
Sierra Nevada Sunrise

Equilibria:
Stochastic Environment

Roger J-B Wets
University of California, Davis

Complementarity & its Extensions
National University of Singapore, December 2012

support: U.S. Army Research Laboratory and the U.S. Army Research Office, grant no. W911NF1010246

Arrow-Debreu model

pure-exchange economy: goods $\in \mathbb{R}^L$, prices $p = (p_1, \ldots, p_L)$, free disposal

agents: $i \in I$, $|I|$ finite ---- initial holdings: $(e_i, i \in I)$

demand functions: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p, x \rangle \le \langle p, e_i \rangle \right\}$

utility fcn: u_i : dom $u_i = X_i \to \mathbb{R}$, usc, concave $\not\Rightarrow X_i$ closed (but convex)

excess supply function: $s(p) = \sum_{i \in I} (e_i - x_i(p))$, market clearing: $s(p) \ge 0$

$$\bar{p} \ge 0 \text{ equilibirum } \Leftrightarrow s(\bar{p}) \ge 0$$

Existence: $x = (x_m, x_g)$, x_m = 'money' allows $p = (1, p_g)$, under

ample survivability: $(e_{im}, e_{ig}) \Rightarrow \exists (\hat{x}_{im}, \hat{x}_{ig}) \in X_i$

such that $\hat{x}_{ig} \le e_{ig}$, $\hat{x}_{im} < e_{im}$ and $\sum_{i \in I} \hat{x}_{ig} < \sum_{i \in I} e_{ig}$

+ indispensability & unactractiveness

Solution Procedures

Walras' law: $\bar{p} \perp s(\bar{p}) \sim \bar{p}_l s_l(\bar{p}) = 0,\ l = 1,\ldots,L,$ $\quad s(p) = s(\alpha p)$ for $\alpha > 0$

scaling: $\bar{p} \in \Delta = \left\{ p \in \mathbb{R}^L_+ \,\middle|\, \sum_l p_l = 1 \right\}$ since $\forall\, \alpha > 0 : \langle \alpha p, x \rangle \leq \langle \alpha p, e_i \rangle$

one possible way

find $\bar{p}$ $(\in \Delta)$ such that $\quad 0 \leq \bar{p} \perp s(\bar{p}) \geq 0,$

0. (very) special instances: via convex programming

1. tâtonnement, : $\dot{p} = -s(p),\ p(0) = p^0$ (Adam Smith, Léon Walras)

variant: 'Global Newton' (S. Smale) :

$$\nabla s(p)\dot{p} = \lambda s(p),\ \operatorname{sgn}(\lambda) = (-1)^L \operatorname{sgn}\det\left(\nabla s(p)\right)$$

requires s single-valued and differentiable,

$e_i \in \operatorname{int} X_i$ or bdry conditions on s

fails, "in general"

source of doubts about economic equilibrium theory

Sunday, December 23, 12

Solution Procedures

2. simplicial methods (based on "pivoting")

   - Scarf (& Hansen) '73: $\Rightarrow$ find fixed point of $p \mapsto s(p) - p$ in Δ

     partitioning Δ in a simplicial complex, pivoting á la Lemke-Howson

   - piece-wise linear homotopy methods: Eaves '74, Saigal, ...

3. homotopy continuation methods

   - homotopy methods $G(x) = 0$, Yorke *et al* ('72, '78)

     variants: Kojima, Meggido and Noma for NCP ('89)

     Newton homotopy: Wu ('05), ...

   - 'interior point' homotopy method: Dang and Ye ('11)

4. Variational Inequality

to be dealt with in glorious detail later

a maxinf approach

recall: $s(p) = \sum_{i\in I}(e_i - x_i(p))$, market clearing: $s(p) \geq 0$

Walrasian: $W(p,q) = \langle q, s(p)\rangle$, $W : \Delta \times \Delta \to \mathbb{R}$ (a bifunction)

Key observation:

$\bar{p} \in$ maxinf W, $W(\bar{p},\bullet) \geq 0$ on $\Delta \Rightarrow \bar{p}$ is an equilibirum point.

under insatiability, $\bar{p}$ an equilibrium $\Rightarrow \bar{p} \in$ maxinf W, $W(\bar{p},\bullet) \geq 0$ on Δ

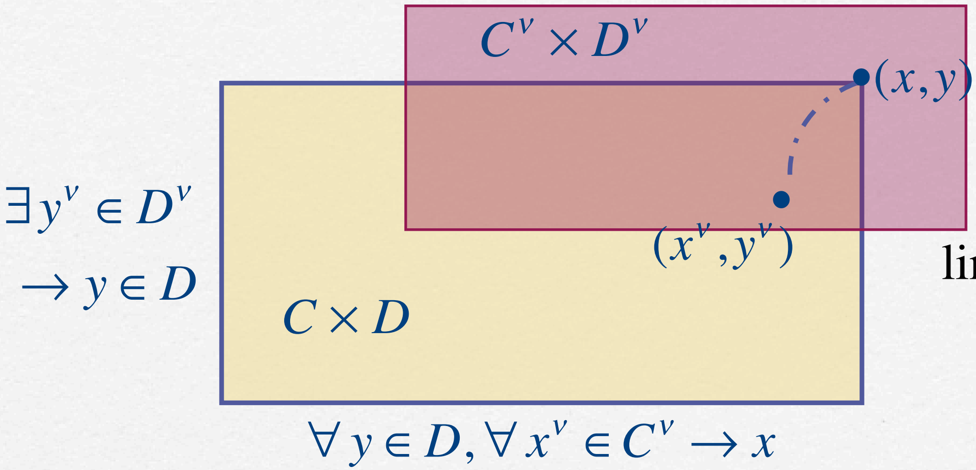
Moreover:

p_ε : ε-equilibrium point if $\forall l$ (good), $s_l(p_\varepsilon) \geq -\varepsilon$

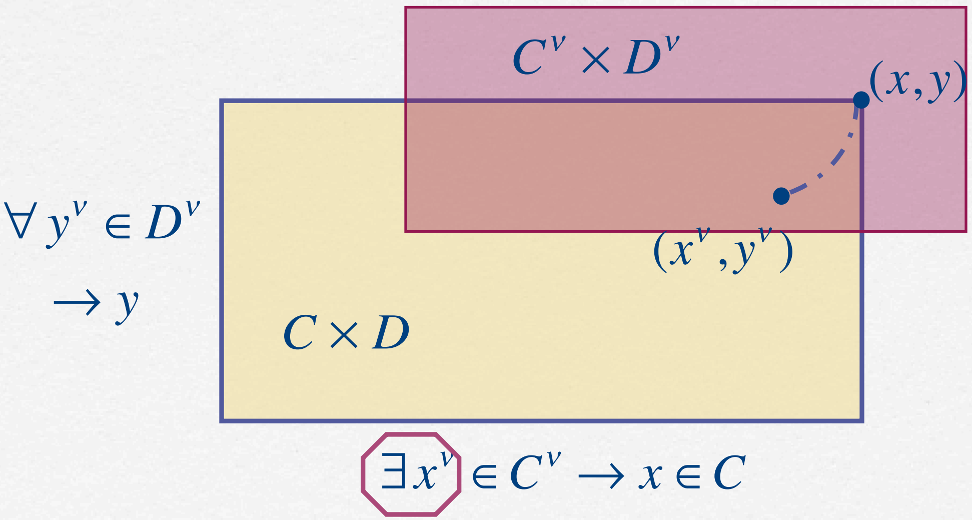
$p_\varepsilon \in$ ε-maxinf W, $W(p_\varepsilon,\bullet) \geq -\varepsilon$ on $\Delta \Rightarrow p_\varepsilon$ is an ε-equilibirum point.

with insat., p_ε an ε-equilibrium $\Rightarrow p_\varepsilon \in$ ε-maxinf W, $W(p_\varepsilon,\bullet) \geq -\varepsilon$ on Δ

Lopsided Convergence



$\limsup_\nu K^\nu(x^\nu, y^\nu) \le K(x, y)$ when $x \in C$



$\liminf_\nu K^\nu(x^\nu, y^\nu) \ge K(x, y)$ when $y \in D$

$K^\nu(x^\nu, y^\nu) \to \infty$ when y $\notin D$

The maxinf "family" ...

- saddle-point problems: Lagrangians, zero-sum games, Hamiltonians
- equilibrium: classical mechanics, Wardrop, economic (Walras, etc.)
- variational inequalities: finance, ecological models, complementarity, PDE
- non-cooperative games: pricing, generalized Nash equilibrium
- finding fixed points: Brouwer-type, Kakutani-type (set-valued), MPEC
- solving inclusions (equivalently, generalized equations): $S(x) \ni 0$
- minimal surface problems, ... , mountain pass solutions, ...
- ... and the dynamic versions, and the stochastic (dynamic) versions.

Ancillary-tightly ~ 'compact in y'

THM. $K^{\nu}_{C^{\nu}\times D^{\nu}} \to_{lop.} K_{C\times D}$ & ancillary-tightly,

$\bar{x} \in$ cluster points of $\{x^{\nu} \in \operatorname{maxinf} K^{\nu}_{C^{\nu}\times D^{\nu}}\}_{\nu\in\mathbb{N}} \Rightarrow \bar{x} \in \operatorname{maxinf} K_{C\times D}$

$K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop\ ancillary-tight}{\to} K_{C\times D}$ if $K^{\nu}_{C^{\nu}\times D^{\nu}} \underset{lop}{\to} K_{C\times D}$ and

(b) $\forall x \in C$, $\exists x^{\nu} \to x$, $\forall y^{\nu} \in D^{\nu}$ and $y^{\nu} \to y$:

$$\liminf K^{\nu}(x^{\nu}, y^{\nu}) \geq K(x,y) \text{ if } y \in D$$

$$K^{\nu}(x^{\nu}, y^{\nu}) \to \infty \text{ if } y \notin D$$

but also $\forall \varepsilon>0$, $\exists B_{\varepsilon}$ compact (depends on $x^{\nu} \to x$):

$$\inf_{B_{\varepsilon}\cap D^{\nu}} K^{\nu}(x^{\nu},\bullet) \leq \inf_{D^{\nu}} K^{\nu}(x^{\nu},\bullet) + \varepsilon,\ \forall \nu \geq \nu_{\varepsilon}$$

certainly satisfied when $D = \Delta$ is compact

Convergence of ε-solutions

including $\varepsilon = 0$

$K^{\nu}_{C^{\nu}\times D^{\nu}} \to K_{C\times D}$ lop. ancillary-tightly,

(i) $x^{\nu} \in \varepsilon$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$, $\bar{x}$ cluster point of $\{x^{\nu}\}_{\nu\in\mathbb{N}}$

$\Rightarrow \bar{x} \in \varepsilon$-maxinf $K_{C\times D}$

(ii) $x^{\nu} \in \varepsilon_{\nu}$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$, $\bar{x}$ cluster point of $\{x^{\nu}\}_{\nu\in\mathbb{N}}$

& $\varepsilon_{\nu} \searrow 0 \Rightarrow \bar{x} \in$ maxinf $K_{C\times D}$ (special case: locally unique)

(iii) $\bar{x} \in$ maxinf $K_{C\times D} \Rightarrow \exists \varepsilon_{\nu} \searrow 0$ & $x^{\nu} \in \varepsilon_{\nu}$-maxinf $K^{\nu}_{C^{\nu}\times D^{\nu}}$

such that $x^{\nu} \to \bar{x}$,

Under tight-lop: convergence of the full ε_{ν}-maxinf sets

and convergence of values

tight-lop when $C = \Delta$ & $D = \Delta$ are compact

...back to our Walrasian

$$W(p,q) = \langle q, s(p) \rangle \text{ on } \Delta \times \Delta,\ p\text{-usc and } q\text{-convex}$$

Augmented Walrasian: σ augmenting function

$$\tilde{W}_r(p,q) = \inf_z \left\{ W(p, q - z) + r *_e \sigma^*(z) \right\} \qquad r\sigma(r^{-1}z)$$

$$= \sup_z \left\{ W(p,z) \,\middle|\, \|z - q\|_\square \le r \right\} \quad \sigma = |\bullet|_\square,\ \iota_{\mathbb{B}} = \sigma^*$$

as $r \to \bar{r} < \infty$, $\tilde{W}_r \to_{\text{lop}} \tilde{W}_{\bar{r}} = W \Rightarrow \varepsilon\text{-maxinf } W_r \to \text{maxinf W}$

choosing $|\bullet|_\square = |\bullet|_\infty, \mathbb{B} = [-1,1]^L$

or $|\bullet|_\square = |\bullet|_2, \mathbb{B} =$ euclidean unit ball

augmented Walrasian strategy

$$W(p,q) = \langle q, s(p) \rangle \text{ on } \Delta \times \Delta, \; p\text{-usc and } q\text{-convex}$$

$$\tilde{W}_r(p,q) = \sup_z \left\{ W(p,z) \middle| \, \|z - q\|_{\square} \le r \right\}$$

$$q^{k+1} = \arg\max_{q \in \Delta} \left[\max_z \langle z, s(p^k) \rangle \middle| \, \|z - q\|_{\square} \le r_k \right]$$

minimizing a linear form on a ball

i.e. finding the largest element of $s(p^k)$

$$p^{k+1} = \arg\min_{p \in \Delta} \left[\max_z \langle z, s(p) \rangle \middle| \, \|z - q^{k+1}\|_{\square} \le r_{k+1} \right]$$

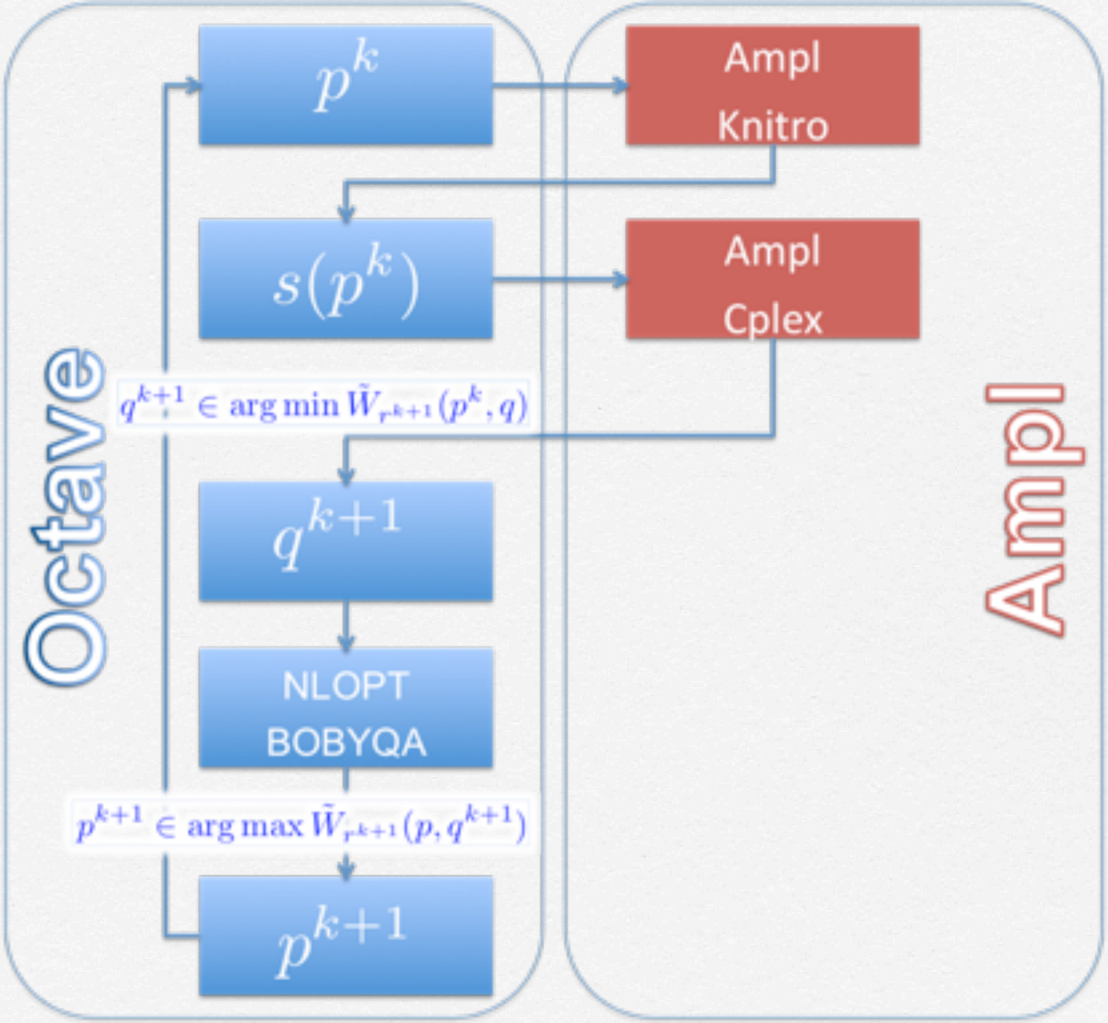
as $r_k \nearrow \infty, \quad p^k \to \bar{p}$

Bagh, Lucero & Wets ≈ '03

first experiments: 10 agents, 150 goods (two blinks)

CMM-implementation

Center for Mathematical Modeling --- Universidad de Chile



Deride,
Jofré
& Wets
'06 -- '12

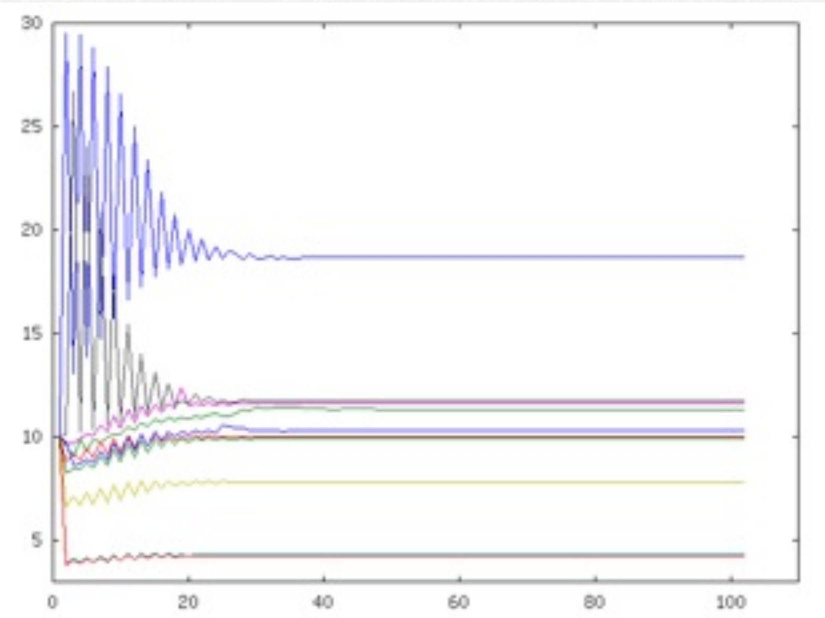
$$\sigma = \frac{1}{2}|\cdot|^2$$

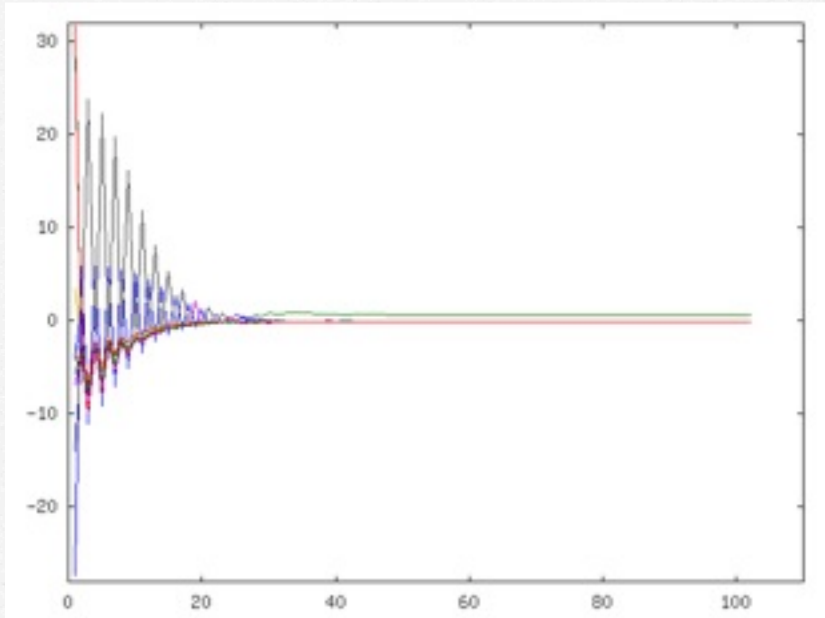
Scarf's example

$$u_i(x) = \left(\sum_{l=1}^{L} (a_{il})^{\beta_i^{-1}} (x_l)^{1-\beta_i^{-1}} \right)^{\beta_i (\beta_i - 1)^{-1}}$$ CES-utility

constant elasticity substitution

$i \in I = 5$ agents, $L = 10$ goods (2000 simplicial pivots)



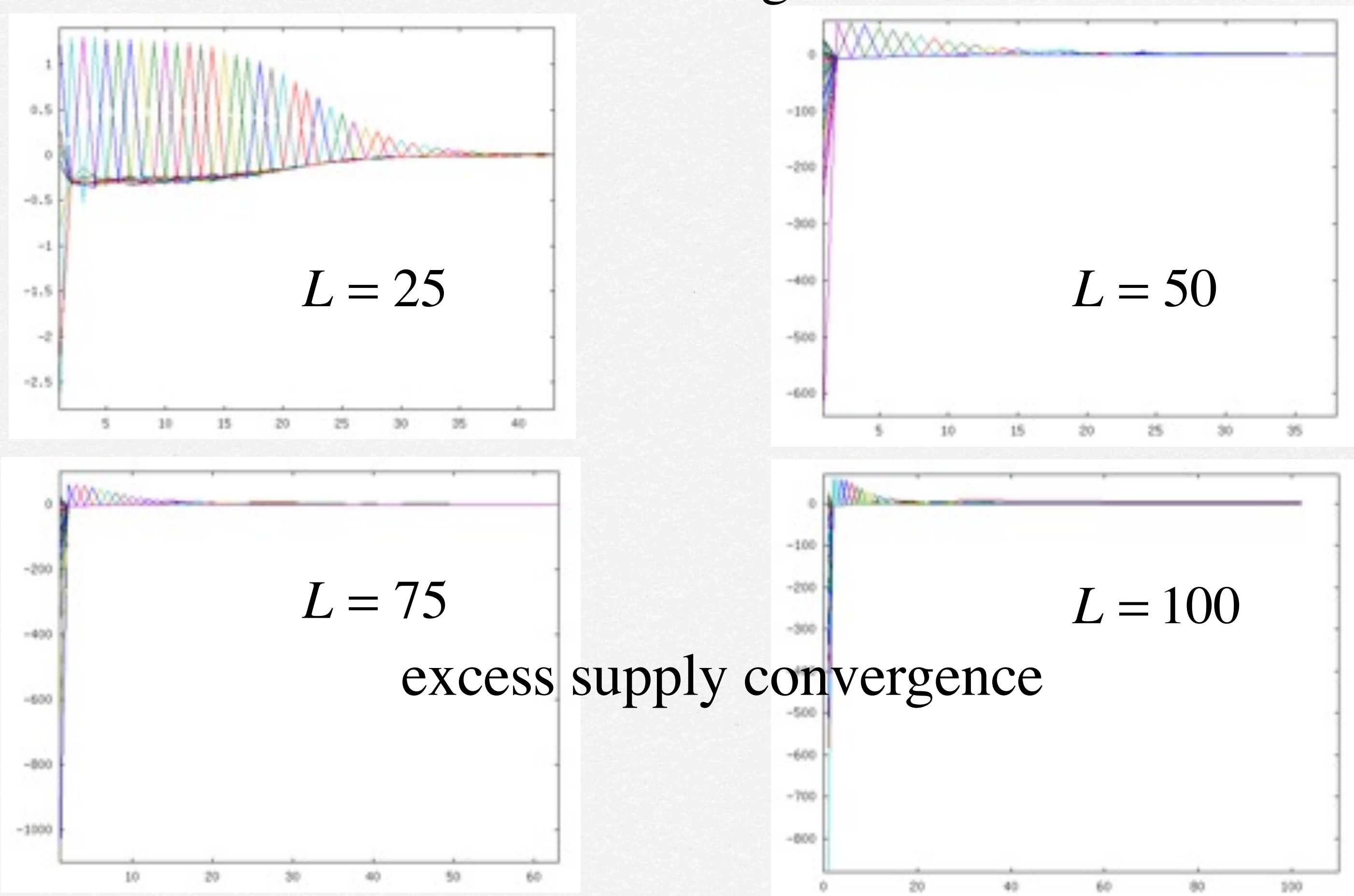


prices and excess supply convergences

just, ... one more example

same CES-utility function ($\neq \beta_i$), $I = 10$ agents, $r_k = 1.21^k$

$L =$ # of goods



A dynamic model

$$\max_{(x^0, y, x^1)} \quad u_i^0(x^0) + u_i^1(x^1)$$

such that $\langle p^0, x^0 + T_i^0 y \rangle \le \langle p^0, e_i^0 \rangle$

$$\langle p^1, x^1 \rangle \le \langle p^1, e_i^1 + T_i^1 y \rangle$$

$$x^0 \in X_i^0,\ y \in Y_i,\ x^1 \in X_i^1$$

with solutions: $\left\{ \left(x_i^0(p), y_i(p), x_i^1(p) \right), i \in I \right\},\ p = (p^0, p^1)$

excess supply: $s^0(p) = \sum_{i \in I} \left(e_i^0 - \left(x_t(p) + T_i^0 y_i(p) \right) \right)$

$$s^1(p) = \sum_{i \in I} \left(\left(e_i^1 + T_i^1 y_i(p) \right) - x^1(p) \right)$$

equilibirum: $\bar{p} \in \Delta_L \times \Delta_L$ such that $s^0(\bar{p}) \ge 0, s^1(\bar{p}) \ge 0 \quad (=)$

Walrasian: $W(p,q) = \left\langle (q^0, q^1), \left(s^0(p), s^1(p) \right) \right\rangle$ on $\Delta_L^2 \times \Delta_L^2$

$\Rightarrow$ augmented Walrasian, ...

Exploiting separability

$$r_i(y,p) = \sup_{x^0 \in X_i^0} \left[u_i^0(x^0) \middle| \left\langle p^0, x^0 + T_i^0 y \right\rangle \le \left\langle p^0, e_i^0 \right\rangle \right]$$

$$+ \sup_{x^1 \in X_i^1} \left[u_i^1(x^1) \middle| \left\langle p^1, x^1 \right\rangle \le \left\langle p^1, e_i^1 + T_i^1 y \right\rangle \right]$$

i-agent problem: find $y_i(p) \in \mathbb{R}_+^{n_i}$ that maximizes $\left\{ r_i(y) \middle| T_i^0 y \le e_i^0 \right\}$

Example: $u_i^{0,1}$ are of Cobb-Douglas type: $u_i(x) = \prod_{l=1}^{L} x_l^{\beta_l}, \beta \in \Delta$

$$x_l^0(p^0,y) = \frac{\beta_l}{p_l^0} \sum_{k=1}^{L} p_k^0 \left(e_k^0 - \left\langle (T_i^0)_k, y \right\rangle \right), \; l = 1,\dots,L, \quad X_i^0 = \mathbb{R}_+^L$$

$$x_l^1(p^1,y) = \frac{\beta_l}{p_l^1} \sum_{k=1}^{L} p_k^1 \left(e_k^1 + \left\langle (T_i^1)_k, y \right\rangle \right), \; l = 1,\dots,L, \quad X_i^0 = \mathbb{R}_+^L$$

substituting $\Rightarrow r_i$ linear in y: i-agent's problem is a linear program!

substantial gain in processing time

Stochastic Environment

a ‘minimal’ model

Chap.7 -- Theory of Value G. Debreu, '59

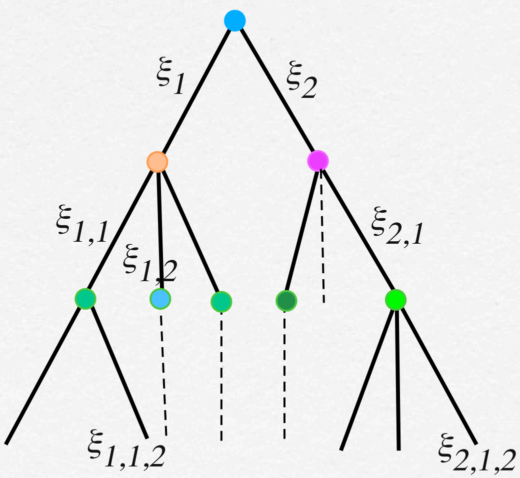
$$\max_{x^0, x^1_\cdot \in \mathcal{M}} E\left\{u_i(x^0, x^1_{\xi^1}, x^2_{\xi^1,\xi^2}, \dots)\right\}$$ i-agent

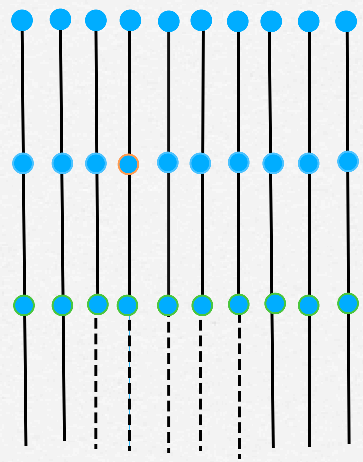
$$\text{such that } \left\langle p^t_\xi, \sum_{\tau \le t}\left(e^\tau_{\xi^1,\dots\xi^\tau} - x^\tau_{\xi^1,\dots\xi^\tau}\right)\right\rangle \ge 0,\ \forall \xi = (\xi^1, \xi^2, \dots),\ \ t = 0,1,\dots$$

$$\sum_{i \in I}\left(\sum_{\tau \le t}\left(e^\tau_{\xi^1,\dots\xi^\tau} - x^\tau_{\xi^1,\dots\xi^\tau}\right)\right) \ge 0, \forall \xi, t$$ Clearing the market

Key Assumption (via K. Arrow): all contingencies available at time 0

$\Rightarrow$ complete market, i.e., all ξ's can be dealt with separately





$\exists \left(p_\xi = (p^0_\xi, p^1_\xi, \dots)\right)$

$(\forall\ \xi)$ equilibrium prices.

but complete markets are few and far in between

a more realistic approach
“Incomplete markets”

i-Agent: stochastic case

an engineer's viewpoint?

$$\max_{x^0, y, x^1_\cdot \in \mathcal{M}} \quad u_i^0(x^0) + E^i\left\{u_i^1(\xi; x_\xi^1)\right\}$$

$$\text{so that} \quad \left\langle p^0, x^0 + T_i^0 y \right\rangle \le \left\langle p^0, e_i^0 \right\rangle$$

$$\left\langle p_\xi^1, x_\xi^1 \right\rangle \le \left\langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y \right\rangle, \quad \forall \xi \in \Xi$$

$$x^0 \in X_i^0, \; y \in \mathbb{R}_+^{n_i}, \; x_\xi^1 \in X_{i,\xi}^1, \quad \forall \xi \in \Xi$$

- $E^i\{.\}$ expectation w.r.t. *i*-agent beliefs

Stochastic program with recourse: 2-stage
Well-developed solution procedures
Well-developed "Approximation Theory"

Simplest-classical assumptions

Ξ finite (support)

$u_i^0 : X_i^0 \to \mathbb{R}, \quad \forall \xi \in \Xi,\ u_i^1(\xi,\bullet) : X_{i,\xi}^1 \to \mathbb{R}$ concave

continuous, numerical experiments: differentiable

$T_{\mathrm{i}}^0, T_{i,\xi}^1$: input-ouput matrices

(savings, production, investment, etc.)

$X_{\mathrm{i}}^0, X_{i,\xi}^1$: closed, convex, non-empty interior (survival sets

$e_{\mathrm{i}}^0 \in \operatorname{int} X_{\mathrm{i}}^0,\ e_{i,\xi}^1 \in \operatorname{int} X_{i,\xi}^1$ for all ξ (or as on first slide)

Market Clearing

Agents: $i \in I$, $|I|$ finite ("large"), $\quad p = \left(p^0, (p^1_\xi, \xi \in \Xi)\right)$

$$\left(\bar{x}_i^0(p), \bar{y}_i(p), \left\{\bar{x}^1_{i,\xi}(p)\right\}_{\xi\in\Xi}\right) \in \arg\max \{i\text{-agent problem}\}$$

excess supply:

$$\sum_{i\in I}\left(e_i^0 - (\bar{x}_i^0(p) + T_i^0 \bar{y}_i(p))\right) = s^0(p^0, \{p^1_\xi\}_{\{\xi\in\Xi\}}) \geq 0$$

$\forall \xi \in \Xi$:

$$\sum_{i\in I}\left(e^1_{i,\xi} + T^1_{i,\xi}\bar{y}_i(p) - \bar{x}^1_{i,\xi}(p)\right) = s^1_\xi(p^0, \{p^1_\xi\}_{\{\xi\in\Xi\}}) \geq 0$$

Existence: via Ky Fan Inequality

$$W(p_{\bullet},q_{\bullet}) = \langle q_{\bullet}, s(p_{\bullet})\rangle$$

$$= \left\langle (q^0, \{q^1_\xi\}_{\xi\in\Xi}), \left(s^0(p^0, \{p^1_\xi\}_{\xi\in\Xi}), \left\{ s^1_\xi(p^0, \{p^1_\xi\}_{\xi\in\Xi}) \right\}_{\xi\in\Xi} \right) \right\rangle$$

$$W : \prod_{1+|\Xi|} \Delta \times \prod_{1+|\Xi|} \Delta \to \mathbb{R}$$ a Ky Fan function: usc, convex

linear w.r.t. $q_{\bullet}$, continuous w.r.t. $p_{\bullet}$

and also $W(p_{\bullet}, p_{\bullet}) \geq 0.$

provided $s(\bullet)$ continuous w.r.t. $p_{\bullet}$

↑

another lecture

Stochastic optimization

---

a short basic introduction

Here-&-Now vs. Wait-&-See

▲ basic process: decision → observation → decision

$$\left(x_i^0, y_i\right) \rightarrow \quad \xi \quad \rightarrow \left(x_{i,\xi}^1\right)$$

▲ here-and-now problem:

not all contingencies available at time 0

$\left(x_i^0, y_i\right)$ can't depend on ξ

▲ wait-and-see problem

implicitly all contigencies available at time 0

choose $\left(x_{i,\xi}^0, y_{i,\xi}^0, x_{i,\xi}^1\right)$ after observing ξ

▲ from here-and-now → wait-and-see

'completing' the i-agent market?

Fundamental Theorem of Stochastic Optimization

A here-and-now problem can be "reduced" to a wait-and-see problem by introducing the

appropriate 'contingency' costs:

i.e., what i-agent be willing to pay for letting x^0, y depend on ξ

Wets '77

Contingencies prices (nonanticipativity)

Here-ξ-now

$$\max E\left\{ f(\xi; z^0, z^1_\xi) \right\}$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0), \forall \xi.$$

Contingencies prices (nonanticipativity)

Here-ξ-now

$$\max E\left\{ f(\xi; z^0, z_\xi^1) \right\}$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z_\xi^1 \in C_\xi^1(z^0), \forall \xi.$$

to wait-and-see

$$\max E\left\{ f(\xi; z_\xi^0, z_\xi^1) \right\}$$

$$z_\xi^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z_\xi^1 \in C_\xi^1(z^0), \forall \xi.$$

$$z_\xi^0 = E\{z_\xi^0\} \; \forall \xi$$

Contingencies prices (nonanticipativity)

Here-&-now

$$\max E\left\{ f(\xi; z^0, z^1_\xi) \right\}$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0), \forall \xi.$$

to wait-and-see

$$\max E\left\{ f(\xi; z^0_\xi, z^1_\xi) \right\}$$

$$z^0_\xi \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0), \forall \xi.$$

$$z^0_\xi = E\{z^0_\xi\} \ \forall \xi$$

$$w_\xi \perp \text{c}^{\text{ste}} \ \text{fcns}$$

$$\Rightarrow E\{w_\xi\} = 0$$

Contingencies prices (nonanticipativity)

with correct $\bar{w}_\xi$

$$\max E\left\{ f(\xi; z^0_\xi, z^1_\xi) - \left\langle \bar{w}_\xi, z^0_\xi \right\rangle \right\}$$

$$z^0_\xi \in C^0, \; z^1_\xi \in C^1_\xi(z^0_\xi)$$

⇓

$\forall \xi \in \Xi$:

$$\max f(\xi; z^0, z^1) - \left\langle \bar{w}_\xi, z^0 \right\rangle$$

to wait-and-see

$$\max E\left\{ f(\xi; z^0_\xi, z^1_\xi) \right\}$$

$$z^0_\xi \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0), \forall \xi.$$

$$z^0_\xi = E\{z^0_\xi\} \; \forall \xi$$

$$w_\xi \perp \text{c}^{\text{ste}} \;\; \text{fcns}$$

$$\Rightarrow E\{w_\xi\} = 0$$

Progressive Hedging

Rockafellar & Wets '91
@ Academia Sinica '86

- Step 0. $w^0(\bullet)$ so that $E\{w^0(\xi)\} = 0,\ \nu = 0$
- Step 1. for all ξ:

$$\left(z_\xi^{0,\nu}, z_\xi^{1,\nu}\right) \in \arg\max f(\xi; z^0, z^1) - \left\langle w_\xi^\nu, z^0 \right\rangle$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_0},\ z^1 \in C^1(\xi, x^0) \subset \mathbb{R}^{n_1}$$

- Step 2. $w_\xi^{\nu+1} = w_\xi^\nu + \rho\left[z_\xi^{0,\nu} - E\{z_\xi^{0,\nu}\}\right],\quad \rho > 0$
  - and return to Step 1, $\nu = \nu + 1$
- Convergence: add proximal term $-\frac{\rho}{2}\left\| z_\xi^{0,\nu} - E\{z_\xi^{0,\nu}\}\right\|^2$

  linear rate in $\left(z^\nu, w^\nu\right)$

Disintegrating: agent's problem

with $p_{\blacklozenge} = \left(p^0, \{p^1_\xi\}_{\xi\in\Xi}\right)$

$$\left(\bar{x}^0_{i,\xi}, \bar{y}^0_{i,\xi}, \bar{x}^1_{i,\xi}\right) \in$$

'*i*-contingency' costs

$$\underset{x^0,y,x^1}{\arg\max}\left\{u_i^0(x^0) - \left\langle \bar{w}_{i,\xi}, (x^0, y)\right\rangle - \frac{\rho}{2}\left\|(x^0, y) - (\bar{x}^0, \bar{y})\right\|^2 + u_i^1(\xi; x^1)\right\}$$

$$\left\langle p^0, x^0\right\rangle \le \left\langle p^0, e_i^0 - T_i^0 y_i\right\rangle$$

$$\left\langle p^1_\xi, x^1\right\rangle \le \left\langle p^1_\xi, e^1_{i,\xi} + T^1_{i,\xi} y\right\rangle,$$

$$x^0 \in C_i^0,\ x^1 \in C^1_{i,\xi}.$$

solved for each ξ **separately**

Incomplete → '*i*-Complete' Market

$\forall \xi \in \Xi$ (separately),

i-agent's problem:

$$\left(x_i^0, y_i, x_{i,\xi}^1\right) \in \arg\max\left\{u_i^{w_{i,\xi}}\left(\xi; x^0, y, x^1\right) \text{ on } \widehat{C}_{i,\xi}(p^0, p_\xi^1)\right\}$$

for $\{w_{i,\xi}\}_{\xi\in\Xi}$ associated with (p^0, p_ξ^1)

clearing the market:

$$s^0(p^0, p_\xi^1) \geq 0, \quad s_\xi^1(p^0, p_\xi^1) \geq 0$$

Arrow-Debreu 'stochastic' equilibrium problem

Cobb-Douglas utilities

$$u_i^t(x) = \prod_{l=1}^{L} x_l^{\beta_l^t}, \beta^t \in \Delta, \; t = 0,1$$

"agent's optimization" (skipping i)

$$r(\xi; y, p) = \alpha^0(p^0)\Big(\sum_{k=1}^{n} p_k^0\big(e_k^0 - \langle T_k^0, y\rangle\big)\Big)$$

taking advantage of separability

$$+ \alpha^1(p_\xi^1)\Big(\sum_{k=1}^{n} p_{k,\xi}^1\big(e_{k,\xi}^1 - \langle (T_\xi^1)_k, y\rangle\big)\Big), \qquad \alpha(p) = \prod_{l=1}^{L}\left(\frac{\beta_l}{p_l}\right)^{\beta_l}$$

$$y_\xi^{v+1} \in \arg\max\left\{ r^v(\xi; y) - \langle w^v, y\rangle - \frac{\rho}{2}\left|y - \bar{y}^v\right|^2 \;\middle|\; T^0 y \le e^0, u \in \mathbb{R}_+^n \right\}$$

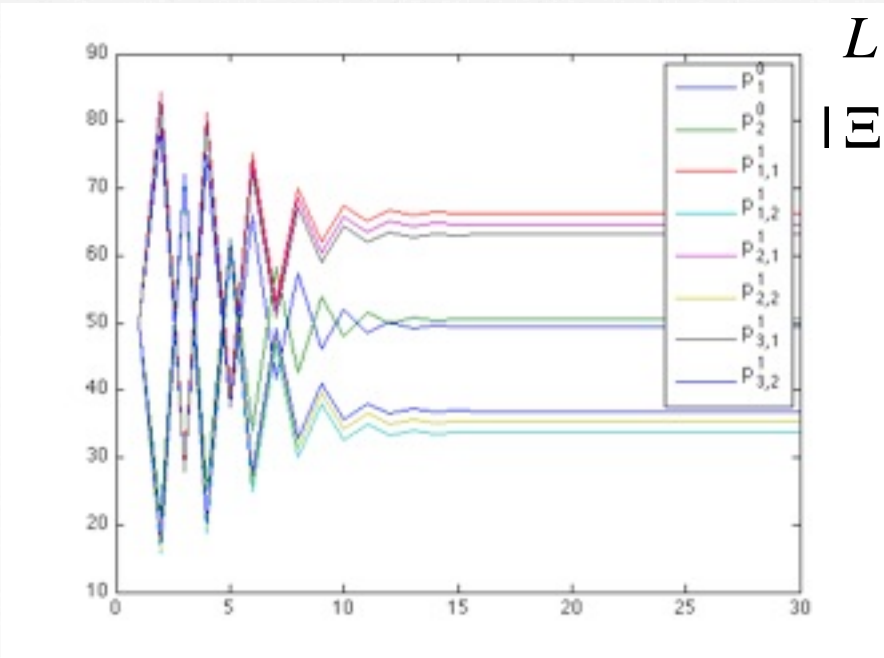
"outer loop", calculating $\big(p^0, (p_\xi^1, \xi \in \Xi)\big)$:

augmented Walrasian

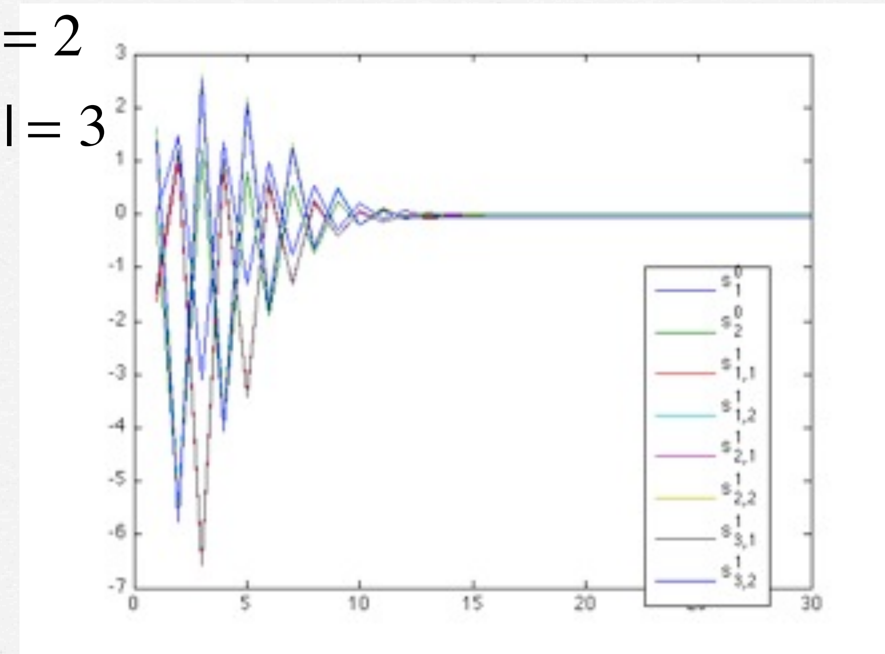
Convergence: exploiting separability

$L = 2$

$|\Xi| = 3$



prices



excess supply

4. Variational Inequality

to be dealt with now in glorious detail

back ... Arrow-Debreu model

i-agent demand: $x_i(p) \in \arg\max\left\{u_i(x) \,\middle|\, \langle p,x\rangle \le \langle p,e_i\rangle\right\}$

$u_i = X_i \to \mathbb{R}^L$, usc, concave $\not\Rightarrow X_i$ closed (but convex)

excess supply: $s(p) = \sum_{i\in I}\left(e_i - x_i(p)\right)$, market clearing: $s(p) \ge 0$

(under ample survivability, indispensability, unactractiveness)

i-agent optimal $x_i(p) \Leftrightarrow \exists \lambda_i \ge 0$ such that

$\lambda_i \perp \langle p, e_i - x_i(p)\rangle, \quad \langle p, e_i - x_i(p)\rangle \ge 0,$

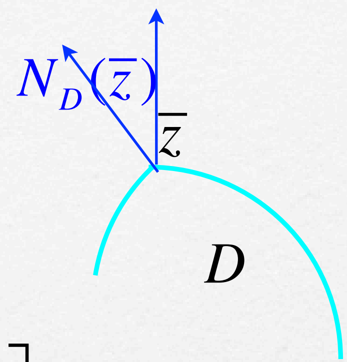
$x_i(p) \in \arg\max_{x\in X_i} u_i(x) - \lambda_i\langle p,x\rangle$

λ_i utility scaling

when $X_i = \mathbb{R}^L_+, u_i$ smooth:

$0 \le x_i(p) \perp \lambda_i p - \nabla u_i(x_i(p) \ge 0$

via Variational Inequality



$$x_i(p) \in \arg\max\left\{u_i(x) \,\middle|\, \langle p,x\rangle \le \langle p,e_i\rangle, x \in X_i\right\}$$

$$\sum\nolimits_i \left(e_i - x_i(p)\right) = s(p) \ge 0$$

$$\Updownarrow$$

$$N_D(\bar{z}) = \left\{v \,\middle|\, \langle v, z-\bar{z}\rangle \le 0, \forall z \in D\right\}$$

$$G\left(p,(x_i),(\lambda_i)\right) = \left[\sum\nolimits_i (e_i - x_i);(\lambda_i p - \nabla u_i(x_i));\langle p, e_i - x_i\rangle\right]$$

$$D = \Delta \times \left(\prod\nolimits_i X_i\right) \times \left(\prod\nolimits_i \mathbb{R}_+\right)$$

suggests LCP, NCP J.-S. Pang, M.Ferris, ..

$$-G\left(\bar{p},(x_i),(\lambda_i)\right) \in N_D\left(\bar{p},(x_i),(\lambda_i)\right)$$

D unbounded $\to \hat{D}$ bounded

geometric V.I.

via smoothing: L. Qi, X.Chen, ...

adding Firms (a parenthesis)

i-agent: $e_i \to e_i + \theta_{ij} z_j$ share θ_{ij} of production $z_j \in Z_j$ of firm $j \in J$

$$\theta_{ij} \geq 0, \ \sum_{i \in I} \theta_{ij} = 1 \quad j = 1, \ldots, J$$

$$z_j(p) \in \arg\max \left\{ \langle p, z \rangle \,\middle|\, z \in Z_j \right\}$$

excess supply: $\sum_{i \in I} (e_i - x_i(p)) + \sum_{j \in J} z_j(p) \geq 0$ (equilibrium)

functional V.I. $-G\left(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i), (\bar{z}_j)\right) \in \partial f\left(\bar{p}, (\bar{x}_i), (\bar{\lambda}_i), (\bar{z}_j)\right)$

$$f\left(p, (x_i), (\lambda_i), (z_j)\right) = \iota_{\mathbb{R}^L_+}(p) - \sum_{i \in I} u_i(x_i) + \sum_{i \in I} \iota_{\mathbb{R}_+}(\lambda_i) + \sum_{j \in J} \iota_{Z_j}(z_j)$$

f convex $\Rightarrow \partial f$ monotone operator (yields existence of solution)

$$G\left(p, (x_i), (\lambda_i), (z_j)\right)$$

$$= \left[\sum_i (e_i - x_i) + \sum_j z_j; (\lambda_i p); \left(\left\langle p, e_i + \sum_j \theta_{ij} z_j - x_i \right\rangle \right); (-p_j) \right]$$

Jofré, Rockafellar & Wets '07

Path Solver .. (M.Ferris, D.Ralph et al)

$$-G(\bar{z}) \in N_D(\bar{z}), \quad \bar{z} = (\bar{p}, (\bar{x}_i), (\bar{\lambda}_i))$$

$$D = \Delta \times \left(\prod_i X_i\right) \times \left(\prod_i \mathbb{R}_+\right) = \left\{ z \middle| Az \geq b \right\}$$

Complementarity problem:

$$-G(z) = A^T y, \; y \geq 0, \; Az - b \perp y$$

with $K = \mathbb{R}^N \times \mathbb{R}_+^M$:

$$(z,y) \in K, \; H(z,y) \in -K^*, \; (z,y) \perp H(z,y)$$

$$H(z,y) = \begin{bmatrix} G(z) + A^T y \\ Az \end{bmatrix} - \begin{pmatrix} 0 \\ b \end{pmatrix}$$

Equivalent nonsmooth mapping

- $0 = H(\mathrm{prj}_K(z,y)) + (z,y) - \mathrm{prj}_K(z,y)$
- with simplified K

  (CP) $0 \le x \perp F(x) \ge 0$ Complementarity Problem

  (NS) $0 = F(x_+) + x - x_+$ Nonlinear system

- $\bar{x}$ sol'n (CP) $\Rightarrow$ $\tilde{x}$ sol'n (NS):

  $\tilde{x}_k = \bar{x}_k$ if $F_k(\bar{x}) = 0$, $\tilde{x}_k = -F_k(\bar{x}_k)$ if $F_k(\bar{x}) > 0$

- $\tilde{x}$ sol'n (NS) $\Rightarrow \tilde{x}_+$ sol'n (CP):

  $\tilde{x}_+ \ge 0, F(\tilde{x}_+) = \tilde{x}_+ - \tilde{x} \ge 0$ & $\tilde{x}_+ \perp \tilde{x}_+ - \tilde{x}$

PATH Solver:

$x = (z, y),\ x_+ = \mathrm{prj}_K(x, y)$

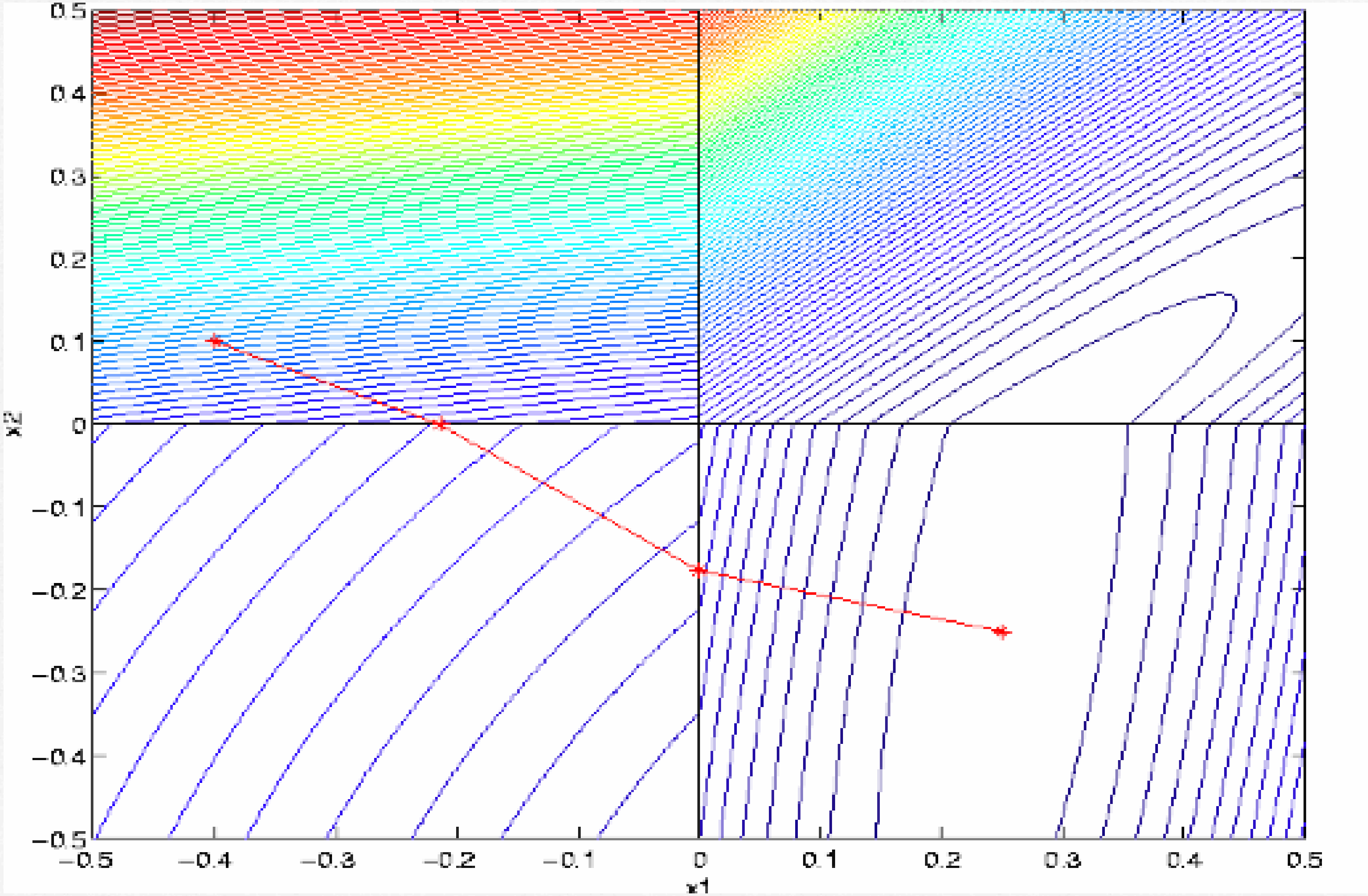
- PATH: Newton method based on nonsmooth normal mapping:

$$H(x_+) + x - x_+$$

- Newton point: solution of piecewise linearization:

$$H(x_+^k) + \left\langle \nabla H(x_+^k), x_+ - x_+^k \right\rangle + x - x_+ = 0$$

The “Newton” step



V.I.-Extensive Formulation

discrete distribution: $|\Xi|$ finite

$$G\left(\left(p^0,p^1_{\xi\in\Xi}\right),\left(x^0_i,x^1_{i,\xi\in\Xi}\right)_{i\in I},\left(\lambda^0_i,\lambda^1_{i,\xi\in\Xi}\right)_{i\in I}\right)=$$

$$\left[\left(\sum_i(e^0_i-x^0_i),\sum_i(e^1_{i,\xi}-x^1_{i,\xi})_{\xi\in\Xi}\right);\left(\left(\lambda^0_i p^0-\nabla u^0_i(x^0_i)\right)_{i\in I},\left(\lambda^1_{i,\xi}p^1_\xi-\nabla u^1_i(\xi;x^1_{i,\xi})\right)_{i\in I,\xi\in\Xi}\right);\left(\langle p^0,e^0_i-x^0_i\rangle,\left(\langle p^1_\xi,e^1_{i,\xi}-x^1_{i,\xi}\rangle\right)_{i\in I,\xi\in\Xi}\right)\right]$$

$$D=\left(\Delta\times\prod_{\xi\in\Xi}\Delta\right)\times\left(\prod_i X^0_i\times\left(\prod_i X^1_{i,\xi}\right)_{\xi\in\Xi}\right)\times\left(\prod_i\mathbb{R}_+\times\left(\prod_i\mathbb{R}_+\right)_{\xi\in\Xi}\right)$$

$$-G(\bar{z})\in N_D(\bar{z})=\left\{v\,\middle|\,\langle v,z-\bar{z}\rangle\le 0,\forall z\in D\right\}$$

$$z=\left(p^0,p^1_{\xi\in\Xi}\right),\left(x^0_i,x^1_{i,\xi\in\Xi}\right)_{i\in I},\left(\lambda^0_i,\lambda^1_{i,\xi\in\Xi}\right)_{i\in I}$$

"Thanks the gods for EMP"

for a special VI handled via smoothing/sampling:
listen to Xiajun Chen

so, let's go: PATH Solver

- Economy: (8 goods), 5 types of agents
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks
- small # of scenarios 280,
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer to time 1

unfortunately, ... PATH Solver let's down

Disaggregation!

initialization: $\forall \, \mathrm{i} \in I$

$w_{i,\xi}$ such that $E\left\{w_{i,\xi}\right\} = 0$

$\forall \xi \;\Rightarrow\;$ equilibrium with $\left\{u_i^{w_{i,\xi}}(\xi;\bullet)\right\}_{i\in I}$

yields: $(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}, \bar{x}^1_{i,\xi}), \quad (\bar{p}^0_{i,\xi}, \bar{p}^1_{i,\xi})$

STOP $\forall i \in I : \left(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}\right) \simeq E\left\{\left(\bar{x}^0_{i,\xi}, \bar{y}_{i,\xi}\right)\right\}$

otherwise adjust $\left[w_{i,\xi}, \xi \in \Xi\right]_{i\in I}$

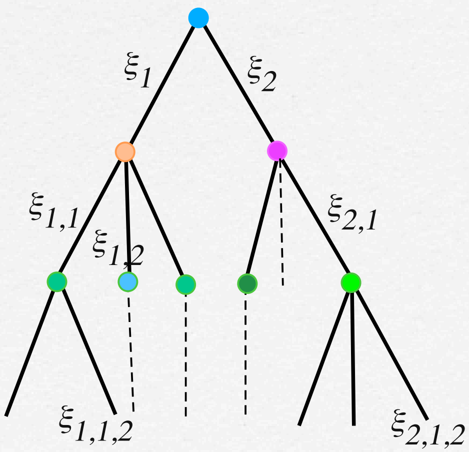
stochastic-pure exchange

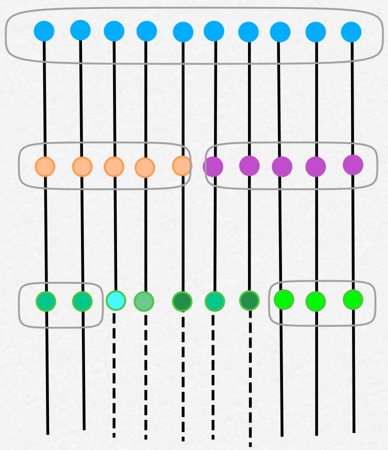
for each $\xi \in \Xi$: the equilibrium problem, I agents with

i-agent's problem: $\max \left\{ u_i^{w_{i,\xi}} \left(\xi; x^0, y, x^1 \right) \text{ on } \widehat{C}_{i,\xi}(p^0, p^1_\xi) \right\}$

$$u_i^{w_{i,\xi}} = u_i^0(x^0) - \left\langle w_{i,\xi}, (x^0, y) \right\rangle - \frac{\rho}{2} \left\| (x^0, y) - (\bar{x}^0, \bar{y}) \right\|^2 + u_i^1(\xi; x^1)$$

clearing the market: $s^0(p^0, p^1_\xi) \geq 0, \;\; s^1_\xi(p^0, p^1_\xi) \geq 0$





but now with $w_{i,\xi} \sim$ constraints

$x^0(\xi) \equiv x^0$ constant

$x^1(\xi | \xi^1) \equiv x^1_1 , x^1(\xi | \xi^2) \equiv x^1_2$

…

Disaggregation with PATH Solver

- Economy: (5 agents - 8 goods)
  - Skilled & unskilled workers
  - Businesses: Basic goods & leisure
  - Banker: bonds (riskless), 2 stocks

on M. Ferris
semi-slow laptop
using EMP-package
4 min + 2 min for
verification

- 2-stages, solved under # of scenarios (280)
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer to time 1
- used for calibration (-> stochastic model)

numerically: `blink' (5000 iterations).

Ja! scenario disaggregation, but ...

i-agent: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p,x \rangle \le \langle p,e_i \rangle \right\},\ i \in I$

with excess supply $s(p)$: $\quad 0 \le p \perp s(p) \ge 0$

Multi-Optimization Problem with Equilibirum Constraint

MOPEC-class ~ maxinf family

$$x_i \in \arg\max_{x \in \mathbb{R}^{n_i}} f_i(p, x_i, x_{-i}), \quad i \in I, \quad x_I = (x_i, i \in I)$$

$$D(p, x_I) \in \partial g(p) \quad [\text{or } \in N_C(p)]$$

with Michael Ferris '11-'?? ... '05?

Examples: Walras, noncooperative games, ...
stochastic (dynamic): decentralized electricity markets,
joint estimation and optimization, financial equilibirum, ...

Financial Markets

Contracts (Assets)

assets (= contract types): $k \in K$, $|K|$ finite

$z_i = z_i^+ - z_i^- = (z_i^1, \ldots, z_i^K)$ assets 'acquired' by i-agent

q_k market price of asset k

D_ξ^k bundle of goods 'delivered' by one unit of asset k

budgetary constraints:

$$\langle p^0, x^0 + T_i^0 y \rangle + \langle q, z \rangle \leq \langle p^0, e_i^0 \rangle$$

$$\langle p_\xi^1, x_\xi^1 \rangle \leq \langle p_\xi^1, e_{i,\xi}^1 + T_{i,\xi}^1 y + D_\xi z \rangle \quad \forall \xi \in \Xi$$

clearing the market:

$$s^0\left(p^0, \left(p_\xi^1\right)_{\xi \in \Xi}\right) \geq 0,\ s^1\left(\xi; p^0, \left(p_\xi^1\right)_{\xi \in \Xi}\right) \geq 0\ \forall \xi, \boxed{\sum_{i \in I} z_i = 0}$$

The BDE-example

Brown-DeMarzo-Eaves (Econometrica '96)

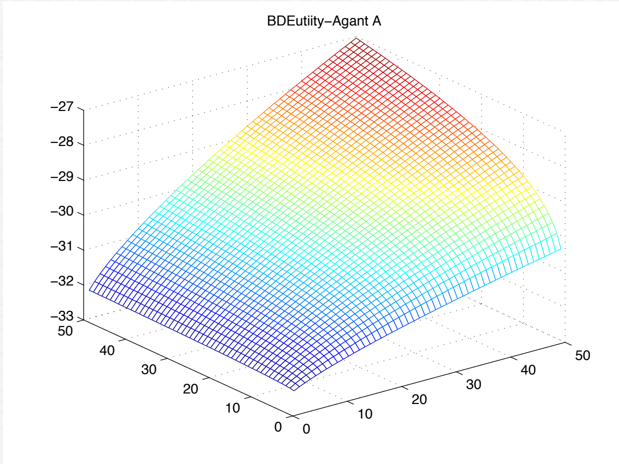
3 agents (2-agent & 3-agent of the same type)

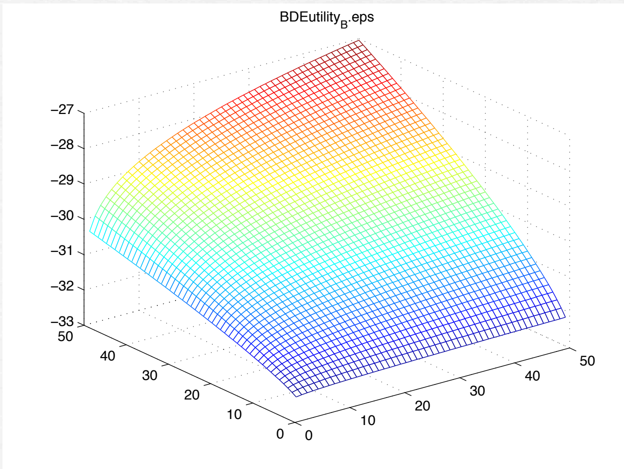
2 goods, $|\Xi| = 3$ (future states), no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x)$$

$$\alpha_1 = (0.25, 0.75), \ \alpha_{2\&3} = (0.75, 0.25)$$

$$\text{asset \#1: } D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ asset \#2: } D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for all } \xi$$





The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x), \quad \alpha_1 = (0.25, 0.75), \quad \alpha_{2\,\&\,3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ

Path Solver solution: $p^0 = (1,\ 0.73)$ (with scaling)

$p_\xi^1 = (1,\ 0.7159;\ 1,\ 0.7181;\ 1,\ 0.7206)$

$q = (1,\ 0.72)$, $z = (72.98, -100; -36.49, 50;\ -36.49, 50)$

sol'n time: not noticable

The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x), \quad \alpha_1 = (0.25, 0.75), \quad \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ

BDE- solution: $p^0 = (1,\ 0.74)$ (with scaling)

$p_\xi^1 = (1,\ 0.7375;\ 1,\ 0.7174;\ 1,\ 0.6633)$

$q = (??,\ ??)$, $z = (0.94, 0;\ 0.03, 0;\ 0.03, 0)$ ⟸ all buyers no sellers

change of variables + add unconstrainted agent:

homotopy continuation method (predictor-corrector steps)

The Cass trick

or the no-arbitrage condition

$$q = \sum_{\xi\in\Xi} w_\xi \left(D_\xi p_\xi^1\right) \text{ for some weights } w_\xi \geq 0,\ \sum_\xi w_x i = 1$$

$$\max u_i\left(x^0, (x_\xi^1)_{\xi\in\Xi}\right) = u_i^0(x^0) + E\left\{u_i^1(\xi; x_\xi^1)\right\}$$

such that $\left\langle p^0, e_i^0 - x^0\right\rangle + \langle q, z\rangle \geq 0$

$$\left\langle p_\xi^1, e_{i,\xi}^1 + D_{i,\xi} z - x_\xi^1\right\rangle \geq 0, \forall \xi \in \Xi$$

$$\tilde{p}^0 = p^0,\ \tilde{p}_\xi^1 = w_\xi p_\xi^1, \xi \in \Xi$$

solved by BDE

Path Solver $\Rightarrow$ BDE-sol'n

no Path Solver sol'n!

via Augmented Walrasian

for 'money' assets (Deride, Jofré & Wets '09)

cf. financial equilibirum: Hens & Pilgrim '06

$\max u_i\left(x^0, (x_\xi^1)_{\xi\in\Xi}\right)$ such that

$$\left\langle \tilde{p}^0, e_i^0 - x^0\right\rangle + \sum_{\xi\in\Xi}\left\langle \tilde{p}_\xi^1, D_\xi z - x_\xi^1\right\rangle \geq 0$$

$$\left\langle \tilde{p}_\xi^1, e_{i,\xi}^1 + D_{i,\xi} z - x_\xi^1\right\rangle \geq 0,\ \forall \xi \in \Xi$$

$$s^0\left(\tilde{p}^0, \left(\tilde{p}_\xi^1\right)_{\xi\in\Xi}\right) \geq 0,\ s^0\left(\xi; \tilde{p}^0, \left(\tilde{p}_\xi^1\right)_{\xi\in\Xi}\right) \geq 0, \forall \xi$$

clearing the market also: $\sum_{i\in I} z_i = 0$

Further readings

- Jofré, A. & R. Wets, Variational convergence of bivariate functions: theoretical foundations. Mathematical Programming (2006).
- Jofré, A., R.T. Rockafellar R.T & R. Wets. Variational Inequalities and economic equilibrium. Mathematics of Operation Research (2006?)
- Jofré, A., RT. Rockafellar & R. Wets, A variational inequality scheme for determining an economic equilibrium of classical or extended type. In "Variational analysis and applications", 553--577, Nonconvex Optimisation and Applications, 79, Springer, New York, 2005.
- Jofré, A. & R. Wets, Continuity properties of Walras equilibrium points. Stochastic equilibrium problems in economics and game theory. Annals of Operations Research, 114 (2002), 229--243.
- S. P. Dirkse and M. C. Ferris. The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems. Optimization Methods and Software, 5:123-156, 1995.