PRICING CONTINGENT CLAIMS: A COMPUTATIONAL COMPATIBLE APPROACH *

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Abstract. In this paper, we develop an 'operational' duality theorem, which can be applied to a contingent claims pricing model. By virtue of this theorem we can find that there is duality between pricing contingent claims and that of finding (strictly) equivalent martingale measures for a given stochastic process, that in our context corresponds to a description of the state of the financial market; in a practical environment, it is reasonable to assume that such *strictly equivalent martingale measures* exist. Also this theorem, allows us to discuss no-arbitrage, hedging, equilibrium equations, and so on.

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1 Formulation

1.1 Problem Description

A *contingent claim*, also called a derivative security, associated with one or more financial contracts, is derived from the values of other basic financial market securities, such as stocks or bonds.

We begin with the discrete time case. In general, the financial environment can be described by the states of a $I\!R^d$-valued stochastic process $\{\xi_t\}_{t=0}^T$, to which we refer as the *environment process*, and for $t = 1, \ldots, T$, let

$$\underset{\to t}{\xi} = (\xi_0, \xi_1, ..., \xi_t) \in I\!R^{N_t}, \quad \text{where} \quad N_t = (t+1)d,$$

so $\underset{\to t}{\xi}$ represents the history of the environment process up to time t. Let's denote the market prices process of the basic securities by

$$S_t(\underset{\to t}{\xi}) = (S_t^1, S_t^2, ..., S_t^n)$$

Without loss of generality, we can choose the risk-free asset to be that with index 1 and convert others to prices relative to S_t^1 by $S_t^i = S_t^i / S_t^1$. Then, $S_t^1 = 1$ and will play the role of our numeraire.

A contingent claim can be expressed in terms of a collection of functions, for $t = 1, \ldots, T$, whose values, at time t given the environment $\underset{\to t}{\xi}$ determine the 'claim', positive or negative, that will have to be 'paid out,' by the writer of the contingent claim, i.e.,

$$\{G_t : I\!R^{N_t} \to R\}$$

One (extremely) simple example is coupon payments but in general G_t can be a quite involved function that takes into account the full, or simply a part, of the past history of the environment process.
The writer of the contingent claim shall set up a portfolio to meet these 'claims', by choosing an investment strategy $\{X_t(\underset{\to t}{\xi}),\ t = 1, \ldots, T\}$, the value of this portfolio at time t is:

$$\langle S_t(\underset{\to t}{\xi}), X_t(\underset{\to t}{\xi}) \rangle$$

It's said to be *self-financing* if

$$\langle S_{t+1}(\underset{\rightarrow}{\xi}_{t+1}), X_{t+1}(\underset{\rightarrow}{\xi}_{t+1})\rangle = \langle S_{t+1}(\underset{\rightarrow}{\xi}_{t+1}), X_t(\underset{\rightarrow}{\xi}_{t})\rangle,$$

i.e., the value of the new allocation X_{t+1} is consistent (equal) with the value of the portfolio associated with the pre-existing allocation X_t.

1.2 Problem Formulation

The writer of the contingent claim seeks to maximize terminal wealth while meeting all the claims of the contract:

$$\begin{aligned}
&\max\ E\{\langle S_T(\underset{\rightarrow}{\xi}_{T}), X_T(\underset{\rightarrow}{\xi}_{T})\rangle\} \\
&\text{so that } \langle S_0(\xi_0), X_0(\xi_0)\rangle \leq G_0(\xi_0), \\
&\qquad \langle S_t(\underset{\rightarrow}{\xi}_{t}), X_t(\underset{\rightarrow}{\xi}_{t}) - X_{t-1}(\underset{\rightarrow}{\xi}_{t-1})\rangle \leq G_t(\underset{\rightarrow}{\xi}_{t}), t = 1, ..., T \\
&\qquad \langle S_T(\underset{\rightarrow}{\xi}_{T}), X_T(\underset{\rightarrow}{\xi}_{T})\rangle \geq 0 \ a.s.
\end{aligned}$$

Usually, G_0 is positive, that could mean that the writer borrows or receives an initial investment, $G_1, .., G_T$ are generally, but not necessarily, negative quantities, claims that will have to be met, i.e., paid out by the writer. The first constraint is quite natural, it means that in any case your initial portfolio value should be less or equal to the initial investment. We can rewrite the second constraint as

$$\langle S_t, X_t\rangle \leq \langle S_t, X_{t-1}\rangle + G_t,$$

$\langle S_t, X_{t-1}\rangle$ is the actual value of portfolio X_{t-1} at time t, remember that you also have to make payment G_t, therefore the portfolio value at time t should end up with a value less than or equal to $\langle S_t, X_{t-1}\rangle + G_t$. Finally, under no circumstance are you willing to lose any money, therefore the terminal wealth should be nonnegative whatever be the observed environment. That's the last constraint.

2 Simple Examples

2.1 Example Stock Trading. *Let S_t be the stock prices at time t, X_t the corresponding quantities of shares of the stocks. Assume the writer of this* *contingent claim (stock contract) borrows* \$5000 *from a buyer to invest in stocks, and has to make ten monthly payments to the buyer of at least* \$520 *a month, then* $G_0 = 5000$, $G_t \equiv -520$, $t = 1, 2, ..., 10$.

Detail. So, at time 0, the portfolio value (buying power) is subject to: $\langle S_0, X_0 \rangle \leq 5000$. In the first month, the stock prices are changing, your actual wealth value is $\langle S_1, X_0 \rangle$, and you also need to pay the contingent claim, therefore this month's portfolio value is subject to: $\langle S_1, X_1 \rangle \leq \langle S_1, X_0 \rangle - 520$, and so on. In the tenth month, $\langle S_{10}, X_{10} \rangle \leq \langle S_{10}, X_9 \rangle - 520$.

And the terminal wealth requires $\langle S_{10}, X_{10} \rangle \geq 0$. Naturally, the writer's goal is to maximize his expected terminal wealth, $\max E\{\langle S_{10}, X_{10} \rangle\}$, under all the preceding constraints.

In practice, some minor changes may be needed. For example, suppose that you open a marginal account with \$5000, then your buying power is $\$2 \cdot 5000$ instead of just \$5000. □

2.2 Example Future Contracts *A future contract holder has the right to purchase or sell a specific amount of a commodity at a future market delivery price.*

Detail. Suppose that contracts are initially written at price P_0 and the next day the price becomes P_1, and suppose $P_1 > P_0$. If one holds a one-unit long position with price P_0, then the profit is simply $P_1 - P_0$, otherwise, if one has taken a short position, the loss is $P_1 - P_0$. Except for the context, the formulation of the problem is similar to that involving stock trading. □

In some specific instances, some constraints may need to be revised and some new constraints may need to be added, but the basic formulation of the problem remains essentially the same.

3 Approaches

Contingent Claim Pricing problems are simply stochastic optimization problems that arise in the context of Mathematical Finance. Many models (Black-Scholes, Whitney,etc) are based on stochastic differential equations under some, rather strong, assumptions on the environment and prices processes. By solving certain partial differential equations many results can be derived, such as the existence of equivalent martingale measures, hedging, etc., but some of these assumptions are too far from 'reality'. Of course, in some instances, they could be valid simplifications. For example, Black-Scholes model's basic assumption is that the market price process is a geometric Brownian motion, but statistical analysis quickly reveals that this is seldom the case and thus not close to 'reality'. Moreover, this model can't explain the famous 'σ-smile' phenomena. Later, in the 1980's, people resorted to some weaker assumptions, such as semi-martingale models, or some new approaches, such as functional analytic approach. But semi-martingales models also came with more complex stochastic differential equations, almost impossible to solve in many situations.. Although the functional analytic approach allows for some elegant results about the first fundamental theorem of asset pricing, cf. [5], [19], under slightly more general assumptions than those used here, it is shown that no-arbitrage is equivalent to the existence of equivalent martingale measures, yet these results are more of a theoreoretical nature and have do not hold much promise for even a potential computational (efficient) procedures. Moreover, this approach cannot deal with financial problems in incomplete market because martingale measures may not be attainable in incomplete market.

In the 1990's, people started to analyze the semi-martingale models by optimization techniques, such as in [3], [2], by duality or Legendre-transform, they derived some properties of the dual problem and its relationship with the original problem, but these approaches yet can't be used for practical computation. The greatest contribution of these approaches is that one may easily think of the possibility of formulating the pricing problems as stochastic optimization problems. Based on [17, 13, 15] on duality in stochastic programming, we can finally analyze the pricing problems by stochastic duality techniques. This duality in a stochastic programming framework makes it possible to connect theory, practice and computation.

In 2001, A. King and L. Korf [8] proposed a similar approach as ours, but they directly used the duality in Rockfellar and Wets [14], [16] in the dual of L^∞ for pricing contingent claims problems, where they had to deal with singular multipliers by some special techniques, introducing 'induced constraints', and in order to use that duality they had to make the assumption that the market price process is essentially bounded, but even if the price is log-normal, this assumption is not satisfied, and they didn't provide a method for practical computation. We deal with musch weaker assumptions, in particular without restricting market prices to belongs to L^1, we derive an 'operational' Duality Theorem, that allows us to establish a duality between pricing contingent claims and finding equivalent martingale measures for a given stochastic process, that in our context corresponds to a description of the state of the financial market. And by virtue of this duality we can discuss no-arbitrage, hedging, equilibrium equation, etc. In practice, for numerical computational purposes, we have to assume that *strictly equivalent martingale measures* exist, we shall explain later that actually this assumption is intrinsically a 'natural' one and it makes actual computation possible. In the follow-up paper [21], we show how to gather information via this duality and how to discretize efficiently the problem at hand, and incidentally, we also propose a novel approach for estimating the price distribution from the historical price data.

This paper is organized as follows. The main result is an operational duality theorem in §4, in the following sections §5 − 7 this theorem is used to bring to the fore the relationship between no-arbitrage and equivalent martingale measures, *strictly equivalent martingale measures.* Hedging and equilibrium equations are discussed, and some interesting examples are provided. In the last sections, a brief overview of the continuous time case is provided as well as a counterexample to the possibility of extending the results involving 'strictly' equivalent meausre to ('pure') equivalent martingale measures.

4 An operational duality theorem

In general, duality theory is always related to the existence of saddle points, which means that both original problem and dual problem have the same optimal value and the optimal values could be attained, or equivalently saddle points of the associated Lagrangian exist. If we want to prove the existence of saddle points, it may require some additional assumptions or some special techniques, such as introducing induced constraints, refer to [15]. In some cases, such as in our problem, we are just going to require that the optimal values of the primal and the dual are the same and that the minimum of our primal is actually attained. That's leads us to develop an operational duality theorem that in some ways is weaker, but in this situation more useful, than the 'standard' duality results. We begin with a brief introduction followed by the duality theorem that will eventually lead to implementable procedure to price contingent claims.

The 'duality' will focus on the interchangeability of 'min' and 'sup' operators in the Lagrangian function; we are not concerned with the existence of an optimal solution for the dual problem. Our basic duality scheme rest on deriving the identity $h^{**} = h$. For stochastic optimization problems of the type

$$\min E\{f(\boldsymbol{\xi}, x(\boldsymbol{\xi}))\} \quad \text{where} \quad \boldsymbol{\xi} \text{ is a random variable,}$$

our aim will thus be to obtain $(Ef)^{**} = Ef$. If we already know that $f^{**} = f$ and $(Ef)^* = E(f^*)$ under some conditions, then one expects that $(Ef)^{**} = E(f^{**}) = Ef$. Therefore, $(Ef)^* = E(f^*)$, the interchangeability of expectation $E\{\cdot\}$ and conjugation '*' becomes the key stone on which rest the duality results.

For a function $\xi \mapsto f(\xi, x(\xi))$, the first question is under what conditions is this function measurable when $\xi \mapsto x(\xi)$ is measurable? Then, under what conditions can expectation E and conjugation '*' be interchanged? The following set-up answers these questions.

Let lsc-fcns(X) denote the space of extended real-valued, lower semicontinuous (lsc) functions from X to $\overline{I\!R}$. Given a probability space $(\Xi, \mathcal{A}, P)$, a *random lsc function* is a function $f : \Xi \to \text{lsc-fcns}(X)$ such that the associated *epigraphical mapping*

$$\xi \mapsto S_f(\xi) = \text{epi}\, f(\xi,\cdot) = \{(x,\alpha) \in X \times I\!R \,|\, f(\xi,x) \geq \alpha\}$$

is a random closed set, i.e., for any open set $O \subset I\!R^{n+1}$, $S_f^{-1}(O) = \{\xi \in \Xi \,|\, S_f(\xi) \in O\}$ belongs to $\mathcal{A}$.† Further properties of random lsc functions are set forth in [18, Chapter 14], see also[1, 9, 10, 11], let's just record some useful properties used in the sequel.

4.1 Proposition *[18, Proposition 14.28, Example 14.29]. When f is a random lsc function, then $\xi \mapsto f(\xi, x(\xi))$ is measurable whenever $\xi \mapsto x(\xi)$ is measurable.*

Any function $f : \Xi \times I\!R^n \to R$ such that $f(\cdot, x)$ is measurable for all x and $f(\xi,\cdot)$ continuous for all ξ is a random lsc function.

Our contingent claims pricing problem is actually a linear optimization problem, and all the functions, the objective function and the functions in the constraints, are random lsc functions.

Denote by $(T, \mathcal{T}, \mu)$ a measurable space; here T is a non-empty set, $\mathcal{T}$ is a $\sigma-$field on T, μ is just some measure on $(T, \mathcal{T})$, not necessarily a probability measure.

4.2 Definition (decomposable spaces, [18, Definition 14.59]) . *A space $\mathcal{L}$ of measurable functions $f : T \to I\!R^n$ is decomposable if for every function $f_0 \in \mathcal{L}$, every set $A \in \mathcal{T}$ with $\mu(A) < \infty$ and any bounded, measurable function $f_1 : A \to I\!R^n$, $\mathcal{L}$ also contains the function $f : T \to I\!R^n$ defined by $f(t) = f_0(t)$ for $t \in T\backslash A$, $f(t) = f_1(t)$ for $t \in A$.*‡

Let $I_f(x) := \int f(t, x(t))\mu(dt)$ be a functional with f a random lsc functions and $x \in \mathcal{L}$ where $\mathcal{L}$ consists of measurable functions. For $u \in \mathcal{L}^*$, the

†The concept of a random lsc function is due to Rockafellar [12] who introduced it in the context of the Calculus of Variations under the name of *normal integrand*.

‡For example, the Lebesgue spaces $\mathcal{L}^p(T, \mathcal{T}, \mu; I\!R^n)$, with $0 < p \leq \infty$, are decomposable.

conjugate of I_f is defined by

$$I_f^*(u^*) = \sup_x \Big\{ \int x \cdot u^* \mu(dt) - I_f(x) \Big\}.$$

When $\mathcal{L}$ is a Banach space, $\mathcal{L}^*$ doesn't necessarily have to be its dual.

4.3 Theorem ([12, Theorem 2]) . *Suppose $\mathcal{L}$ and $\mathcal{L}^*$ are decomposable. Let f be a convex random lsc function, i.e., $x \mapsto f(t,x)$ is also convex for all $t \in T$, and such that $t \mapsto f(t, x(t)$ is summable for at least one $x \in \mathcal{L}$ and $t \mapsto f^*(t, u^*(t))$ is summable for at least one $u^* \in \mathcal{L}^*$. Then I_f on $\mathcal{L}$ and I_{f^*} on $\mathcal{L}^*$ are proper convex functions conjugate to each other.*

By the preceding theorem and 'perturbation' theory then we can develop our duality theory for stochastic optimization problems. We shall use the same notations as in [14]. We are interested in the two-stage recourse problem:

$$\begin{aligned} \min\ & f_{10}(x_1) + E\{f_{20}(\xi, x_1, x_2(\xi))\} \\ \text{so that}\ & f_{1i}(x_1) \leq 0, i = 1, ..., m_1, \\ & f_{2i}(\xi, x_1, x_2(\xi)) \leq 0, i = 1, ..., m_2, \\ & x_1 \in C_1, x_2(\xi) \in C_2. \end{aligned}$$

where $x_1 \in I\!R^{n_1}, x_2(\xi) \in \mathcal{L}^\infty_{n_2}$, C_1 and C_2 are bounded, closed, convex and nonempty, C_2 doesn't depend on ξ, all functions are random lsc functions, everywhere defined and summable with respect to P our probability measure.

Duality is developed by embedding the problem in a class of 'perturbed' problems. Let,

$$X = I\!R^{n_1} \times \mathcal{L}^\infty_{n_2} \quad \text{and} \quad U = I\!R^{m_1} \times \mathcal{L}^1_{m_2},$$

where $\mathcal{L}^p_n$ denotes the usual Lebesgue space of $I\!R^n$-valued functions over $(\Xi, \mathcal{A}, P)$. Notice that the only difference is that here 'perturbation' functions belong to $I\!R^{m_1} \times \mathcal{L}^1_{m_2}$ instead of $I\!R^{m_1} \times \mathcal{L}^\infty_{m_2}$ as in [14]. However, argument will proceed along similar lines as in [14]. The function $F : X \times U \to (-\infty, \infty]$ is defined as follows,

$$F(x, u) = F_1(x_1, u_1) + E\{F_2(\xi, x_1, x_2(\xi), u_2(\xi))\}$$

where

$$F_1(x_1,u_1)=\begin{cases} f_{10}(x_1) & \text{if } x_1\in C_1 \text{ and } f_{1i}(x_1)\le u_{1i},\ i=1,2,..m_1,\\ \infty & \text{otherwise.}\end{cases}$$

and

$$F_2(\xi,x_1,x_2(\xi),u_2(\xi))=\begin{cases} f_{20}(\xi,x_1,x_2) & \text{if } x_2\in C_2, \text{ and}\\ & f_{2i}(\xi,x_1,x_2)\le u_{2i},\ i=1,2,..m_2,\\ \infty & \text{otherwise.}\end{cases}$$

Define,

$$\langle u,y\rangle = u_1\cdot y_1 + E\{u_2(\xi)\cdot y_2(\xi)\},\quad \text{for}\quad y\in Y:=I\!R^{m_1}\times \mathcal{L}^{\infty}_{m_2},$$

then the Lagrangian function is defined by

$$L(x,y)=\inf_u\{\langle u,y\rangle + F(x,u)\},$$

and it's easy to calculate that

$$L(x,y)=\begin{cases} L_1(x,y)+E\{L_2(s,x_1,x_2(\xi),y_2(\xi))\}, & x\in X_0, y\in Y_0,\\ \infty, & x\notin X_0,\\ -\infty, & x\in X_0, y\notin Y_0,\end{cases}$$

where

$$X_0=\left\{x=(x_1,x_2)\in X \,\middle|\, x_1\in C_1,\ x_2(\xi)\in C_2 \text{ a.s.}\right\},$$
$$Y_0=\left\{y=(y_1,y_2)\in Y \,\middle|\, y_1\ge 0,\ y_2(\xi)\ge 0 \text{ a.s.}\right\},$$
$$L_1(x_1,y_1)=f_{10}(x_1)+\sum_{i=1}^{m_1} y_{1i}f_{1i}(x_1),$$
$$L_2(\xi,x_1,x_2,y_2)=f_{20}(\xi,x_1,x_2)+\sum_{i=1}^{m_2} y_{2i}f_{2i}(\xi,x_1,x_2).$$

Let,

$$\inf P := \inf_{x\in X}\sup_{y\in Y} L(x,y),\qquad \sup D := \sup_{y\in Y}\inf_{x\in X} L(x,y),$$

set,

$$I_h(z)=\int h(\xi,z(\xi))P(d\xi).$$

One more assumption is needed to derive our results.

4.4 Assumption *$I_{f_{20}^*}$ is well defined, i.e., $I_{f_{20}^*}(w) < \infty$ for some $w \in \mathcal{L}_n^1$ for $n = n_1 + n_2$.*

Since obviously, $I_{f_{20}}(w) < \infty$ for some $w \in \mathcal{L}_n^\infty$, by Theorem 4.4, the integral functionals $I_{f_{20}}$ and $I_{f_{20}^*}$ are conjugate to each other with respect to the natural paring between $\mathcal{L}_n^\infty$ and $\mathcal{L}_n^1$. In particular,

$$f_{20}(\underset{\rightarrow T}{\xi}, X_T) = E\{\langle S_T(\underset{\rightarrow T}{\xi}), X_T(\underset{\rightarrow T}{\xi})\rangle\}$$

in our contingent claim model, it is easy to see that $f_{20}^*(\underset{\rightarrow T}{\xi}, S_T(\underset{\rightarrow T}{\xi})) = 0$ for any $\underset{\rightarrow T}{\xi}$. Therefore $I_{f_{20}^*}(S_T) < \infty$ and $(I_{f_{20}})^* = I_{f_{20}^*}$.

We now all set to state the main results.

4.5 Theorem *Define $\varphi(u) = \inf_{x \in X} F(x, u), u \in U$. Then φ is a proper convex function on U which is lsc with respect to the weak topology, and the infimum is always attained. In particular, $\varphi^{**} = \varphi$, $\min P = \sup D > -\infty$.*

The proof requires the following lemma.

4.6 Lemma *The functional F on $X \times U$ is lsc, convex and not identically ∞, the lower semi-continuity being not only with respect to the norm topology, but also with respect to the weak topology on $X \times U$ induced by the pairing introduced earlier,*

$$\langle u, y\rangle = u_1 \cdot y_1 + E\{u_2(\xi) \cdot y_2(\xi)\}, \quad \textit{for} \quad y \in Y := I\!R^{m_1} \times \mathcal{L}_{m_2}^\infty,$$

Proof. We just need to prove the lower semi-continuity with respect to the weak topology; for the remaining properties one can refer to [14, Proposition 3]. That $I_{F_2}(z) < \infty$ for any $z \in \mathcal{L}_n^\infty$ is immediate. Let $h(\xi, z) = f_{20}(\xi, x_1, x_2)$, since $h \le F_2$, one has $h^* \ge F_2^*$. Taking any $w \in \mathcal{L}_n^1$ such that $I_{h^*}(w) < \infty$, one also has $I_{F_2^*}(w) < \infty$. Hence, by Theorem 4.4 $I_{F_2^*}$ and I_{F_2} are conjugate to each other, and, in particular, are lsc with respect to the weak topology from which follows the lower semi-continuity of F. □

Proof of the theorem. The argument is similar to that of the proof of [14, Theorem 3], only some minor adjustments are required. We begin by showing that

$$X_0' = \{x_2 \in \mathcal{L}^\infty_{n_2} | x_2(\xi) \in C_2 \text{ a.s.}\}$$

is compact in the weak topology induced on $\mathcal{L}^\infty_{n_2}$ by $\mathcal{L}^1_{n_2}$. Certainly X_0' is relatively compact in this topology, inasmuch as C_2 is bounded. There remains to verify that X_0' is also closed, and consequently compact. Consider the function h on $\Xi \times I\!R^{n_2}$ defined by:

$$h(\xi, x_2) = \begin{cases} 0 & \text{if } x_2 \in C_2, \\ \infty & \text{if } x_2 \notin C_2. \end{cases}$$

This is a convex random lsc function, because C_2 is a nonempty, closed, convex set. The corresponding integral functional I_h on $\mathcal{L}^\infty_{n_2}$ satisfies

$$I_h(x_2) = \begin{cases} 0 & \text{if } x_2 \in X_0', \\ \infty & \text{if } x_2 \notin X_0'. \end{cases}$$

In particular, $I_h(x_2) < \infty$ for at least one $x_2 \in \mathcal{L}^\infty_{n_2}$. On the other hand, the conjugate integrand

$$h^*(\xi, v_2) = \sup_{x_2 \in R^{n_2}} \{x_2 \cdot v_2 - h(\xi, x_2)\}$$

has $h^*(s, 0) \equiv 0$, and hence $I_{h^*}(v_2) < \infty$ for at least one $v_2 \in \mathcal{L}^1_{n_2}$, namely $v_2 = 0$. It follows that I_h on $\mathcal{L}^\infty_{n_2}$ and I_{h^*} on $\mathcal{L}^1_{n_2}$ are convex functionals conjugate to each other, and this implies, among other things, that I_h is lower semicontinuous with respect to the weak topology induced on $\mathcal{L}^\infty_{n_2}$ by I_h on $\mathcal{L}^1_{n_2}$. But, $X_0' = \{x_2 \in \mathcal{L}^\infty_{n_2} \,\big|\, I_h(x_2) \leq 0\}$, is just the level set $\operatorname{lev}_0 I_h$ of this lsc function and hence closed as claimed.

This, in turn implies that X_0 is compact, and hence in the definition of

$$\varphi(u) = \inf_{x \in X} F(x, u),$$

the infimum is always attained, since F is lsc in the weak topology, cf. Lemma 4.6, and

$$F(x, u) < \infty \text{ implies } x \in X_0.$$

Thus, like F, φ is not identically ∞ and nowhere has the value $-\infty$, i.e., φ is proper, and the level sets $\operatorname{lev}_\alpha \varphi = \{u \in U \mid \varphi(u) \leq \alpha\}$ are the projection on U of the corresponding level sets of F:

$$\operatorname*{lev}_\alpha F = \{(x, u) \in X \times U | F(x, u) \leq \alpha\}.$$

But, the projection of $\operatorname{lev}_\alpha F$ on U is closed in the weak topology. This holds because (i) $\operatorname{lev}_\alpha F$ is closed by the lower semicontinuity of F, and (ii) for all α this projection on X is contained in the compact set X_0. Therefore φ is lower semicontinuous in the weak topology induced on U by Y. Inasmuch as φ is a proper convex function on U which is lower semicontinuous in a topology compatible with the paring between U and Y, we have $\varphi^{**} = \varphi$. In terms of the biconjugate φ^{**}, one has

$$\varphi^{**}(0) = \sup D,$$

whereas,

$$\varphi(0) = \inf P,$$

therefore:

$$\min P = \sup D > -\infty,$$

and the infimum is always attained. □ .

4.7 Remark *The key step of proof is to prove that $I_{F_2^*}$ and I_{F_2} are conjugate to each other, or equivalently, that the expectation operator 'E' and '*' are commutative. The choice of random lsc functions is predicated to render this interchange possible.*

4.8 Remark *If we allow for $x_1 \in \mathcal{L}^{n_1}$, instead of $x_1 \in I\!R^{n_1}$, all the preceding goes through, we just need to adjust some notations. In terms of our contingent claim model, it means that ξ_0 does not necessarily have to be fixed, i.e., it could also be random.*

If the probability space Ξ only has finite support, we are then dealing with a 'discrete case' duality result. But note that in our proof of duality, we don't have to consider constraint qualifications (such as strictly feasible, for example) as is usual. Thus, we have the following even in the finite dimensional case, of the form

$$\begin{aligned} &\min\ f_0(x) \\ \text{so that } &f_i(x) \leq 0,\ x \in I\!\!R^n,\ i = 1, ..., m, \\ &x \in C. \end{aligned}$$

where $f_i, i = 0, 1, ..., m$, are convex, lsc, proper, and C is a nonempty convex set, if we add the condition that $X \in C$ is bounded, or this is implied by the constraints, then one still has a duality result: $\min P = \sup D$. We, actually, have $\inf_{x \in X} \sup_{y \in Y} L(x, y) = \sup_{y \in Y} \inf_{x \in X} L(x, y)$ and the optimal value for primal problem can be attained, we don't guarantee the existence of multiplier y, we just say that 'inf' and 'sup' are commutative if X is bounded. Although, a strict feasibility condition in the standard duality theory also results in commutativity, the boundedness of the feasibility set is, usually, much easier to check.

5 No-arbitrage and EMM

Arbitrage (= free-lunch) usually boils down to the possibility of positive returns without any investments. This section, and the next one, is concerned with arbitrage, and a slightly modified version of arbitrage, in a general framework. Duality allows us to derive some useful conditions between no-arbitrage and *Equivalent Martingale Measures* (EMM).
It's noteworthy that in some special case, cf. 6.2, one can't have positive returns without any investments, one might be able to end up with excessively large returns with only a very small investment. This is like an *'almost' arbitrage*. It seems that there is no clear boundary between arbitrage and no-arbitrage, the details are given in Example 6.2.

No-arbitrage means that if you begin with zero wealth also the terminal wealth should end up to zero (in all circumstances). This means that the optimal value of the following problem is zero:

$$
\begin{aligned}
&\max \; E\{\langle S_T(\underset{\to T}{\xi}), X_T(\underset{\to T}{\xi})\rangle\}\\
&\text{so that } \langle S_0(\xi_0), X_0(\xi_0)\rangle \le 0,\\
&\qquad \langle S_t(\underset{\to t}{\xi}), X_t(\underset{\to t}{\xi}) - X_{t-1}(\underset{\to t-1}{\xi})\rangle \le 0, \quad t = 1, \dots, T, \text{a.s.}\\
&\qquad \langle S_T(\underset{\to T}{\xi}), X_T(\underset{\to T}{\xi})\rangle \ge 0, \quad \text{a.s.}
\end{aligned}
$$

Let's begin with the following simple observation: there is no arbitrage if and only if the optimal value of the following *modified* problem is zero,

$$
\begin{aligned}
&\max E\{\langle S_T(\underset{\to T}{\xi}), X_T(\underset{\to T}{\xi})\rangle\}\\
&\text{so that } \langle S_0(\xi_0), X_0(\xi_0)\rangle \le 0,\\
&\qquad \langle S_t(\underset{\to t}{\xi}), X_t(\underset{\to t}{\xi}) - X_{t-1}(\underset{\to t-1}{\xi})\rangle \le 0, \quad t = 1, \dots, T, \quad \text{a.s.}\\
&\qquad \langle S_T(\underset{\to T}{\xi}), X_T(\underset{\to T}{\xi})\rangle \ge 0, \quad \text{a.s.}\\
&\qquad \|X_t\|_\infty \le M, t = 0, 1, \dots, T.
\end{aligned}
$$

where M is a positive constant. If there is no arbitrage, i.e., the optimal value of the original problem is zero. If the optimal value of the original problem is greater than zero, actually it should then be ∞: assume the optimal solution $X^* \in \mathcal{L}^\infty$, then just choose some constant $C > 0$ large enough so that $X^*/C < M$, then the optimal value of the second program should be greater than $E\{\langle S_T, X_T^*\rangle\}/C > 0$. Therefore, there is no arbitrage if and only if the optimal value of this modified program is also 0. Let's rewrite max $E\{\langle S_T, X_T\rangle\}$ as $\min -E\{\langle S_T, X_T\rangle\}$. For this problem, that comes with a bounded feasibility sets, by the duality theory, one has, $\sup D = \min P = 0$, where

$$
\begin{aligned}
\sup D = \sup_{y\in\mathcal{Y}_0} \inf_{X\in\mathcal{X}_0} E\Big\{& y_0\langle S_0(\xi_0), X_0(\xi_0)\rangle\\
&+ \sum_{t=1}^{T} y_t\langle S_t, X_t - X_{t-1}\rangle - (y_{T+1}+1)\langle S_T, X_T\rangle\Big\} = 0,
\end{aligned}
$$

where

$$
\mathcal{X}_0 = \big\{X = (X_0, \dots, X_T) \,\big|\, \|X_t\|_\infty \le M,\ t = 0, 1, \dots, T\big\},
$$

$$\mathcal{Y}_0 = \{y = (y_0, \ldots, y_T) \mid y_0 \geq 0, y_1 \geq 0, \ldots, y_T \geq 0\}.$$

Grouping terms with respect to the X_t, simplifying and taking iteratively conditional expectations with respect to $\underset{\rightarrow}{\xi}_{T-1}, \ldots, \underset{\rightarrow}{\xi}_1, \xi_0$, yields

$$\begin{aligned}
\sup D = & \sup_{y \in \mathcal{Y}_0} \inf_{x \in \mathcal{X}_0} E\Big\{ \langle y_0 S_0 - E\{y_1 S_1 | \xi_0\}, X_0 \rangle \\
& + E\{\langle y_1 S_1 - E\{y_2 S_2 | \underset{\rightarrow}{\xi}_1\}, X_1 \rangle\} + \cdots + \langle (y_T - y_{T+1} - 1) S_T, X_T \rangle \Big\} \\
& = \sup_{y \in \mathcal{Y}_0} \Big\{ - M(E\{| \, y_0 S_0 - E\{y_1 S_1 | \xi_0\} \, |\} \\
& \quad + E\{| \, y_1 S_1 - E\{y_2 S_2 | \underset{\rightarrow}{\xi}_1\} \, |\} + \cdots + E\{| \, (y_T - y_{T+1} - 1) S_T \, |\}) \Big\} \\
& = 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
\inf_{y \in \mathcal{Y}_0} \Big\{ & E\{ \Big| y_0 S_0 - E\{y_1 S_1 | \xi_0\} \Big| \} + E\{ \Big| y_1 S_1 - E\{y_2 S_2 | \underset{\rightarrow}{\xi}_1\} \Big| \\
& + \cdots + E\{ \Big| (y_T - y_{T+1} - 1) S_T \Big| \Big\} = 0
\end{aligned}$$

With $\hat{y}_{T+1} = y_{T+1} + 1$, let

$$\begin{aligned}
h(\xi, y_0, \ldots, \hat{y}_{T+1}) = & \Big| y_0 S_0 - E\{y_1 S_1 | \xi_0\} \Big| + \Big| y_1 S_1 - E\{y_2 S_2 | \underset{\rightarrow}{\xi}_1\} \Big| \\
& + \cdots + \Big| (y_T - \hat{y}_{T+1}) S_T \Big|.
\end{aligned}$$

Then,

$$\inf_{y \in \mathcal{Y}_0} E\{h(\xi, y_0, \ldots, \hat{y}_{T+1})\} = 0.$$

We now claim that $\forall \varepsilon > 0, \forall \delta > 0$,

$$\exists \, y_0, \ldots, \hat{y}_{T+1} \in \mathcal{Y}_0 \quad \text{such that} \quad P(h(\xi, y_0, \ldots, \hat{y}_{T+1}) > \varepsilon) < \delta.$$

If it's not true, then for some fixed $\varepsilon_0 > 0, \delta_0 > 0$ and any $y \in \mathcal{Y}_0$,

$$P(h(\xi, y_0, \ldots, \hat{y}_{T+1}) > \varepsilon_0) > \delta_0.$$

Then, for any $y \in \mathcal{Y}_0$,

$$E\{h(\xi, y_0, \ldots, \hat{y}_{T+1})\} > \varepsilon_0 \cdot P(h(\xi, y_0, \ldots, \hat{y}_{T+1}) > \varepsilon_0) > \delta_0 > 0,$$

and this means

$$\inf_{y\in\mathcal{Y}_0} E\{h(\xi, y_0, \dots, \hat{y}_{T+1})\} > 0,$$

a contradiction.

Let's now choose

$\varepsilon_1 = 1, \delta_1 = 1$ and $y_0^1, \dots, \hat{y}_{T+1}^1$ such that $P(h(\xi, y_0^1, \dots, \hat{y}_{T+1}^1) > 1) < 1$,

$\varepsilon_2 = 1/2, \delta_2 = 1/2$ and $y_0^2, \dots, \hat{y}_{T+1}^2$ such that $P(h(\xi, y_0^2, \dots, \hat{y}_{T+1}^2) > 1/2) < 1/2$,

$\dots\dots$,

$\varepsilon_\nu = 1/\nu, \delta_\nu = 1/\nu$ and $y_0^\nu, \dots, \hat{y}_{T+1}^\nu$ such that $P(h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu) > 1/\nu) < 1/\nu$.

Thus, for any $\varepsilon > 0$ and $\nu > 1/\varepsilon$,

$$P(h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu) > \varepsilon) < P(h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu) < 1/\nu \to 0.$$

This means that $h(\cdot, y_0^\nu, \dots, \hat{y}_{T+1}^\nu)$ converges to 0 in probability. Therefore, one can find a subsequence, for simplicity's sake say $\{y_0^\nu, \dots, \hat{y}_{T+1}^\nu,\ \nu \in \mathbb{N}\} \in \mathcal{Y}_0$ such that $\{h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu), \nu \in \mathbb{N}\}$ converges to 0 a.s., or equivalently,

$$h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu) \to 0 \quad \text{approximately uniformly.}$$

Therefore, for any $\delta > 0$, $\exists\, \Xi'$ such that $P(\Xi') > 1 - \delta$,

$$h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu)\} \to 0 \quad \text{uniformly on} \quad \Xi'.$$

In other words, $\forall\, \varepsilon \in (0, 1/2)$, $\delta > 0$, $\exists\, \Xi'$ and ν_ε such that $P(\Xi') > 1 - \delta$,

$$\text{for all} \quad \xi \in \Xi', \quad h(\xi, y_0^\nu, \dots, \hat{y}_{T+1}^\nu)\} < \varepsilon,\ \forall \nu \geq \nu_\varepsilon.$$

Recalling that $S_t^1 \equiv 1$ for $t = 0, 1, 2, \dots, T+1$, one has

$$y_T^\nu > 1 + y_{T+1}^\nu - \varepsilon \geq 1 - \varepsilon > 1/2 \text{ on } \Xi',$$
$$y_{T-1}^\nu > E\{y_T^\nu |\, \underset{\to T-1}{\xi}\} - \varepsilon > (1-\varepsilon)(1-\delta) - \varepsilon \sim 1 - 2\varepsilon > 1/2 \text{ on } \Xi',$$

for δ sufficiently small. By a similar argument,

$$y_t^\nu > 1/2 \quad \text{for} \quad t = 0, 1, \dots, T-2 \quad \text{on} \quad \Xi',$$

and from the above, it follows that

$$|y^\nu_{T-1}S_{T-1} - E\{y^\nu_T S_T|\underset{\rightarrow}{\xi}_{T-1}\}| = y^\nu_{T-1}|S_{T-1} - E\{y^\nu_T/y^\nu_{T-1}S_T|\underset{\rightarrow}{\xi}_{T-1}\}| < \varepsilon,$$
$$\ldots\ldots$$
$$|y^\nu_0 S_0 - E\{y^\nu_1 S_1|\xi_0\}| = y^\nu_0|S_0 - E\{y^\nu_1/y^\nu_0 S_1|\xi_0\}| < \varepsilon.$$

Hence,

$$|S_{T-1} - E\{y^\nu_T/y^\nu_{T-1}S_T|\underset{\rightarrow}{\xi}_{T-1}\}| < 1/y^\nu_{T-1}\varepsilon < 2\varepsilon \quad \text{on} \quad \Xi',$$
$$\ldots\ldots$$
$$|S_0 - E\{y^\nu_1/y^\nu_0 S_1|\xi_0\}| < 1/y^\nu_0\varepsilon < 2\varepsilon \quad \text{on} \quad \Xi',$$

with $y^\nu_0, y^\nu_1, \ldots, y^\nu_T \in \mathcal{L}^\infty$. Therefore,

$$\exists \text{ a constant } N > 0, y^\nu_{t+1}/y^\nu_t > n \quad \text{and} \quad y^\nu_{t+1}/y^\nu_t \in \mathcal{L}^\infty.$$

In conclusion, one has the following:

5.1 Theorem *If there is no arbitrage, then* $\exists\{y^\nu_0, \ldots, y^\nu_T, y^\nu_{T+1}\}$ *such that*

$$|S_0 - E\{y^\nu_1/y^\nu_0 S_1|\xi_0\}|,\ \ldots,\ |S_{T-1} - E\{y^\nu_T/y^\nu_{T-1}S_T|\underset{\rightarrow}{\xi}_{T-1}\}|,\ |y^\nu_T - (y^\nu_{T+1}+1)|$$

converge to 0 *approximately uniformly. In other words, for any* $\varepsilon > 0$, $\delta > 0$, *there exists* $u_t \in \mathcal{L}^\infty$ *and constant* $N > 0$ *with* $u_t > N$ *and* Ξ', $P(\Xi') > 1 - \delta$ *such that*

$$|S_{t-1} - E\{u_t S_t|\underset{\rightarrow}{\xi}_{t-1}\}| < \varepsilon \text{ on } \Xi' \quad \text{for } t = 1, \ldots, T.$$

5.2 Remark . *The preceding condition is just a necessary one to have no arbitrage, not a sufficient one, see 6.1 in the next section.* $\delta > 0$ *can't be omitted in certain instances, i.e., the last inequality may not hold on* Ξ, *the entire probability space, cf. 6.3.*

5.3 Theorem *If for any* $\varepsilon > 0$, $\delta > 0$, *there exists* $u_t \in \mathcal{L}^\infty$ *and constant* $N > 0$ *with* $u_t > N$ *and* Ξ', $P(\Xi') > 1 - \delta$ *such that*

$$S_{t-1} = E\{u_t S_t|\underset{\rightarrow}{\xi}_{t-1}\} \quad \text{on} \quad \Xi', \ \ t = 1, \ldots, T,$$

then there is no arbitrage.

Proof. For any $\varepsilon > 0$, since $|S_0 - E\{S_1|\xi_0\}| + \cdots + |S_{T-1} - E\{S_T| \underset{\to T-1}{\xi}\}|$ is summable, one can find $\delta > 0$, such that whenever $P(A) < \delta$,

$$E\{\{|S_0 - E\{S_1|\xi_0\}| + \cdots + |S_{T-1} - E\{S_T| \underset{\to T}{\xi}\}|\} \cdot \underset{\{A\}}{1\!\!1} < \varepsilon.$$

From the assumptions, for this ε and δ, there exists u_t that satisfies the given condition. Let

$$y_T = u_T \cdot y_{T-1}, \ldots, y_1 = u_1 \cdot y_0, y_0 = 1/N^T \quad \text{on} \quad \Xi',$$

and on $\Xi \setminus \Xi'$, $y_0 \equiv 1, \ldots, y_T \equiv 1$. Then, $y_T = u_T u_{T-1} \cdots u_1/N^T \geq 1$ on Ξ'. Let $y_{T+1} = y_T - 1$ then, $y_{T+1} \geq 0$ and

$$\begin{aligned}
&E\Big\{|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + |y_1 S_1 - E\{y_2 S_2|(\xi_0, \xi_1)\}| \\
&\qquad + \cdots + |(y_T - y_{T+1} - 1)S_T|\Big\} \\
&= E\Big\{(|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + |y_1 S_1 - E\{y_2 S_2|(\xi_0, \xi_1)\}| \\
&\qquad + \cdots + |(y_T - y_{T+1} - 1)S_T|) \cdot 1\!\!1_{\{\Xi'\}}\Big\} \\
&+ E\Big\{(|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + |y_1 S_1 - E\{y_2 S_2|(\xi_0, \xi_1)\}| \\
&\qquad + \cdots + |(y_T - y_{T+1} - 1)S_T|) \cdot 1\!\!1_{\{\Xi \setminus \Xi'\}}\Big\} \leq 0 + \varepsilon = \varepsilon.
\end{aligned}$$

Since ε is arbitrary,

$$\inf_{y \in \mathcal{Y}_0} \Big\{E\Big|\, y_0 S_0 - E\{y_1 S_1|\xi_0\}\Big| + E\Big|\, y_1 S_1 - E\{y_2 S_2|(\xi_0, \xi_1)\}\Big| + \cdots + E\Big|\, y_T - y_{T+1} - 1)S_T\Big|\Big\} = 0,$$

and this means no arbitrage. ◻

5.4 Remark *For discrete probability spaces, the condition in Theorem 5.3 is also necessary.*

Recall that a measure $\hat{P}$ on a measurable space $(\Xi, \mathcal{A})$ is *absolutely continuous* with respect to another measure P, one writes $\hat{P} << P$, if $\hat{P}(A) = 0$ for each $A \in \mathcal{A}$ such that $P(A) = 0$. Also, $\hat{P}$ and P are said to be *equivalent* if $\hat{P} << P$ and $P << \hat{P}$, or equivalently, the Radon-Nikodym derivative $d\hat{P}/dP > 0$. Finally, $\hat{P}$ and P are said to be *strictly equivalent* if there exists $\varepsilon > 0$ such that $d\hat{P}/dP > \varepsilon$.

5.5 Remark *If the infimum is actually attained, then*

$$y_T = 1 + y_{T+1} \geq 1 \text{ a.s.},$$

$$y_t S_t(\underset{\rightarrow t}{\xi}) = E\{y_{t+1} S_{t+1}(\underset{\rightarrow t+1}{\xi}) | \underset{\rightarrow t}{\xi}\} \text{ a.s.} \quad \textit{for} \quad t = 1, .., T-1,$$

or

$$S_t = E\left\{\frac{y_{t+1}}{y_t} S_{t+1} | \underset{\rightarrow t}{\xi}\right\}, \text{ a.s.}$$

Since $S_t^1 \equiv 1$ *for* $t = 1, 2, \ldots, T,$,

$$y_t = E\{y_{t+1} | \underset{\rightarrow t}{\xi}\} \quad \textit{or} \quad E\left\{\frac{y_{t+1}}{y_t} | \underset{\rightarrow t}{\xi}\right\} = 1,$$

and since also $y_t \in \mathcal{L}^\infty$ *then,*

$$y_{t+1}/y_t > 1/||y_t||_\infty > 0.$$

Such y_t *may be called* strictly equivalent martingale multipliers*, then* $u := y_T/y_0$ *is a martingale measure for* $\{S_t\}_{t=0}^T$.

We know that in some situations (6.3), under no arbitrage, strictly equivalent martingale measures may not exist, but what about the existence of equivalent martingale measures? Actually, the first fundamental theorem about arbitrage-free market [20, Chapter V, §2] tells us that no arbitrage is equivalent to the existence of equivalent martingale measures. There are two ways to prove this theorem, one way is to prove the existence by a separation theorem, see [4, 5]. Another way is to construct an equivalent martingale measure by the Esscher transformation, see [19]. We don't want to go through the details of the proofs. Here we just record below the general results for further reference and comparison's sake.

First of all, let's introduce some notation. Let

$K(P)$ be the (topological) support of a probability measure P, the smallest closed set carrying P,

$L(P)$ be the closed convex hull of $K(P)$,

$L^o(P)$ be the relative interior of $L(P)$,

Q_t be the regular conditional distributions of $S_t - S_{t-1}$ given S_{t-1}.

5.6 Theorem [20, theorem A^*]. *For our arbitrage-check optimization problem, the following assertions are equivalent:*
(a) *there is no arbitrage, i.e., the optimal value is zero;*
(b) *equivalent martingale measures (EMM) exist;*
(c) $0 \in L^o(Q_t)$.

5.7 Remark *Assertion (c) means that* $S_{t-1}(\underset{\rightarrow t-1}{\xi})$ *is included in the relative interior of* $\text{con}(S_t(\cdot, \underset{\rightarrow t-1}{\xi}))$ *a.s.;* con *denotes the* convex hull. *Therefore, if we want to know if arbitrage exists or not, we just need to check if the present price lies in the interior of convex hull of all the possible future prices.*

5.8 Corollary *No arbitrage for multi-stage problems is equivalent to no arbitrage for any two-stage subproblems.*

Proof. It is a immediate consequence of the preceding theorem. □

Although strictly equivalent martingale measures don't always exist, yet by Theorem 5.3 and Remark 5.4 we know that in the discrete cases they always exist. In practice, for numerical computational purposes, one always has to discretize our continuous distribution problem, therefore it's not out of line to assume that *strictly equivalent martingale measures* exist.

Notice that S is a n-dimensional vector, $y's$ are scalars, which reminds us of the relatively complete market problem to find the common Brownian measure via Girsanov Theorem, refer to [6]. But we are dealing with a more general framework! And by the same argument as above we know that for each S_1^i, maybe we could find martingale measures just for this stock (or coupon-bond) but you may not get the common martingale measures for all of them, which means that possiblky you can't have any positive returns if you just buy only one of the financial instruments, but you might be able to make a profit if you invest in a combination of all of them.

6 Some examples

6.1 Example *Assume ξ_0 fixed, $S_0 \equiv 1$, and S_1 uniformly distributed on $[0,1]$, then clearly $S_0 \in \mathrm{con}(S_1)$, but arbitrage exists.*

Detail. Suppose that there is no arbitrage, i.e.,

$$\inf_{y_0,y_1,y_2 \in \mathcal{L}^\infty} \{E|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + E|(y_1 - y_2 - 1)S_1|\} = 0.$$

Then,

$$y_1 \sim 1 + y_2 \geq 1, y_0 \sim E\{y_1\},$$

$$\begin{aligned} E|y_0 S_0 - E\{y_1 S_1|\xi_0\}| \sim E|y_1(1 - S_1)| &= \int_0^1 y_1(1-x)dx, \\ &\geq \int_0^1 (1-x)dx, \\ &= 1/2 > 0. \end{aligned}$$

So, we are lead to conclude that even in this simple set up arbitrage can occur. □

6.2 Example *Assume that S_0 is uniformly distributed on: $(0,1)$ and*

$$\begin{aligned} S(u, \xi_0 = x) &= 1, \quad \textit{with probability:} \quad 1 - x, \\ S(d, \xi_0 = x) &= 0, \quad \textit{with probability:} \quad x. \end{aligned}$$

where u is up, d is down. The claim is that

$$\inf_{y_0,y_1,y_2 \in \mathcal{L}^\infty} \{E|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + E|(y_1 - y_2 - 1)S_1|\} = 0.$$

Detail. Indeed, we just have to solve the following equations,

$$\begin{aligned} y_0(x) \cdot x &= 1 \cdot (1-x) y_1(u,x) + 0, \\ y_0(x) &= y_1(u,x) \cdot (1-x) + y_1(d,x) \cdot x. \end{aligned}$$

then,

$$y_0(x) = (1-x)/x \cdot y_1(u,x),$$
$$y_0(x) = x/(1-x) \cdot y_1(d,x).$$

If the infimum could be attained, then

$$y_1 \geq 1, \quad y_0(x) = (1-x)/x \cdot y_1(u,x) \geq (1-x)/x,$$

that turns out not to be summable with respect to x, contradicting that $y_0 \in \mathcal{L}^\infty$. And via successive approximation, it's clear that the infimum is 0. It means that although $S_0(x)$ is in the relative interior of $\mathrm{con}(S_1(\cdot,x)))$, there is no arbitrage and no strictly equivalent transition probability exists. We can just choose

$$y_0(x) = 4/(x(1-x)), \quad y_1(u,x) = 4/(1-x)^2, \quad y_1(d,x) = 4/x^2.$$

and then,

$$y_0 S_0 = E\{y_1 S_1 | \xi_0\}, \quad y_1(u,x), y_1(d,x) \geq 1.$$

This means that equivalent transition probability exists, but strictly equivalent martingale measure don't exist because y_0 is not integrable. But if we assume that S_0 is uniformly distributed on $(\varepsilon, 1-\varepsilon), 1 > \varepsilon > 0$ instead of $(0,1)$, it's immediate that strictly equivalent martingale measures exist. Later, we shall use this simple fact to construct another example for the continuous time case.

The interesting part of this example is: if we change a little bit the first constraint, say, $\langle S_0, X_0 \rangle \leq 1$ instead of ≤ 0, then the optimal value is infinity! The optimal solution are $X_0^* = (0, 1/x))$, X_1^* which can be any vector such that $\langle S_1, X_1 - X_0 \rangle = 0$. The solution suggests that you should use all your money($1) to buy this stock, then your expectation return is unbounded! Actually, $\langle S_1, X_1 \rangle = (1-x)/x$, the expectation is obviously infinity. That is because when $x > 1/2$ you may lose some money at most $1, but when $x < 1/2$ you may earn a lot of money ($>> 1$). Or say, you may lose money less that 1 with one half chance, and you could have returns much greater that 1 with probability 1/2. This is like arbitrage, except that you need to invest a small amount of money and you might lose money with positive probability. For this reason, one should refer to it as *almost arbitrage.* □

6.3 Example *Assume* S_0 *uniform on* $(0,1)$*, and for each point* $x \in (0,1)$*,* $S_1(\cdot, x)$ *is also uniform on* $(0,1)$*, then obviously* $S_0(x)$ *is in the relative interior of* $\operatorname{con} S_1(\cdot, x)$*, and we will prove later that there is no arbitrage, i.e.,*

$$\inf_{y_0, y_1, y_2 \in \mathcal{L}^\infty} \{E|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + E|(y_1 - y_2 - 1)S_1|\} = 0.$$

Detail. If the infimum could be attained, one would have,

$$\int_0^1 y_1(u,x)du = y_0(x), \quad \int_0^1 y_1(u,x)udu = xy_0(x), \quad y_1 \geq 1, \text{ a.s.}$$

Therefore

$$\int_0^1 y_1(u,x)(x-u)du \equiv 0, \text{ a.s.}, \quad x \in (0,1),$$
$$y_1 \geq 1, \text{ a.s.}$$

Since $y_1 \in \mathcal{L}^\infty$, we can assume that $y_1 < N$ for some constant $N > 0$, then we have:

$$\int_0^1 y_1(u,x)xdu \leq \int_0^1 Nxdu = Nx,$$
$$\int_0^1 y_1(u,x)udu \geq \int_0^1 udu = \frac{1}{2}.$$

Therefore, $Nx > 1/2$ for any $x \in (0,1)$ and that's impossible. In conclusion even though $S_0(x)$ is in the relative interior of $\operatorname{con}(S_1(\cdot, x))$, equivalent transition probability measures don't exist, and *afortiori* strictly equivalent martingale measures don't exist.

Proof of no-arbitrage:

$$\int_0^x (x-u)du = \frac{x^2}{2},$$
$$\int_x^1 (x-u)du = -\frac{(1-x)^2}{2}.$$

Let

$$\tilde{y_1}(u,x) = \begin{cases} (1-x)^2, & x \geq u, \\ x^2, & x < u. \end{cases}$$

then,

$$\int_0^1 \tilde{y_1}(u,x)(x-u)du \equiv 0, \quad \text{for any} \quad x \in (0,1),$$

If we let $y_1 = \varepsilon^{-2}\tilde{y_1}$ on $x \in (\varepsilon, 1-\varepsilon)$ for some small ε, $y_1 = 1$ otherwise, let $y_2 = y_1 - 1 \geq 0$, $y_0(x) = \int_0^1 y_1(u,x)du$, then obviously $E|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + E|(y_1 - y_2 - 1)S_1|$ goes to 0 as ε goes to 0, therefore the infimum is 0 and that means no-arbitrage. □

7 Hedging

Hedging is the process of reducing the financial risks. In our model, hedging is to meet all the contingent claims. Equivalently, the contingent claims problem has at least one feasible solution. The hedging problem could be formulated as follows:

$$\begin{aligned} \min \ & 0 \\ \text{so that } & \langle S_0(\xi_0), X_0(\xi_0)\rangle \leq G_0(\xi_0) \\ & \langle S_t(\underset{\to t}{\xi}), X_t(\underset{\to t}{\xi}) - X_{t-1}(\underset{\to t-1}{\xi})\rangle \leq G_t(\underset{\to t}{\xi}), t = 1, ..., T, \\ & \langle S_T(\underset{\to T}{\xi}), X_T(\underset{\to T}{\xi})\rangle \geq 0, a.e. \\ & \|X_t\|_\infty \leq M_t, \quad t = 0, 1..., T. \end{aligned}$$

where $M_t > 0$ is a constant. Let's refer to this problem as $(\mathcal{P}_H)$. call this problem $(\mathcal{P}_H)$. As we did earlier, we add the last constraint in order to use our duality theory. Obviously, the original problem, cf. §1, is feasible, i.e., hedging is possible, if and only if problem $(\mathcal{P}_H)$ is feasible for large enough M_t. The dual problem $(\mathcal{D}_H)$ is the following,

$$\begin{aligned} \sup \mathcal{D}_H = -\inf_{y \in \mathcal{Y}_0} E\{ & \{M_0|y_0 S_0 - E\{y_1 S_1|\xi_0\}| + M_1|y_1 S_1 - E\{y_2 S_2| \underset{\to 1}{\xi}\}| \\ & + ... + M_{T+1}|(y_T - y_{T+1})S_T|\} + y_0 G_0 + y_1 G_1 + ... + y_T G_T\} \end{aligned}$$

or still,

$$\sup \mathcal{D}_H = -\inf_{y\in\mathcal{Y}_0} E\{\{M_0|y_0S_0 - E\{y_1S_1|\xi_0\}| + M_1|y_1S_1 - E\{y_2S_2|\underset{\to 1}{\xi}\}|$$
$$+ ... + M_T|(y_{T-1}S_{T-1} - E\{y_TS_T|\underset{\to t-1}{\xi}\}$$
$$+ y_0G_0 + y_1G_1 + ... + y_TG_T\}$$

It is easy to see that the optimal value of the primal problem $(\mathcal{P}_H)$ is either 0 or ∞ when $(\mathcal{P}_H)$ is not feasible. Thus, by duality also the optimal value of the dual problem $(\mathcal{D}_H)$ is either 0 or ∞, and obviously if $(\mathcal{P}_H)$ is feasible for some M_t's, then the given contingent claims problem is feasible. If $(\mathcal{P}_H)$ is not feasible for any choice of M_t, then also the original problem is not feasible, i.e., one has:

(I) $\min \mathcal{P}_H = 0$ (or $\sup \mathcal{D}_H = 0$) for some selected M_t if and only if the given contingent claims problem is feasible;

(II) $\min \mathcal{P}_H = \infty$ (or $\sup \mathcal{D}_H = \infty$) for any choice of M_t's if and only if the contingent claim problem is not feasible.

For simplicity sake, we can just consider a two-stage problem. For any martingale multipliers y_0, y_1 such that

$$|y_0S_0 - E\{y_1S_1|\xi_0\}| \equiv 0,$$

if the contingent claim problem is feasible , or $\sup \mathcal{D}_H$ is zero, one has,

$$E\{y_0G_0 + y_1G_1\} = E\{y_1(G_1 + G_0)\} \geq 0.$$

Observing that $y_1 \geq 0$, hence $Ey_1 \geq 0$ and $Ey_1 = 0$ if and only if $y_1 = 0$ a.s., and when $y_1 = 0$ a.s., obviously $E\{y_1(G_1 + G_0)\} \geq 0$. We just need to consider $Ey_1 > 0$, let $u = y_1/Ey_1$, then $E\{u\} = 1$, u is a martingale measure, therefore a necessary condition for hedging is:

$$E_u\{(G_1 + G_0)\} \geq 0, \quad \text{for any martingale measures.}$$

For multi-stage, by a similar argument, one arrives at the following necessary conditions for hedging:

$$G_0/S_0^i + G_1/S_1^i + ... + G_T/S_T^i \geq 0, \;\; i = 1, 2, ..., n,$$
$$E_u\{(G_1 + G_0 + ... + G_T)\} \geq 0, \quad \text{for any martingale measures.}$$

8 Equilibrium equation

Suppose there is no arbitrage, or that equivalent martingale measures exist, and the original problem (the writer's problem) is also feasible, then the writer of a contract would try to maximize terminal expected wealth $E\{\langle S_T, X_T\rangle\} \geq 0$. In terms of the dual problem,

$$E\{y_T G_T + y_{T-1} G_{T-1} + ... + y_0 G_0\} \geq 0$$

for any martingale multipliers y_t. The buyer of this contract always looks for larger 'pay-backs',

$$E\{y_T G_T + y_{T-1} G_{T-1} + ... + y_0 G_0\} \leq 0$$

for any equivalent martingale measure y_t. This brings us to the *equilibrium equation*:

$$\inf_{y \in \mathcal{Y}_0} E\left\{y_T G_T + y_{T-1} G_{T-1} + ... + y_0 G_0\right\} = 0.$$

Observe that for martingale multipliers $\{y_t\}$,

$$\begin{aligned} &E\{y_T G_T + y_{T-1} G_{T-1} + ... + y_0 G_0\} \\ &= E\{y_0\} E\{G_T y_T / E\{y_0\} + G_{T-1} y_{T-1} / E\{y_0\} + ... + G_0 y_0 / E\{y_0\}\}, \end{aligned}$$

and

$$E\{y_T / E\{y_0\}\} = \cdots = E\{y_0 / E\{y_0\}\} = 1,$$

$$y_T / E\{y_0\}, ..., y_0 / E\{y_0\} > 0 \;\; a.s.$$

If we consider the set, $\{E\{y_0\} = 1,\, y_t \geq 0,\, t = 0, \ldots, T\}$, it means that $y_T/E\{y_0\}$,..., $y_0/E\{y_0\}$ are just martingale measures, not necessarily equivalent, then the infimum should be attained on this closed set. Therefore the *equilibrium equation* for pricing the contingent claims is

$$\text{for some martingale measure } \; u : E_u\{G_T + G_{T-1} + ... + G_0\} = 0.$$

9 The Black-Scholes Equation

For continuous time, the constraint $\langle S_1, X_1 - X_0 \rangle \leq G_t$ becomes

$$\langle S_{t+\triangle t}, X_{t+\triangle t} - X_t \rangle = dG_t.$$

Here we proceed with the '=' version recalling that S_t^1 represents the risk-free asset; in any case one can always put extra money into the risk-free asset to obtain the equality.

For some special case such as European option, $dG_t = 0$, the terminal wealth is a function of the terminal price S_T, i.e., $g(S_T)$, and one has,

$$\langle S_{t+\triangle t}, X_{t+\triangle t} - X_t \rangle = 0$$

which means the the portfolio is *self-financing*. Let's consider a simple two-dimensional case: $S_t = (r_t, s_t)$, $X_t = (c_t, x_t)$, where r_t is the risk-free rate at time t —in order to derive Black-Scholes equation's we don't need to fix the numeraire— $r_{t+\triangle t} = r_t + r \cdot \triangle t$, where r is a constant rate, and let $V_t = \langle S_t, X_t \rangle$ be the wealth at time t. Then, the terminal wealth $V(T, S_T) = g(s_T)$. Moreover, assume s_t is log-normal such that

$$ds_t = s_t(\mu \triangle t + \sigma dz_t),$$

where z_t is a one-dimensional Brownian motion, μ and σ are constants, and s_0 is given. One can derive the Black-Scholes equation from Itô's formula [6],

$$\begin{aligned} V_{t+\triangle t} &= \langle S_{t+\triangle t}, X_{t+\triangle t} \rangle \\ &= \langle S_{t+\triangle t}, X_t \rangle \\ &= \langle S_t, X_t \rangle + \langle S_{t+\triangle t} - S_t), X_t \rangle \\ &= V_t + x_t ds_t + r(V_t - x_t s_t) dt \end{aligned}$$

and, from definition of s_t above,

$$\begin{aligned} dV_t &= x_t ds_t + r(V_t - x_t s_t) dt \\ &= rV_t dt + (\mu - r) x_t s_t dt + \sigma x_t s_t dz_t. \end{aligned}$$

Let $F(t, s_t)$ be the value of the option at time t with $F(T, S_t) = g(s_T)$, again by Itô's formula:

$$\begin{aligned} dF &= F_t(t, s_t)dt + F_x(t, s_t)ds_t + \tfrac{1}{2}F_{xx}(t, s_t)(ds_t)^2 \\ &= F_t dt + F_x(us_t dt + \sigma s_t dz_t) + \tfrac{1}{2}F_{xx}\sigma^2 s_t^2 dt, \end{aligned}$$

using the definition of ds_t and where F_t, F_x represent the partial derivatives with respect to the first and second variables.

Comparing the coefficients of dz_t and d_t, one obtains

$$\begin{aligned} &x_t = F_x, \\ &F_t + rF_x s_t + \frac{1}{2}F_x x\sigma^2 s_t^2 = rF, \qquad \text{(Black-Scholes Equation)} \\ &F(T, s_T) = g(s_T). \end{aligned}$$

The solution has this form, $F(0, s_0) = e^{-rT}E^{\widetilde{P}}g(s_T)$, where $\widetilde{P}$ is the equivalent martingale measure for $\{S_t\}_{t=0}^{t=T}$. Here, we can see that the key point is the same, i.e., to find the equivalent martingale measures, but our approach via stochastic optimization methodology is significantly more general.

10 Multipliers: Continuous Martingale Measures

As shown in the last section, the usual paradigm way is to assume that S_t satisfies some stochastic differential equation, say $dS_t = S_t(\mu dt + \sigma dW_t)$, where W_t is a vector Brownian motion, then one derives some partial differential equations such as the Black-Scholes equation. Or more generally, assuming that S_t is a semi-martingale [7], one can also derive some more involved stochastic differential equations and some similar results. In this paper, we just posit: For a filtration $\{\mathcal{F}_t\}_{t=0}^{t=T}$, $S_t \in \mathcal{F}_t$, S_t is continuous and $S_t \in \mathcal{L}_1^n$. The difficulty is that there is no 'good theory' for the extension of discrete time martingales to continuous time martingales, we have an example that may provide a 'hint' on how to potentially improve these results.

One might be tempted to conjecture that if there exists strict martingale multipliers for any discrete times instead of stopping tines, then there exists strict martingale multipliers for the continuous time case. Actually, that's not in the cards, one has the following counterexample.

10.1 Example *Let S_0 be uniform distributed on: $(\varepsilon, 1-\varepsilon), 1 > \varepsilon > 0$, and the support of the distribution of $S_{t+\Delta t}(\cdot, S_t = x)$ is $(x - \varepsilon\Delta t, x + \varepsilon\Delta t)$ with density $f_t(\cdot, x)$, $0 \le t \le 1$. Then, S_1 may take values on $(0,1)$. Suppose $f_t(\cdot, x)$ satisfies:*

- *(i)* $\int_{x-\varepsilon\Delta t}^{x+\frac{1}{2}\varepsilon\Delta t} f_t(y,x)dy = O(x^2)$,
- *(ii)* $f_t(\cdot, x) > 0,$ *for any* $x \in (0,1)$.

Detail. Condition (i) means that if $S_t = x$ is approximately 0, then $S_{t+\Delta t}$ is greater than $x + \frac{1}{2}\varepsilon\Delta t$ with approximately probability 1, therefore the expectation $\int_{x-\varepsilon\Delta t}^{x+\frac{1}{2}\varepsilon\Delta t} y f_t(y,x)dy > x + \frac{\varepsilon}{3}\Delta t$ when x is small. Condition (ii) guarantees the existence of equivalent martingale measures for the discrete time case. It is not difficult to find some density functions that satisfy these two conditions. For the continuous time case, if there exists equivalent martingale measures $u_t(\cdot, x)$, then similarly to Example 6.2, from $S_{1-\Delta t}$ to S_1, one has

$$\int_{x-\varepsilon\Delta t}^{x+\varepsilon\Delta t} (y-x) f_1(y,x) u_1(y,x)dy = 0, \text{ for any } x \in (0,1),$$
$$u_1(\cdot, x) > 1, \text{ for any } x \in (0,1).$$

By substitution, with $y = y + x$,

$$\int_{-\varepsilon\Delta t}^{\varepsilon\Delta t} y f_1(y+x,x) u_1(y+x,x)dy = 0, \text{ for any } x \in (0,1),$$
$$u_1(\cdot, x) > 1, \text{ for any } x \in (0,1).$$

Suppose that $u_1 < N$ for some constant $N > 0$, then:

$$\left| \int_{-\varepsilon\Delta t}^{\frac{\varepsilon\Delta t}{2}} y f_1(y+x,x) u_1(y+x,x)dy \right| \le N \int_{-\varepsilon\Delta t}^{\frac{\varepsilon\Delta t}{2}} f_1(y+x,x)dy = NO(x^2),$$
$$\int_{\frac{\varepsilon\Delta t}{2}}^{\varepsilon\Delta t} y f_1(y+x,x) u_1(y+x,x)dy \ge \int_{\frac{\varepsilon\Delta t}{2}}^{\varepsilon\Delta t} y f_1(y+x,x)dy > \frac{\varepsilon}{3}\Delta t.$$

Therefore, one has $NO(x^2) > \frac{\varepsilon}{3}\Delta t$, which is obviously impossible, since when Δt is small, x can take values in $(\varepsilon\Delta t, 1 - \varepsilon\Delta t)$. Hence, no continuous martingale multipliers exist. □

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