol{\xi})\}$ P–$a.s.$

Nonanticipativity <u>constraints</u>:

$\mathcal{N}_a = \big\{x : \Xi \to \mathbb{R}^n\big\} \subset$ linear subspace of constant fcns

$\Longrightarrow\ \exists\, w : \Xi \to \mathbb{R}$ "multipliers" $\perp \mathcal{N}_a$ ($\Rightarrow \mathbb{E}\{w(\boldsymbol{\xi})\} = 0$) such that

$$x^* \in \operatorname{argmin} Ef \quad \Longrightarrow \quad x^* \in \operatorname{argmin} \big\{\mathbb{E}\{f\big(\boldsymbol{\xi}, x(\boldsymbol{\xi})\big) + \langle w(\boldsymbol{\xi}), (x(\boldsymbol{\xi}) - \mathbb{E}\{x(\boldsymbol{\xi})\})\rangle\big\}$$

$$\Longrightarrow \quad x^* \in \operatorname{argmin} \big\{\mathbb{E}\{f\big(\boldsymbol{\xi}, x(\boldsymbol{\xi})\big) + \langle w(\boldsymbol{\xi}), x(\boldsymbol{\xi})\rangle\}\big\}$$

$$P\text{–}a.s. \quad \Longrightarrow \quad x^* \in \operatorname*{argmin}_{x \in E} \{f\big(\xi, x\big) + \langle w(\xi), x\rangle\}\big\}, \quad \xi \in \Xi$$

$w(.)$: contingencies equilibrium prices, $\sim$ 'insurance' prices

PH: Implementation issues

implementation: choice of ρ ... scenario ($\times$), decision (+) dependent

(heuristic) extension to problems with integer variables

non-convexities: e.g. ground-water remediation with non-linear PDE recourse

asynchronous

partitioning (= different information feeds)

$$\min \mathbb{E}\{f(\boldsymbol{\xi},x)\}\ , \quad f(\boldsymbol{\xi},x) = f_0(x) + \iota_{C(\xi,x)}(x)$$

$S = \{\Xi_1, \Xi_2, \ldots, \Xi_K\}$ a partitioning of Ξ, $\quad p_k = P(\Xi_k)$

$\mathbb{E}\{f(\boldsymbol{\xi},x)\} = \sum_n p_n \mathbb{E}\{f(\boldsymbol{\xi},x) \mid \Xi_n\}$ (Bundling)

defining $g(k,x) = \mathbb{E}\{f_0(\boldsymbol{\xi},x) \mid \Xi_n\}$ if $x \in C_k = \bigcap_{\xi\in\Xi_k} C_\xi$

solve the problem as: $\min \sum_{n=1}^{N} p_k g(k,x)$

Multistage Stochastic Programs

$\min_{x\in\mathcal{N}^a} \mathbb{E}\{f(\boldsymbol{\xi},x(\boldsymbol{\xi}))\}, \quad x(\xi)=\left(x^1(\xi),\dots,x^T(\xi)\right)$

filtration : $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_T = \mathcal{A}$, $\mathcal{A}_0$ trivial

$x\in\mathcal{N}^a$ if x^t $\mathcal{A}_{t-1}$-measurable $\approx \sigma$-field($\overset{\rightarrow v-1}{\boldsymbol{\xi}}$)

(here $\boldsymbol{\xi}^0$ deterministic, $x^1(\boldsymbol{\xi})\equiv x^1$)

under usual $\mathbb{C}.\mathbb{Q}.$ (convex case): $\bar{x}\in\mathcal{X}$ optimal if

$$\boxed{\begin{array}{l}\exists\,\bar{w}\perp\mathcal{N}^a, \bar{w}\in\mathcal{X}^* \text{ such that } \bar{x}\in\arg\min_{x\in\mathcal{X}} Ef(x)-\mathbb{E}\{\langle\bar{\boldsymbol{w}},x\rangle\}\\ \bar{w}\perp\mathcal{N}^a \Leftrightarrow \mathbb{E}\left\{\bar{w}(\boldsymbol{\xi})\,\middle|\,\mathcal{A}_{t-1}\right\}=0, \forall t=1,\dots,T\end{array}}$$

$\bar{w}$ non-anticipativity prices

at which to buy the right to adjust decision (after observation)

can be viewed as insurance premiums, ...

A dual PH-strategy

single-stage case

minimize $Ef(x) := \sum_{\xi \in \Xi} p_\xi f(\xi, x)$ over all $x \in \mathbb{R}^n$

Strategy: better estimates of the w-variables
"aggregation" of the solutions" to

minimize $f^a(\xi, x)$ over all $\xi \in \mathbb{R}^n$ for fixed $\xi \in \Xi$

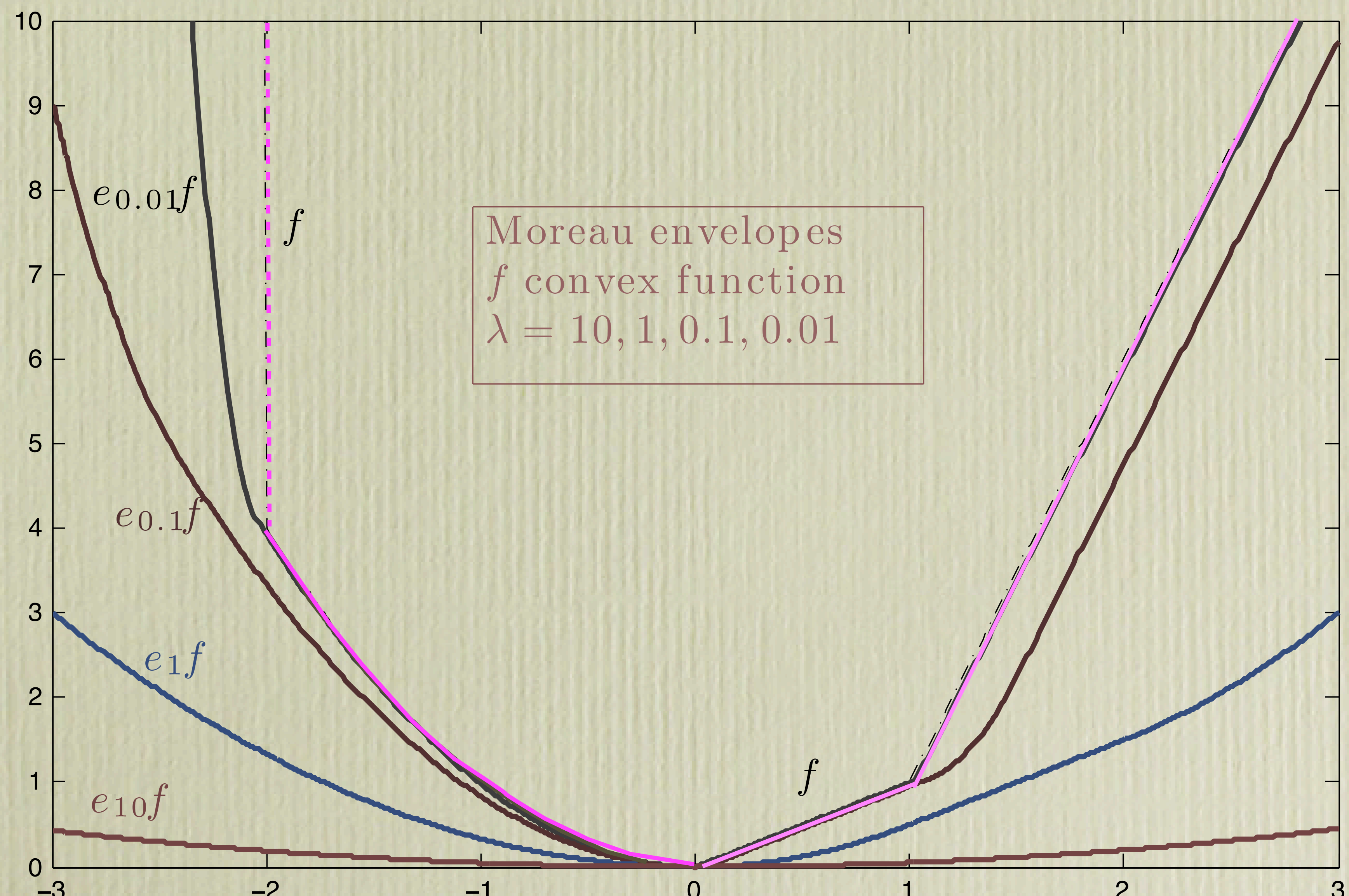
where f^a approximates f.

Moreau envelopes

a.k.a. Moreau-Yosida approximations

$$\operatorname{epi} f \# \operatorname{epi} g = \inf_u \{ f(u) + g(u-x) \} \quad e_\lambda f(x) \text{ with } g = \frac{1}{2\lambda} \left| \cdot \right|^2$$

epi-sums



Approximating problem

$\min F_\lambda(x) := \sum_{\xi\in\Xi} p_\xi f_\lambda(\xi, x), x \in \mathbb{R}^n$
$f_\lambda(\xi, x) = \inf_u \left\{ f(\xi, u) + \frac{1}{2\lambda} |u - x|^2 \right\}$
$F_\lambda \neq (F)_\lambda$ but $F_\lambda \xrightarrow{p} F, F_\lambda \xrightarrow{e} F$, finite-valued

Dual: $\max G_\lambda(w) = -\sum_{\xi\in\Xi} p_\xi f_\lambda^*(\xi, w_\xi),$
such that $\sum_{\xi\in\Xi} p_\xi w_\xi = 0$

Solution strategy:
$\min \sum_{k=0}^{\nu-1} \beta_k \alpha_k$ such that $\sum_{k=0}^{\nu-1} \beta_k \bar{w}^k = 0$
$\sum_{k=0}^{\nu-1} \beta_k = 1,\ \beta_k \geq 0,\ k = 0, \dots, \nu - 1$
for a 'desirable' collection of $\{\bar{w}^k\}$
Check for optimality, if not, generate $\bar{w}^\nu$

Algorithmic procedure

Step 0. Initialize by setting $\nu = 1$ picking $\{\widehat{w}^0_\xi\}_{\xi\in\Xi}$ in such a way that
$\bar{w}^0 := \sum p_\xi \widehat{w}^0_\xi = 0$ & $\alpha_0 = \sum_{\xi\in\Xi} p_\xi[\lambda/2|\widehat{w}^0_\xi|^2 + \sup_u(\widehat{w}^0_\xi u - f(\xi,u))]$

Step 1. $\min_\beta \sum_{k=0}^{\nu-1} \beta_k\alpha_k$, such that $\sum_{k=0}^{\nu-1} \beta_k \bar{w}^k = 0$,
$\sum_{k=0}^{\nu-1} \beta_k = 1,\ \beta_k \geq 0,\ k = 0,\dots,\nu-1$
Let $(z^\nu, \theta^\nu) \in \mathbb{R}^{n+1}$ be the associated multipliers

Step 2. For each $\xi \in \Xi$, let

$$u^\nu_\xi \in \operatorname*{argmin}_u \left\{ f(\xi,u) + \tfrac{1}{2\lambda}|z^\nu - u|^2 \right\}$$

$$\widehat{w}^\nu_\xi = \lambda^{-1}(z^\nu - u^\nu_\xi), \quad \bar{w}^\nu = \lambda^{-1}\textstyle\sum_{\xi\in\Xi} p_\xi(z^\nu - u^\nu_\xi)$$

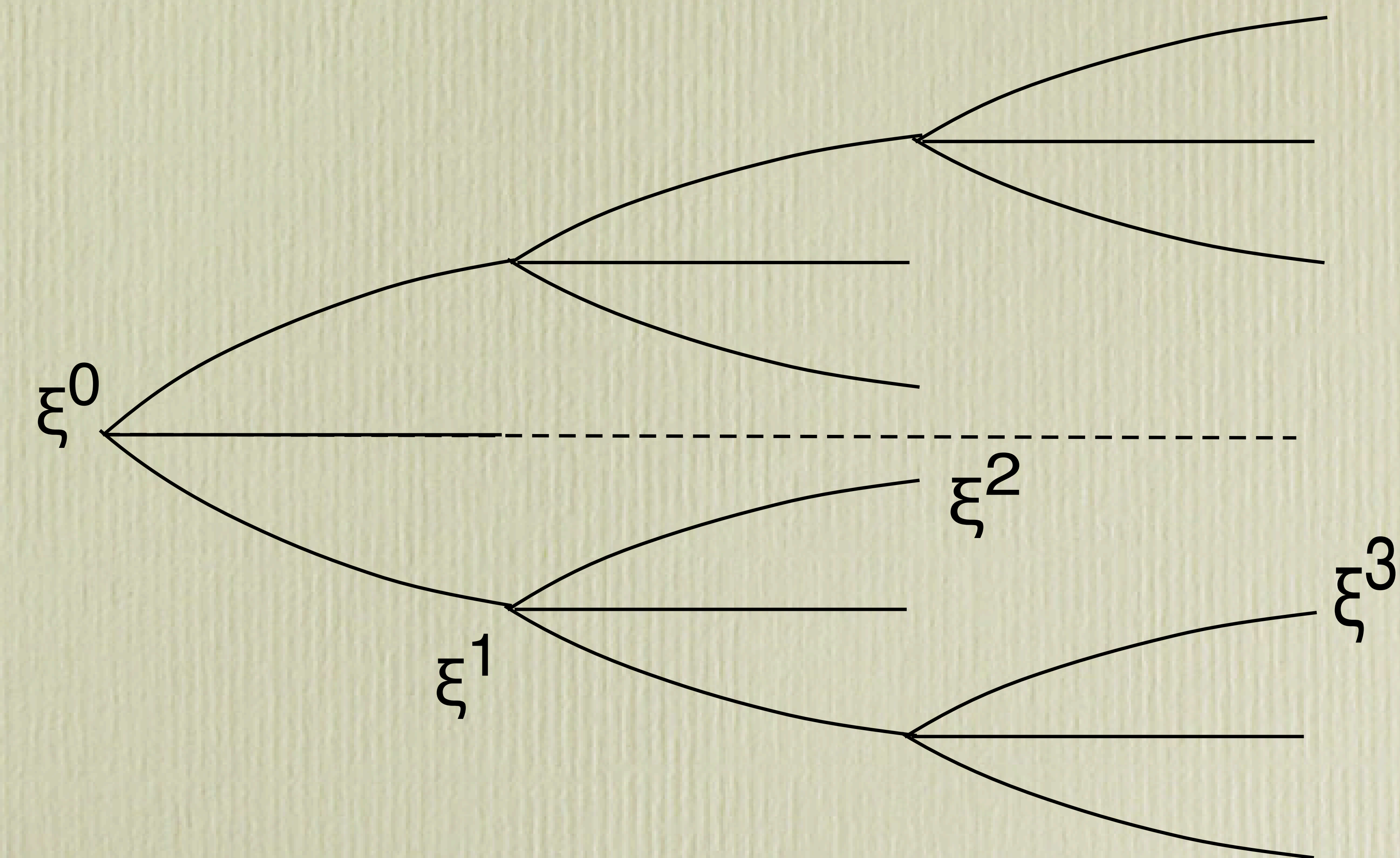
$$\alpha^\nu_\xi = z^\nu \widehat{w}^\nu_\xi - f(\xi, z^\nu - \lambda\widehat{w}^\nu_\xi) - \tfrac{\lambda}{2}|\widehat{w}^\nu_\xi|^2, \quad \alpha_\nu = \textstyle\sum_{\xi\in\Xi} p_\xi \alpha^\nu_\xi$$

Step 3. If $\alpha_\nu < \bar{w}^\nu z^\nu + \theta^\nu$ return to Step 1 with $\nu + 1 \to \nu$
If $\alpha_\nu \geq \bar{w}^\nu z^\nu + \theta^\nu$; z^ν is optimal

Adjust λ if appropriate; always generates bounds for original problem.

Bundling

Discrete Scenario Tree



Algorithm: (1) Nested Sequential SLP
(2) Progressive Hedging + Bundling

Bundling Decomposition

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\}, \qquad f + \iota_{C(\xi)}, \quad \xi \in \Xi$

$\iota_C(x) = 0$ when $x \in C$, $= \infty$ otherwise

Ξ discrete or discretization (not based on best approximation of Ξ)

$\{\Xi_k, k = 1, \ldots, K\}$ a partition of Ξ

$p_k = \int_{\Xi_k} P(d\xi),\ k = 1, \ldots, K$

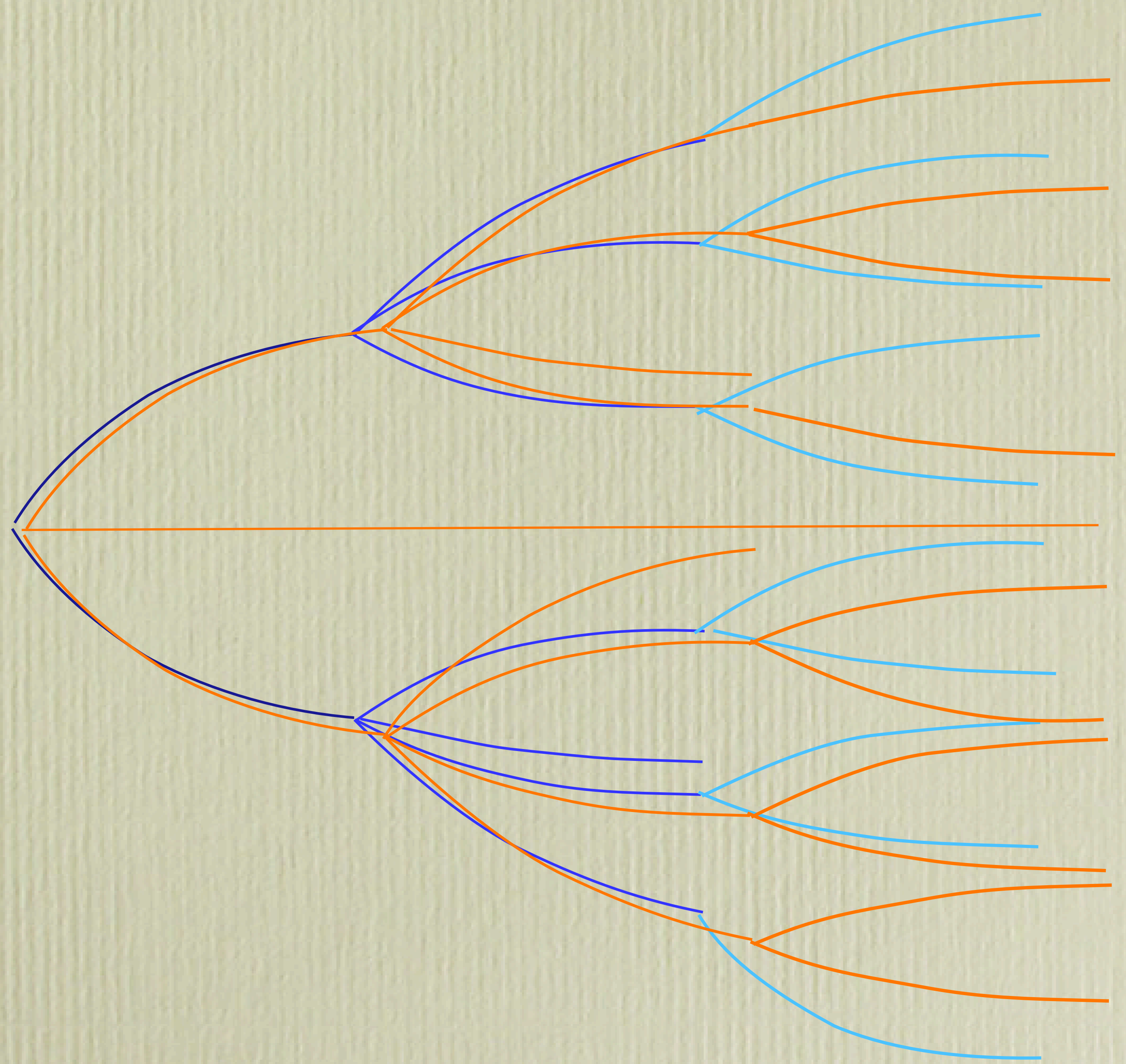
$$\mathbb{E}\{f(\boldsymbol{\xi}, x) = \sum_{k=1}^{K} p_k \mathbb{E}\{f(\boldsymbol{\xi}, x) \,\big|\, \Xi_k\}$$

$$g(k, x) = \mathbb{E}\{f(\boldsymbol{\xi}, x) \,\big|\, \Xi_k\}$$

$$\min \sum_{k=1}^{K} p_k g(k, x)$$

Bayes' Rule

Scenario Tree Decomposition



Ξ_1, Ξ_2

⭐ PH with Bundling ⭐

0. w_k^0 such that $\mathbb{E}\left\{w_k^0\right\} = 0,\ \nu = 0.$ Pick $\rho > 0$

1. for all k : $\widehat{x}^1,\ \widehat{x}^2 = (x_\xi^2, \xi \in \Xi_k)$

$$(\widehat{x}_k^{1,\nu}, x_k^{2,\nu}) \in \arg\min g(k; \widehat{x}^1, \widehat{x}^2) - \langle w_k^\nu, \widehat{x}^1 \rangle$$

$$\widehat{x}^1 \in C^1 \subset \mathbb{R}^{n_1},\ x^2 \in C^2(\xi, \widehat{x}^1) \subset \mathbb{R}^{n_2}$$

2. $\overline{x}^{1,\nu} = \sum_{k=1}^K p_k \widehat{x}_k^{1,\nu}$. Stop if $\left|\widehat{x}_k^{1,\nu} - \overline{x}^{1,\nu}\right| = 0$ (approx.)

otherwise $w_k^{\nu+1} = w_k^\nu + \rho\left[\widehat{x}_k^{1,\nu} - \overline{x}^{1,\nu}\right]$, return to 1. with $\nu = \nu + 1$

Convergence: add a proximal term

$$f(\xi; \widehat{x}^1, \widehat{x}^2) - \langle w_k^\nu, \widehat{x}^1 \rangle - \frac{\rho}{2}\left|\widehat{x}^1 - \overline{x}^{1,\nu}\right|^2$$

linear rate in $(\widehat{\mathrm{x}}^{1,\nu}, w^\nu)$ ... still eminently parallelizable

Arrow-Debreu model

pure-exchange economy: goods $\in \mathbb{R}^L$, prices $p = \left(p_1, \ldots, p_L\right)$, free disposal

agents: $i \in I$, $|I|$ finite ---- initial holdings: $(e_i, i \in I)$

demand functions: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p, x \rangle \leq \langle p, e_i \rangle \right\}$

utility fcn: u_i : dom $u_i = X_i \to \mathbb{R}$, usc, concave $\not\Rightarrow X_i$ closed (but convex)

excess supply function: $s(p) = \sum_{i \in I} \left(e_i - x_i(p)\right)$, market clearing: $s(p) \geq 0$

$$\bar{p} \geq 0 \text{ equilibrium } \Leftrightarrow s(\bar{p}) \geq 0$$

Existence: $x = (x_m, x_g)$, $x_m =$ 'money' allows $p = (1, p_g)$, under

ample survivability: $\left(e_{im}, e_{ig}\right) \Rightarrow \exists\, (\hat{x}_{im}, \hat{x}_{ig}) \in X_i$

such that $\hat{x}_{ig} \leq e_{ig}$, $\hat{x}_{im} < e_{im}$ and $\sum_{i \in I} \hat{x}_{ig} < \sum_{i \in I} e_{ig}$

+ indispensability & unactractiveness

Solution Procedures

Walras' law: $\bar{p} \perp s(\bar{p}) \sim \bar{p}_l s_l(\bar{p}) = 0,\ l = 1,\ldots,L, \qquad s(p) = s(\alpha p)$ for $\alpha > 0$

scaling: $\bar{p} \in \Delta = \left\{ p \in \mathbb{R}^L_+ \middle| \sum_l p_l = 1 \right\}$ since $\forall\, \alpha > 0 : \langle \alpha p, x \rangle \leq \langle \alpha p, e_i \rangle$

one possible way

find $\bar{p}$ $(\in \Delta)$ such that $\quad 0 \leq \bar{p} \perp s(\bar{p}) \geq 0,$

0. (very) special instances: via convex programming

1. tâtonnement, : $\dot{p} = -s(p),\ p(0) = p^0$ (Adam Smith, Léon Walras)

variant: 'Global Newton' (S. Smale) :

$$\nabla s(p)\dot{p} = \lambda s(p),\ \operatorname{sgn}(\lambda) = (-1)^L \operatorname{sgn} \det\big(\nabla s(p)\big)$$

requires s single-valued and differentiable,

$e_i \in \operatorname{int} X_i$ or bdry conditions on s

fails, “in general”

source of doubts about economic equilibrium theory

Solution Procedures

2. simplicial methods (based on "pivoting")

   - Scarf (& Hansen) '73: $\Rightarrow$ find fixed point of $p \mapsto s(p) - p$ in Δ

     partitioning Δ in a simplicial complex, pivoting á la Lemke-Howson

   - piece-wise linear homotopy methods: Eaves '74, Saigal, ...

3. homotopy continuation methods

   - homotopy methods $G(x) = 0$, Yorke *et al*. ('72, '78)

     variants: Kojima, Meggido and Noma for NCP ('89)

     Newton homotopy: Wu ('05), ...

   - 'interior point' homotopy method: Dang and Ye ('11)

a maxinf approach

recall: $s(p) = \sum_{i \in I} \big(e_i - x_i(p)\big)$, market clearing: $s(p) \geq 0$

Walrasian: $W(p,q) = \langle q, s(p) \rangle$, $W : \Delta \times \Delta \to \mathbb{R}$ (a bifunction)

Key observation:

$\bar{p} \in$ maxinf W, $W(\bar{p}, \bullet) \geq 0$ on $\Delta \Rightarrow \bar{p}$ is an equilibrium point.

under insatiability, $\bar{p}$ an equilibrium $\Rightarrow \bar{p} \in$ maxinf W, $W(\bar{p}, \bullet) \geq 0$ on Δ

Moreover:

p_ε : ε-equilibrium point if $\forall l$ (good), $s_l(p_\varepsilon) \geq -\varepsilon$

$p_\varepsilon \in$ ε-maxinf W, $W(p_\varepsilon, \bullet) \geq -\varepsilon$ on $\Delta \Rightarrow p_\varepsilon$ is an ε-equilibrium point.

with insat., p_ε an ε-equilibrium $\Rightarrow p_\varepsilon \in$ ε-maxinf W, $W(p_\varepsilon, \bullet) \geq -\varepsilon$ on Δ

... even more moreover

recall: $\max\left\{u_i(x) \,\middle|\, \langle p,x\rangle \le \langle p,e_i\rangle\right\}$ i-agent problem, $i \in I$

$u_i^\nu \to_{hypo} u_i$, $e_i^\nu \to e_i$ & p_ε^ν equilibrium points and $\varepsilon \searrow 0$

$\Rightarrow$ every cluster point of $\left\{p_\varepsilon^\nu\right\}_{(\nu,\varepsilon)}$ is an equilibrium point!

Consequence of $W_\varepsilon^\nu \to_{\mathrm{lop}} W$ lopsided-convergence

$\left\{C, C^\nu \subset \mathbb{R}^n\right\}$, $\left\{D, D^\nu \subset \mathbb{R}^m\right\}$ $K : C \times D \to \mathbb{R}$, $K^\nu : C^\nu \times D^\nu \to \mathbb{R}$

$K^\nu \to_{\mathrm{lop}} K$ (lopsided convergence) if

multi-hypoconvergence
$\Rightarrow$ lopsided (Gürkan & Pang)

(a) $\forall \left(y \in D, (x^\nu \in C^\nu) \to x \in C\right)$,

$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y)$ for some $(y^\nu \in D^\nu) \to y \in D$

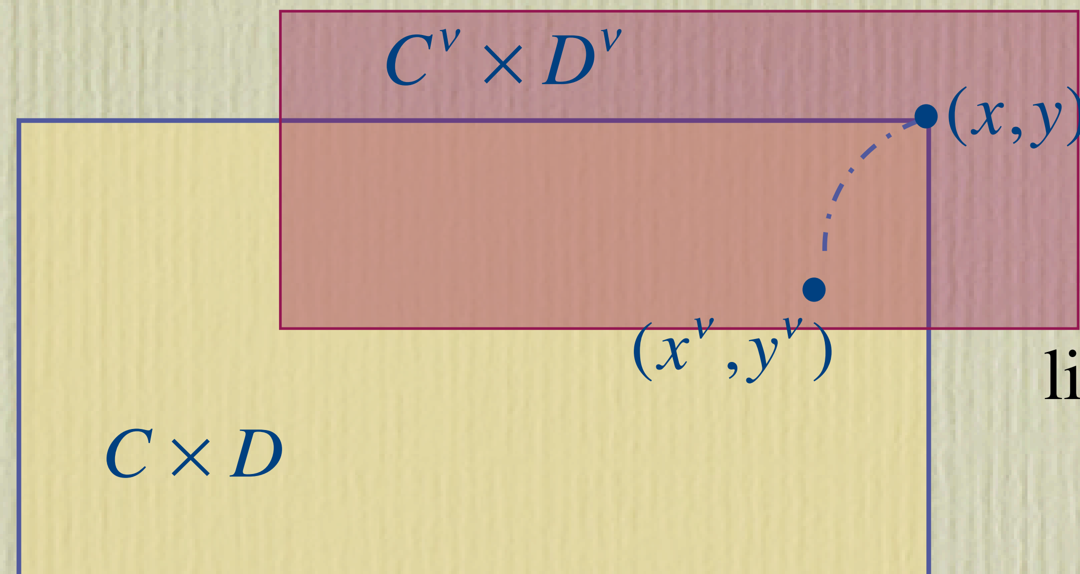
(b) $\forall\, x \in C$, $\exists (x^\nu \in C^\nu) \to x$ such that for any $(y^\nu \in D^\nu) \to y$

$\liminf_\nu K^\nu(x^\nu, y^\nu) \le K(x,y)$ when $y \in D$, $K^\nu(x^\nu, y^\nu) \to \infty$ $y \notin D$

Attouch & Wets '83, Jofré & Wets '09

Lopsided Convergence

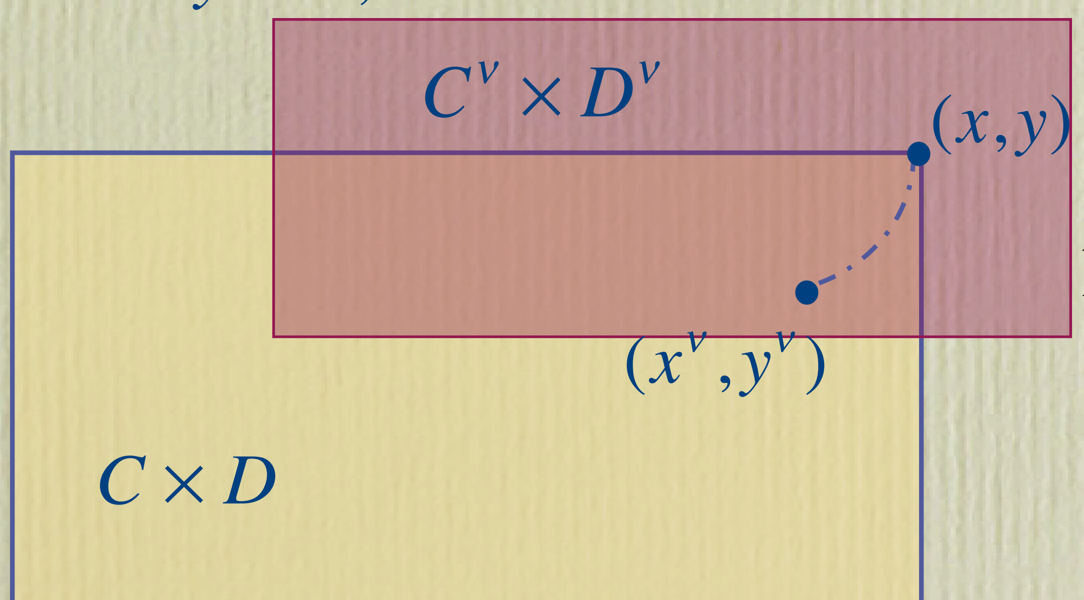
$\exists y^\nu \in D^\nu$
$\to y \in D$



$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y)$ when $x \in C$

$\forall y \in D, \forall x^\nu \in C^\nu \to x$

$\forall y^\nu \in D^\nu$
$\to y$



$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y)$ when $y \in D$

$K^\nu(x^\nu, y^\nu) \to \infty$ when y $\notin D$

$\exists x^\nu \in C^\nu \to x \in C$

The maxinf "family" ...

- saddle-point problems: Lagrangians, zero-sum games, Hamiltonians
- equilibrium: classical mechanics, Wardrop, economic (Walras, etc.)
- variational inequalities: finance, ecological models, complementarity, PDE
- non-cooperative games: pricing, generalized Nash equilibrium
- finding fixed points: Brouwer-type, Kakutani-type (set-valued), MPEC
- solving inclusions (equivalently, generalized equations): $S(x) \ni 0$
- minimal surface problems, ... , mountain pass solutions, ...
- ... and the dynamic versions, and the stochastic (dynamic) versions.

Ancillary-tightly - 'compact in y'

THM. $K^\nu_{C^\nu \times D^\nu} \to_{lop.} K_{C\times D}$ & ancillary-tightly,

$\bar{x} \in$ cluster points of $\{x^\nu \in \operatorname{maxinf} K^\nu_{C^\nu \times D^\nu}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \operatorname{maxinf} K_{C\times D}$

$K^\nu_{C^\nu \times D^\nu} \xrightarrow[\text{lop ancillary-tight}]{} K_{C\times D}$ if $K^\nu_{C^\nu \times D^\nu} \xrightarrow[lop]{} K_{C\times D}$ and

(b) $\forall x \in C$, $\exists x^\nu \to x$, $\forall y^\nu \in D^\nu$ and $y^\nu \to y$:

$$\liminf K^\nu(x^\nu, y^\nu) \geq K(x,y) \quad \text{if } y \in D$$

$$K^\nu(x^\nu, y^\nu) \to \infty \quad \text{if } y \notin D$$

but also $\forall \varepsilon > 0$, $\exists B_\varepsilon$ compact (depends on $x^\nu \to x$):

$$\inf_{B_\varepsilon \cap D^\nu} K^\nu(x^\nu, \bullet) \leq \inf_{D^\nu} K^\nu(x^\nu, \bullet) + \varepsilon, \ \forall \nu \geq \nu_\varepsilon$$

certainly satisfied when $D = \Delta$ is compact

Convergence of ε-solutions

including $\varepsilon = 0$

$K^{\nu}_{C^{\nu}\times D^{\nu}} \to K_{C\times D}$ lop. ancillary-tightly,

(i) $x^{\nu} \in \varepsilon$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$, $\bar{x}$ cluster point of $\{x^{\nu}\}_{\nu\in\mathbb{N}}$

$\Rightarrow \bar{x} \in \varepsilon$-maxinf $K_{C\times D}$

(ii) $x^{\nu} \in \varepsilon_{\nu}$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$, $\bar{x}$ cluster point of $\{x^{\nu}\}_{\nu\in\mathbb{N}}$

& $\varepsilon_{\nu} \searrow 0 \Rightarrow \bar{x} \in$ maxinf $K_{C\times D}$ (special case: locally unique)

(iii) $\bar{x} \in$ maxinf $K_{C\times D} \Rightarrow \exists \varepsilon_{\nu} \searrow 0$ & $x^{\nu} \in \varepsilon_{\nu}$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$

such that $x^{\nu} \to \bar{x}$,

Under tight-lop: convergence of the full ε_{ν}-maxinf sets

and convergence of values

tight-lop when $C = \Delta$ & $D = \Delta$ are compact

...back to our Walrasian

$W(p,q) = \langle q, s(p) \rangle$ on $\Delta \times \Delta$, p-usc and q-convex

Augmented Walrasian: σ augmenting function

$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p, q-z) + r *_e \sigma^*(z) \right\} \qquad r\sigma(r^{-1}z)$$

$$= \sup_z \left\{ W(p,z) \,\middle|\, \|z-q\|_\square \le r \right\} \quad \sigma = |\bullet|_\square,\ \iota_{\mathbb{B}} = \sigma^*$$

as $r \to \bar{r} < \infty$, $\boxed{\tilde{W}_r \to_{\text{lop}} \tilde{W}_{\bar{r}} = W} \Rightarrow$ ε-maxinf $W_r \to$ maxinf W

choosing $|\bullet|_\square = |\bullet|_\infty, \mathbb{B} = [-1,1]^L$

or $|\bullet|_\square = |\bullet|_2, \mathbb{B} =$ euclidean unit ball

Friday, June 21, 13

augmented Walrasian strategy

$$W(p,q) = \langle q, s(p)\rangle \text{ on } \Delta \times \Delta,\ \ p\text{-usc and } q\text{-convex}$$

$$\tilde{W}_r(p,q) = \sup_z \left\{ W(p,z) \,\middle|\, \|z-q\|_\square \le r \right\}$$

$$q^{k+1} = \operatorname*{arg\,max}_{q\in\Delta} \left[\max_z \left\langle z, s(p^k)\right\rangle \,\middle|\, \|z-q\|_\square \le r_k \right]$$

minimizing a linear form on a ball

i.e. finding the largest element of $s(p^k)$

$$p^{k+1} = \operatorname*{arg\,min}_{p\in\Delta} \left[\max_z \left\langle z, s(p)\right\rangle \,\middle|\, \|z-q^{k+1}\|_\square \le r_{k+1} \right]$$

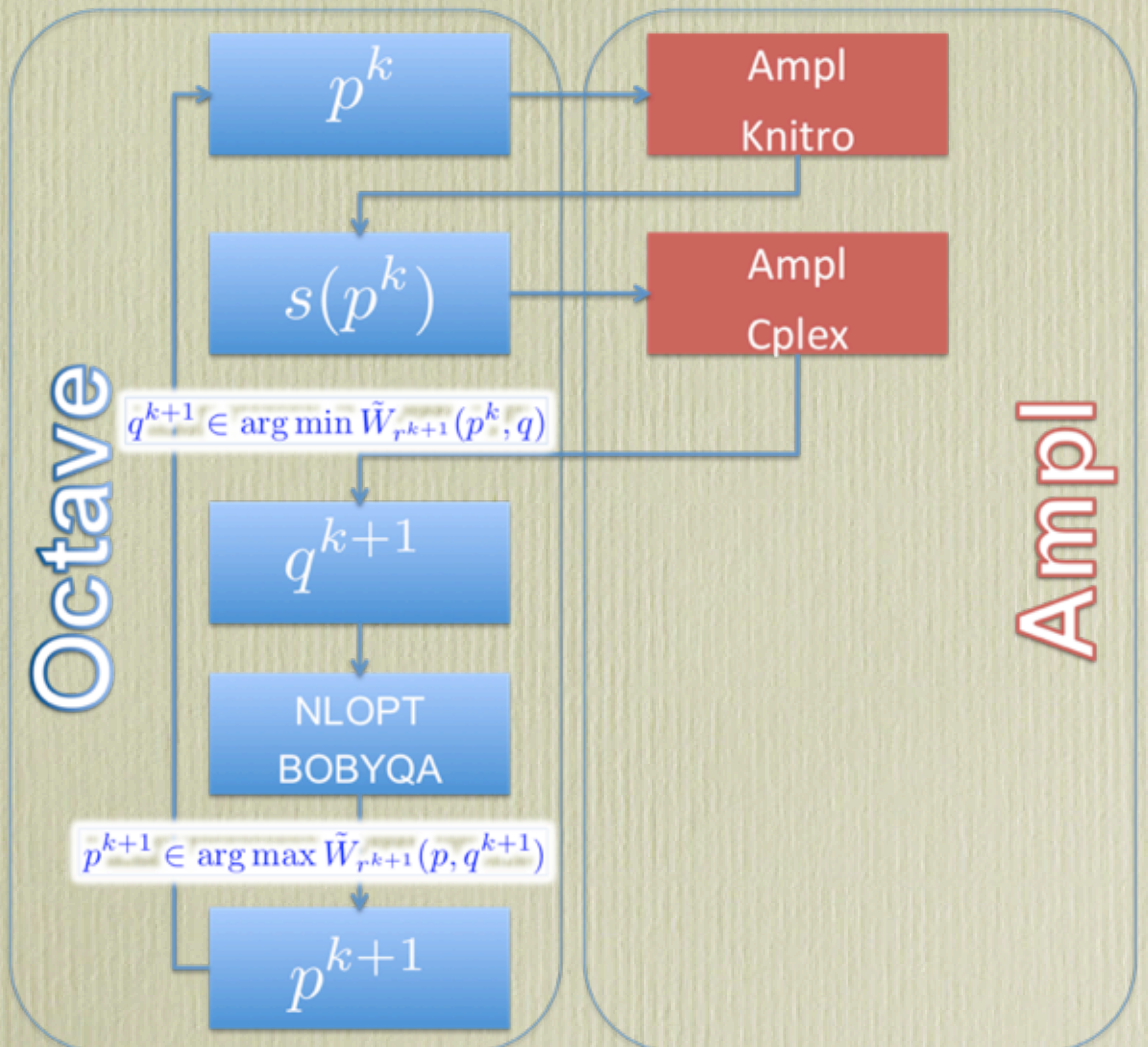
$$\text{as} \quad r_k \nearrow \infty, \quad p^k \to \bar{p}$$

Bagh, Lucero & Wets ≈ '03

first experiments: 10 agents, 150 goods (two blinks)

CMM-implementation

Center for Mathematical Modeling --- Universidad de Chile



Deride,
Jofré
& Wets
'06 -- '12

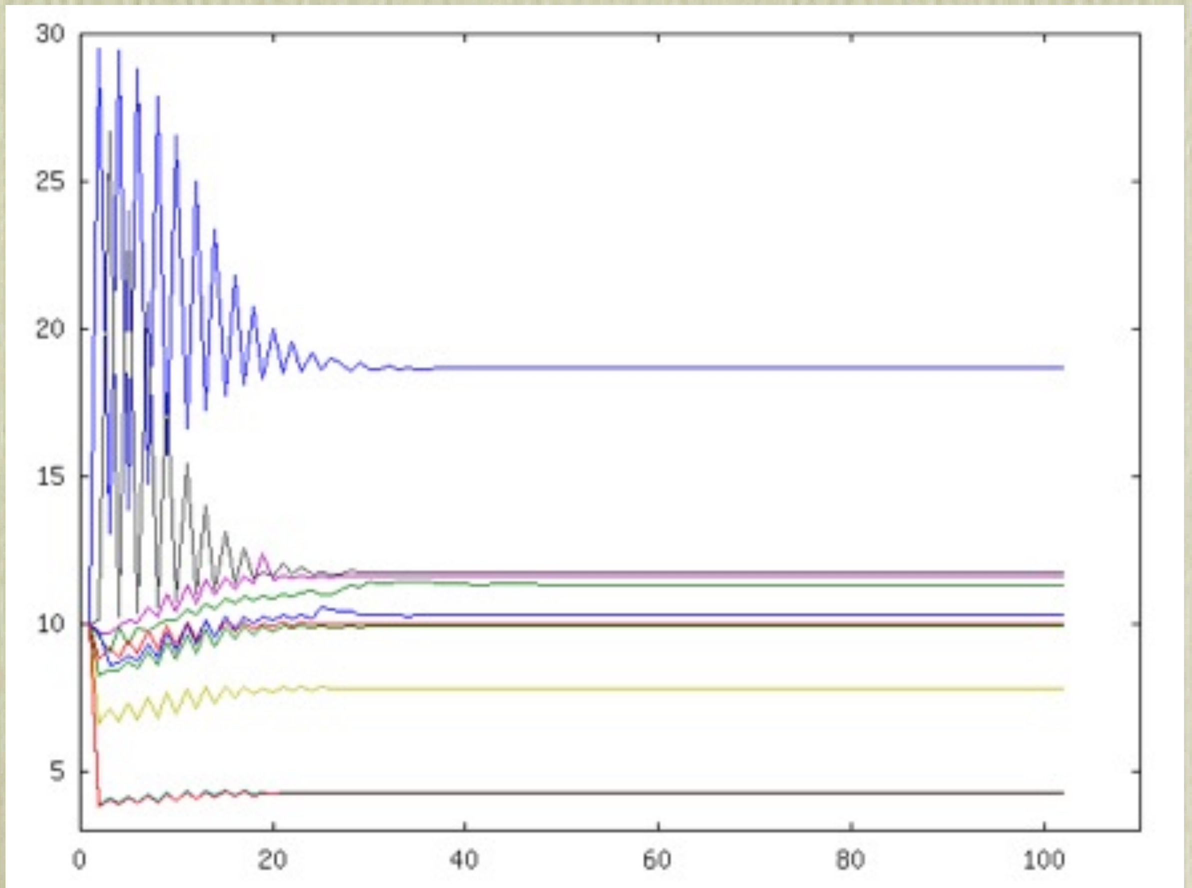
$$\sigma = \frac{1}{2}|\bullet|^2$$

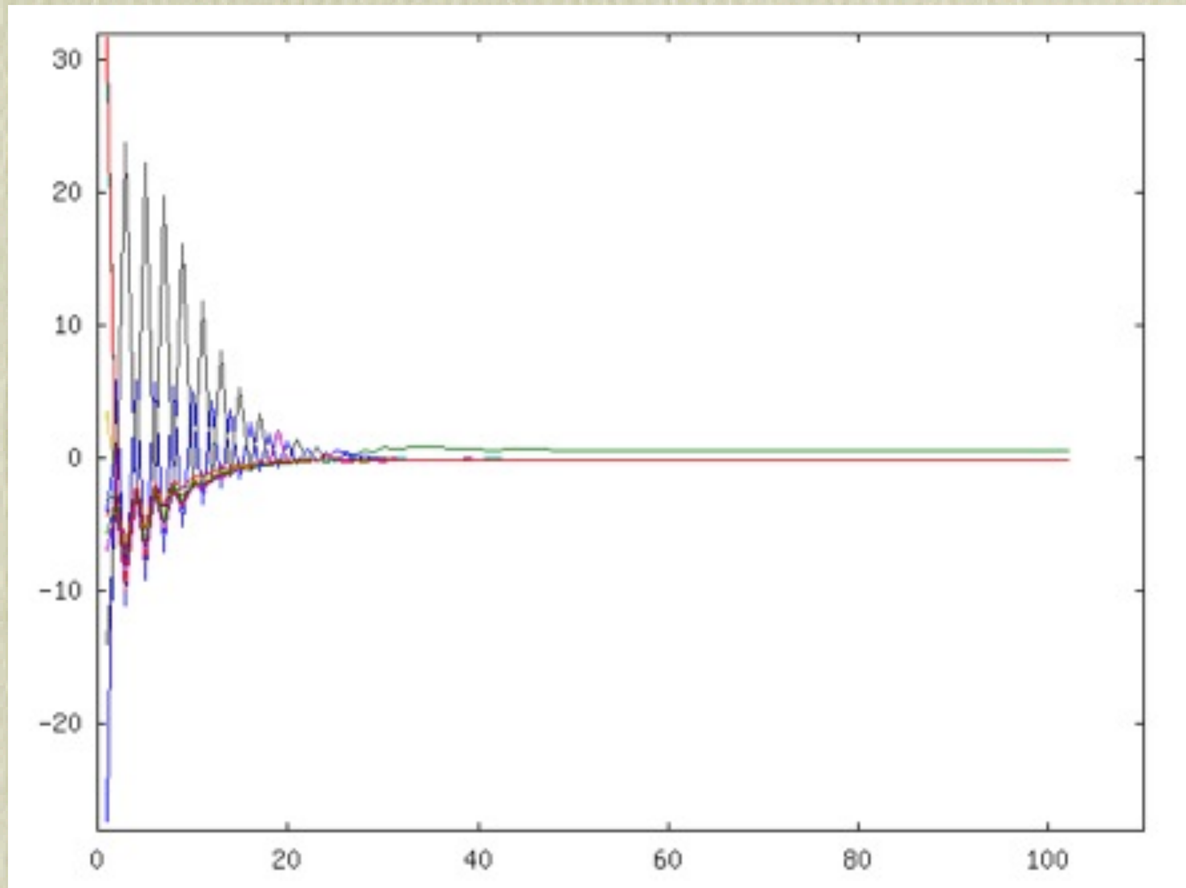
Scarf's example

$$u_i(x) = \left(\sum_{l=1}^{L} (a_{il})^{\beta_i^{-1}} (x_l)^{1-\beta_i^{-1}} \right)^{\beta_i (\beta_i - 1)^{-1}}$$ CES-utility

constant elasticity substitution

$i \in I = 5$ agents, $L = 10$ goods (2000 simplicial pivots)



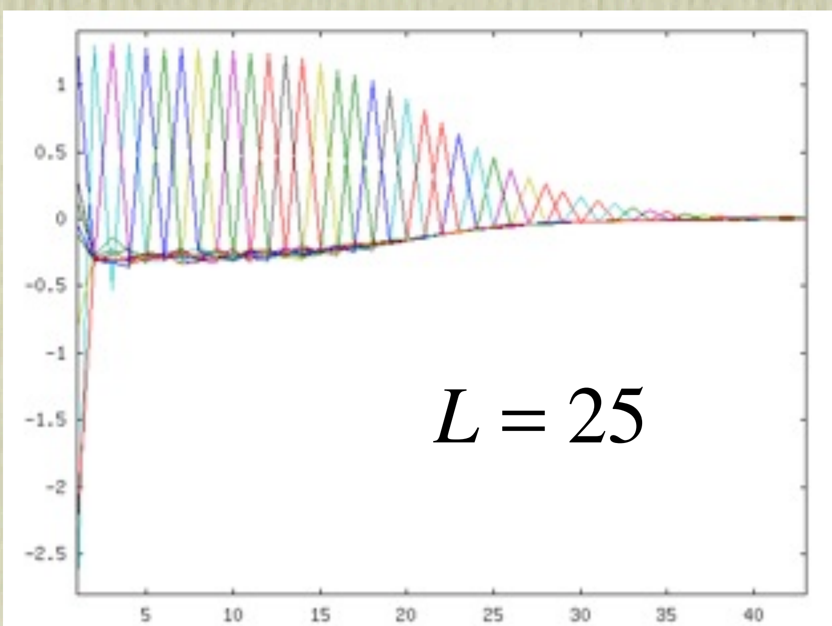


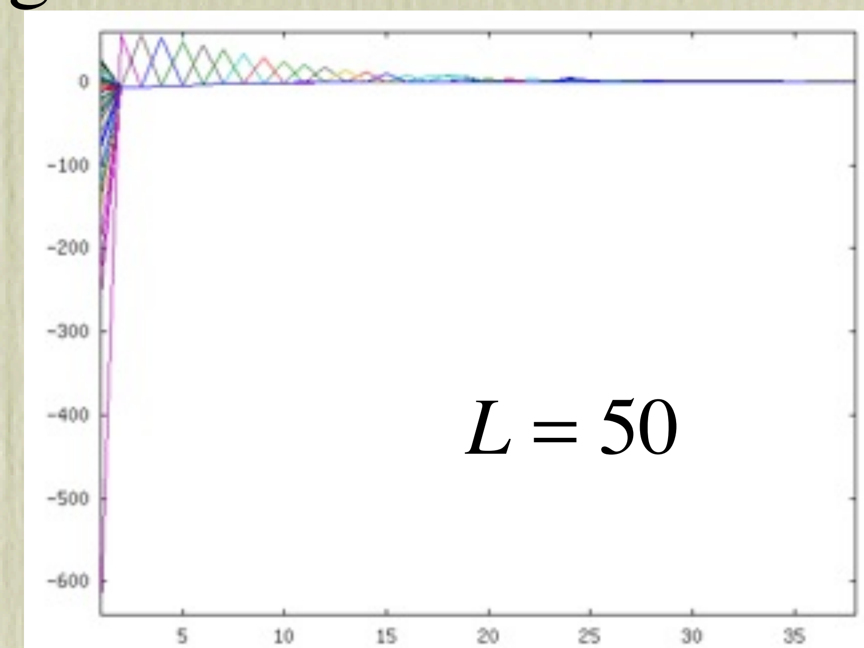
prices and excess supply convergences

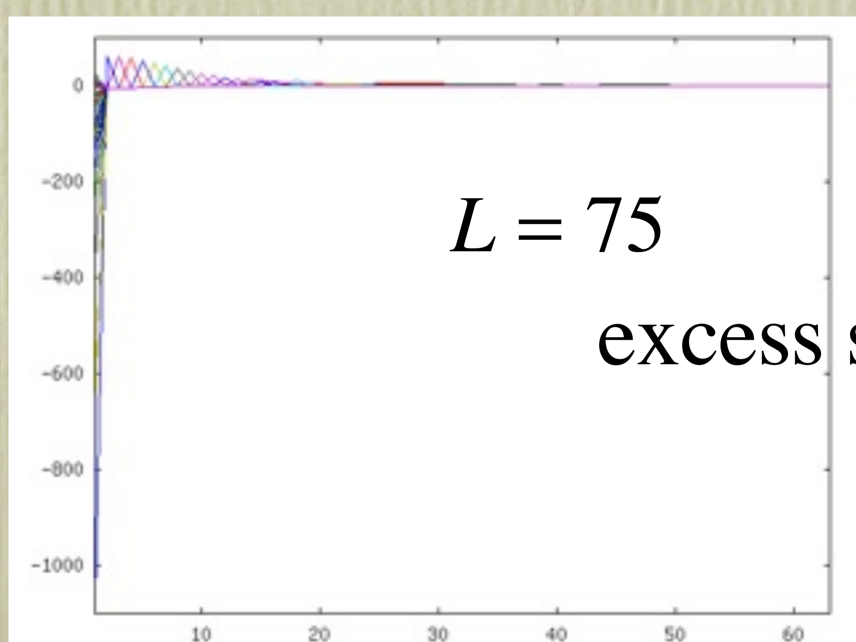
just, ... one more example

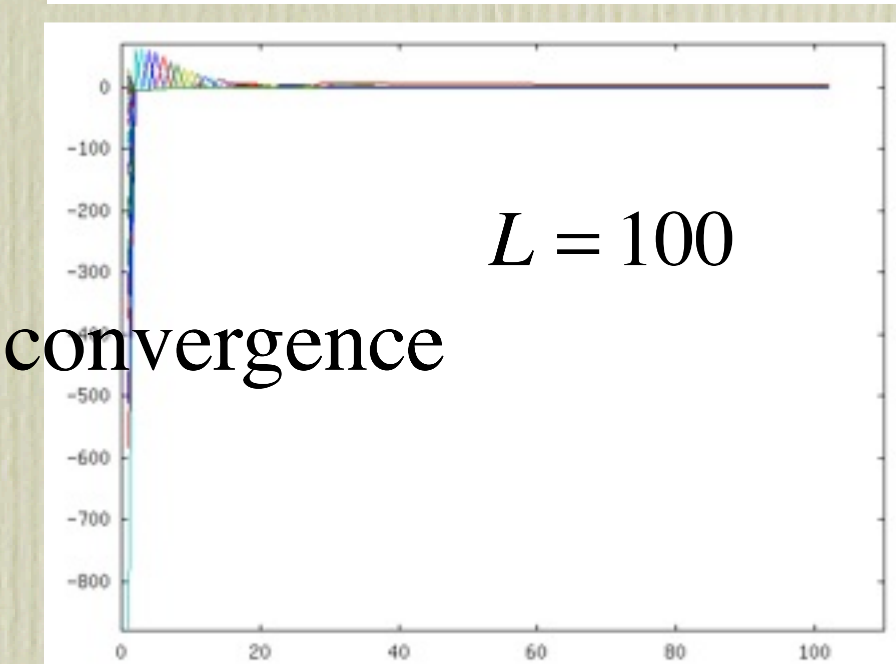
same CES-utility function ($\neq \beta_i$), $I = 10$ agents, $r_k = 1.21^k$

$L =$ # of goods









excess supply convergence

A dynamic model

$$\max_{(x^0, y, x^1)} \quad u_i^0(x^0) + u_i^1(x^1)$$

such that $\left\langle p^0, x^0 + T_i^0 y \right\rangle \le \left\langle p^0, e_i^0 \right\rangle$

$$\left\langle p^1, x^1 \right\rangle \le \left\langle p^1, e_i^1 + T_i^1 y \right\rangle$$

$$x^0 \in X_i^0,\ y \in Y_i,\ x^1 \in X_i^1$$

with solutions: $\left\{ \left(x_i^0(p), y_i(p), x_i^1(p) \right), i \in I \right\},\ p = (p^0, p^1)$

excess supply: $s^0(p) = \sum_{i \in I} \left(e_i^0 - \left(x_t(p) + T_i^0 y_i(p) \right) \right)$

$$s^1(p) = \sum_{i \in I} \left(\left(e_i^1 + T_i^1 y_i(p) \right) - x^1(p) \right)$$

equilibrium: $\bar{p} \in \Delta_L \times \Delta_L$ such that $s^0(\bar{p}) \ge 0, s^1(\bar{p}) \ge 0 \quad (=)$

Walrasian: $W(p,q) = \left\langle (q^0, q^1), \left(s^0(p), s^1(p) \right) \right\rangle$ on $\Delta_L^2 \times \Delta_L^2$

$\Rightarrow$ augmented Walrasian, ...

Exploiting separability

$$r_i(y,p) = \sup_{x^0 \in X_i^0}\left[u_i^0(x^0) \middle| \left\langle p^0, x^0 + T_i^0 y \right\rangle \le \left\langle p^0, e_i^0 \right\rangle \right]$$

$$+ \sup_{x^1 \in X_i^1}\left[u_i^1(x^1) \middle| \left\langle p^1, x^1 \right\rangle \le \left\langle p^1, e_i^1 + T_i^1 y \right\rangle \right]$$

i-agent problem: find $y_i(p) \in \mathbb{R}_+^{n_i}$ that maximizes $\left\{ r_i(y) \middle| T_i^0 y \le e_i^0 \right\}$

Example: $u_i^{0,1}$ are of Cobb-Douglas type: $u_i(x) = \prod_{l=1}^{L} x_l^{\beta_l}, \beta \in \Delta$

$$x_l^0(p^0, y) = \frac{\beta_l}{p_l^0} \sum_{k=1}^{L} p_k^0 \left(e_k^0 - \left\langle (T_i^0)_k, y \right\rangle \right),\ l = 1,\ldots,L, \quad X_i^0 = \mathbb{R}_+^L$$

$$x_l^1(p^1, y) = \frac{\beta_l}{p_l^1} \sum_{k=1}^{L} p_k^1 \left(e_k^1 + \left\langle (T_i^1)_k, y \right\rangle \right),\ l = 1,\ldots,L, \quad X_i^0 = \mathbb{R}_+^L$$

substituting $\Rightarrow r_i$ linear in y: i-agent's problem is a linear program!

substantial gain in processing time

Stochastic Environment

a 'minimal' model

Chap.7 -- Theory of Value G. Debreu, '59

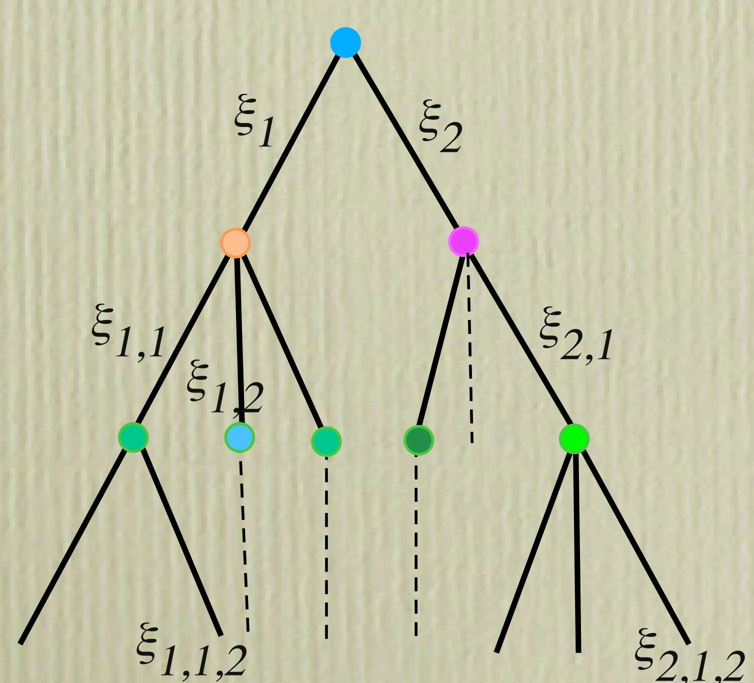
$$\max_{x^0, x^1_{\cdot} \in \mathcal{M}} E\left\{u_i(x^0, x^1_{\xi^1}, x^2_{\xi^1,\xi^2}, \ldots)\right\}$$ *i*-agent

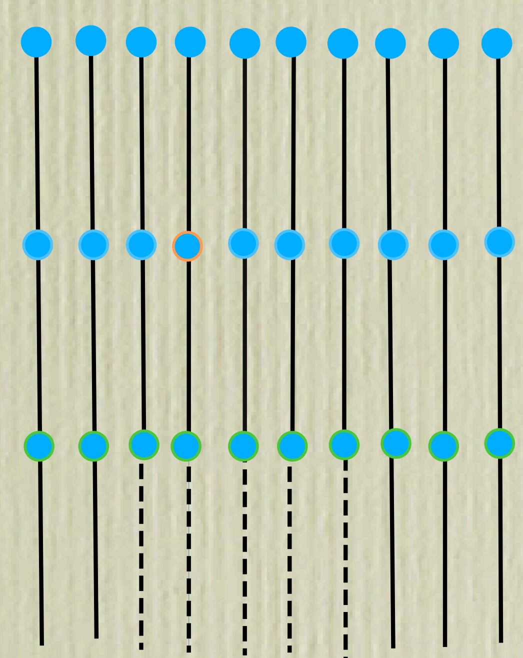
such that $\left\langle p^t_\xi, \sum_{\tau \le t}\left(e^\tau_{\xi^1,\ldots\xi^\tau} - x^\tau_{\xi^1,\ldots\xi^\tau}\right)\right\rangle \ge 0,\ \forall \xi = (\xi^1, \xi^2, \ldots),\ \ \mathrm{t} = 0,1,\ldots$

$$\sum_{i \in I}\left(\sum_{\tau \le t}\left(e^\tau_{\xi^1,\ldots\xi^\tau} - x^\tau_{\xi^1,\ldots\xi^\tau}\right)\right) \ge 0, \forall \xi, t$$ Clearing the market

Key Assumption (via K. Arrow): all contingencies available at time 0

$\Rightarrow$ complete market, i.e., all ξ's can be dealt with separately





$\exists\ \left(p_\xi = (p^0_\xi, p^1_\xi, \ldots)\right)$

$(\forall\ \xi)$ equilibrium prices.

i-Agent: stochastic case

an engineer's viewpoint?

$$\max_{x^0, y, x^1_\bullet \in \mathcal{M}} \; u_i^0(x^0) + E^i\left\{u_i^1(\xi; x_\xi^1)\right\}$$

$$\text{so that } \left\langle p^0, x^0 + T_i^0 y \right\rangle \le \left\langle p^0, e_i^0 \right\rangle$$

$$\left\langle p_\xi^1, x_\xi^1 \right\rangle \le \left\langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y \right\rangle, \quad \forall \xi \in \Xi$$

$$x^0 \in X_i^0,\; y \in \mathbb{R}_+^{n_i},\; x_\xi^1 \in X_{i,\xi}^1, \quad \forall \xi \in \Xi$$

- $E^i\{.\}$ expectation w.r.t. i-agent beliefs

Stochastic program with recourse: 2-stage
Well-developed solution procedures
Well-developed "Approximation Theory"

Simplest-classical assumptions

Ξ finite (support)

$u_i^0 : X_i^0 \to \mathbb{R}, \quad \forall \xi \in \Xi,\ u_i^1(\xi,\bullet) : X_{i,\xi}^1 \to \mathbb{R}$ concave

continuous, numerical experiments: differentiable

$T_{\mathrm{i}}^0, T_{i,\xi}^1$: input-ouput matrices

(savings, production, investment, etc.)

$X_{\mathrm{i}}^0, X_{i,\xi}^1$: closed, convex, non-empty interior (survival sets

$e_{\mathrm{i}}^0 \in \operatorname{int} X_{\mathrm{i}}^0,\ e_{i,\xi}^1 \in \operatorname{int} X_{i,\xi}^1$ for all ξ (or as on first slide)

Market Clearing

Agents: $i \in I,\ |I|$ finite ("large"), $\quad p = \left(p^0, (p^1_\xi, \xi \in \Xi)\right)$

$$\left(\overline{x}_i^{\,0}(p), \overline{y}_i(p), \left\{\overline{x}^1_{i,\xi}(p)\right\}_{\xi\in\Xi}\right) \in \arg\max\left\{i\text{-agent problem}\right\}$$

excess supply:

$$\sum_{i\in I}\left(e_i^0 - (\overline{x}_i^{\,0}(p) + T_i^{\,0}\overline{y}_i(p))\right) = s^0(p^0, \{p^1_\xi\}_{\{\xi\in\Xi\}}) \geq\ 0$$

$\forall \xi \in \Xi:$

$$\sum_{i\in I}\left(e^1_{i,\xi} + T^1_{i,\xi}\overline{y}_i(p) - \overline{x}^1_{i,\xi}(p)\right) = s^1_\xi(p^0, \{p^1_\xi\}_{\{\xi\in\Xi\}}) \geq\ 0$$

Existence: via Ky Fan Inequality

$$W(p_\bullet, q_\bullet) = \langle q_\bullet, s(p_\bullet) \rangle$$

$$= \left\langle (q^0, \{q^1_\xi\}_{\xi\in\Xi}), \left(s^0(p^0, \{p^1_\xi\}_{\xi\in\Xi}), \left\{ s^1_\xi(p^0, \{p^1_\xi\}_{\xi\in\Xi}) \right\}_{\xi\in\Xi} \right) \right\rangle$$

$W : \prod_{1+|\Xi|} \Delta \times \prod_{1+|\Xi|} \Delta \to \mathbb{R}$ a Ky Fan function: usc, convex

linear w.r.t. $q_\bullet$, continuous w.r.t. $p_\bullet$

and also $W(p_\bullet, p_\bullet) \geq 0$.

provided $s(\bullet)$ continuous w.r.t. $p_\bullet$

another lecture series

Incomplete → '*i*-Complete' Market

$\forall \xi \in \Xi$ (separately),

i-agent's problem:

$$\left(x_i^0, y_i, x_{i,\xi}^1\right) \in \arg\max \left\{ u_i^{w_{i,\xi}}\left(\xi; x^0, y, x^1\right) \text{ on } \widehat{C}_{i,\xi}(p^0, p_\xi^1) \right\}$$

for $\{w_{i,\xi}\}_{\xi \in \Xi}$ associated with (p^0, p_ξ^1)

clearing the market:

$$s^0(p^0, p_\xi^1) \geq 0, \quad s_\xi^1(p^0, p_\xi^1) \geq 0$$

Arrow-Debreu 'stochastic' equilibrium problem

Cobb-Douglas utilities

$$u_i^t(x) = \prod_{l=1}^{L} x_l^{\beta_i^t}, \; \beta^t \in \Delta, \; t = 0,1$$

"agent's optimization" (skipping i)

$$r(\xi; y, p) = \alpha^0(p^0)\Big(\sum_{k=1}^{n} p_k^0\big(e_k^0 - \langle T_k^0, y\rangle\big)\Big)$$

taking advantage of separability

$$+\, \alpha^1(p_\xi^1)\Big(\sum_{k=1}^{n} p_{k,\xi}^1\big(e_{k,\xi}^1 - \langle (T_\xi^1)_k, y\rangle\big)\Big), \qquad \alpha(p) = \prod_{l=1}^{L}\left(\frac{\beta_l}{p_l}\right)^{\beta_l}$$

$$y_\xi^{\nu+1} \in \arg\max\left\{ r^\nu(\xi; y) - \langle w^\nu, y\rangle - \frac{\rho}{2}\left|y - \bar{y}^\nu\right|^2 \;\middle|\; T^0 y \le e^0, u \in \mathbb{R}_+^n \right\}$$

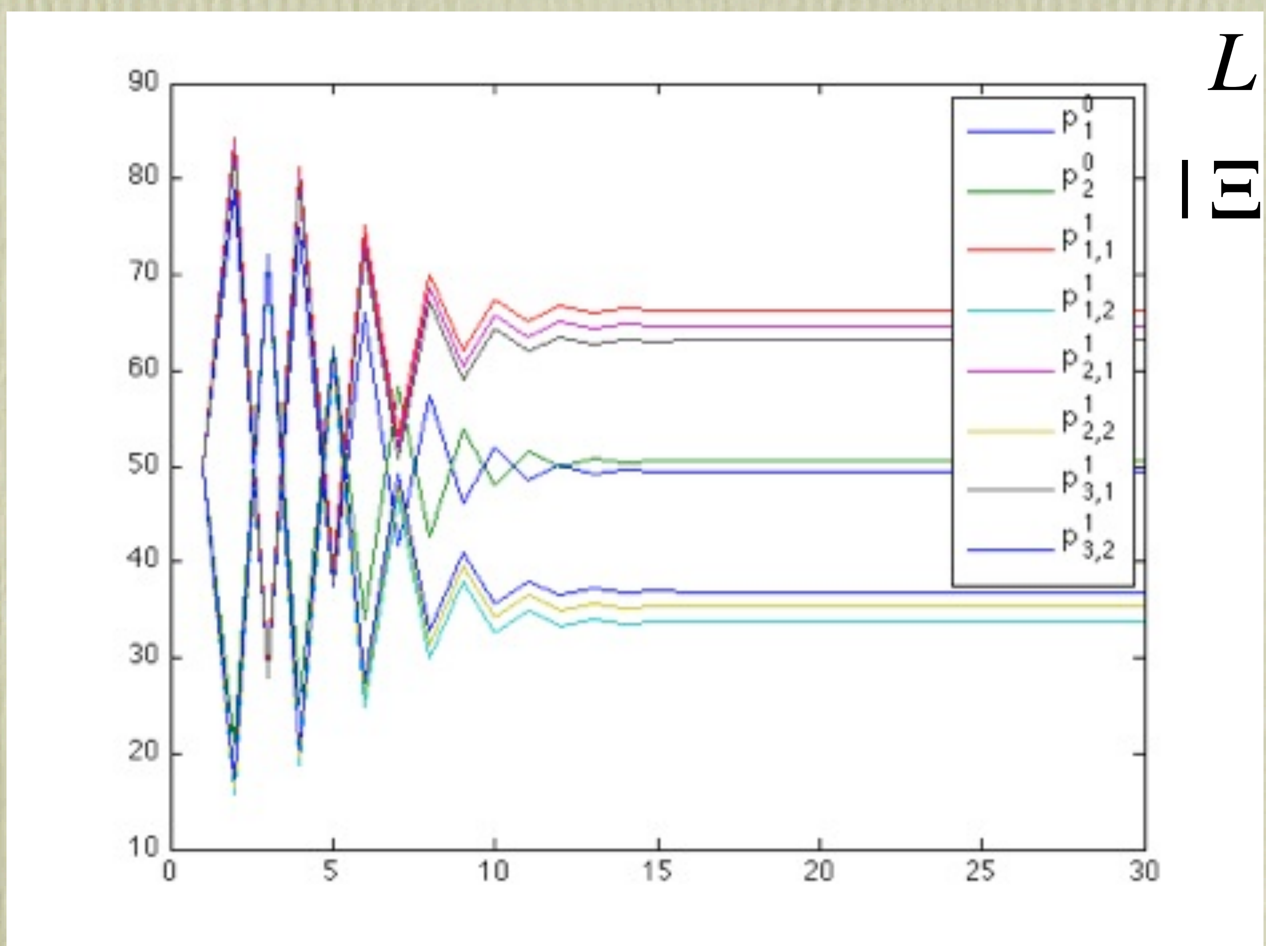
"outer loop", calculating $\big(p^0, (p_\xi^1, \xi \in \Xi)\big)$

augmented Walrasian

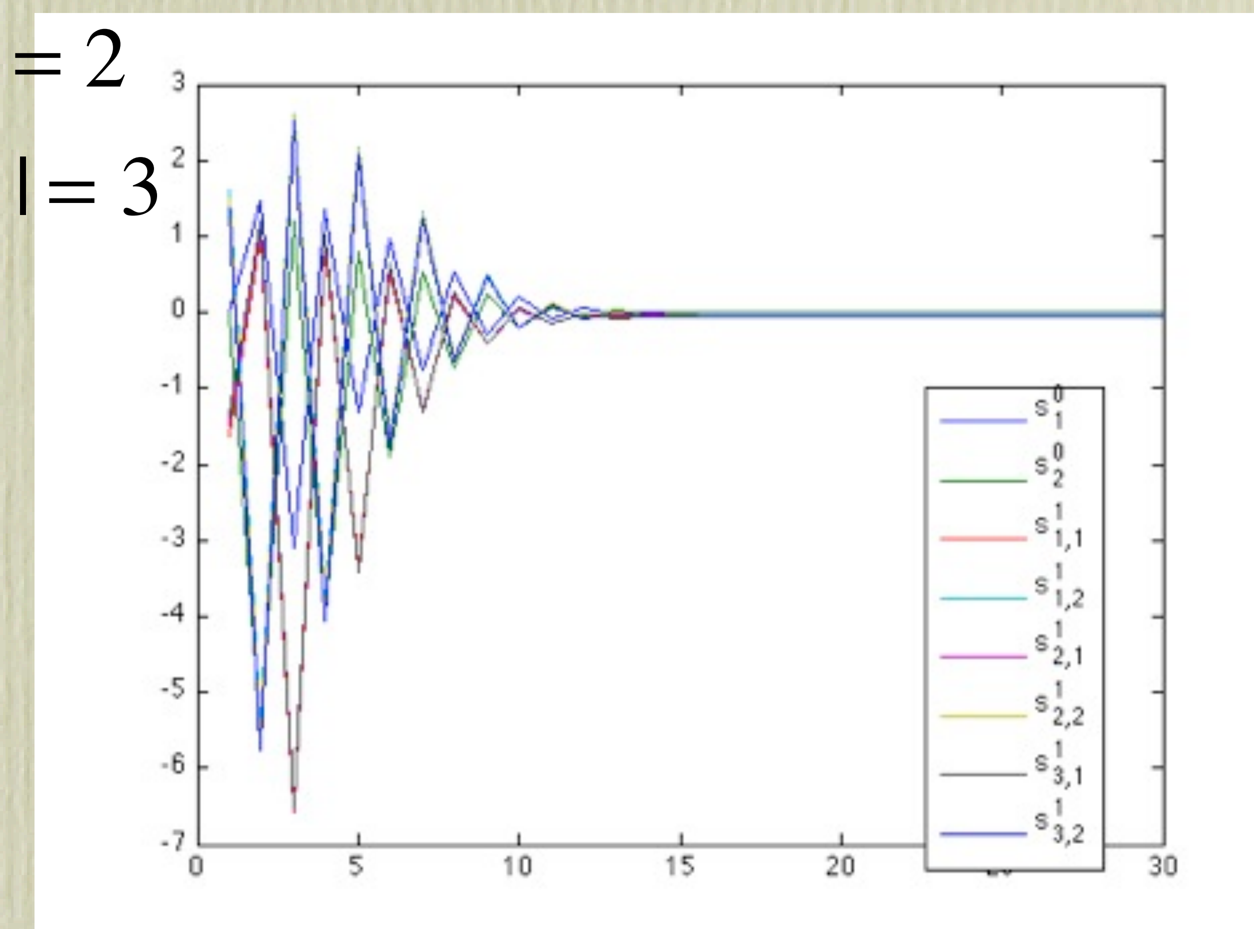
Convergence: exploiting separability

$L = 2$

$|\Xi| = 3$



prices



excess supply

4. Variational Inequality

to be dealt with <u>now</u> in glorious detail

back ... Arrow-Debreu model

i-agent demand: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p,x \rangle \le \langle p,e_i \rangle \right\}$

$u_i = X_i \to \mathbb{R}^L$, usc, concave $\not\Rightarrow X_i$ closed (but convex)

excess supply: $s(p) = \sum_{i\in I} \big(e_i - x_i(p)\big)$, market clearing: $s(p) \ge 0$

(under ample survivability, indispensability, unactractiveness)

i-agent optimal $x_i(p) \iff \exists \lambda_i \ge 0$ such that

$\lambda_i \perp \langle p, e_i - x_i(p) \rangle, \quad \langle p, e_i - x_i(p) \rangle \ge 0,$

$x_i(p) \in \arg\max_{x\in X_i} u_i(x) - \lambda_i \langle p, x \rangle$

λ_i utility scaling

when $X_i = \mathbb{R}^L_+, u_i$ smooth:

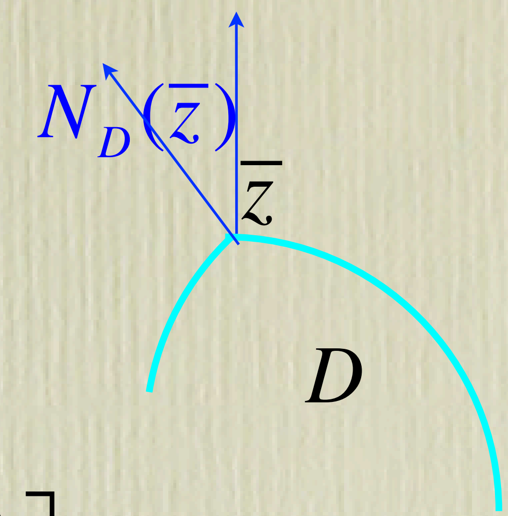
$0 \le x_i(p) \perp \lambda_i p - \nabla u_i(x_i(p) \ge 0$

via Variational Inequality

$$x_i(p) \in \arg\max\left\{u_i(x) \,\middle|\, \langle p,x\rangle \le \langle p,e_i\rangle, x \in X_i\right\}$$

$$\Updownarrow \quad \sum_i \left(e_i - x_i(p)\right) = s(p) \ge 0$$

$$N_D(\bar{z}) = \left\{v \,\middle|\, \langle v, z-\bar{z}\rangle \le 0, \forall z \in D\right\}$$



$$G\left(p,(x_i),(\lambda_i)\right) = \left[\sum_i (e_i - x_i); \left(\lambda_i p - \nabla u_i(x_i)\right); \langle p, e_i - x_i\rangle\right]$$

$$D = \Delta \times \left(\prod_i X_i\right) \times \left(\prod_i \mathbb{R}_+\right)$$

$$\boxed{-G\left(\bar{p},(x_i),(\lambda_i)\right) \in N_D\left(\bar{p},(x_i),(\lambda_i)\right)}$$

suggests
LCP, NCP
J.-S. Pang,
M.Ferris, ..

D unbounded $\rightarrow \hat{D}$ bounded

geometric V.I.

via smoothing: L. Qi, X.Chen, ...

adding Firms (a parenthesis)

i-agent: $e_i \to e_i + \theta_{ij} z_j$ share θ_{ij} of production $z_j \in Z_j$ of firm $j \in J$

$$\theta_{ij} \geq 0,\ \sum_{i \in I} \theta_{ij} = 1 \quad j = 1, \ldots, J$$

$$z_j(p) \in \arg\max \left\{ \langle p, z \rangle \,\middle|\, z \in Z_j \right\}$$

excess supply: $\sum_{i \in I} \big(e_i - x_i(p)\big) + \sum_{j \in J} z_j(p) \geq 0$ (equilibrium)

functional V.I. $\boxed{-G\big(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i), (\bar{z}_j)\big) \in \partial f\big(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i), (\bar{z}_j)\big)}$

$$f\big(p, (x_i), (\lambda_i), (z_j)\big) = \iota_{\mathbb{R}^L_+}(p) - \sum_{i \in I} u_i(x_i) + \sum_{i \in I} \iota_{\mathbb{R}_+}(\lambda_i) + \sum_{j \in J} \iota_{Z_j}(z_j)$$

f convex $\Rightarrow \partial f$ monotone operator (yields existence of solution)

$$G\big(p, (x_i), (\lambda_i), (z_j)\big)$$

$$= \left[\sum_i (e_i - x_i) + \sum_j z_j; (\lambda_i p); \left(\left\langle p, e_i + \sum_j \theta_{ij} z_j - x_i \right\rangle \right); (-p_j) \right]$$

Jofré, Rockafellar & Wets '07

Path Solver .. (M.Ferris, D.Ralph et al)

$$-G(\bar{z}) \in N_D(\bar{z}), \quad \bar{z} = (\bar{p},(\bar{x}_i),(\bar{\lambda}_i))$$

$$D = \Delta \times \left(\prod_i X_i\right) \times \left(\prod_i \mathbb{R}_+\right) = \left\{ z \middle| Az \geq b \right\}$$

Complementarity problem:

$$-G(z) = A^T y, \ \ y \geq 0, \ \ Az - b \perp y$$

with $K = \mathbb{R}^N \times \mathbb{R}^M_+$:

$$\boxed{(z,y) \in K, \ \ H(z,y) \in -K^*, \ \ (z,y) \perp H(z,y)}$$

$$H(z,y) = \begin{bmatrix} G(z) + A^T y \\ Az \end{bmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Equivalent nonsmooth mapping

- $0 = H(\mathrm{prj}_K(z, y)) + (z, y) - \mathrm{prj}_K(z, y)$
  - with simplified K

    (CP) $0 \le x \perp F(x) \ge 0$ Complementarity Problem

    (NS) $0 = F(x_+) + x - x_+$ Nonlinear system

- $\bar{x}$ sol'n (CP) $\Rightarrow$ $\tilde{x}$ sol'n (NS):

  $\tilde{x}_k = \bar{x}_k$ if $F_k(\bar{x}) = 0$, $\tilde{x}_k = -F_k(\bar{x}_k)$ if $F_k(\bar{x}) > 0$
- $\tilde{x}$ sol'n (NS) $\Rightarrow \tilde{x}_+$ sol'n (CP):

  $\tilde{x}_+ \ge 0, F(\tilde{x}_+) = \tilde{x}_+ - \tilde{x} \ge 0$ & $\tilde{x}_+ \perp \tilde{x}_+ - \tilde{x}$

PATH Solver:

$x = (z, y),\ x_+ = \mathrm{prj}_K(x, y)$

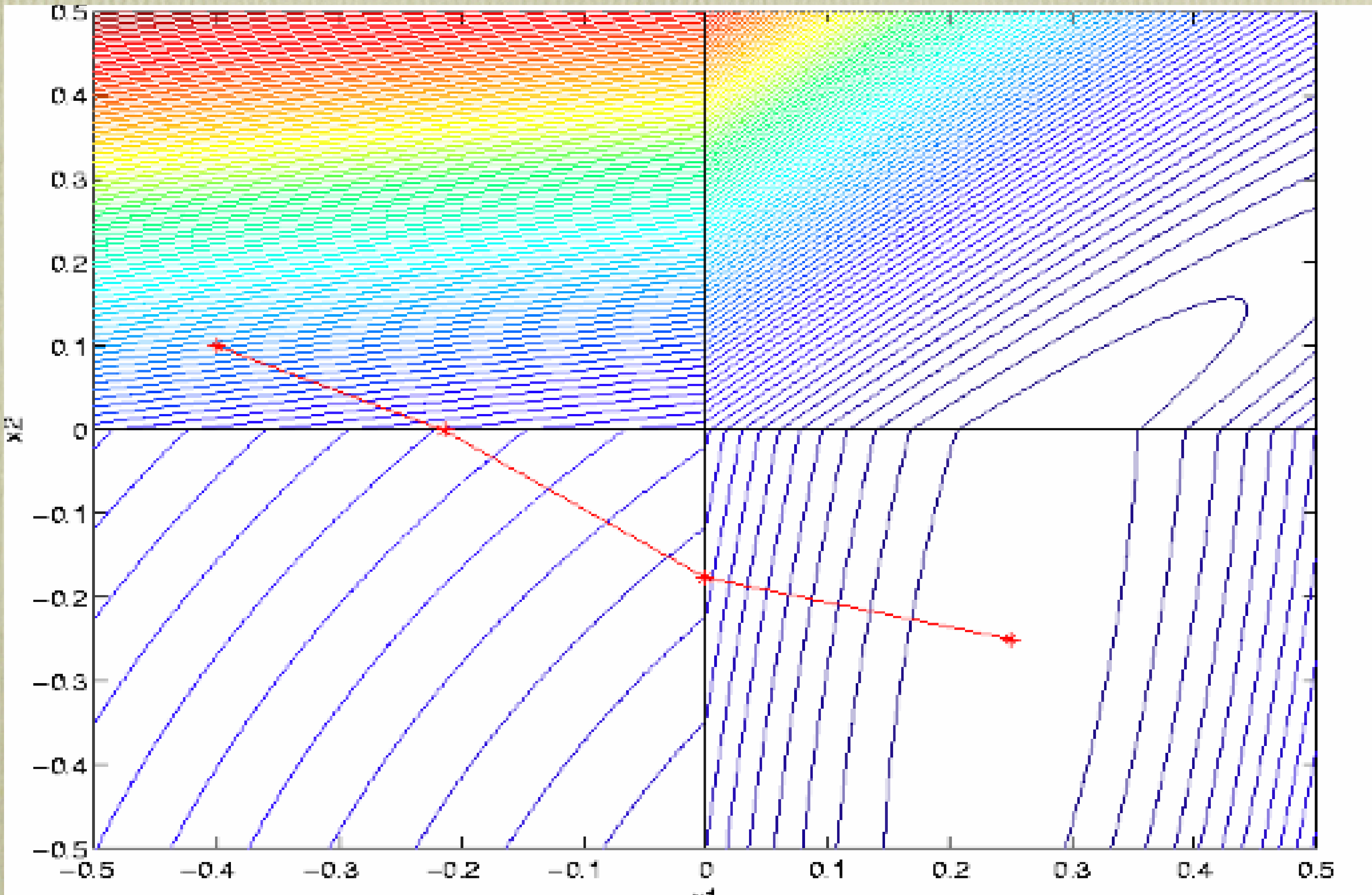
- PATH: Newton method based on nonsmooth normal mapping:

$$H(x_+) + x - x_+$$

- Newton point: solution of piecewise linearization:

$$H(x_+^k) + \left\langle \nabla H(x_+^k), x_+ - x_+^k \right\rangle + x - x_+ = 0$$

The “Newton” step



V.I.-Extensive Formulation

discrete distribution: $|\Xi|$ finite

$$G\Big(\big(p^0, p^1_{\xi\in\Xi}\big), \big(x^0_i, x^1_{i,\xi\in\Xi}\big)_{i\in I}, \big(\lambda^0_i, \lambda^1_{i,\xi\in\Xi}\big)_{i\in I}\Big) =$$

$$\Big[\Big(\sum_i (e^0_i - x^0_i), \sum_i (e^1_{i,\xi} - x^1_{i,\xi})_{\xi\in\Xi}\Big); \Big(\big(\lambda^0_i p^0 - \nabla u^0_i(x^0_i)\big)_{i\in I}, \big(\lambda^1_{i,\xi} p^1_\xi - \nabla u^1_i(\xi; x^1_{i,\xi})\big)_{i\in I,\xi\in\Xi}\Big); \Big(\langle p^0, e^0_i - x^0_i\rangle, \big(\langle p^1_\xi, e^1_{i,\xi} - x^1_{i,\xi}\rangle\big)_{i\in I,\xi\in\Xi}\Big)\Big]$$

$$D = \Big(\Delta \times \prod_{\xi\in\Xi} \Delta\Big) \times \Big(\prod_i X^0_i \times \big(\prod_i X^1_{i,\xi}\big)_{\xi\in\Xi}\Big) \times \Big(\prod_i \mathbb{R}_+ \times \big(\prod_i \mathbb{R}_+\big)_{\xi\in\Xi}\Big)$$

$$-G(\overline{z}) \in N_D(\overline{z}) = \big\{ v \,\big|\, \langle v, z - \overline{z}\rangle \le 0, \forall z \in D \big\}$$

$$z = \big(p^0, p^1_{\xi\in\Xi}\big), \big(x^0_i, x^1_{i,\xi\in\Xi}\big)_{i\in I}, \big(\lambda^0_i, \lambda^1_{i,\xi\in\Xi}\big)_{i\in I}$$

"Thanks the gods (& M. Ferris) for EMP"

for a special VI handled via smoothing/sampling: later

so, let's go: PATH Solver

- Economy: (8 goods), 5 types of agents
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- small # of scenarios 280,
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments

unfortunately, ... PATH Solver lets us down

Back to the drawing board

Disaggregation!

initialization: $\forall\, \mathrm{i} \in I$

$w_{i,\xi}$ such that $E\left\{w_{i,\xi}\right\} = 0$

$\forall \xi \;\Rightarrow\;$ equilibrium with $\left\{u_i^{w_{i,\xi}}(\xi;\bullet)\right\}_{i\in I}$

yields: $(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}, \bar{x}^1_{i,\xi}),\quad (\bar{p}^0_{i,\xi}, \bar{p}^1_{i,\xi})$

STOP $\forall i \in I : \left(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}\right) \simeq E\left\{\left(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}\right)\right\}$

otherwise adjust $\left[w_{i,\xi}, \xi \in \Xi\right]_{i\in I}$

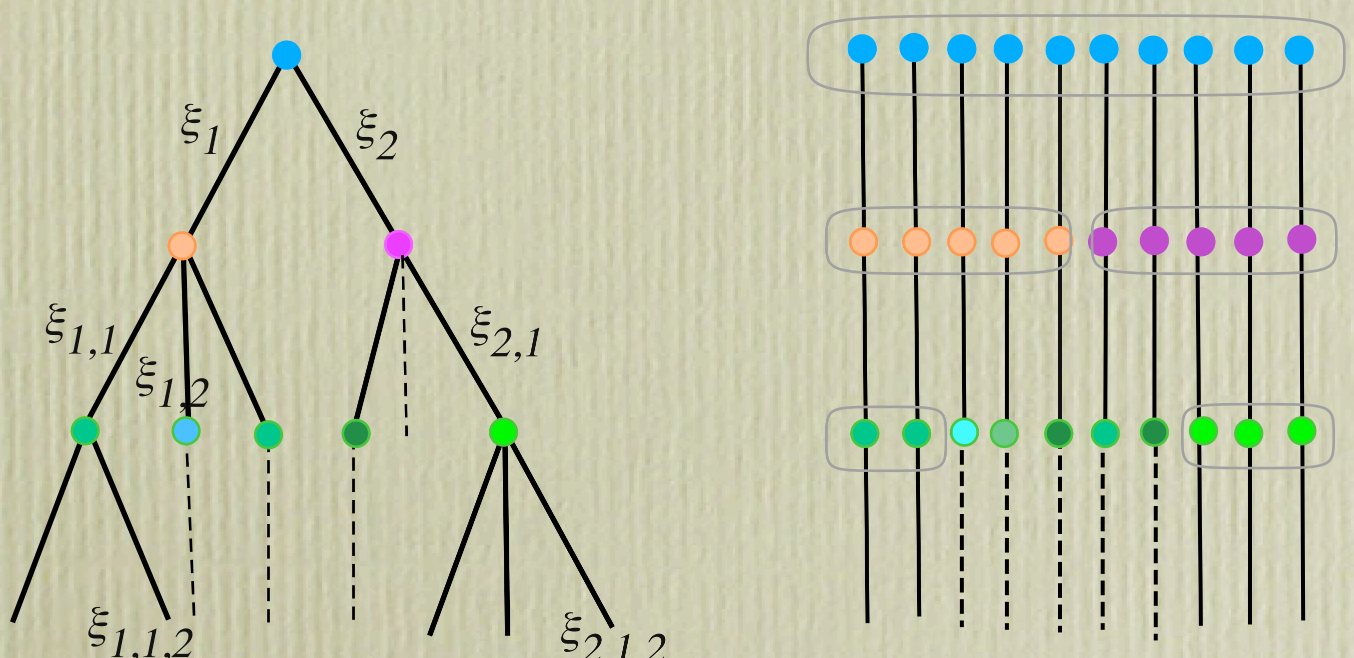
stochastic-pure exchange

for each $\xi \in \Xi$: the equilibrium problem, I agents with

i-agent's problem: $\max \left\{ u_i^{w_{i,\xi}} \left(\xi; x^0, y, x^1 \right) \text{ on } \widehat{C}_{i,\xi}(p^0, p_\xi^1) \right\}$

$$u_i^{w_{i,\xi}} = u_i^0(x^0) - \langle w_{i,\xi}, (x^0, y) \rangle - \frac{\rho}{2} \|(x^0, y) - (\bar{x}^0, \bar{y})\|^2 + u_i^1(\xi; x^1)$$

clearing the market: $s^0(p^0, p_\xi^1) \geq 0, \;\; s_\xi^1(p^0, p_\xi^1) \geq 0$



but now with $w_{i,\xi} \sim$ constraints

$x^0(\xi) \equiv x^0$ constant

$x^1(\xi | \xi^1) \equiv x_1^1, x^1(\xi | \xi^2) \equiv x_2^1$

$\ldots$

Disaggregation with PATH Solver

- Economy: (5 agents - 8 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks

on M. Ferris
semi-slow laptop
using EMP-package
4 min + 2 min for
verification

- 2-stages, solved under # of scenarios (280)
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer to time 1
- used for calibration (-> stochastic model)

numerically: `blink' (5000 iterations).

Ja! scenario disaggregation, but ...

i-agent: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p, x \rangle \le \langle p, e_i \rangle \right\},\ i \in I$

with excess supply $s(p)$: $\quad 0 \le p \perp s(p) \ge 0$

Multi-Optimization Problem with Equilibirum Constraint

MOPEC-class ~ maxinf family

$$x_i \in \arg\max_{x \in \mathbb{R}^{n_i}} f_i(p, x, x_{-i}), \quad i \in I, \quad x_I = (x_i, i \in I)$$

$$D(p, x_I) \in \partial g(p) \quad [\text{or } \in N_C(p)]$$

with Michael Ferris '11-'?? ... '05?

Examples: Walras, noncooperative games, ...
stochastic (dynamic): decentralized electricity markets,
joint estimation and optimization, financial equilibrium, ...

Contracts (Assets)

assets (= contract types): $k \in K$, $|K|$ finite

$z_i = z_i^+ - z_i^- = (z_i^1, \ldots, z_i^K)$ assets 'acquired' by i-agent

q_k market price of asset k

D_ξ^k bundle of goods 'delivered' by one unit of asset k

budgetary constraints:

$$\left\langle p^0, x^0 + T_i^0 y \right\rangle + \langle q, z \rangle \leq \left\langle p^0, e_i^0 \right\rangle$$

$$\left\langle p_\xi^1, x_\xi^1 \right\rangle \leq \left\langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y + D_\xi z \right\rangle \quad \forall \xi \in \Xi$$

clearing the market:

$$s^0\left(p^0, \left(p_\xi^1\right)_{\xi \in \Xi}, q\right) \geq 0, \; s^1\left(\xi; p^0, \left(p_\xi^1\right)_{\xi \in \Xi}, q\right) \geq 0 \; \forall \xi, \; \boxed{\sum_{i \in I} z_i = 0}$$

The BDE-example

Brown-DeMarzo-Eaves (Econometrica '96)

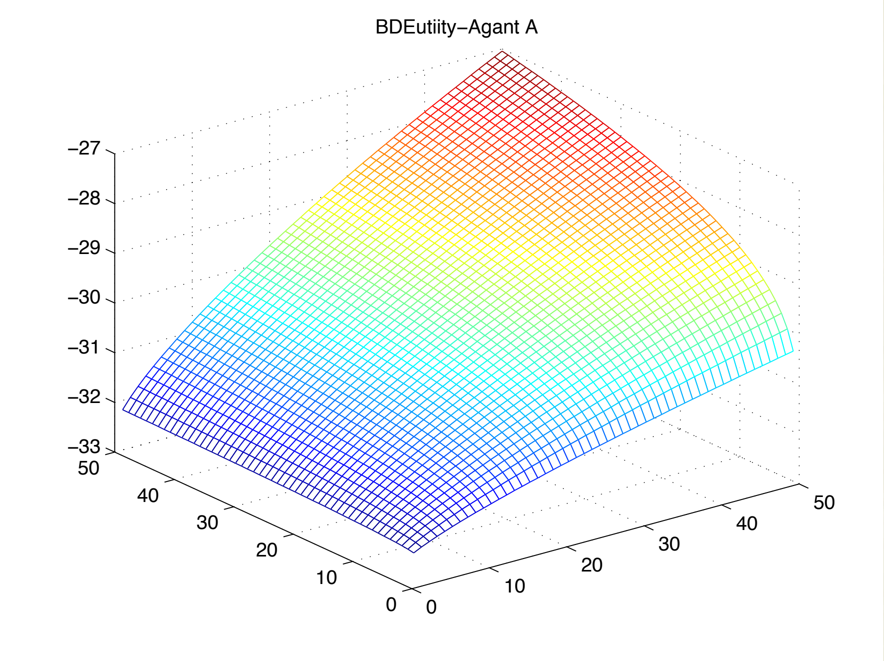
3 agents (2-agent & 3-agent of the same type)

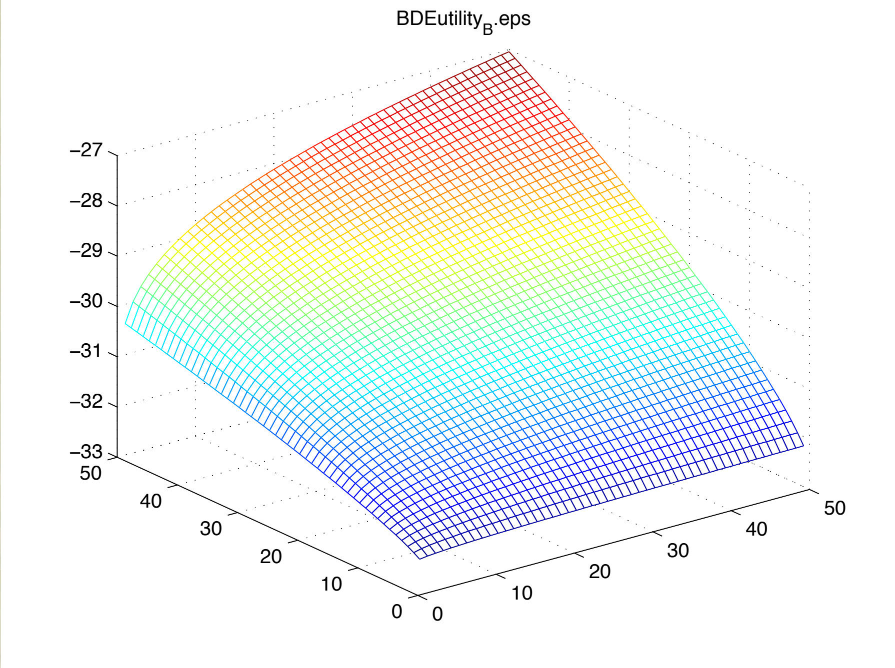
2 goods, $|\Xi| = 3$ (future states), no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x)$$

$$\alpha_1 = (0.25, 0.75),\ \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ





The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x), \quad \alpha_1 = (0.25, 0.75), \ \alpha_{2\,\&\,3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ

Path Solver solution: $p^0 = (1,\ 0.7338)$ (with scaling & $z\_i \leq 100$)

$p_\xi^1 = (1,\ 0.7158;\ 1,\ 0.7182;\ 1,\ 0.7205)$

$q = (0.9188,\ 0.6600)$, $z = (72.9868, \text{-}100; \text{-}36.4934, 50;\ \text{-}36.4934, 50)$

sol'n time: not noticeable

value transfer for #1-agent: @$t = 0$: -1.0649,

@$t = 1$, scn-1: 1.403, scn-2: 1.168, scn-3: 0.933,

The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi;x) = -\left(5.7 - \prod_{l=1}^{2}(x_l)^{\alpha_{i,l}}\right) = u_1^0(x),\ \alpha_1 = (0.25, 0.75),\ \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ for all ξ

BDE- solution: $p^0 = (1,\ 0.74)$ (with scaling)

$p_\xi^1 = (1,\ 0.7375;\ 1,\ 0.7174;\ 1,\ 0.6633)$

$q = (??,\ ??),\quad z = (0.94, 0;\ 0.03, 0;\ 0.03, 0)$ ⟸ all buyers no sellers

change of variables + add unconstrained agent:

homotopy continuation method (predictor-corrector steps)

The Cass trick

or the no-arbitrage condition

$$q = \sum_{\xi\in\Xi} w_\xi \left(D_\xi^T p_\xi^1\right) \text{ for some weights } w_\xi \geq 0,\ \sum_\xi w_\xi = 1$$

$$\max u_i\left(x^0, (x_\xi^1)_{\xi\in\Xi}\right) = u_i^0(x^0) + E\left\{u_i^1(\xi; x_\xi^1)\right\}$$

$$\text{such that } \left\langle p^0, e_i^0 - x^0 \right\rangle + \langle q, z\rangle \geq 0$$

$$\left\langle p_\xi^1, e_{i,\xi}^1 + D_{i,\xi} z - x_\xi^1 \right\rangle \geq 0, \forall \xi \in \Xi$$

$$\tilde{p}^0 = p^0,\ \tilde{p}_\xi^1 = w_\xi p_\xi^1, \xi \in \Xi$$

solved by BDE

Path Solver $\Rightarrow$ BDE-sol'n

no Path Solver sol'n!

via Augmented Walrasian

for 'money' assets (Deride, Jofré & Wets '09)

cf. financial equilibrium: Hens & Pilgrim '06

$$\max u_i\left(x^0, (x_\xi^1)_{\xi\in\Xi}\right) \text{ such that}$$

$$\left\langle \tilde{p}^0, e_i^0 - x^0 \right\rangle + \sum_{\xi\in\Xi} \left\langle \tilde{p}_\xi^1, D_\xi z - x_\xi^1 \right\rangle \geq 0$$

$$\left\langle \tilde{p}_\xi^1, e_{i,\xi}^1 + D_{i,\xi} z - x_\xi^1 \right\rangle \geq 0,\ \forall \xi \in \Xi$$

$$s^0\left(\tilde{p}^0, \left(\tilde{p}_\xi^1\right)_{\xi\in\Xi}\right) \geq 0,\ s^0\left(\xi; \tilde{p}^0, \left(\tilde{p}_\xi^1\right)_{\xi\in\Xi}\right) \geq 0, \forall \xi$$

clearing the market $\boxed{\text{also: } \sum_{i\in I} z_i = 0}$

Further readings

- Jofré, A. & R. Wets, Variational convergence of bivariate functions: theoretical foundations. Mathematical Programming (2006).
- Jofré, A., R.T. Rockafellar R.T & R. Wets. Variational Inequalities and economic equilibrium. Mathematics of Operation Research (2006?)
- Jofré, A., RT. Rockafellar & R. Wets, A variational inequality scheme for determining an economic equilibrium of classical or extended type. In "Variational analysis and applications", 553--577, Nonconvex Optimisation and Applications, 79, Springer, New York, 2005.
- Jofré, A. & R. Wets, Continuity properties of Walras equilibrium points. Stochastic equilibrium problems in economics and game theory. Annals of Operations Research, 114 (2002), 229--243.
- S. P. Dirkse and M. C. Ferris. The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems. Optimization Methods and Software, 5:123-156, 1995.

Primary Objective: constructive theory

- Exhibits and exploits the interrelation between these problems
- Existence theory: (mostly, not exclusively)
  - Aubin & Ekeland, "Applied Nonlinear Analysis" (Chap. 6), 1984
  - Facchinei & Pang, "Finite Dimensional Variational Inequalities and Complementarity problems" (2003)
  - Iusem & Sosa (+ Kasay), "Existence of solutions to equilibrium problems" (2005-....)
- Approximation theory $\Rightarrow$ algorithmic strategies + existence

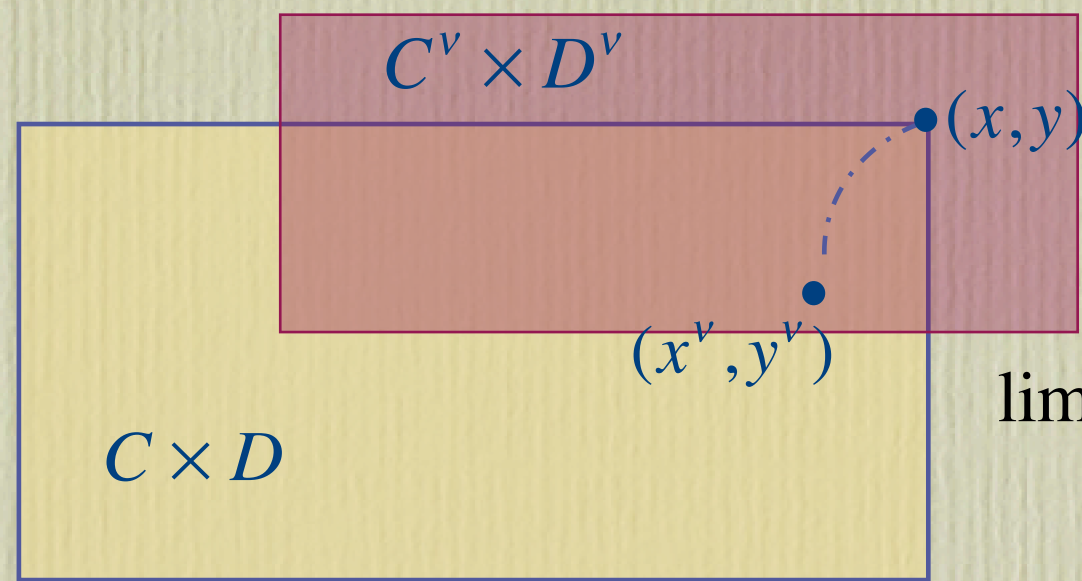
Saddle functions
Epi/hypo convergence

- *Lagrangians (concave/convex)*
- *zero-sum games*
- *Hamiltonians*

EPI/HYPO-Convergence

$\exists y^\nu \in D^\nu$

$\to y \in D$



$\forall\, y \in D, \forall\, x^\nu \in C^\nu \to x$

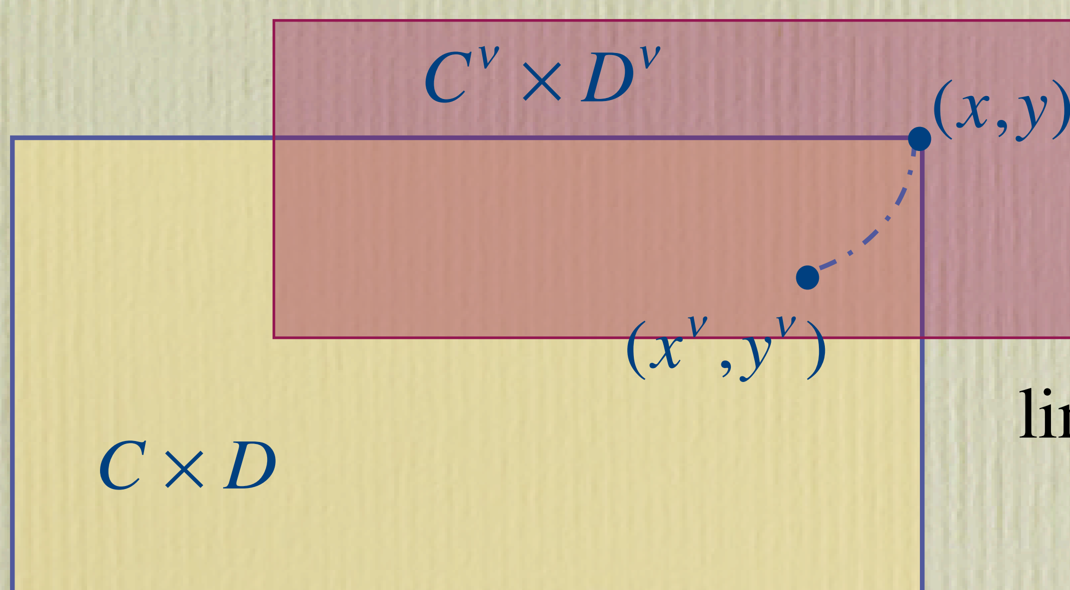
just y dependent: epi-convergence
just x dependent: hypo-convergence

$$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x,y) \text{ when } x \in C$$

epi- in y & hypo- in x $\Rightarrow$ epi/hypo
but not a necessary condition!

$\exists x^\nu \in C^\nu$

$\to x \in C$



$$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x,y) \text{ when } y \in D$$

$\forall\, x \in C, \forall\, y^\nu \in D^\nu \to x \in D$

Saddle Points: vs-Convergene

$$K^\nu \xrightarrow{e/h} K : C \times D \to \mathbb{R}, \varepsilon_\nu \searrow 0, \quad \left(x^\nu, y^\nu\right) \in \ \varepsilon_\nu\text{-sdl}(K^\nu)$$

$$\left(\bar{x}, \bar{y}\right) = \lim_{\nu \in N \subset \mathbb{N}} \left(x^\nu, y^\nu\right), \ N \sim \text{ subsequence}$$

$$\Rightarrow \left(\bar{x}, \bar{y}\right) \in \text{sdl}(K) \ \& \ K\left(\bar{x}, \bar{y}\right) = \lim_{\nu \in N \subset \mathbb{N}} K^\nu\left(x^\nu, y^\nu\right)$$

in the convex/concave case $\Rightarrow$ convergence primal/dual solutions

ancillary tight ($\sim$ y-compact): $\forall \varepsilon > 0, \exists\, B_\varepsilon$ compact, ν_ε

$$\forall \nu \geq \nu_\varepsilon, \sup_{B_\varepsilon \cap D^\nu} K^\nu(x^\nu, \bullet) \geq \sup_{D^\nu} K^\nu(x^\nu, \bullet) - \varepsilon$$

e/h-convergence + ancillary tight $\Rightarrow$ sv-convergence saddle points

Zero-Sum Games

$x^* \in \arg\max_{x\in X} u(x,y^*), \quad y^* \in \arg\min_{y\in Y} u(x^*,y)$

$(x^*,y^*) \in \text{sdl}(u)$

if X,Y convex, compact ($\Rightarrow$ tight)

$\forall y, x \mapsto u(x,y)$ concave, usc, $\quad \forall x, y \mapsto u(x,y)$ convex, lsc

$\Rightarrow$ the zero-sum game $\mathcal{G} = \{(X,u),(Y,-u)\}$ has a solution

moreover, $X^\nu \to X, Y^\nu \to Y$, $u^\nu \underset{e/h}{\to} u$ (with same properties)

$\Rightarrow$ their solutions (x^ν, y^ν) cluster to solution of $\mathcal{G}$

also the case for approximate solutions

Max-Inf $\neq$ Min-Sup

Variational Inequalities

- $G : C \to \mathbb{R}^n$ $\quad C \subset \mathbb{R}^n$ non-empty, convex set
- find $\bar{u} \in C$ such that $-G(\bar{u}) \in N_C(\bar{u})$

$$v \in N_C(\bar{u}) \Leftrightarrow \langle v, u - \bar{u} \rangle \le 0, \forall u \in C$$

- let $C^\nu \to C$, $\quad G^\nu : C^\nu \to \mathbb{R}^n$ continuous
- S^ν solution set of approximating problems

  S solution of the limit problem. Does $S^\nu \to S$?

V.I.: The gap function

- Let $\boxed{K(u,v) = \langle G(u), v-u \rangle}$ on dom $K = C \times C$
  - then $-G(\bar{u}) \in N_C(\bar{u})$ if and only if
  - $\bar{u} \in$ maxinf point of K with $K(\bar{u},\bullet) \geq 0$
- $K^\nu(u,v) := \langle G^\nu(u), v-u \rangle$, dom $K^\nu = C^\nu \times C^\nu$
  - $u^\nu \in \arg\max\text{–}\inf K^\nu$ with $K^\nu(u^\nu,\bullet) \geq 0$
  - $\boxed{K^\nu \xrightarrow{?} K \quad \text{and} \quad \ldots}$
  - $\bar{u} \in$ cluster points $\{u^\nu\} \Rightarrow ?\ \bar{u} \in \arg\min\text{–}\sup K$

Non-Cooperative Games

- $a \in \mathcal{A}$, payoff: $u_a(x_a, x_{-a}) : \mathbb{R}^N \to \bar{\mathbb{R}}$, $\therefore$ includes $x_a \in C(x_{-a})$
- Generalized Nash equilibrium: $(\bar{x}_a, a \in \mathcal{A})$ such that

$$\forall a \in \mathcal{A},\ \bar{x}_a \in \arg\max u_a(x_a, \bar{x}_{-a})$$

- Nikaido-Isoda function:

$$N(x,y) = \sum_{a\in\mathcal{A}} u_a(x_a, x_{-a}) - \sum_{a\in\mathcal{A}} u_a(y_a, x_{-a})$$

- $\bar{x} = (\bar{x}_a, a \in \mathcal{A})$ is a Nash equilibrium

$$\Leftrightarrow \bar{x} \in \arg\operatorname{maxinf} N,\ \ N(\bar{x}, \bullet) \geq 0$$

Approximating games

- Nikaido-Isoda functions of approximating games

$$N^\nu(x,y) = \sum_{a\in \mathsf{A}} u_a^\nu(x_a, x_{-a}) - \sum_{a\in \mathsf{A}} u_a^\nu(y_a, x_{-a})$$

- $x^\nu \in \arg\max\text{–}\inf N^\nu$, $\bar{x} \in$ cluster points $\{x^\nu\}$

$$\boxed{N^\nu \xrightarrow{?} N \quad \text{and} \quad \ldots}$$

- $\Rightarrow ?\ \bar{x} \in \arg\max\text{–}\inf N \sim$ equilibrium point

Ky Fan functions & inequality

$K : C \times C \to \mathbb{R}$ Ky Fan function if

(a) $\forall\, y \in C: x \mapsto K(x,y)$ usc on C

(b) $\forall\, x \in C: y \mapsto K(x,y)$ convex on C

K Ky Fan fcn, dom $K = C \times C$ + C compact

$$\boxed{\Rightarrow \arg\max\!-\!\inf K \neq \emptyset}$$

if $K(x,x) \geq 0$ on dom K, $\bar{x} \in \arg\max\!-\!\inf K$

$$\boxed{\Rightarrow \inf_y K(\bar{x},y) \geq 0.}$$

Improvements: Iusem, Kasay, Sosa (locals)

Lignola, Nessah, Tian, X. Yu, ...

Ky Fan's inequality: an extension

$$K^\nu \to K \text{ lopsided tightly with } C^\nu \to C,$$
$$K^\nu \text{ Ky Fan } \Rightarrow K \text{ Ky Fan fcn}$$
$$\& \text{ if } \forall \nu : \arg\max\text{–}\inf K^\nu \neq \emptyset$$
$$\bar{x} \in \text{ cluster-pts } \{\arg\max\text{–}\inf K^\nu\}$$
$$\Rightarrow \bar{x} \in \arg\max\text{–}\inf K \;\&\; K(\bar{x},\bullet) \geq 0$$

Application: guideline for approximation schemes
truncations, coercivity, ...

MOPEC Family

Checking Lopsided Tightness

Linear Complementarity Problems

LCP: find $z \geq 0$, $Mz + q \geq 0$ and $(Mz+q) \perp z$

$$K(z,v) = \langle Mz+q, v-z \rangle \text{ on } \mathbb{R}^n_+ \times \mathbb{R}^n_+, \text{ Ky Fan fcn}$$

approx. $z \in \left[0, r^\nu\right], M^\nu z + q^\nu \geq 0$ and $(M^\nu z + q^\nu) \perp z$

$$K^\nu(z,v) = \left\langle M^\nu z + q^\nu, v - z \right\rangle \text{ on } \left[0, r^\nu\right] \times \mathbb{R}^n_+$$

- $K^\nu \to_{lop} K$ when $M^\nu \to M, q^\nu \to q, r^\nu \nearrow \infty$
- $K^\nu \to_{lop} K$ ancillary tightly when also

$$P^\nu = \left\{ z \in [0, r^\nu] \,\middle|\, M^\nu z + q^\nu \geq 0 \right\} \to P = \left\{ z \geq 0 \,\middle|\, Mz + q \geq 0 \right\}$$

$\Rightarrow$ cluster points of sol'ns of approx. solve LCP

$$\left(\textit{note}: \text{int}\, P \neq \emptyset, \text{ no row of } [M, q] \; = \; 0 \Rightarrow P^\nu \to P \right)$$

- $K^\nu \to_{lop} K$ tightly (study of quadratic forms)

Variational Inequalities

- $-G(u) \in N_C(u)$, G continuous, C convex, compact
- bifunction: $K(u,v) = \langle G(u), v-u \rangle$ on $C \times C$, Ky Fan fcn & $K(u,u) \geq 0$
- THM: $C^\nu \to C \Rightarrow C^\nu$ compact $\nu \geq \bar{\nu}$, G^ν continuous

  $G^\nu \to_{cont} G : G^\nu(x^\nu) \to G(x), \ \forall x^\nu \in C^\nu \to x$

  $K^\nu(u,v) = \left\langle G^\nu(u), v-u \right\rangle$ on dom $K^\nu = C^\nu \times C^\nu$

  lop-converge ancillary tightly to $K \Rightarrow$ sol'ns converge

Continuous convergence (?):

sol'ns $S^\nu = G^\nu + N_{C^\nu} \ni 0 \to$ sol'ns $S = G + N_C \ni 0$

Fixed Points (Set-Valued)

find $x \in C$ (convex) : $\quad x \in S(x),\ S : C \rightrightarrows \mathrm{C} \ \subset \mathbb{R}^n$, osc (gph S closed)

$$K(x,v) = \sup\left\{ \langle x - v, z - x \rangle \middle| z \in S(x) \subset C \right\}$$

K a Ky Fan fcn, convex in v, usc in x (sup-projection) + $K(x,x) \geq 0$

Approx. bifunctions: $K^\nu(x,v) = \sup\left\{ \langle x - v, z - x \rangle \middle| z \in S^\nu(x) \subset C^\nu \right\}$

THM. $C^\nu \to C$, gph $S^\nu \to$ gph S (as sets), C compact. Then,

$\forall \varepsilon_\nu \searrow 0, \quad \bar{x} \in$ cluster points $\left\{ x^\nu \in \varepsilon_\nu\text{-maxinf } K^\nu \right\}$ is a maxinf point of K,

i.e., a fixed point of S. (lop-convergence is tight)

an Application (J.S. Pang) - Cognitive radio multi-user game

$f : C \to C \subset \mathbb{R}^n$ continuous, C compact, convex, $\bar{x}$ fixed point

Pertubation (ε-enlargement): $S(\bullet;\varepsilon) : C \rightrightarrows C$, osc , $S(\bullet;0) = f$

For ε near 0: existence? $\exists\, x^\varepsilon \in S(x^\varepsilon, \varepsilon) = S^\varepsilon(x),\ x^\varepsilon \to \bar{x}$?

Lop- & Epi/Hypo-convergence

1. $L^\nu \underset{lop}{\to} L \not\Rightarrow L^\nu \underset{e/h}{\to} L$
2. $L^\nu \underset{e/h}{\to} L$ & convex-concave $\Rightarrow$ $L^\nu \underset{lop}{\to} L$
3. epi/hypo- = hypo/epi-convergence
4. $L^\nu \underset{e/h}{\to} L \Rightarrow$ convergence of saddle points

   $\Rightarrow$ convergence of approximate sadde points

   (without ancillary tightness)
5. Existence requires tightness-conditions (~coercivity, e.g.)

Uniqueness of
lop- & epi/hypo-limits

Convincing Examples (?)

- Lagrangians: $L^\nu(x,y) = f_0^\nu(x) + \sum_{i=1}^{m} y_i f_i^\nu(x)$ on $X^\nu \times \left(\mathbb{R}^s \times \mathbb{R}^{m-s}\right)$
- Lopsided convergence (maxinf-paradigm) - sufficient conditions
  - $f_0^\nu, f_1^\nu, \ldots, f_m^\nu$ hypo-converge to $f_0, f_1, \ldots, f_m$ on $X^\nu \to X$
  - the collection $\left\{f_i^\nu, \nu \in \mathbb{N}\right\}$ is equi-usc, $i = 0, \ldots, m$
  - Constraint Qualification: $S^\nu = \left\{x \,\middle|\, f_i^\nu \geq 0,\ i = 1, \ldots, m\right\} \to S$
- lop-limit L is unique concave-convex case (epi/hypo): int $S \neq \emptyset$

Variational Inequalities

$C^\nu \to C, \quad G^\nu: C^\nu \to \mathbb{R}^n$ continuous, C^ν convex

$-G^\nu(x) \in N_{C^\nu}, \ \nu \in \mathbb{N}$

THM: $C^\nu \to C \Rightarrow C$ compact $\nu \geq \bar{\nu}$, G^ν continuous

$G^\nu \to_{cont} G : G^\nu(x^\nu) \to G(x), \ \forall x^\nu \in C^\nu \to x$

$K^\nu(u,v) = \left\langle G^\nu(u), v - u \right\rangle$ on dom $K^\nu = C^\nu \times C^\nu$

lop-converge ancillary tightly to $K \Rightarrow$ sol'ns converge

lop-limit: $-G(x) \in N_C(x)$ uniquely determined

MPEC (generalized?)

$\max g(x)$ such that $\mathrm{x} \in S(x)$, g continuous, $S : C \rightrightarrows C$ convex

bifunction: $K(x,v) = g(x) + \sup_z \left\{ \langle x - v, z - x \rangle \middle| z \in S(x) \right\}$

V.I.-constraint: $S(x) = N_C(x) + G(x) + Ix$ on C

LCP: $S(x) = \langle Mx + q, v - x \rangle + Ix$ on $\mathbb{R}^n_+$

$\bar{x} \in \arg\max \inf K \Rightarrow \bar{x}$ solves MPEC.

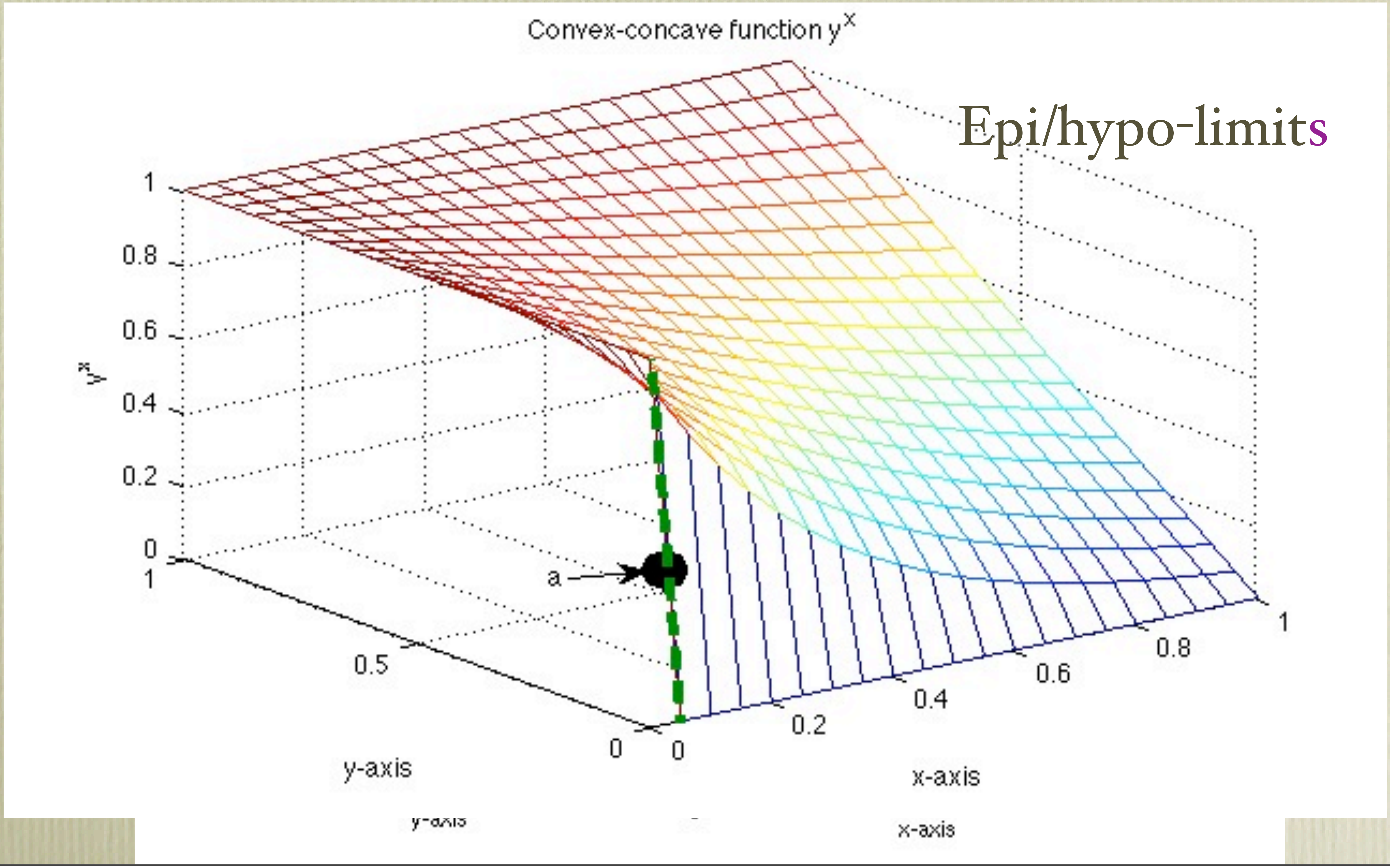
approximating bifunctions: $S^\nu : C^\nu \rightrightarrows C$

$$K^\nu(x,v) = g^\nu(x) + \sup_z \left\{ \langle x - v, z - x \rangle \middle| z \in S^\nu(x) \right\}$$

$C^\nu \to C$, gph $S^\nu \to$ gph S, g^ν hypo-converges to g

then $K^\nu \xrightarrow[lop]{} K$ & K unique

$K^{\nu}(x,y) \equiv y^{x}$ Uniqueness fails!



III. Random Sets & Mappings

Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) \ni 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu} \sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample P^ν (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

Examples:

$\min f = f_0 + \iota_C$, optimality: $"0 \in \partial f(\bar{x}) = S(x)"$ $\quad \sim 0 = \nabla f(\bar{x}))$

generally, $\partial\big(f + g\big) \neq \partial f + \partial g$

$\mathbb{C}.\mathbb{Q}.$ (Constraint Qualification): $-N_C(\bar{x}) \cap \partial^\infty f_0(\bar{x}) = \{0\}$

$v \in \partial^\infty f_0(\bar{x}) =$ horizon subgradient if

$\exists\, \mathrm{x}^\nu \to \bar{x}$ with $f(x^\nu) \to f(\bar{x}), v^\nu \in \hat{\partial} f(x^\nu), \lambda_\nu \searrow 0 \;\&\; \lambda_\nu v^\nu \to v$

with $\mathbb{C}.\mathbb{Q}.$ $\quad \bar{x}$ locally optimal $\Rightarrow \boxed{\partial f_0(\bar{x}) + N_C(\bar{x}) = S(\bar{x}) \ni 0}$

f convex ($\Rightarrow$ regular), $\partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0$

$\Rightarrow$ globally optimal (without $\mathbb{C}.\mathbb{Q}$)

When f_0, C are convex: $-\partial f_0(\bar{x}) \in N_C(\bar{x})$,

a functional variational inequality

“Variational” Approximations

(E, d) Polish, in paricular $E = \mathbb{R}^n$

$(\text{cl-sets}(E), d\!l)$ complete metric space; Polish if $E = \mathbb{R}^n$

$d\!l(C^\nu, C) \to 0 \iff C^\nu \to C$

<u>osc-mappings</u> = closed graph

(osc-maps$(S), d\!l)$ complete, metric space;

Polish if dom $\subset \mathbb{R}^n$, rge $\subset \mathbb{R}^m$

Convergence:

$S^\nu \xrightarrow{g} S$ if $d\!l(\text{gph}\, S^\nu, \text{gph}\, S) \to 0 \implies (S^\nu)^{-1}(0) \rightrightarrows_v S^-1(0)$

Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \ldots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

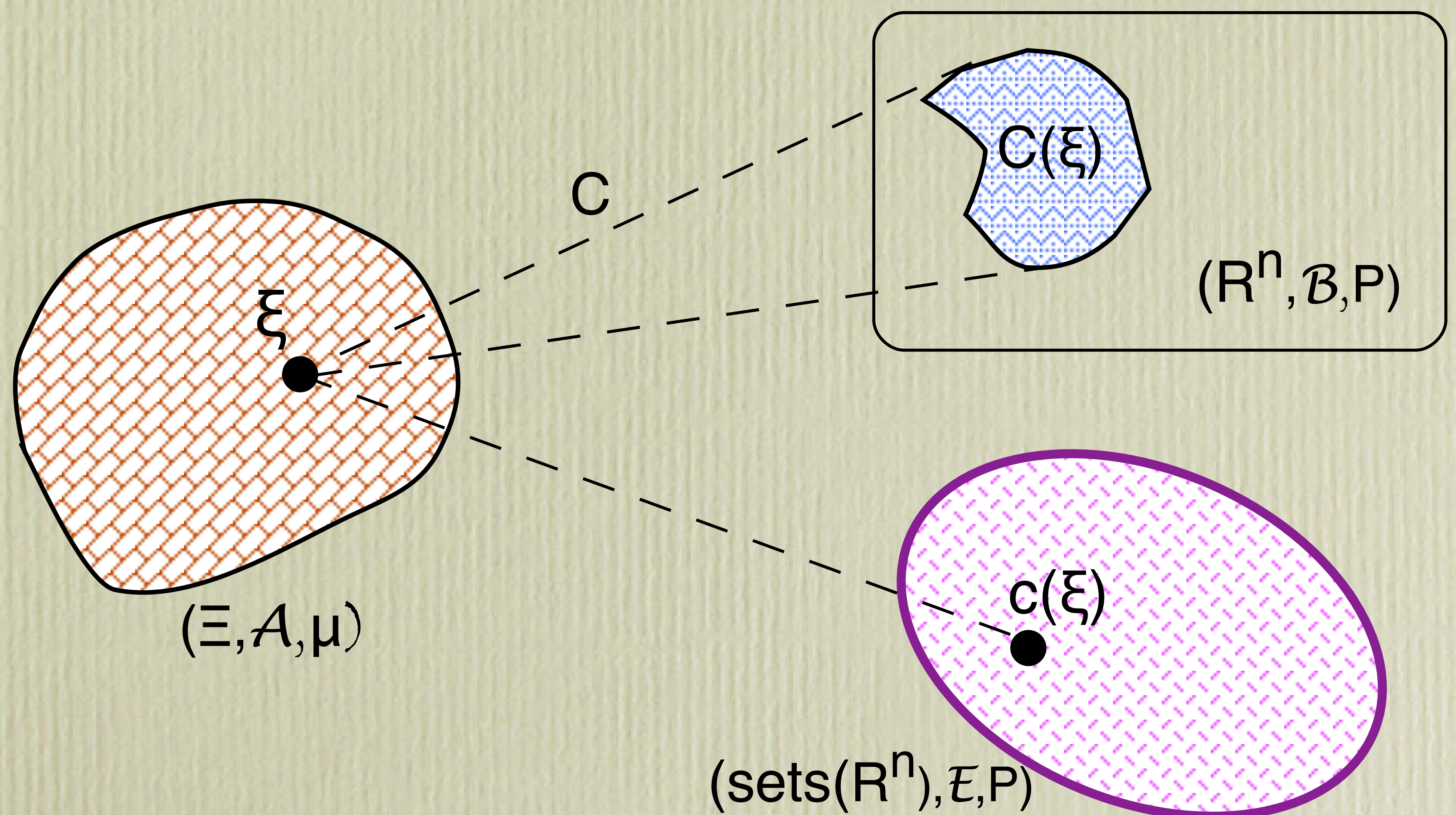
$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \ldots \xi^\nu$ sample P^ν (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

Set- & “Single”-valued



Random Closed Sets

$(\Xi, \mathcal{A}, P)$, $\quad \Xi \subset \mathbb{R}^N$ & E Polish, for example $\mathbb{R}^n$

$C : \Xi \rightrightarrows E,\ C(\xi) \subset E$ closed set for all $\xi \in \Xi$

& $C^{-1}(O) = \{\xi \,|\, C(\xi) \cap O \neq \emptyset\} \in \mathcal{A},\ \forall O \subset E$, open

$\quad \Rightarrow$ dom $C = C^{-1}(E) \in \mathcal{A}$, measurability ~ hit open sets

==

$c : \Xi \to$ cl-sets(E), $c(\xi) \sim C(\xi)$, $\mathcal{F}_O = \{F \subset E \text{ closed} \,|\, F \cap O \neq \emptyset\}$

$\left(\text{sets}(E), \mathcal{E}\right)$, $\mathcal{E}$ Effrös field $= \sigma\text{-}\{\mathcal{F}_O \in \text{ sets}(\mathbb{R}^n), O \text{ open}\}$,

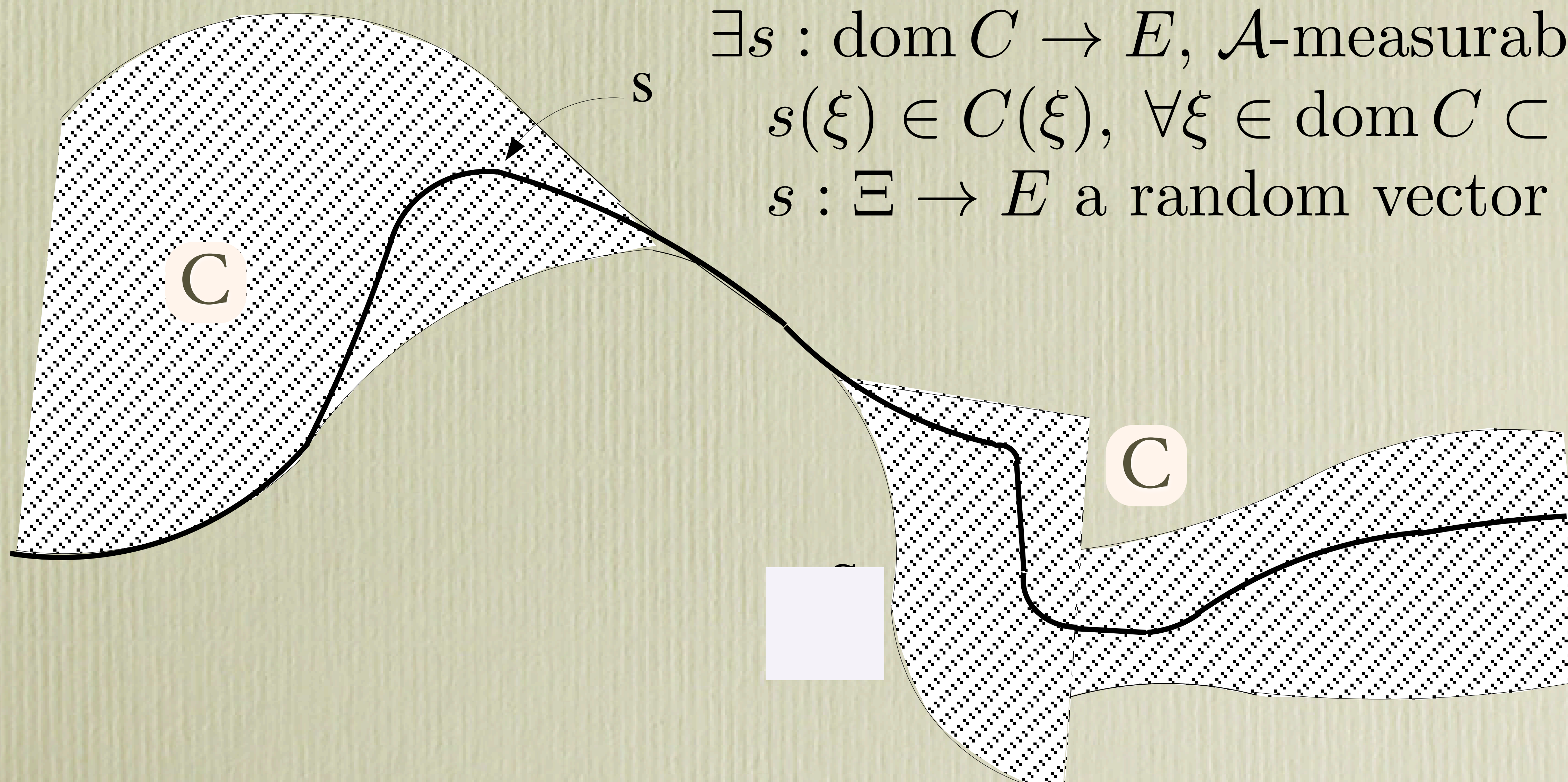
$\quad C$ measurable $\Leftrightarrow c$ measurable $[c^{-1}(\mathcal{F}_O) \in \mathcal{A}]$

$\mathcal{E} = \mathcal{B}$ Borel field when E Polish (complete separable metric space)

Measurable selection

- a random closed set C always admits a measurable selection!

$$\exists s : \operatorname{dom} C \to E,\ \mathcal{A}\text{-measurable},$$
$$s(\xi) \in C(\xi),\ \forall \xi \in \operatorname{dom} C \subset \Xi$$
$$s : \Xi \to E \text{ a random vector}$$



Castaing Representation

- C is a random closed set (& dom C measurable) $\Leftrightarrow$ it admits a Castaing representation: $\exists$ a countable family

$$\left\{ s^\nu : \text{ dom } C \to E, \text{ meas.-selections} \right\}$$

$$\text{cl} \bigcup_{\nu \in \mathbb{N}} s^\nu(\xi) = C(\xi), \forall \xi \in \text{ dom } C \subset \Xi$$

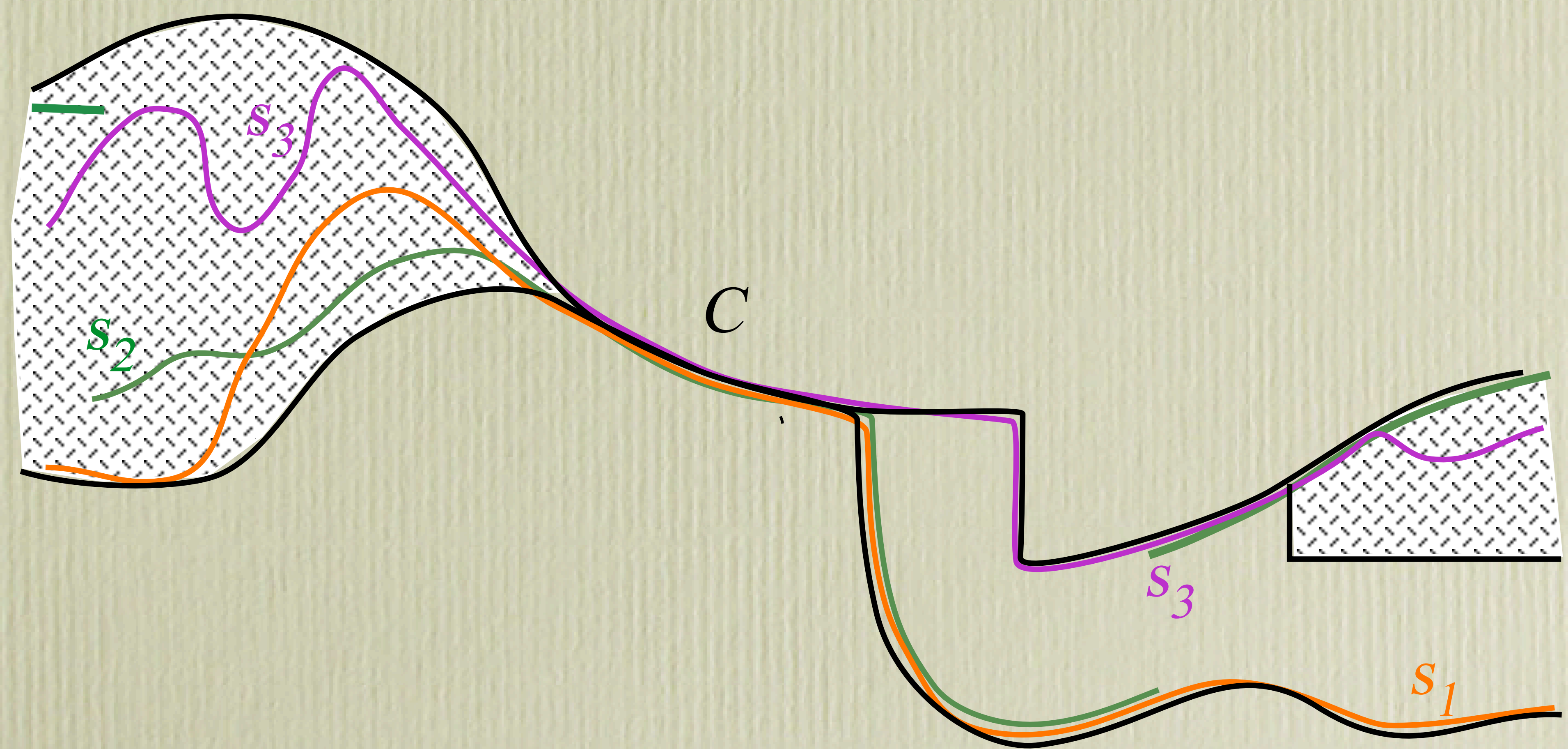
- Graph measurability

$$(\Xi, \mathcal{A}) \ P\text{-complete for some } P,$$

(negligible sets are P-measurable)

$$C \text{ random set } \Leftrightarrow \text{ gph } C \ \mathcal{A} \otimes \mathcal{B}_n\text{-measurable}$$

Castaing Representation



Convergence of Random Elements

(review)

$$\boldsymbol{\xi} : (\Omega, \mathcal{F}, \mu) \to (\Xi, \mathcal{A}, P), \quad \boldsymbol{\xi}^\nu \xrightarrow{\star} \boldsymbol{\xi}$$

- a.s. (almost sure) convergence:

$$P\big\{\xi \,\big|\, \lim_\nu \boldsymbol{\xi}^\nu(\omega) = \xi \neq \boldsymbol{\xi}(\omega),\, \omega \in \Omega\big\} = 0$$

- convergence in probability:

$$P(|\boldsymbol{\xi}^\nu - \boldsymbol{\xi}| > \varepsilon) \to 0 \text{ for all } \varepsilon > 0$$

- convergence in distribution: $P^\nu \xrightarrow{\mathcal{D}} P$

Outer/Inner Limits

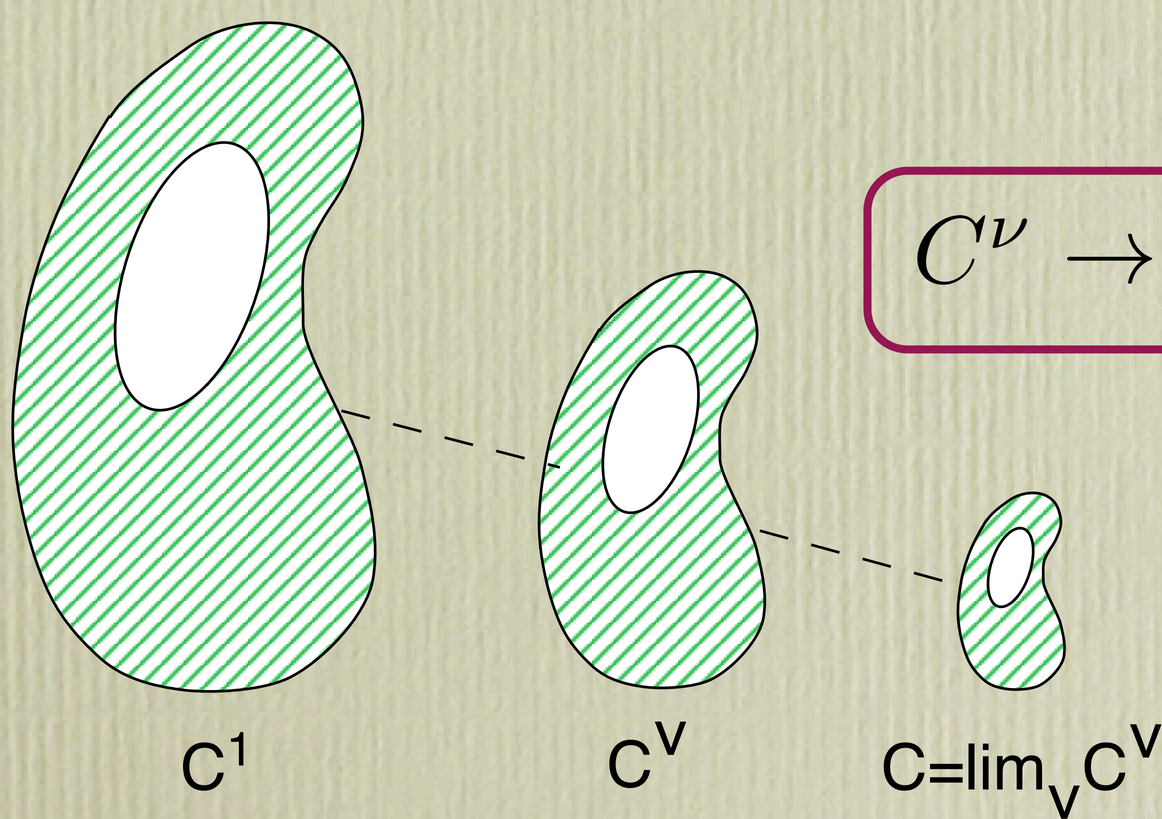
(review)

outer limit: $\mathrm{Lo}_\nu C^\nu = \left\{ x \in \text{ cluster-points}\{x^\nu\},\, x^\nu \in C^\nu \right\} = \mathrm{Ls}_\nu C^\nu$

inner limit: $\mathrm{Li}_\nu C^\nu = \left\{ x = \lim_\nu x^\nu, x^\nu \in C^\nu \subset \mathbb{R}^n \right\} \subset \mathrm{Lo}_\nu C^\nu$

limit: $C^\nu \to C$ if $C = \mathrm{Li}_\nu C^\nu = \mathrm{Lo}_\nu C^\nu$ (Painlevé - Kuratowski)

All limit sets are closed



$$\boxed{C^\nu \to C \iff d\!l(C^\nu, C) \to 0}$$

Characterizing a.s. convergence

$\Big\{C; C^\nu : \Xi \rightrightarrows \mathbb{R}^n,\ \nu \in \mathbb{N}\Big\}$ random closed sets. Then,

1. $C^\nu \to C$ *a.s.*, $d\!\!l(C^\nu, C) \to 0$ *a.s.*, $\mathrm{Lo}_\nu(C^\nu) \subset C \subset \mathrm{Li}_\nu(C^\nu)$ *a.s.*,

2. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$, $d(x, C^\nu(\xi)) \to d(x, C(\xi))$,

3. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$,

$$\lim_{\rho \nearrow \infty} \mathrm{Lo}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\big) \subset C(\xi) \subset \lim_{\rho \nearrow \infty} \mathrm{Li}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\big).$$

"Proof 1. $\Leftrightarrow$ 2."

$C^\nu \to C \iff \forall x \in \mathbb{R}^n,\ d(x, C^\nu) \to d(x, C)$ provided $E = \mathbb{R}^n$.

$C^\nu \to C$ if and only if the hit-miss criterion is satisfied

C hits $\mathbb{B}^o(x,\rho)$ then C^ν hits $\mathbb{B}^o(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \subset \mathrm{Li}_\nu C^\nu \iff d(x,C) \geq \mathrm{limsup}_\nu\, d(x, C^\nu), \forall x$

C misses $\mathbb{B}(x,\rho)$ then C^ν misses $\mathbb{B}(x,\rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \supset \mathrm{Lo}_\nu C^\nu \iff d(x,C) \geq \mathrm{liminf}_\nu\, d(x, C^\nu), \forall x$

a.s.-Convergence via Castaing Representations

- $\left\{C^\nu : \Xi \rightrightarrows \mathbb{R}^d, \nu \in \mathbb{N}\right\}$ random closed sets

- $C^\nu \to C$ P-a.s. and dom $C^\nu =$ dom C. Then,

  $\exists$ Castaing representations of $C^\nu \to$ a Castaing representation of C

  If $s : \Xi \to E$ is a measurable selection of C, then

  $\exists\, s^\nu : \Xi \to E$ selections of C^ν converging P-a.s. to s

- ('Egorov's Theorem': $C^\nu \to C$ μ-a.s. $\Leftrightarrow C^\nu \to C$ almost uniformly)

Building Castaing representations

$C : \Xi \rightrightarrows \mathbb{R}^n$, a random closed set. Let

$$A = \left\{ a_k = (a_k^1, \ldots, a_k^n, a_k^{n+1}) \,\middle|\, a_k^i \in \mathbb{Q}^n \text{ \& aff. independent } \right\}$$

for $\emptyset \neq D = D^0$ closed, define $\operatorname{prj}_D a_k = \operatorname{prj}_{D^n} a_k^{n+1}$
where $D^l = \operatorname{prj}_{D^{l-1}} a_k^l$ for $l = 1, \ldots, n$

$\operatorname{prj}_D a_k$ is a singleton: intersection of n+1 "aff. independent" spheres.
Moreover, $\left\{ \operatorname{prj}_D a_k, a_k \in A \right\}$ also dense in D

$s_k : \Xi \to \mathbb{R}^n$ with $s_k(\xi) = \operatorname{prj}_{C(\xi)} a_k$ is a measurable selection of C

$\square$ When D is a random closed set, so is $\xi \mapsto \operatorname{prj}_{D(\xi)} a$, $a \in \mathbb{R}^n$
repeat the argument $n+1$ times to obtain s_k measurable. $\square$

Converging Castaing representations

$C^\nu : \Xi \rightrightarrows \mathbb{R}^n$ random closed sets converging P-$a.s.$ to C, $\operatorname{dom} C^\nu = \operatorname{dom} C$.
Then, $\exists \{ s_k^\nu, k \in \mathbb{N} \}$ Castaing representations of C^ν converging for each k to a Castaing representation $\{ s_k, k \in \mathbb{N} \}$ of C.

$\square$ All Castaing representations are built via our earlier "projections".
Then, $\forall \xi \in \Xi_1, s_k^\nu(\xi) \to s_k(\xi)$, $P(\Xi_1) = 1$ the set of $a.s.$-convergence.
Since, P-$a.s.$ convergence of $C^\nu \to C \Longrightarrow$ (rely on 2. earlier)

$$d(a_k^1, s_k^\nu(\xi)) = d(a_k^1, C^\nu(\xi)) \to d(a_k^1, C(\xi)) = d(a_k^1, s_k(\xi)), \forall \xi \in \Xi_1. \quad \square$$

(a) Convergence of Castaing representations $\not\Rightarrow$ convergence of random sets!
(b) v meas-selection of $C \Rightarrow \exists v^\nu$ meas-selection of C^ν converging $a.s.$ to v.

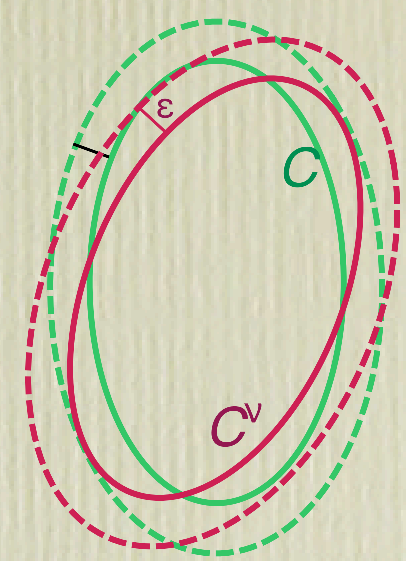
Convergence in probability

Let $\varepsilon^o C = \{ x \in \mathbb{R}^m \,|\, d(x,C) < \varepsilon \}$, C^ν, C random sets

$\Delta_{\varepsilon,\nu} = \left(C^\nu \setminus \varepsilon^o C \right) \cup \left(C \setminus \varepsilon^o C^\nu \right)$

μ-a.s. convergence: $\mu\left\{ \xi \,\middle|\, C^\nu(\xi) \to C(\xi) \right\} = 1$

in probability: $P\left[\Delta^{-1}_{\varepsilon,\nu}(K) \right] \to 0, \forall \varepsilon > 0, K \in \mathcal{K} =$ cpct-sets



C^ν converges to C in probability

$\Leftrightarrow P(dl(C^\nu, C) > \varepsilon) \to 0$ for all $\varepsilon > 0$

$\Leftrightarrow$ every subsequence of $\{C^\nu\}_{\nu \in \mathbb{N}}$

contains a sub-subsequence converging μ-a.s to C

i.e., in probability $\Rightarrow$ in distribution $\left[\int h(\xi)\, dl(C^\nu(\xi), C(\xi))\, P(d\xi) \to 0 \right]$

$P^\nu \xrightarrow{\mathcal{D}} P \;\sim$ distribution fcns converge

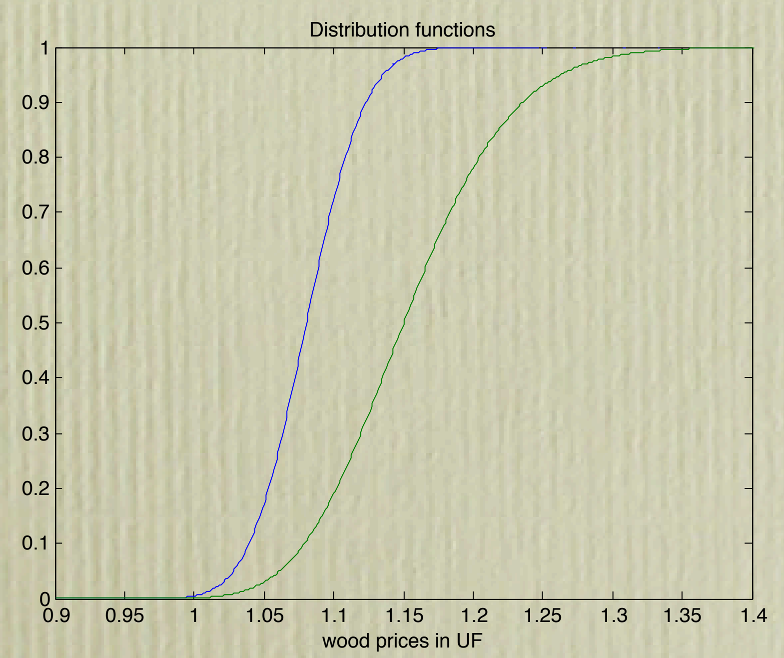
(review)

P^ν, P defined on $(\mathbb{R}, \mathcal{B})$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\ \forall h$ continuous

$F^\nu(z) = P^\nu\big((-\infty, z)\big), \quad F(z) = P\big((-\infty, z)\big)$, cumulative distributions

$P^\nu \xrightarrow{\mathcal{D}} P \iff F^\nu \xrightarrow{p} F$ on cont $F = \{$ all continuity points of $F\}$



$$\boxed{P^\nu \xrightarrow{\mathcal{D}} P \iff F^\nu \xrightarrow{h} F}$$

Friday, June 21, 13

$P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

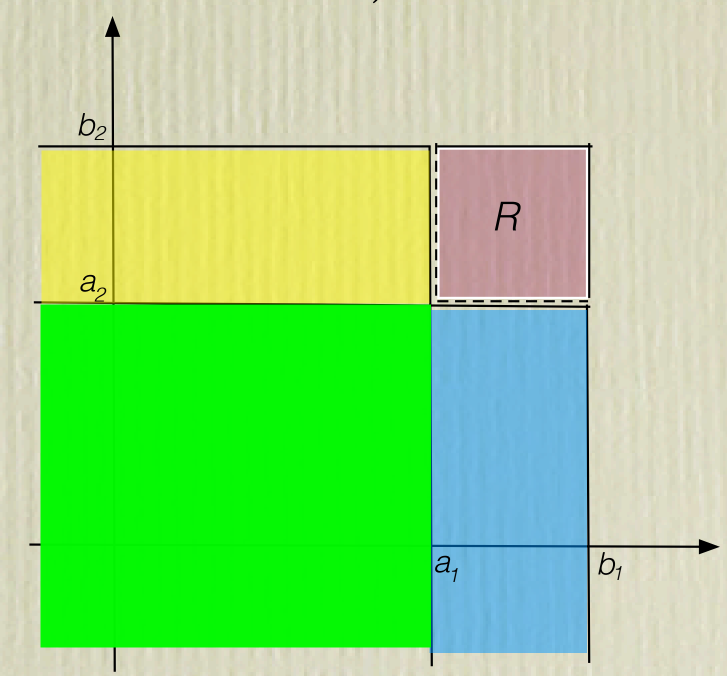
(review)

P^ν, P defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\,P^\nu(d\xi) \to \int h(\xi)\,P(d\xi)\;\; \forall h$ continuous

$F^\nu(z) = P^\nu(\xi_i \le z_i, i = 1, \dots, n), \quad F(z) = P(\xi_i \le z_i, i = 1, \dots, n)$

1. $z \le \tilde{z} \implies F(z) \le F(\tilde{z})$ "increasing"
2. $\lim_{z\to\infty} F(z) = 1,\;\; \lim_{z_j \to -\infty} F(z) \to 0,$
3. F is usc (upper sc) $\limsup_{z' \to z} F(z') \le F(z),$



4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of R
   $\forall R \subset \mathbb{R}^n,\; P(\boldsymbol{\xi} \in R) = \sum_{v \in V} \mathrm{sgn}(v) F(v), \quad \mathrm{sgn}(v \in V) = (-1)^{\#a \text{ in } v}$

$$\boxed{P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F}$$

Distribution of a random set

Borel σ-field: $\mathcal{B} = \sigma\text{-}\{\mathcal{F}^K \mid K \text{ compact}\}$ or $\sigma\text{-}\{\mathcal{F}_O \mid O \text{ open}\}\ldots$

Distribution $(P, \mathcal{B})$ regular, $\quad \mathcal{K}$ compact subsets E

determined by values on $\{\mathcal{F}^K \mid K \in \mathcal{K}\}$ or $\{\mathcal{F}_K \mid K \in \mathcal{K}\}$

Distribution function (Choquet capacity):

(2) $T : \mathcal{K} \to [0,1], T(\varnothing) = 0$ and $\forall \{K^\nu, \nu \in \{0\} \cup \mathbb{N}\} \subset \mathcal{K}$:

a) $T(K^\nu) \searrow T(K)$ when $K^\nu \searrow K \qquad$ (~ usc on $\mathbb{R}^n$) *(1,3)*

(4) b) $\{D_\nu : \mathcal{K} \to [0,1]\}_{\nu \in \mathbb{N}}$ where $D_0(K^0) = 1 - T(K^0)$

$D_1(K^0; K^1) = D_0(K^0) - D_0(K^0 \cup K^1)$ and for $\nu = 2, \ldots$

$D_\nu(K^0; K^1, \ldots, K^\nu) = D_{\nu-1}(K^0; K^1, \ldots, K^{\nu-1}) - D_{\nu-1}(K^0 \cup K^\nu; K^1, \ldots, K^{\nu-1})$

(~ rectangle condition on $\mathbb{R}^n$)

Existence-Uniqueness of T

P on $\mathcal{B}$ determines a unique distribution function T on $\mathcal{K}$

$$T(K) = P(\mathcal{F}_K)$$

$$D_\nu(K^0; K^1, \ldots, K^\nu) = P(\mathcal{F}^{K^0} \cap \mathcal{F}_{K^1} \cap \cdots \cap \mathcal{F}_{K^\nu})$$

T on $\mathcal{K}$ determines a unique probability measure P.

Proof. via Choquet Capacity Theorem *(Matheron)*

(refined) via probabilistic arguments *(Salinetti-Wets)*

$C : \Xi \rightrightarrows \mathbb{R}^d$ a random closed set

$(P, \mathcal{B})$ induced probability measure:

$$P(\mathcal{F}_G) = P\left[C^{-1}(G)\right] \quad \forall G \in \mathcal{B}, \quad T(K) = P\left[C^{-1}(K)\right] \quad \forall K \in \mathcal{K}$$

Convergence in Distribution

random sets C^ν converge in distribution to C when

induced P^ν narrow-converge to $P: P^\nu \to_n P \; = P^\nu \overset{\mathcal{D}}{\to} \; P$

$\Leftrightarrow T^\nu \to_p T$ on $\mathcal{K}_{T\text{-cont}}$ (convergence of distribution functions)

$\mathcal{K}_{T\text{-cont}}$?

a) $\forall C^\nu, \nu \in N, \exists$ converging subsequence (pre-compact)

b) $K^\nu \nearrow K =$ cl $\bigcup_\nu K^\nu$ regularly if int $K \subset \bigcup_\nu K^\nu$

c) distribution (fcn) continuity: $\lim_\nu T(K^\nu) = T(\text{cl} \bigcup_\nu K^\nu)$

d) convergence $T^\nu \to_p T$ on C_T continuity set $\Rightarrow P^\nu \to_n P$

e) $P^\nu \to_n P \Leftrightarrow T^\nu \to_p T$ on $C_T^{ub} = C_T \cap \mathcal{K}^{ub}$

$\mathcal{K}^{ub} =$ finite union of rational ball, positive radius

f) $\varepsilon \mapsto T(K + \varepsilon\mathbb{B})$: countable number of discontinuities

a detour about rates

$T^\nu \to_p T$ on $C_T \Leftrightarrow P^\nu \to_n P$ (Polish space: E,d)

P^ν, P defined on $\mathcal{B}$

probability sc-measures on cl-sets(E): λ

(i) $\lambda \geq 0$, (ii) $\lambda \nearrow$ $\lambda(C^1) \leq \lambda(C^2)$ if $C^1 \subset C^2$

(iii) λ is τ_f-usc on cl-sets(E), (iv) $\lambda(\emptyset) = 0, \lambda(E) = 1$

(v) λ modular: $\lambda(C^1) + \lambda(C^2) = \lambda(C^1 \cup C^2) + \lambda(C^1 \cap C^2)$

P and $\lambda = P_{\text{cl-sets}}$ define each other uniquely (E complete $\Rightarrow$ tight)

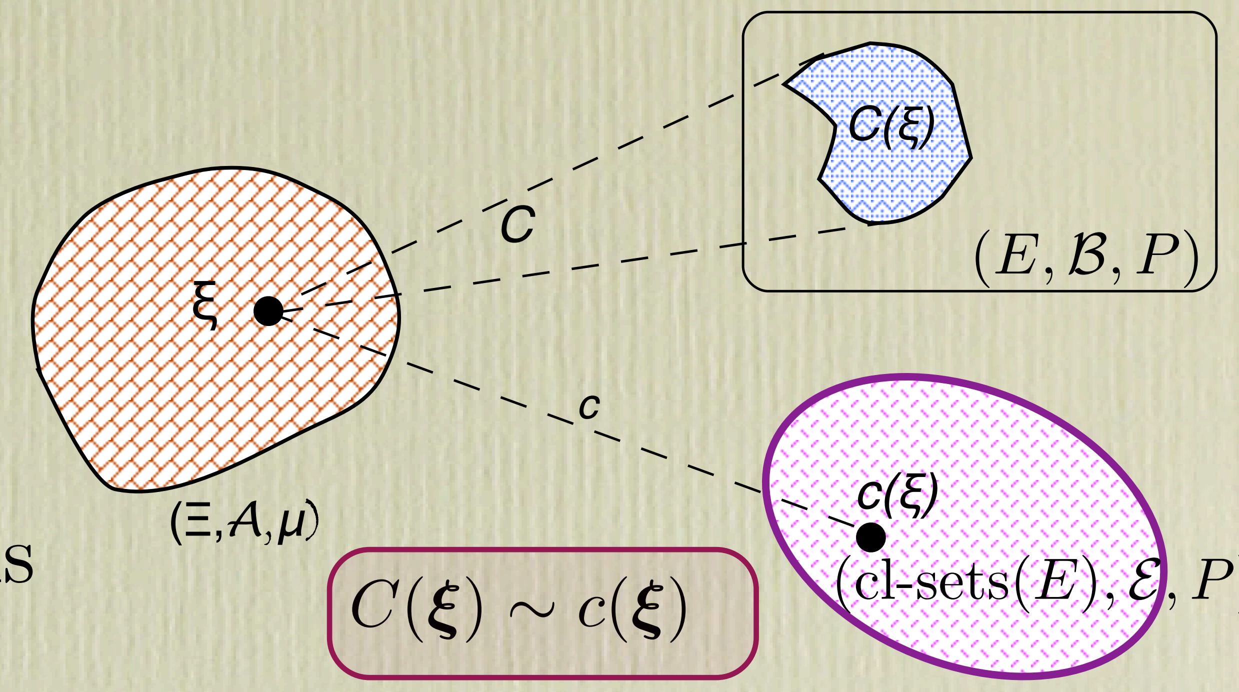
$\{P^\nu, \nu \in \mathbb{N}\}$ tight: $P^\nu \to_n P \Leftrightarrow \lambda^\nu \to_h \lambda$ ($\sim -\lambda^\nu \to_e -\lambda$) on cl-sets($E$)

tightness ~ equi-usc of $\{\lambda^\nu\}_{\nu \in \mathbb{N}}$ at $\emptyset$

rates: $dl(\lambda^\nu, \lambda) \to 0$ (for $\mathbb{R}$-valued r.v., "~" Skorohod distance)

Random sets

(review)



C 'covered' by countable selections
Castaing represnetation

a.s convergence: $P\{\xi \,|\, d\!l(C^\nu(\xi), C(\xi)) \to 0\} = 0$

$\Rightarrow$ in probability: $\forall \varepsilon > 0,\ P\{\xi \,|\, d\!l(C^\nu(\xi), C(\xi)) > \varepsilon\} \to 0$

$\Rightarrow$ in distribution $T : \text{cpct-sets}(E) \to [0,1],\ T(\emptyset) = 0,$
(a) $T(K^\nu) \searrow T(K)$ for $K^\nu) \searrow K$, (b) 'rectangle cond'n'
$P^\nu \overset{\mathcal{D}}{\to} P \iff T^\nu \to T$ on cpct-sets$(\mathbb{R}^n)$
or, even, on finite union of closed rational balls.

Expectation

Artstein, Vitale, Hart, Wets,
Cressis, Hiai, Weyl, ...

“Simple” random sets

$C : \Xi \rightrightarrows \mathbb{R}^n$ is a *simple* random set if $\operatorname{rge} C$ is finite.
C is a closed random set $\iff$ $C = P$-$a.s.$ limit of simple random sets.

$\square$ $\Leftarrow$: the limit of a sequence of random sets is a random set
$\Rightarrow$: let $C^\nu = C \cap \nu\mathbb{B}$, unif. bounded closed random set, $C = \operatorname{Lm}_\nu C^\nu$
build (via "prj") Castaing representations $\{r_k^\nu\}_{k\in\mathbb{N}}$ of the C^ν
let $\{s_k^\nu\}_{k\in\mathbb{N}'} = \bigcup_{\upsilon\le\nu} \{r_k^\upsilon\}_{\upsilon\in\mathbb{N}}$, also Castaing for C^ν
$D_k^\nu = \bigcup_{j\le k} s_j^\nu$ $d\!l$-converge uniformly to C^ν as $k\to\infty$
since each $s_k^\nu = \lim_{l\to\infty} s_{kl}^\nu$ uniformly, s_{kl}^ν simple random variables
$\Delta_{kl}^\nu = \bigcup_{j\le k} s_{jl}^\nu$ is a simple random set, $C(\xi) = \operatorname{Lm}_\nu \operatorname{Lm}_k \operatorname{Lm}_l \Delta_{kl}^\nu(\xi)$

$$\Delta_{kl}^\nu \xrightarrow{u} D_k^\nu \xrightarrow{u} C^\nu \text{ allows diagonalization to find } \Delta_{k^\nu l^\nu}^\nu \to C. \quad \square$$

Sierpiński-Lyapunov Theorems

$(\Xi, \mathcal{A})$ a measure space

Sierpiński (1922). Suppose P is an atomless probability measure.
Given $A_0, A_1 \in \mathcal{A}$ with $0 \leq P(A_0) \leq P(A_1) \leq 1$, then
$\forall \lambda \in [0,1]$, $\exists$ $A_\lambda \in \mathcal{A}$ such that $P(A_\lambda) = (1-\lambda)P(A_0) + \lambda P(A_1)$.
In particular, it implies $\forall\, \lambda \in [0,1]$, $\exists$ $A \in \mathcal{A}$ such that $P(A) = \lambda$;
choose $A_0 = \emptyset$ and $A_1 = \Xi$.

Lyapunov (1940) $\mu : \mathcal{A} \to \mathbb{R}^n$ atomless, σ-additive measure.
For $A \in \mathcal{A}$, define $\operatorname{rge} \mu(A) = \left\{ \mu(B) \,\middle|\, B \subset A \cap \mathcal{A} \right\}$. Then,
$\operatorname{rge} \mu(\Xi) \subset \mathbb{R}^n$ is convex and if μ is also bounded, it's compact.

Expectation: simple random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a simple random set, i.e., $\operatorname{rge} C = \left\{ z^k \in \mathbb{R}^n \,\middle|\, k \in K, |K| \text{ finite } \right\}$
Given $\bar{r}, \bar{s} \in EC = \mathbb{E}\{C(\boldsymbol{\xi})\} \quad \Longrightarrow$
$\exists$ simple selections $r, s : \Xi \to \mathbb{R}^n$ with $\mathbb{E}\{r(\boldsymbol{\xi})\} = \bar{r}, \mathbb{E}\{s(\boldsymbol{\xi})\} = \bar{s}$.
Let $\lambda \in [0,1]$. Define $v : \Xi \to \mathbb{R}^n$ as follows:

1. partition Ξ into subsets $A_=$ and $\mathcal{A}_{\neq}$
2. $A_= = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = s(\xi) \right\} \in \mathcal{A}$
3. $A = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = z_k, s(\xi) = z_l, k \neq l \right\} \in \mathcal{A}_{\neq}$, a finite collection
4. split each $A \in \mathcal{A}_{\neq}$, $P(A_r) = \lambda P(A)$ & $A_s = A \setminus A_r$ (Sierpiński)

$$\text{set } v(\xi) = \begin{cases} r(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_r \,\cup\, A_= \\ s(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_s \end{cases}$$

then $\bar{v} = \mathbb{E}\{v(\xi)\} = \lambda \bar{r} + (1-\lambda)\bar{s} \quad \Longrightarrow \quad EC$ convex.
Clearly EC is bounded and it's easy to show it's also closed $\Longrightarrow$ compact.

Expectation of random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a closed random set

$\Longrightarrow C = P$-$a.s.$ limit of simple random sets,

say $C^\nu \underset{a.s.}{\to} C$ with $C^\nu \nearrow$ without loss of generality

$EC^\nu = \mathbb{E}\{C^\nu(\boldsymbol{\xi})\} \nearrow$ are convex, compact $\Rightarrow$

$EC = \mathbb{E}\{C(\boldsymbol{\xi})\} = \bigcup_\nu EC^\nu$

$\Longrightarrow EC$ convex

$\Longrightarrow EC$ closed if C is integrably bounded

$\Longrightarrow$ compact if $\operatorname{rge} C$ is bounded

Random set: Expectation

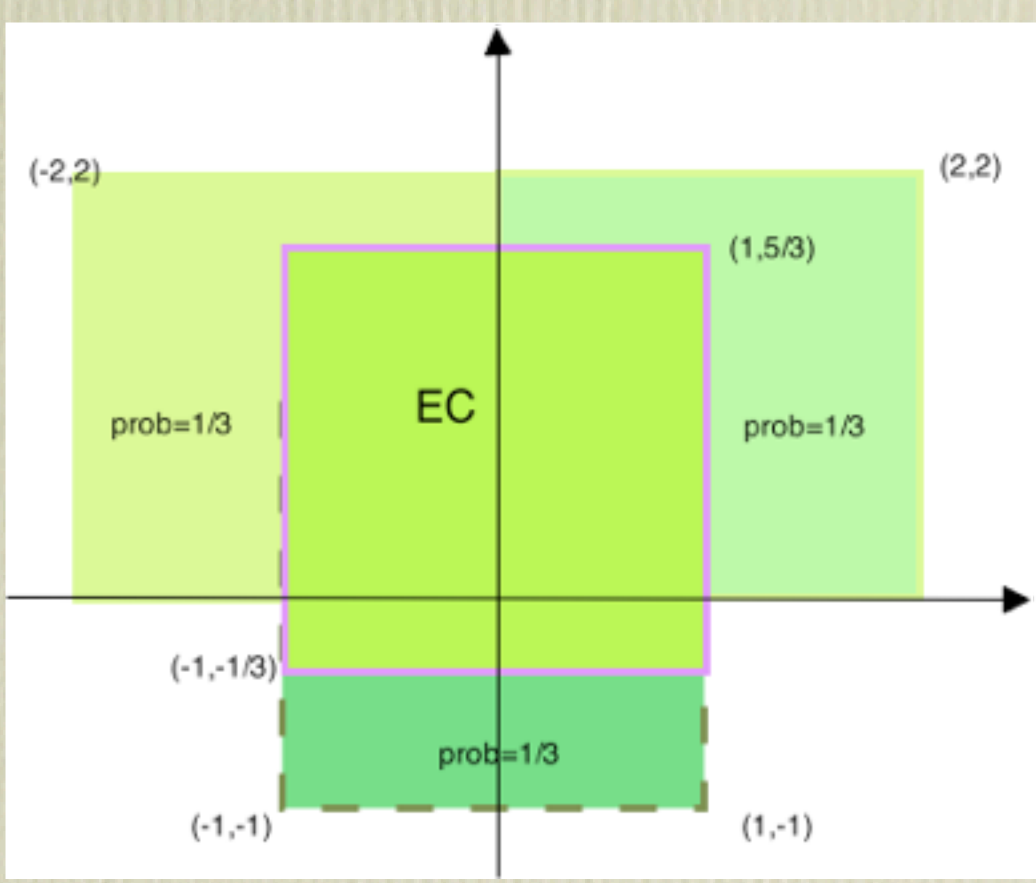
$$EC = \mathbb{E}\{C(\xi)\} = \left\{ \int_{\Xi} s(\xi)\,P(d\xi) \,\middle\|\, s(\bullet)\ P\text{-summable selection} \right\}$$

..not necessarily closed even when C is closed-valued

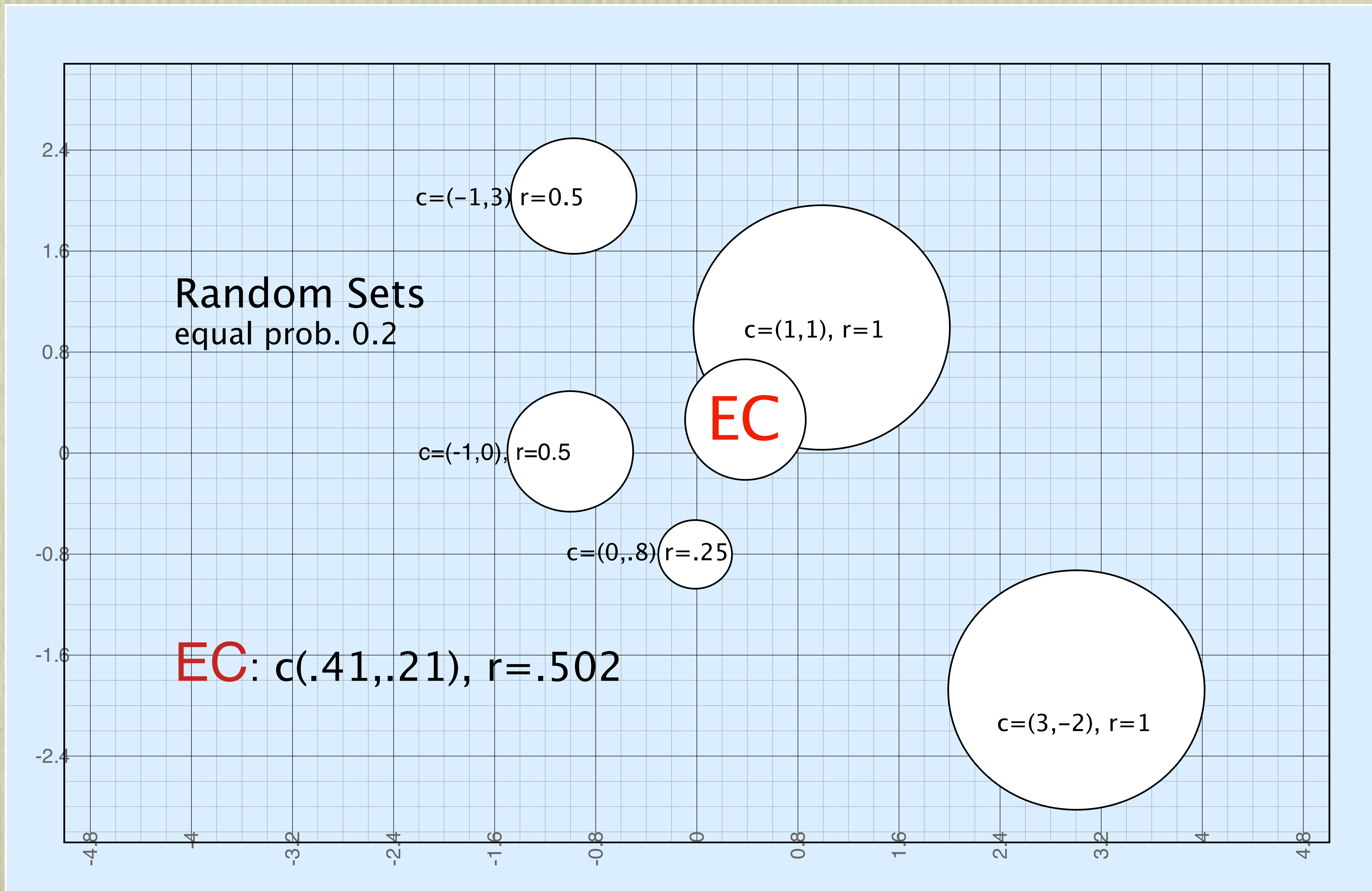
Convexity:

C P-atom convex $\Rightarrow EC$ is convex

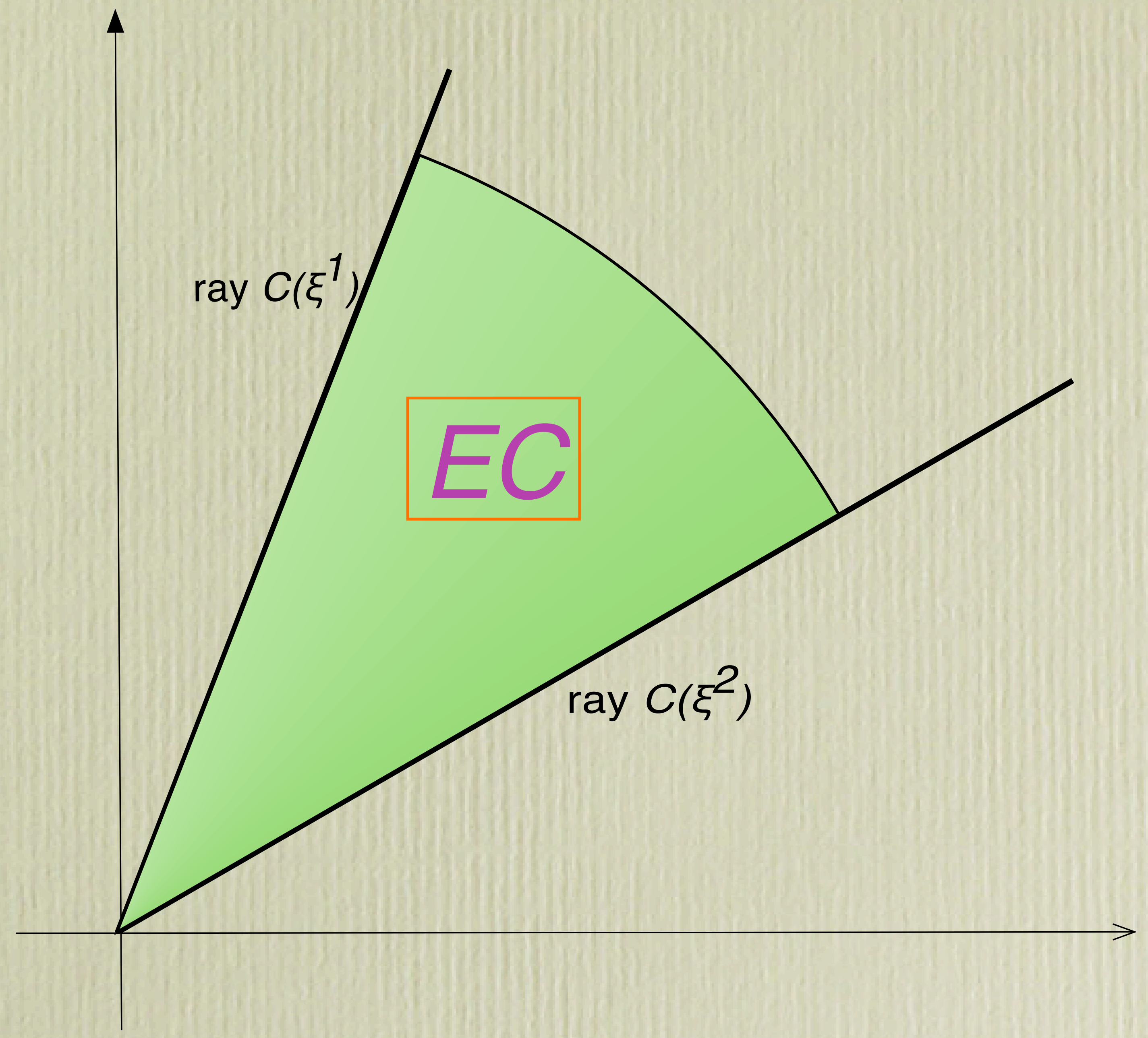
(certainly when P is atomless).



Expectation: Bounded Random Sets



Expectation: Unbounded Random Sets



Some properties: $\mathbb{E}\{C(\boldsymbol{\xi})\}$

- measure P atomless, then $EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ is convex *(Richter, Lyapounov,...)*
- P is P-atom convex $\Longrightarrow$ EC is convex; [an atom contains no (measurable) subset of positive probability]
- C a random set, $\emptyset \neq EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ contains no line, then

  $$\operatorname{con} EC = \mathbb{E}\{\operatorname{con} X(\boldsymbol{\xi})\}$$

  this essentially requires that $C(\xi) \subset$ a pointed cone
- in general, the expectation of a (closed-valued) random set is *not* closed
- if $|C| = \mathbb{E}\big\{\sup\left[\,|s(\xi)|\,\big|\, s(\xi) \in C(\xi)\,\right]\big\} < \infty$ then EC is closed; C is then *integrably bounded.*

LLN: Random Sets (Artstein & Hart)

$C : \Xi \rightrightarrows \mathbb{R}^m$ measurable, $\{\xi^\nu, \nu \in \mathbb{N}\}$ iid Ξ-valued random variables

$C(\xi^\nu)$ iid random sets (i.e. induced P^ν independent and identical)

$EC = \mathbb{E}\{C(\bullet)\} = \left\{ \int_\Xi x(\xi)\mu(d\xi) \,\middle|\, x\ \mu\text{-summable } C(\xi)\text{-selection} \right\}$

independence $\Rightarrow$ all (measurable) selections are independent

$\{C(\xi^\nu) : \Xi \quad \mathbb{R}^m\ \nu \in \mathbb{N}\}$ iid with $EC \neq \emptyset$. Then, with

$$C^\nu(\xi^\infty) = \nu^{-1}\left(\sum_{k=1}^{\nu} C(\xi^k)\right) \to \bar{C} = \text{cl con } EC \quad \mu^\infty\text{-a.s.}$$

$\text{Lo}_\nu C^\nu(\xi^\infty) \subset \bar{C} \iff \limsup_\nu \sigma_{C^\nu} \leq \sigma_{\bar{C}}$ support functions

$\text{Li}_\nu C^\nu(\xi^\infty) \supset \bar{C}$ relies on LLN for (vector-valued) selections

Random mappings

$S : \Xi \times E \rightrightarrows \mathbb{R}^m, \quad E \subset \mathbb{R}^n$

$\mathcal{A} \otimes \mathcal{B}^n$-jointly measurable: $S^{-1}(O) \in \mathcal{A} \otimes \mathcal{B}^n$, O open

$\Rightarrow \forall\, x : \xi \mapsto S(\xi, x)$ a random set

random closed set when S is closed-valued

$ES : E \rightrightarrows \mathbb{R}^m$ with $ES(x) = \mathbb{E}\{S(\boldsymbol{\xi}, x)\}$ expected mapping

ES convex-valued when $\xi \mapsto S(\xi, \cdot)$ P-atom convex

Law of Large Numbers for random sets

applies pointwise

Sample Average Approximation (SAA)

stochastic variational problem: $\bar{S}(x) = \mathbb{E}\{S(\boldsymbol{\xi},x)\} \ni 0$

$S : \Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ random set-valued mapping

$\boldsymbol{\xi}$ random vector with values $\xi \in \Xi \subset \mathbb{R}^N$

solution (a 'stationary point') $\quad \bar{x} \in \bar{S}^{-1}(0)$

—o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o

sample $\overset{\rightarrow v}{\xi} = (\xi^1, \ldots, \xi^v)$ of $\boldsymbol{\xi}$

$\frac{1}{v}\left(\sum_{k=1}^{v} S(\xi^k, x)\right) = S^v(\overset{\rightarrow v}{\xi}, x) \ni 0$, approximating system?

i.e., $\left(S^v\right)^{-1}(0) \overset{?}{\rightarrow} \bar{S}^{-1}(0)$ a.s.

A few examples ...
(from the MOPEC fanily)

Stochastic Optimization

min $Ef(x) = \mathbb{E}\{f(\boldsymbol{\xi},x)\}$ --stationary point-- $\partial Ef(x) \ni 0$

assuming $\mathbb{E}\{\partial f(\boldsymbol{\xi},x)\} = \partial Ef(x)$ (not generally correct)

could $\partial Ef(x) \ni 0$ get replaced (?) by

$$\nu^{-1}\left(\sum_{k=1}^{\nu} \partial f(\xi^k,x)\right) \ni 0 \text{ from sample } \overset{\rightarrow\nu}{\xi}$$

dom $Ef \approx \bigcap_{\xi\in\Xi} \operatorname{dom} f(\xi,\cdot)$,

unless $\xi \mapsto \operatorname{dom} f(\xi,\cdot)$ constant,

interchanging $\mathbb{E}$ & ∂ is only exceptionally valid

Stochastic V.I. (variational inequality)

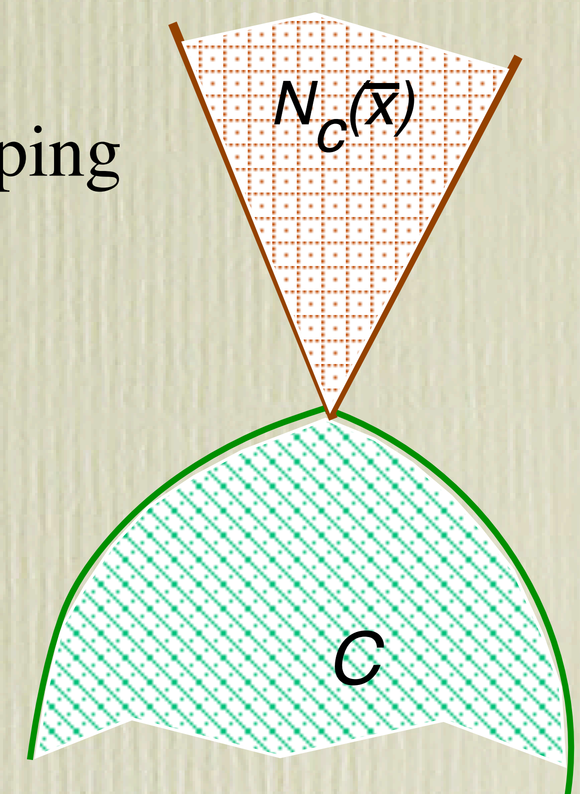
Network flow equilibrium with stochastic demand and link capacities
Economic equilibrium in a stochastic environment

$\xi = (\xi^1, \xi^2, \ldots)$, $G^\nu(\cdot, x)$ σ-$(\xi^1, \ldots \xi^\nu)$ measurable

$-G^\nu(\xi, x) \in N_C(x)$, C compact, convex

$N_{C(x)} + G^\nu(\xi, x) = S^\nu(x) \ni 0$, S^ν closed set-valued mapping

$G^\nu(\xi, \cdot) \to^? G(\xi, \cdot)$



$x^\nu(\xi)$ solution of $-G^\nu(\xi, x) \in N_C(x)$ for sample $\xi \approx \vec{\xi}^{\,\nu}$

does $x^\nu(\xi) \to$ a solution of $-G(\xi, x) \in N_C(x)$? a.s.

what if C depends on (ξ, ν) : sequence of random sets $C^\nu(\xi)$?

Walras Equilibrium: stochastic environment

$$(c_a^1, y_a, c_{a,\xi}^2) = \arg\max_{x^1, y \in \mathbb{R}^L, x_\cdot^2 \in \mathcal{M}} u_a^1(x^1) + \mathbb{E}^a \left\{ u_a^2(\boldsymbol{\xi}, x^2(\boldsymbol{\xi})) \right\}$$

$$\text{such that } \left\langle p^1, x_a^1 + T_a^1 y \right\rangle \leq \left\langle p^1, e_a^1 \right\rangle$$

$$\left\langle p_\xi^2, x_{a,\xi}^2 \right\rangle \leq \left\langle p_\xi^2, e_{a,\xi}^2 + T_{a,\xi}^2 y \right\rangle, \ \forall \xi \in \Xi$$

$$x_a^1 \in X_a^1, \ x_{a,\xi}^2 \in X_{a,\xi}^2, \quad \forall \xi \in \Xi$$

$\mathbb{E}^a \left\{ \bullet \right\}$ expecttion with respect to a-beliefs, Ξ finite support

2-stage stochastic programs with recourse

solution procedures & approximation theory "well-estblished"

$T_a^1, T_{a,\xi}^2$: input-output matrices (production, investments)

$e_a^1 \in \operatorname{int} X_a^1, \ e_{a,\xi}^2 \in \operatorname{int} X_{a,\xi}^2$ for all ξ

Market Clearing -Equilibrium

excess supply: agent-a: $\left(c_a^1, y_a^1, \left\{ c_{a,\xi}^2 \right\}_{\xi\in\Xi} \right)$

$$\sum_{a\in\mathcal{A}} \left(e_a^1 - (c_a^1 + T_a^1 y_a) \right) = s^1\left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right) \geq 0$$

$$\forall \xi,\ \sum_{a\in\mathcal{A}} \left((e_{a,\xi}^2 + T_{a,\xi}^2) - c_{a,\xi}^2 \right) = s_\xi^2\left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right) \geq 0$$

Variational inequality: $-G(p,(x_a),(\lambda_a)) \in N_D(p,(x_a),(\lambda_a))$,

$$p = \left(p^1, \{p_\xi^2\}_{\xi\in\Xi} \right),\ x = \left(x^1, \{x_\xi^2\}_{\xi\in\Xi} \right),\ \lambda = \left(\lambda^1, \{\lambda_\xi^2\}_{\xi\in\Xi} \right)$$

$$S(\xi,(p,x,\lambda)) = G(\xi,(x,p,\lambda)) + N_{D(\xi)}(p,x,\lambda)),$$

$$\mathbb{E}\left\{ S(\xi,(p,x,\lambda)) \right\} \ni 0$$

Sample Average Approximations

$\boldsymbol{\xi} = (\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \ldots)$ iid, sample $\vec{\xi}^{\nu} = (\xi^1, \ldots, \xi^\nu)$

SAA-mapping: given $S : \Xi \times E \rightrightarrows \mathbb{R}^m$ random mapping

$$S^\nu : \Xi^\infty \times E \rightrightarrows \mathbb{R}^m \text{ with}$$

$$\forall \xi \in \Xi^\infty, x \in E : S^\nu(\xi, x) = \frac{1}{\nu} \sum_{k=1}^{\nu} S(\xi^k, x) = S^\nu(\vec{\xi}^{\nu}, x)$$

S^ν depends only on $\vec{\xi}^{\nu}$

SAA-mappings S^ν are random mappings

not necessarily closed-valued

(the sum of closed sets is not necessarily closed)

Pointwise limits: SAA-mappings

$ES(x) = \mathbb{E}\{S(\boldsymbol{\xi},x)\} \neq \emptyset$, then

$\forall x \in X:\ S^{\nu}(\xi,x) \to \mathrm{cl\,con}\, ES(x) =: \bar{S}(x)\ \ \mu^{\infty}$-a.s.

If $S(\cdot,x)$ is P-atom convex, $S^{\nu}(\xi,\cdot) \to \mathrm{cl}\, ES(x) =: \bar{S}(x)\ \ \mu^{\infty}$-a.s.

Proof: LLN for random sets. □

So far ...

➪ *generalized equations*

$S : \Xi \times D \rightrightarrows E$, set-valued $S(\xi, x) \subset E$, inclusion $\mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

iid-sample $\vec{\xi}^{\,\nu} = \xi^1, \dots, \xi^\nu$ and $x \mapsto S(\xi, x)$ osc

SAA-mapping $S^\nu : \Xi^\infty \times D \rightrightarrows E$, random osc mappings

$S^\nu(\xi, x) = \frac{1}{\nu} \sum_{k=1}^{\nu} S(\xi^k, x) \approx S^\nu(\vec{\xi}^{\,\nu}, x)$, $\forall \xi \in \Xi^\infty$

$\forall x \in D$, $S(\cdot, x)$, closed random set,

let $\bar{S} = \operatorname{cl} \operatorname{con} ES$, $\quad ES(x) = \mathbb{E}\{S(x, \boldsymbol{\xi})\}$

Artstein-Hart LLN applies: $S^\nu \xrightarrow{p} \bar{S}$ $a.s.$ when $E = \mathbb{R}^m$

but $\xrightarrow{p}$ $\not\Rightarrow$ $(S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

recall: $\bar{S}(x) = \operatorname{cl} ES(x)$ when P-atom convex, $ES(x)$ closed if $\xi \mapsto S(\xi, x)$ is integrably bounded and compact if $\operatorname{rge} S(\cdot, x)$ is bounded.

Consistent Approximations ?

$S^\nu(\xi,\cdot) \underset{\text{point}}{\longrightarrow} \bar{S}\ \ \mu^\infty\text{-a.s.} \Rightarrow ?\quad S^\nu(\xi,\cdot)^{-1}(0) \rightrightarrows_\nu \bar{S}^{-1}(0)$

sometimes!

graphical rather than pointwise convergence is required

$S^\nu(\xi,\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S}\ \ \mu^\infty$-a.s. is needed

relationship between graphical and pointwise convergence?

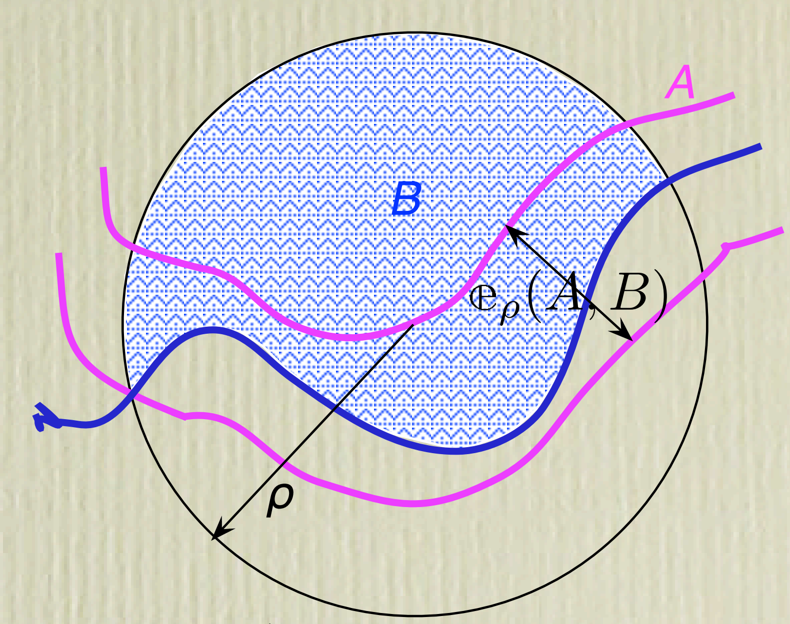
Semicontinuity: osc/isc

(review)

$S : D \rightrightarrows \mathbb{R}^m$ continuous at $\bar{x}$ if $\lim_{x^\nu \to \bar{x}} \mathbb{d}\big(S(x^\nu), S(\bar{x})\big) \to 0$

$$\mathbb{d}(S(x^\nu), S(\bar{x})) \to 0 \iff \mathbb{d}_\rho(S(x^\nu), S(\bar{x})) \to 0$$

$$\iff \hat{\mathbb{d}}_\rho(S(x^\nu), S(\bar{x})) \to 0. \forall\, \rho > \bar{\rho} \geq 0$$



$$\hat{\mathbb{d}}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\big[\,\mathbb{e}_\rho\big(S(x^\nu), S(\bar{x})\big), \mathbb{e}_\rho\big(S(\bar{x}), S(x^\nu)\big)\,\big]$$

S is osc (outer semicontinuous) at $\bar{x}$ if $\mathbb{e}_\rho\big(S(x^\nu), S(\bar{x})\big) \to 0$ as $x^\nu \to \bar{x}$

S is isc (inner semicontinuous) at $\bar{x}$ if $\mathbb{e}_\rho\big(S(\bar{x}), S(x^\nu)\big) \to 0$ as $x^\nu \to \bar{x}$

Equi-osc mappings

$S : D \rightrightarrows \mathbb{R}^m$, $D \subset \mathbb{R}^n$ is osc if gph S is closed

osc at $\bar{x}$: given any $\rho > 0, \epsilon > 0$

$$\exists\, V \in \mathcal{N}(\bar{x}) : e_\rho\big(S(x), S(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$\{S^\nu : D \rightrightarrows \mathbb{R}^m\}$ are equi-osc at $\bar{x}$

given any $\rho > 0, \epsilon > 0$

$$\exists\, V \in \mathcal{N}(\bar{x}) : e_\rho\big(S^\nu(x), S^\nu(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$V = V(\rho, \epsilon)$ doesn't depend on ν.

Graphical versus Pointwise Convergence

$D, D^\nu : X \rightrightarrows \mathbb{R}^m$. Then, $D^\nu \underset{\text{point}}{\longrightarrow} D$ and $D^\nu \underset{\text{gph}}{\longrightarrow} D$ (at x)

$\Leftrightarrow \{D^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically) (at x)

~ Arzela-Ascoli Theorem for set-valued mappings

S random mapping, μ^∞-a.s., $S^\nu(\xi, \cdot) \underset{\text{point}}{\longrightarrow} \mathrm{cl}\,\mathrm{con}\, ES = \bar{S}$

then $S^\nu \underset{\text{gph}}{\longrightarrow} \bar{S} \Leftrightarrow \{S^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically)

gph-convergence of SAA-mappings

$S : \Xi \times X \rightrightarrows \mathbb{R}^m$ random mapping, $(\Xi, \mathcal{A}, P)$

P^∞-a.s.: $S^\nu(\boldsymbol{\xi},\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S}$ at $\bar{x} \Leftrightarrow$ SAA-mappings $\{S^\nu(\boldsymbol{\xi},\cdot)\}$ equi-osc at $\bar{x}$

$\Rightarrow$ sol'ns of $S^\nu(\xi,\cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: P^∞-a.s.

$S(\boldsymbol{\xi},\cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\boldsymbol{\xi},\cdot)\}$ equi-osc

Law of large Numbers for Random Mappings

S random osc mapping: $\Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

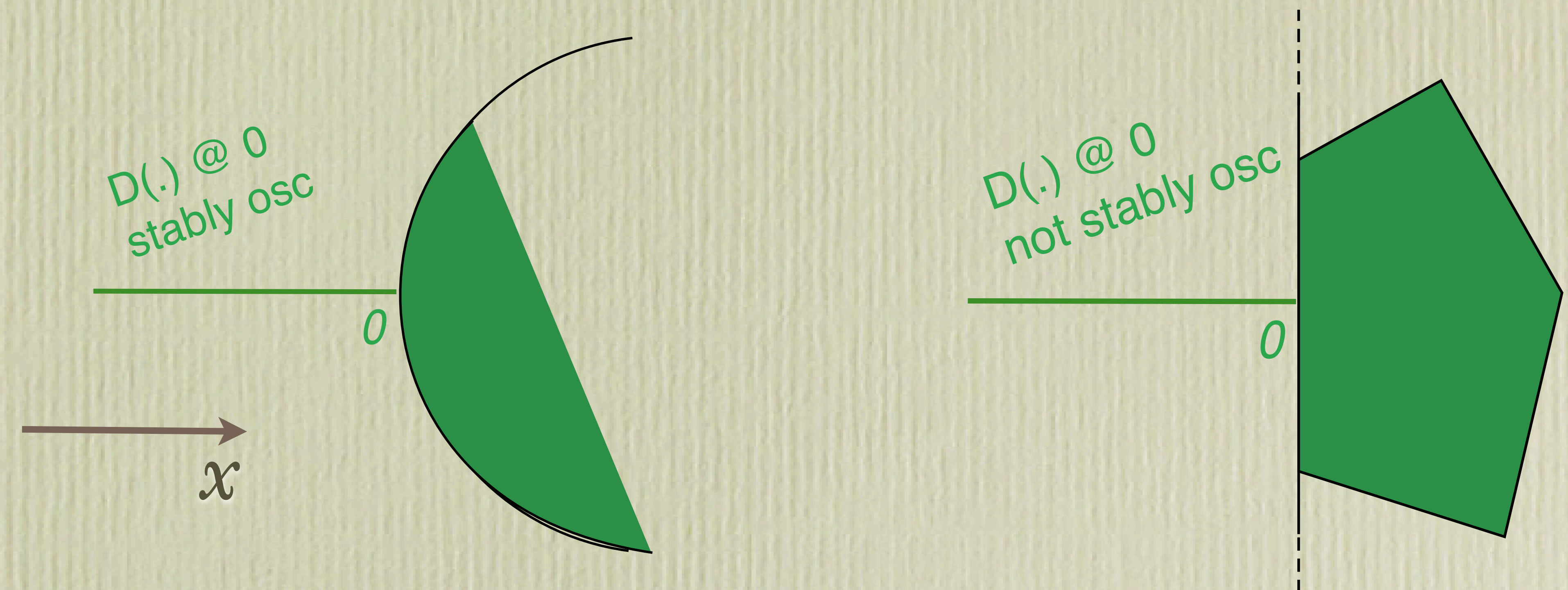
stably osc & steady under averaging

$\boldsymbol{\xi}^1, \boldsymbol{\xi}^2, \ldots,$ iid random variables (values in Ξ), distribution P

Then, $\nu^{-1}\sum_{k=1}^{\nu} S(\xi^k,\cdot) \to_{\text{gph}} \bar{S} = \operatorname{cl}\operatorname{con} E\{S(\boldsymbol{\xi}^0,\cdot)\}$ P^∞-a.s.

Friday, June 21, 13

Stably osc

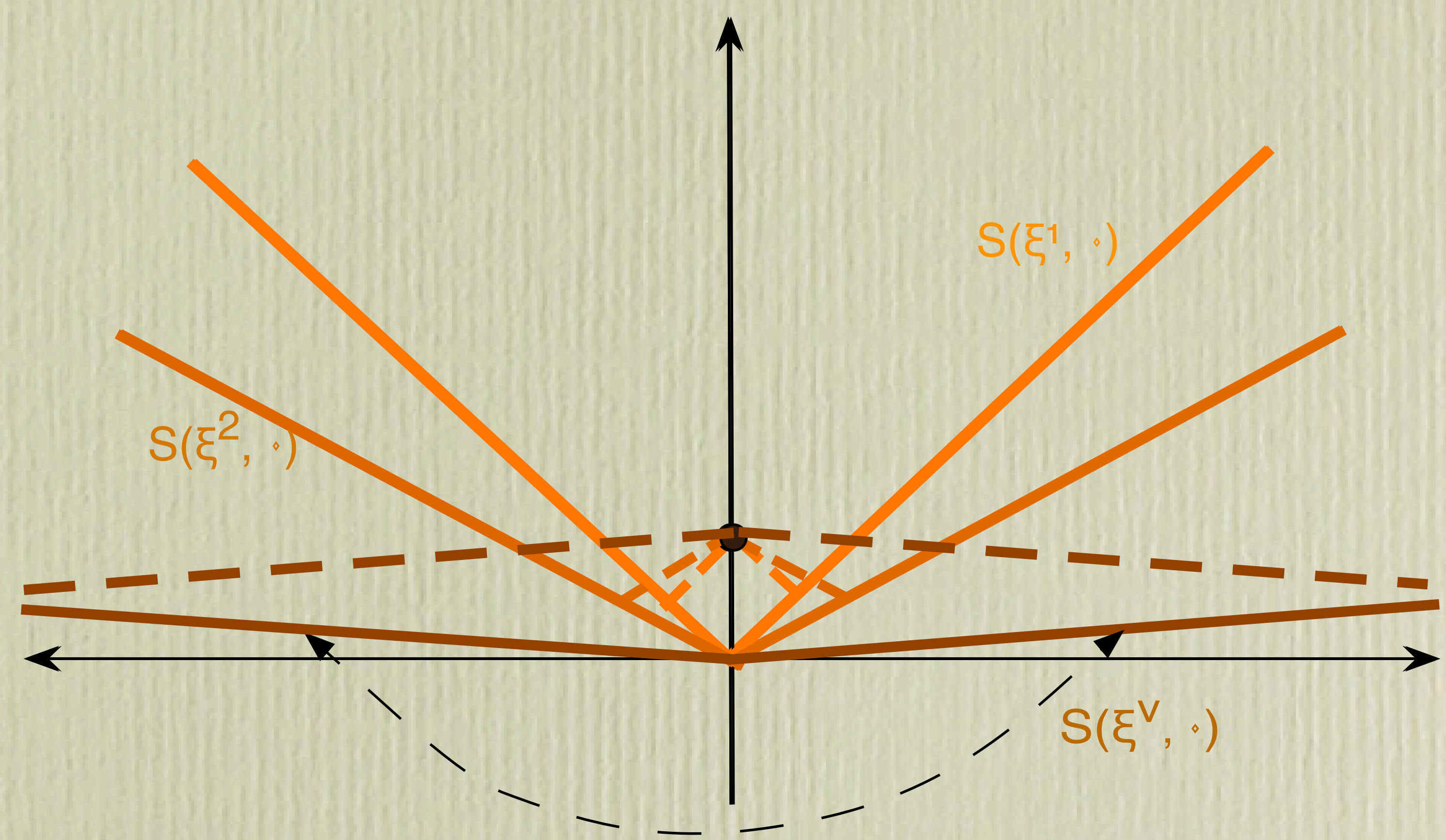


S stably osc near $\bar{x}$ if μ-a.s.,

$$\forall \rho > 0, \varepsilon > 0,\ \exists W \in \mathcal{N}(\bar{x})\ \&\ \eta\mathbb{B}\ (\eta > 0):$$

$$e_\rho\big(S(\xi, x'), S(\xi, x)\big) < \varepsilon, \forall\, x' \in x + \eta\mathbb{B},\ x \in W$$

Steady under averaging



$$u \in S^\nu(\overset{\to\nu}{\xi}, x) \cap \rho\mathbb{B} \Rightarrow \exists\, \hat{\rho} \geq \rho,\ u^k \in S(\xi^k, x) \cap \hat{\rho}\mathbb{B} \text{ such that}$$

$$u = \nu^{-1}(u^1 + \cdots + u^\nu);\ S^\nu(\overset{\to\nu}{\xi}, x) \cap \rho\mathbb{B} \subset \frac{1}{\nu}\left[\sum_{k=1}^{\nu} S(\xi^k, x) \cap \hat{\rho}\mathbb{B}\right]$$

Steady under averaging & Stably osc

rge $S \subset B$ bounded $\Rightarrow$ steady under averaging

S cone-valued and rge $S \subset$ pointed cone K. Then,

$\bar{S} = ES$ and $\Rightarrow$ steady under averaging.

S, R steady under averaging $\Rightarrow$ so is $S + R$

$R(\xi, x) = R(x) \Rightarrow R$ steady under averaging

rge S bounded + R constant $\Rightarrow$ steady under averaging

$G(\xi, x) + N_C(x) \Rightarrow$ steady under averaging (V.I.)

$G : \Xi \times X \to \mathbb{R}^n$ is bounded

S, R stably osc $\Rightarrow S + R$ stably osc

although D^1, D^2 osc $\not\Rightarrow D^1 + D^2$ osc

$\mathbb{B}$ closed, convex $\quad x \mapsto N_{\mathbb{B}}(x)$ osc

but not stably osc ($x^\nu \in$ int $\mathbb{B} \to \bar{x} \in$ bdry $\mathbb{B}$)

Implementing SAA ** locally

$EG(x) = \mathbb{E}\{G(\xi, x)\} \in S(x)$

(V.I.: $S = N_C$, applied to option pricing, ...)

$G^\nu(\vec{\xi}^\nu, \cdot) = \nu^{-1} \sum_{k=1}^{\nu} G(\xi^k, x)$. Assume $G^\nu(\vec{\xi}^\nu, \cdot)$, $EG \in C^1(\mathbb{R}^n; \mathbb{R}^n)$,

$\bar{x}$ strongly regular solution [Robinson] of $EG(x) \in S(x)$,

$\exists V \in \mathcal{N}(\bar{x}), \rho > 0$ such that $\forall z \in \rho\mathbb{B}$:

$$z + EG(\bar{x}) + \nabla EG(\bar{x})(x - \bar{x}) \in S(x)$$

has a unique solution $\bar{x}(z) \in V$, Lipschitz continuous on $\rho\mathbb{B}$, and

$\left\| G^\nu(\vec{\xi}^\nu, \cdot) - EG \right\| \to 0$ μ-a.s. Then, for ν sufficiently large

on a neighborhood of $\bar{x}$, $G^\nu(\vec{\xi}^\nu, \cdot) \in S(x)$ has a unique solution

$$\bar{x}(\vec{\xi}^\nu) \to \bar{x} \quad \mu\text{-a.s.}$$

Implementing SAA ** example

stochastic program with recourse (simple): ξ uniform on [1,2]

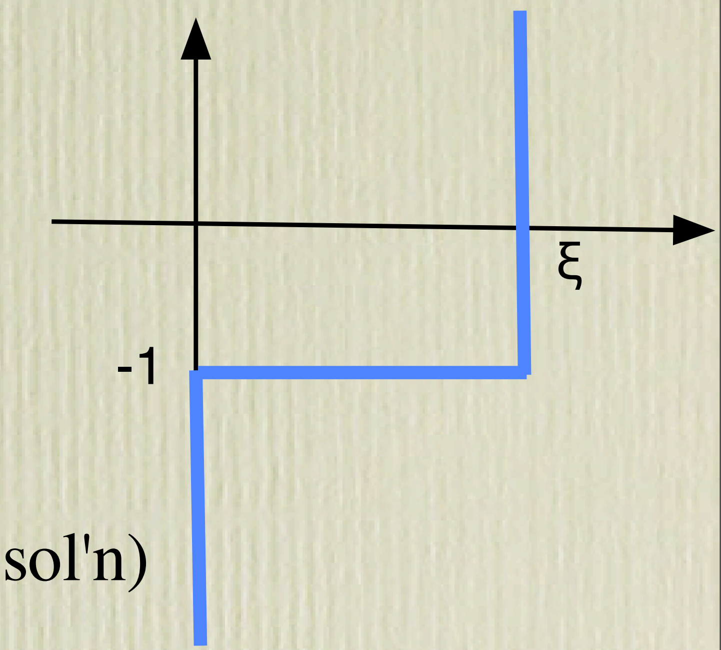
$$\min_{x,y_\xi} \mathbb{E}\left\{-x \,\middle|\, x+y_\xi \le \xi, x\in[0,2], y_\xi \ge 0\right\} = \min\left(Ef(x) = \mathbb{E}\{f(\boldsymbol{\xi},x)\}\right)$$

$$f(\xi,x) = -x + \iota_{[0,2]} + \iota_{(-\infty,\xi]} = -x + \iota_{[0,\xi]}$$

to solve $0 \in \partial Ef(x)$ gets replaced by $0 \in \nu^{-1}\sum_{k=1}^{\nu} S(\xi^k,x) = S^\nu(\overset{\to\nu}{\xi},x)$

$$S(\xi,x) = \partial f(\xi,x) = -1 + N_{[0,\xi]}(x), \qquad \text{dom } \mathrm{S}(\xi,\cdot) = [0,\xi]$$

$$= \begin{cases} (-\infty,-1] & \text{when } x=0, \\ -1 & \text{for } x\in(0,\xi), \\ [1,\infty) & \text{when } x=\xi \end{cases}$$



Solution of $0 \in S^\nu(\overset{\to\nu}{\xi},x)$: $x^\nu = \min\left\{\xi^1,\ldots,\xi^\nu\right\} \to_{\text{a.s.}} \bar{x} = 1$ (opt. sol'n)

but x^ν is never a feasible solution,

$\nexists y_\xi \ge 0$ such that $x^\nu + y_\xi \le \xi$ when $\xi \in [1,x^\nu)$

Problem: $\partial Ef(x) \ne \mathbb{E}\{\partial f(\boldsymbol{\xi},x)\}$ *** interchange is not valid.