TERM AND VOLATILITY STRUCTURES

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Abstract. All valuations (discounted cash flow, instrument pricing, option pricing) and other financial calculations require an estimate of the evolution of the risk-free rates as implied by the term and volatility structures. This presumes that one has, if not perfect knowledge, at least very good estimates of these market term structures. In this paper we review and compare the existing methodologies for deriving zero-curves (spot rates, forward rates and discount factors) and volatility estimates.

Keywords: spot and forward rate curve, discount factors curve, term structures, best fit, max-error criterion, error-sum criterion,

Date: December 8, 2004. Revised February 3, 2005

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Term and volatility structures are the corner stone to practically all valuations of fixed income financial instruments and consequently affects, or should affect, significantly the trading and the management of financial holdings; they are also used to help shape monetary policy by Central Banks. Part I is devoted to a review and a comparison of the methods that have been suggested to construct the term structure associated with a given collection of fixed income financial instruments. Part II addresses the problem of determining the (associated) volatility structure that has been given only scant attention in the literature.

Part I
TERM STRUCTURE

1 An example

Let's begin with the following simple, but fundamental, issue: Find the zero-curves (= term structure) associated with a given portfolio. We use the term *zero-curve* in a generic sense to designate all or any one of the following financial curves: *spot rates, forward rates, discount factors and discount rates*; any zero curve completely determines the others. Indeed,

$$d_{t,m} = e^{-s_{t,m}m}, \quad s_{t,m} = m^{-1} \ln d_{t,m}, \quad f_{t,m} = s_{t,m} + m\dot{s}_{t,m},$$

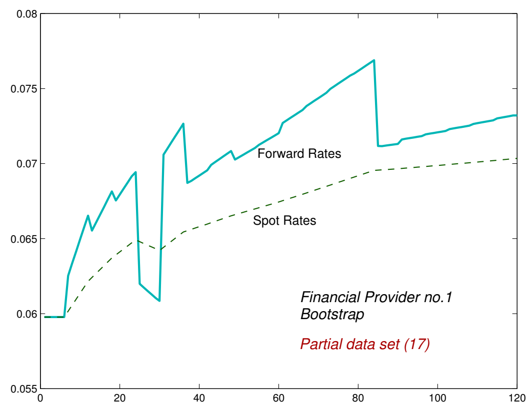
where $s_{t,m}$ is the spot rate at time t for all bonds of maturity m, $d_{t,m}$ is the corresponding discount factor, and $f_{t,m}$ is the (instantaneous) forward rate a time t, again for bonds of maturity m; the upper dot on $\dot{s}_{t,m}$ stands for the time derivative.

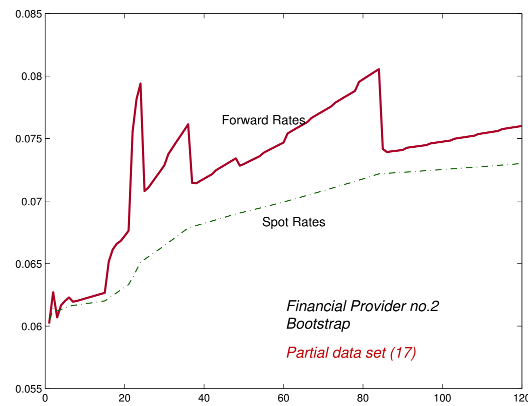
Since the valuation of all fixed income financial instruments rest on starting with the 'right' zero-curves, it may come as a surprise that the zero-curves associated with a well-defined portfolio, derived using different methodologies, might vary significantly? This can best be illustrated by an example: On their web-site, TechHackers described a portfolio that includes Eurodollar Deposits, Eurodollar Futures and Swaps from June 10, 1997; see Tables A–1,... A–3 in Appendix A. The seven pairs of *spot and forward rates* curves in Figure 1 are derived using

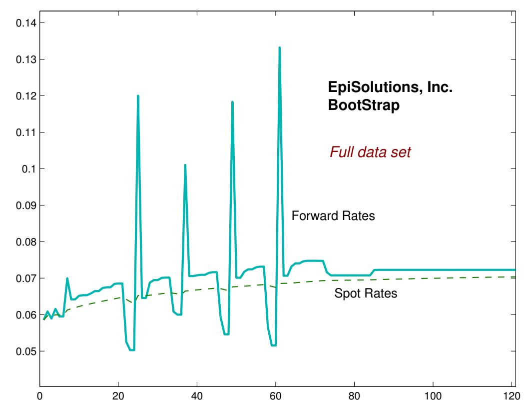
- three different implementations of BootStrapping,
- Svenson's extension of the Nelson-Siegel model,
- the updated Adams-Van Deventer maximum smoothness approach,
- Kortanek's derivation of the forward rates via geometric programming,

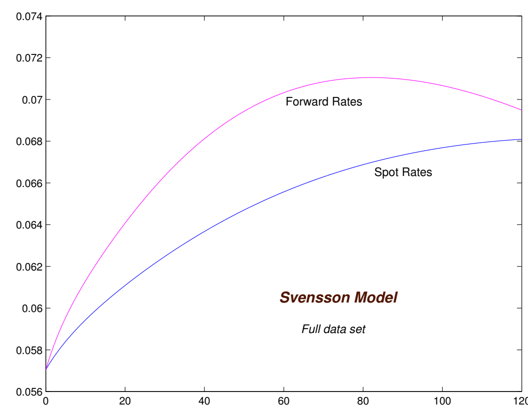
and finally, a novel approach based on Approximation Theory, implemented by EpiSolutions Inc. (ESI: `www.episolutionsinc.com`),

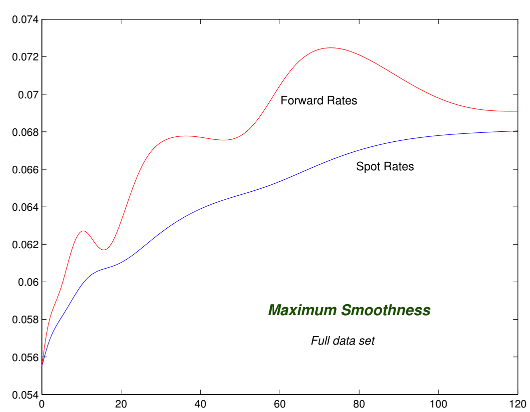
- relying on the construction of *EpiCurves*.

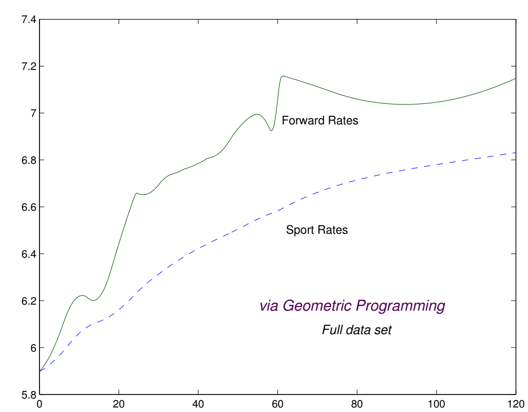












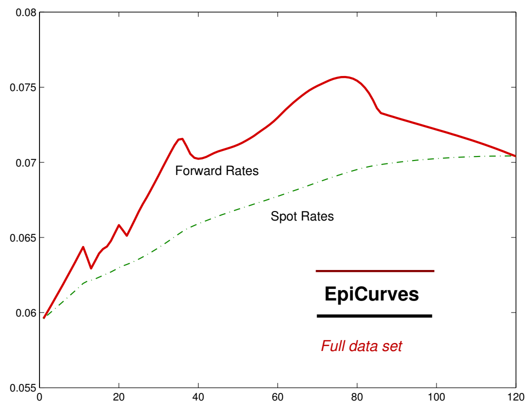


Figure 1: Spot and Forward Rate curves: Seven different functionalities

One might wonder if one should have any confidence in any one of these pairs! There are even noticeable differences between the spot rates curves. To understand the underlying reasons for these differences, one needs to examine the hypotheses under which these zero-curves were obtained. To do so, we begin with a description and an analysis of the BootStrapping methodology, to be followed by each one the alternatives listed earlier. We shall then proceed with a discussion of the criteria that might be used in evaluating the 'quality' of the generated zero-curves and conclude with a comparison of the results based on these criteria.

2 BootStrapping

The valuation of fixed-income securities, and derivatives written on fixed-income securities, requires an estimation of the underlying risk-free term structure of interest rates. In principle, the term structure of interest rates is defined by a collection of zero-coupon bond prices (and their respective maturities), spanning the horizon over which a fixed-income security is to be valued. However, unless a zero-coupon bond exists for *every* maturity for which a discount factor is desired, some form of estimation will be required to produce a discount factor for any 'off-maturity' time. In practice, zero-coupon bond prices are available for a limited number of maturities (typically $\leq$ 1 year). If zero-coupon bonds for other maturities are available, a lack of liquidity may prevent the determination of an accurate or reliable price. As a result, the zero-curve is typically built from a combination of liquid securities, both zero-coupon and coupon-bearing, for which prices are readily available. This can include Treasury Bills, Deposits, Futures, Forward-Rate Agreements, Swaps, Treasury Notes and Treasury Bonds. The list need not be limited to these securities just noted, though current vendor solutions *are* limited to these securities. Given a spanning set of securities, the zero-curve is then built using one of two forms of *BootStrapping*.

Under one BootStrapping method, the first step is to construct a larger set of spanning securities, by creating an 'artificial' security that matures on every date for which a cash flow is expected, and on which no security in the original set matures. For example, given a 5-year and a 6-year swap (paying semi-annually), a 5.5-year swap would be constructed, with a fixed-rate somewhere in between (based on some sort of interpolation) the fixed-rates for the 5-year and the 6-year swaps. Then, 'standard' BootStrapping may be applied to the expanded set of securities, giving discount factors for each maturity and cash flow date in the security set. This 'textbook' description of the BootStrapping method can be found in [10, §4.4], for example.

Another BootStrapping approach, is to make an assumption about how the instantaneous (or periodic) forward rates evolve between maturities in the security set. One assumption might be that forward rates stay constant between maturities, another might be that they increase or decrease in a linear fashion. Whatever the form of the forward rate evolution, *some assumption must be made.* Under this approach, instead of solving for a single discount factor for each successive security, a forward rate (or a parameter governing forward rate evolution) is obtained that will give the appropriate discount factor(s) between two maturity dates. For the example of the 5-year and 6-year swaps, given that the discount factors through the 5-year maturity have already been calculated, a forward rate is determined for the period between 5 and 6 years, that gives a 5.5-year discount factor and a 6-year discount factor that —when combined with the previous discount factors— will value the fixed side of the 6-year swap at par.

The results of either approach 'look' somewhat similar, a set of discount factors and corresponding dates spanning the horizon from today to the last maturity date in the security set. However, when using this set of discount factors as a basis for the valuation of other fixed-income securities, it will rarely be true that the cash flows of these securities will fall directly on the discount factor dates of the newly created zero-curve. In this case, a discount factor or a zero-coupon rate must be interpolated from the spot-curve. Typically available interpolation methodologies for this include linear, log-linear, exponential, cubic-spline, or any of a number of variations on fitting the zero-curve with a polynomial.

As long as two securities do not share the same maturity date, any combination of securities may theoretically be used in constructing a zero-curve, even with the BootStrapping methodology. A limitation of the currently available technology, is that the user *must* typically 'switch' from one security type to another during the BootStrapping process. For example, given a set of deposits, futures, and swaps, the currently available methods will not allow for inclusion of a deposit and a futures contract, where the underlying deposit maturity date of the futures contract is *prior* to the maturity date of the deposit (similarly, no 'overlap' is allowed between futures contracts and swaps). At first one might not consider this as a serious limitation, as users may very well wish to describe different 'sections' of the zero-curve using certain types of securities. Almost all web-based, commercially available zero-curve construction technologies rely on a form of *BootStrapping*, in combination with a variety of interpolation methods. Furthermore, they are limited to using certain types of securities, and are also limited in the ways in which these securities may be combined.

DFS-Portfolio (Deposits, Futures and Swaps): We now return to the example found on the TechHackers web-site. A detailed description of the portfolio is given in Tables A–1, ..., A–3 in the Appendix, but at this point at brief description will suffice. There are 36 instruments in the full data set, broken down as follows: 6 Eurodollar Deposits with term-to-maturities ranging from overnight to 12 months, 24 Eurodollar Futures with 90-day deposit maturities ranging from 3 months to 6 years, and 6 Swaps with term-to-maturities ranging from 2 to 10 years. The yields of the deposits vary from 5.55% to 6.27% with the higher yields corresponding to those with larger maturity. Similarly, the 24 futures have yields that vary from 5.89% to 7.27%. The yields for the swaps vary from 6.20% to 6.86%, again with the yields increasing as maturities get larger.

BootStrapping Sub-portfolio. In this example, there is overlap between the maturity dates for the EuroDollars Deposits (1 day, 1 month, 2 m., 3 m., 6 m., 12 m.) and the EuroDollars Futures (3 months, 6+ m., 9 m., 12 m., 15 m., ...). Thus *not all instruments can be included* in a 'BootStrapping sub-portfolio.' One possible choice, the default setting for Financial Provider no.1, is to switch from Eurodollar Deposits to Eurodollar Futures at the earliest possible time, i.e., include the first available Futures contract. Similarly, switching from Eurodollar Futures to Swaps occurs at soon as possible, i.e., at a time corresponding to the Swap with the lowest maturity. With this selection criteria, the data set used to generate the 'BootStrapping Sub-portfolio' is a subset of the full DFS-Portfolio data set that includes: the first five Eurodollar Deposits, the 2nd through the 7th Eurodollar Futures contracts and all 6 Swaps – for a total of 17 instruments.

The user may choose any time they wish for making these switches, the point is that they *must be made somewhere* along the line. Clearly, Financial Provider no.2 relied on a different selection criterion to constitute its BootStrapping sub-portfolio.

BootStrapping Results. The first two pairs of Spot and Forward Rate curves in Figure 1 are those derived from functionalities made available by two Financial Providers. Although all the Spot Rate curves appear to be relatively similar (except for the time span from month 20 to month 25), the Forward Rate curves are quite dissimilar. Of course, this can be traced back to the different BootStrapping implementations that rely on the selection of different sub-portfolios, as indicated earlier.

The implementation of the BootStrapping technique at EpiSolutions Inc. (ESI) —based on the simple precept that the instantaneous forward rates are constant between (adjacent) maturity dates— for the same BootStrapping sub-portfolio as Financial Provide no. 1 yields the pair of zero-curves in Figure 2; this pair of Spot- and Forward-Rates curves are similar to those obtained by Financial Provider no.1. The same approach, but with the *full* data set, i.e., when there is no mutilation of the given DFS-portfolio, yields the pair of zero-curves in the first graph of line 2 of Figure 1.

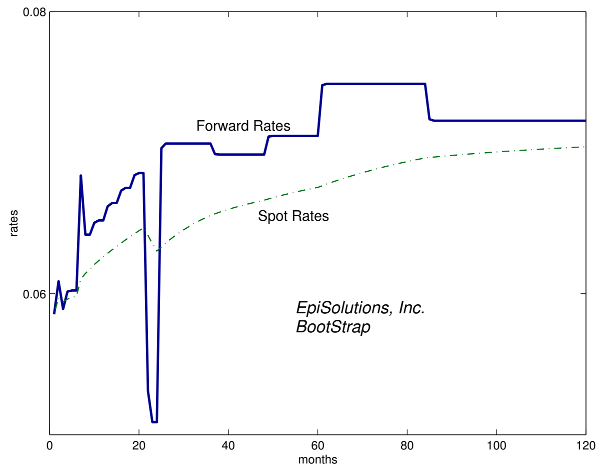


Figure 2: ESI-BootStrapping for the BootStrapping Sub-portfolio

Generalized Bootstrap Method. Deaves and Parlar [6] suggested an approach that overcomes the need to mutilate a portfolio to be able to apply the 'textbook'-version of BootStrapping described above. Their approach, unusual and interesting, consists in calculating the spot-rates r_t associated with each one of the time-dates when there is a cash flow (of any type). In our example, this corresponds to coupon-, settling-, delivery- and maturity-dates; in fact, there are 73 such dates. However, there are only 36 instruments that would generate the following 36 pricing-out equations

$$0 = \sum_{t \in \mathcal{T}} \tfrac{1}{2} p_{kt} e^{-r_t \cdot t}, \ \forall\, k = 1, \ldots, 36,$$

where the instruments are indexed by k. $\mathcal{T}$ is an ordered set of all the time-dates at which cash flow occurs and p_{kt} is the cash flow (positive or negative) for instrument k at time t. It's also assumed that there no repeated maturity-dates for the instruments in the (given) portfolio. This non-linear system is solvable, but has 37 more variables than equations and, consequently, will have multiple solutions. In order to generate a system with (hopefully) a unique solution, Deaves and Parlar add 37 linear equations that relate the rates at the 37 time-dates that are not maturity-dates, say $r_{37}, \ldots, r_{73}$, to the rates at maturity-dates: say $r_1, \ldots, r_{36}$. To do so, they rely on a Maple (symbolic) cubic spline fit functionality; this cubic spline, $cs(\dot{)}$ consists of 36 different cubics: one for each of the time-intervals between the different, increasing maturity dates. For each one of these time intervals,

$$cs(t) = \sum_{j=1}^{3} 6c_{0j} r_j + (\sum_{j=1}^{3} 6c_{1j} r_j)t + (\sum_{j=1}^{3} 6c_{2j} r_j)t^2 + (\sum_{j=1}^{3} 6c_{3j} r_j)t^3,$$

where the 144 coefficients $\{c_{lj},\ l = 0, \ldots, 3,\ j = 1, \ldots, 36\}$ are those calculated by the (symbolic) cubic spline fit. From this, one can obtain a linear equation for each one of the 37 non-maturity-dates rates. For our example, the fourth cash flow non-maturity-date is 0.5147 years after June 10, 1997 (t=0) and the corresponding linear equation is

$$
\begin{aligned}
r_{40} = {} & .0000356 r_1 - .0002056 r_2 + .0013674 r_3 - .0124579 r_4 + .015465 r_5 - .322488 r_6 \\
& + 1.3053364 r_7 + .0162911 r_8 - .0133038 r_9 + .01005185 r_{10} - .0001131 r_{11} + .0000272 r_{12} \\
& - .768069 e^{-5} r_{13} + .783936 e^{-5} r_{14} - .606228 e^{-5} r_{15} + .556858 e^{-7} r_{16} - .1349305 e^{-7} r_{17} \\
& + .379412 e^{-8} r_{18} - .4295145 e^{-8} r_{19} + .3402209 e^{-8} r_{20} - .2493217 e^{-10} r_{21} + .61491 e^{-11} r_{22} \\
& - .174375 e^{-11} r_{23} + .2176 e^{-11} r_{24} - .1767913 e^{-11} r_{25} + .998711 e^{-14} r_{26} - .2495984 e^{-14} r_{27} \\
& + .708697 e^{-15} r_{28} - .1002684 e^{-14} r_{29} + .835819 e^{-15} r_{30} - .352076 e^{-17} r_{31} + .88496 e^{-18} r_{32} \\
& - .2281451 e^{-18} r_{33} + .420647 e^{-19} r_{34} - .1130008 e^{-20} r_{35} + .414996 e^{-22} r_{36}.
\end{aligned}
$$

The 37 equations allow us to replace the 37 non-maturity-date rates in the 36 basic (non-linear) equations and reduce the system to one involving only 36 equations and 36 unknowns! Such a system can be solved by relying on either Newton's method, one of its variants (Quasi-Newton methods), or any other appropriate method; the only drawback is that the Jacobian of the resulting system would be dense and it wouldn't be possible to take advantage of sparsity; Deaves and Parlar rely on Maple's `fsolve` function.

The solution of this system yields the 73 spot-rates at all 'cash-flow' dates. Some further interpolation(s) would be required to obtain the yield curve at all times 't'. Although, Deaves and Parlar don't suggest any specific interpolation strategy, presumably, one would rely on the cubic spline cs, after substituting the values for $r_1, \ldots, r_{36}$, to obtain the spot rates at any time t; forward rates and discount factors would be derived, in turn, from this spot-rates curve.

We tried to implement this approach following the steps suggested by Deaves and Parlar, i.e., by relying on the functionalities provided by Maple. Such an implementation is labor intensive, and after a couple of days, we realized that for this relatively simple 36 instruments example, from modeling to the point where you can input the 36-by-36 (non-linear) system might very well take a week's work. That is expensive and the delay would be much too long for most potential applications. Moreover, even for this example, the time required to derive the (symbolic) cubic spline was non-negligible. The idea of dealing in this manner with a portfolio with 100+ instruments and 2,000 to 3,000 cash-flow dates, is to say the least, intimidating; there is also the requirement that no instruments can have the same maturity-dates. *All* these obstacles can be overcome but to do so one needs to get involved in a *major* implementation effort and rely on much more sophisticated procedures to obtain the cubic spline (cs) and to solve (dense) non-linear systems than the `spline` and `fsolve` Maple-functionalities. Because we could not handle portfolios of reasonable (practical) size, it has not been possible to include this intriguing approach in our analysis.

3 Nelson-Siegel and Svensson's extension

Under all the limitations mentioned in the previous section, BootStrapping strives to accomplish the (presumably) desirable goal of pricing the securities used in the zero-curve estimation exactly. In most cases, however, it does not produce believable forward rates and even the spot rates-curve fails to be smooth, the norm for such curves. There can be many technical reasons for this, that are not inherent to the BootStrapping method itself, such as liquidity, tax effects, and/or missing data points. Regardless, practitioners would like a method of estimating zero-curves that is robust and reasonable, across many different markets and market conditions. One criterion that is especially desirable for those using zero-curves for strategic planning purposes, is *smoothness*, particularly with regard to the evolution of the implied forward-rate curve. There are many ways to estimate zero-curves whose associated forward-rate curves are smooth, one of which is to posit a functional form that is smooth by definition, and try to find best-fit parameters for this function. In 1987, Nelson and Siegel [17] did just this by proposing the following formula for the evolution of the instantaneous forward rate (at time t)

$$f_{t,m} = \beta_{t,0} + \beta_{t,1} e^{-m/\tau_{t,1}} + \beta_{t,2}(m/\tau_{t,1}) e^{-m/\tau_{t,1}}$$

where m is the time-to-maturity and $\beta_{t,0}, \beta_{t,1}, \beta_{t,2}, \tau_{t,1}$ are the parameters to be estimated to fit as well as possible the available data. Integrating the forward rate curve —and dropping the index t— yields the spot rate

$$s_m = \beta_0 + (\tau_1/m)(\beta_1 + \beta_2)(1 - e^{-m/\tau_1}) - \beta_2 e^{-m/\tau_1}.$$

Typically, this expression for the spot rates will generate a curve with one 'hump', to allow for a second 'hump' Svensson [19] proposed, in 1994, an extension to this formula that generally increased the flexibility for fitting a given set of securities and their market prices

$$f_m = \beta_0 + \beta_1 e^{-m/\tau_1} + \beta_2(m/\tau_1) e^{-m/\tau_1} + \beta_3(m/\tau_2) e^{-m/\tau_2}$$

where $\beta_0, \beta_1, \beta_2, \tau_1, \beta_3, \tau_2$ are the parameters to be estimated. The corresponding spot rate curve is

$$s_m = \beta_0 + \beta_1 \frac{-\tau_1}{m}(1 - e^{-m/\tau_1}) + \beta_2 \left(\frac{-\tau_1}{m}(1 - e^{-m/\tau_1}) - e^{-m/\tau_1} \right) + \beta_3 \left(\frac{-\tau_2}{m}(1 - e^{-m/\tau_2}) - e^{-m/\tau_2} \right).$$

One or the other of these two formulas is in use at a large number of central banks [2], where the resulting zero-curves are used to help shape monetary policy; clearly, a strategic planning activity.

Both the Nelson-Siegel and the Svensson models yield smooth zero-curves. That, and the fact that one can explicitly calculate spot- and forward-rates at any time time in terms on a relatively simple formula are certainly desirable attributes. But, these are also the basis for the shortcomings of this approach. The fact that the rate-curves come with a precise analytic expression that depends on either four or six parameters might very well result in a 'fit' that doesn't take full account of the market structure, i.e., it will be difficult to match market prices with a high level of precision: there is an inherent lack of flexibility.

Moreover, because the fitting these parameters leads to an highly non-convex optimization problem, one essentially has to resort to global optimization techniques that are somewhat unreliable unless the search is exhaustive. Equivalently, one could look for a 'good' critical point by solving repeatedly a collection of nonlinear systems by relying on some heuristics to trim down the number of systems that need to be considered. This is the approach suggested by Svensson and implemented in various institutions [2]; a description of our implementation can be found in §7. Thus, the downside of this numerical approach is, on one end, the uncertainty about having considered a sufficient number of (discrete) possibilities and, on the other end, the need for the user to 'intervene' appropriately usually resulting in a time consuming operation.

4 Maximum smoothness

The definition of smooth for a curve might almost be subjective, but it's certainly application dependent. In the world of zero-curves, one is certainly not going to be satisfied with a 'smooth' spot rates curve, or a 'smooth' discount factors curve, when an associated zero-curve, for example the forward rates curve, is seesawing; that this can actually occur was already pointed out in [18]. In fact, it suffices return to Figure 1 and look at the Forward Rate curves generated by BootStrapping, even those derived from the 17-instruments sub-portfolio.

In [1], Adams and Van Deventer rely on a criterion used in engineering applications, cf. [9], in their derivations of spot and forward rates curves with 'maximum smoothness.' They propose finding a forward rates curve, fw, such that for each instrument in the portfolio:

$$P_i = \exp\left(-\int_0^{t_i} fw(s)\,ds\right), \quad i = 1, \dots, I,$$

where P_i is the (today's) price and t_i the maturity date of instrument i, and

$$\int_0^T [fw''(s)]^2\,ds \qquad \text{is minimized,}$$

$[0,T]$ is the time span in which we are interested; usually T is the largest maturity of the instruments in the portfolio. It is shown that the solution is a 4th order spline, certainly smooth, whose coefficients can be easily computed; 'maximum' smoothness is achieved in terms of a criterion attributed to Vasicek.

The Achilles' heel of this approach, at least as laid out in [1], is that the only instruments that can be included in the 'maximum smoothness' portfolio are zero-coupon bonds, and zero-coupon bonds with maturities exceeding one year are extremely rare. In order to obtain zero-curves that span more than a few months, one possibility is to fabricate artificial (long term) zero-coupon bonds that have similar financial characteristics to those instruments found in the portfolio; this requires interpolations of some type. Moreover, prices P_i are present day prices and so no future contracts can be included in the 'maximum smoothness' portfolio. Presumably, this can also be skirted by some adjustments. In the final analysis, like for 'standard' BootStrapping, we have to create a sub-portfolio and then enrich it by artificial instruments in order to be able to apply the suggested method.

From our previous examples and analysis, it's clear that if one is going to derive zero-curves by taking into account more than just a few well chosen instruments, and one is going to aim at an acceptable level of smoothness, there is going to be a 'price' to pay for this. In Function Theory, the smoothness of a curve is identified in terms of the number of times it's continuously differentiable. A curve $z : [0,T] \to I\!R$ is said to be of class $\mathcal{C}^q$ if its qth derivative is continuous. So, if z is of class $\mathcal{C}^2$ it means that it can be differentiated twice and the second derivative is continuous. If it's of class $\mathcal{C}^0$ then z is just continuous, and if it's of class $\mathcal{C}^\infty$ then all derivatives, of any order, exist and are continuous. It is evident that $\mathcal{C}^0 \supset \mathcal{C}^1 \supset \cdots \supset \mathcal{C}^\infty$. One might wish the zero-curves to be of class $\mathcal{C}^\infty$, but it is clear that this is a much smaller family of curves than those that are just continuous, or just continuously differentiable: $\mathcal{C}^1$. Consequently, by insisting that our zero-curve be of class $\mathcal{C}^\infty$, we might very well have excluded those curves that have the 'accuracy' properties we are looking for. Hence, we usually must be content with smooth curves that are less than *infinitely* smooth.

5 Forward-rates via geometric programming

A completely novel and tantalizing approach was developed by K. Kortanek [13], in collaboration with G. Medvedev [14, 15] and H. Yunianto [16], that's focused on obtaining the forward-rates curve. Their motivation, certainly in part, came from the shortcomings of the Nelson and Siegel, and Svensson models. In [14, 15], a forward rate model was developed allowing for non-stochastic perturbations

$$\frac{dfw(t)}{dt} = \alpha + \beta fw(t) + v(t), \quad fw(0) = r_0, \quad t \in [0,T],$$

where r_0, α, β and the perturbation function $v : [0,T] \to I\!R$ are the parameters that need to be estimated; the interval $[0,T]$ could be unbounded. The function v is assumed to be piecewise constant, i.e.,

$$v(t) = v_i \quad \text{if} \quad t \in (t_{i-1}, t_i]$$

where the subintervals $(t_{i-1}, t_i]$ partition $(0,T]$. For $t \in (t_{i-1}, t_i]$, one has [15, (9.12)],

$$fw(t) = r_0 e^{\beta t} + \frac{\alpha}{\beta}(e^{\beta t} - 1) + \frac{e^{\beta t}}{\beta} \sum_{j=1}^{i-1} \left(e^{\beta t_j - 1} - e^{\beta t_j}\right) v_j + \frac{e^{\beta(t - t_{j-1})} - 1}{\beta} v_j.$$

If the interval $[0.T]$ has been partitioned in I intervals, fitting the forward-rates curve will require estimating $I + 3$ parameters. In setting up this estimation problem, one can introduce a number of 'natural' bounds on all or some of the variables; for example, requiring $\beta \in [\beta_l, \beta_u] \subset I\!R_-$ would result in a mean reversion property for the spot rates.

Since this formula provides us with the forward rates for any time t, the price of any bond generating a given cash flow, coupons and notional with maturity less than or equal to T, can be expressed in terms of these parameters, cf. [16, §3.2]. The expression gets a bit involved because one has to take into account coupons that get paid a times that fall between the end points of the intervals $(t_{i-1}, t_i]$, but this is no more than keeping the 'bookkeeping' straight. Anyway, one ends up with an expression that is a *posynomial* of the following type:

$$P(t_k) = e^{a_k r_0} e^{b_k \alpha} (\prod_{j=1}^{J_k} e^{a_j v_j}) e^{b_{J_k} v_{J_k}} + (q_k \sum_{l=1}^{n_k} e^{a_{kl} r_0} e^{b_{kl} \alpha}) \prod_{j=1}^{J_k - 1} e^{c_j v_j} e^{a_{kl} v_j} e^{b_{J_k l} v_{J_k}}$$

for a bond of notional 1 and maturity t_k that falls in the interval $(t_{J_k - 1}, t_{J_k}]$, n_k is the number of coupons before maturity, and the 'coefficients' $a_j, b_j, \ldots$ are themselves functions of α and β; precise definitions can be found in [13] and [16, §3 & §4].

The strategy is now to find a best fit between the estimated prices $P(t_k)$ and the observed prices $\widehat{P}(t_k)$ that respects the *no-arbitrage* conditions. Taking advantage of some elegant simplifications, Kortanek show that the problem to be solved is a *geometric program*, i.e., an optimization problem whose objective and constraints are posynomials. This would fall in the extended family of convex optimization problems except for the presence of so-called 'reversed constraints' that are part of the formulation if the collection of securities being considered includes coupon paying bonds. Formulating a practical problem as a geometric program is called *GP modeling*. "GP modeling is not just a matter of using a software package or trying out some algorithm; it involves some knowledge, as well as creativity, to be done effectively [3]." This is carried out effectively by Kortanek *et al*, cf. [16].

We haven't implemented ourselves this procedure but have relied on the results provided by K. Kortanek, cf. Figure 1. On a number of test problems, this approach came the closest to the quality of the zero-curves generated by the method described in the next section, but our lack of an independent implementation means that we could not include it in the comparisons of §7. Further results, variants and extensions of this method can be found in [16].

6 EpiCurves

EpiCurves, introduced in [21], come from a large, but specific sub-family of curves that are of class $\mathcal{C}^q$ for some $q = 1, 2, \ldots$. EpiCurves could be viewed as 'constrained' splines, however their derivation doesn't follow the standard spline-fitting techniques. To simplify the presentation, suppose we are interested in finding a $\mathcal{C}^2$-curve z on $[0, T]$ given by

$$z(t) = z_0 + v_0 t + \frac{1}{2} a_0 t^2 + \int_0^t \int_0^\tau \int_0^s x(r)\, dr\, ds\, d\tau, \quad t \in [0, T],$$

where

- $x : (0, T) \to I\!R$ is an arbitrary piecewise continuous function that corresponds to the 3rd derivative of z;
- a_0, v_0, z_0 are constants that can be viewed as integration constants.

Once the function x (3rd derivative) and the constants a_0, v_0, z_0 have been chosen, the function z is completely determined.

Let's now go one step further. Instead of allowing for any choice for x, let's restrict the choice of x to piecewise constant functions of the following type: split $[0, T]$ into N sub-intervals of length T/N and let the function x be constant on each one of these intervals, with

$$x(t) = x_k, \quad \text{when} \quad t \in (t_{k-1}, t_k], \quad k = 1, \ldots, N$$

where $t_0, t_1, \ldots, t_L$ are the end points of the N sub-intervals. The corresponding curve z on $[0, T]$ is completely determined by the choice of

$$a_0, v_0, z_0 \quad \text{and} \quad x_1, x_2, \ldots, x_N,$$

i.e., by the choice of a finite number of parameters, exactly $N + 3$.

For $k = 1, \ldots, N$, $t \in (t_{k-1}, t_k]$ with $\delta = T/N$ and $\tau = t - t_{k-1}$,

$$
\begin{aligned}
z(t) &= z_0 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{\tau^3}{6} x_k \\
&\qquad + \sum_{j=1}^{k-1} x_j \left[\frac{\delta^3}{2} \Big(\frac{1}{3} + (k-j-1)(k-j) \Big) + \delta^2 \tau (k-j-0.5) + \frac{\delta \tau^2}{2} \right], \\
z'(t) &= v_0 + a_0 t + \sum_{j=1}^{k-1} x_j \Big(\delta^2 (k-j-0.5) + \delta \tau \Big) + \frac{\tau^2}{2} x_k, \\
z''(t) &= a_0 + \delta \sum_{j=1}^{k-1} x_j + \tau x_k, \\
z'''(t) &= x_k.
\end{aligned}
$$

By restricting the choice of x to piecewise constant functions, the resulting z-curves are restricted to those curves in $\mathcal{C}^2$ that have (continuous) piecewise linear second derivatives. Designate this family of curves by $\mathcal{C}^{2,pl}$ where pl stands for piecewise linear; whenever appropriate we use the more complete designation $\mathcal{C}^{2,pl}([0.T], N)$ with $[0, T]$ the range on which these curves are defined and N the number of pieces, but usually the context will make it evident on which interval these curves are defined. Clearly, not all $\mathcal{C}^2$-curves are of this type. However, Approximation Theory for functions, tells us that any $\mathcal{C}^2$-curve can be approximated *arbitrarily closely* by one whose second derivative is a continuous piecewise linear function, i.e., a curve in $\mathcal{C}^{2,pl}([0, T], N)$, by letting $N \to \infty$. This provides us with the justification one needs to restrict the search for 'serious' zero-curves to those in this particular sub-family of $\mathcal{C}^2$-curves.

Later on, the implementation will impose further restrictions on the choice of the coefficients, not to guarantee 'smoothness' in itself since every curve in $\mathcal{C}^{2,pl}$ is clearly (mathematically) smooth, but to generate zero-curves that would be called 'smooth' by a practitioner.

In summary, the building of *EpiCurves* starts by selecting the level of smoothness desired ($z \in \mathcal{C}^q$), and then a zero-curve is built whose qth derivative is a continuous piecewise linear function. This requires fixing a finite number of parameters; actually $N + q + 1$ parameters. If the resulting curve does not meet certain accuracy criteria, the step size (T/N) is decreased by letting $N \to \infty$.

6.1 Zero-curves from spot rates

To set the stage for finding the zero-curves associated with a collection of instruments generating cash flow streams, let's consider an *EpiCurves* approach to fitting spot rates to obtain a spot rates (yield) curve. The data come in a pair of arrays,

$$s = (s_1, s_2, \dots, s_L), \qquad m = (m_1, m_2, \dots, m_L),$$

that give us the spot rates for a collection of instruments of different maturities, for example, Treasury Notes. The task is to find a spot rates curve that 'fits' these data points. That is easy enough. Assuming that $m_1 < m_2 < \cdots < m_L$, one could simply derive a spot rates curve by linear interpolation between adjacent pairs. That's actually a perfect fit. Generally, this is not a 'smooth' curve. This usually generates a forward rates curves that can be quite jagged. So this 'simple' solution almost never produces zero-curves that practitioners would consider acceptable. Of course, one can use another interpolation method, such as via quadratic or cubic splines, that generates significantly better results. Another possibility is to set-up an artificial portfolio with coupon-bonds whose yields would match the given spot rates. The problem is then reduced to one of finding the zero-curves associated with the cash flow stream of this (artificial) portfolio. This is dealt with in the next subsection. But, this latter approach, found in the packages of some financial technology providers, circles around the problem, at least one too many times, before dealing with it.

The use of the *EpiCurves* technology provides an elegant solution that generates smooth zero-curves. The strategy is to find a spot rates curve of the type described in the previous subsection, say again a $\mathcal{C}^{2,pl}$-curve that will match the given spot rates. One must accept the possibility that we won't be able to find, for a fixed N, a $\mathcal{C}^{2,pl}([0,T],N)$-curve that fits perfectly the given data. So, the problem becomes one of finding the 'best' possible fit. Best possible can be defined in a variety of ways but it always comes down to minimizing the 'error', i.e., the distance between the *EpiCurves* and the given spot rates. Mathematically, the problem can is

$$\text{find} \quad z \in \mathcal{C}^{2,pl}([0,T],N) \text{ so that } \|s - z(m_1 : m_L)\|_p \quad \text{is minimized },$$

where $z(m_1 : m_L) = (z(m_1), z(m_2), \dots, z(m_L))$ and $\|a\|_p$ is the ℓ^p-norm of the vector a. With $p = 1$, one would be minimizing the sum of the (absolute) errors, with $p = 2$ one minimize the sum of the squares of the errors, and with $p = \infty$, it would be the maximum (absolute) error that would be minimized. An implementation by EpiSolutions Inc., has $p = 1$ and thus minimizes the sum of the errors, since

$$\|s - z(m_1 : m_L)\|_1 = \sum_{l=1}^{L} |s_l - z(m_l)|.$$

The resulting optimization problem can then be reduced to a linear programming problem, since, as explained in the previous subsection, the functions z in $\mathcal{C}^{2,pl}$ are completely determined by a finite number of parameters.

To illustrate the results, we apply both linear interpolation and the *EpiCurves* technology to obtain a spot rates curve that fits the spot rates (for T-bills and Treasury notes) of October 1982:

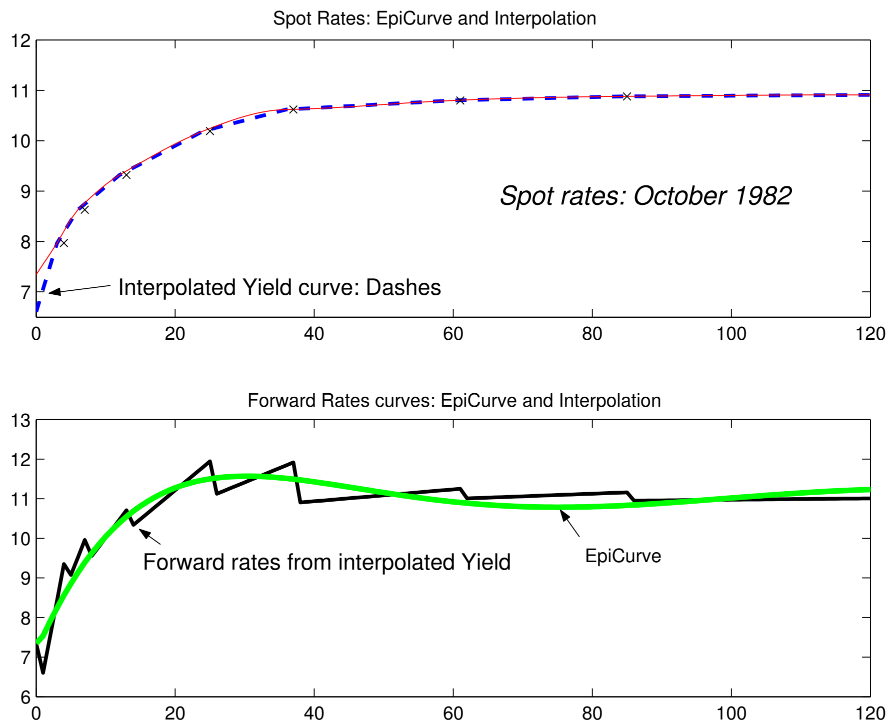


Figure 5: Linear interpolation vs. EpiCurves.

$$
\begin{aligned}
m &= (3, 6, 12, 24, 36, 60, 84, 120, 240, 360),\\
s &= (7.97, 8.63, 9.32, 10.19, 10.62, 10.8, 10.88, 10.91, 10.97, 11.17);
\end{aligned}
$$

the time unit is 1 month. The spot and forward rates curves can be found in Figure 5. It's barely possible to see the difference between the spot rates curves, but the difference between the forward rates curve is more than noticeable. The difference, of course, can be traced back to the intrinsic smoothness of the spot rates curve when it's generated as an *EpiCurve*.

Let also consider the case spot rates for January 1982, the maturities-array m is the same, but now

$$s = (12.92, 13.90, 14.32, 14.57, 14.64, 14.65, 14.67, 14.59, 14.57, 14.22).$$

Running *EpiCurve* yields the result in Figure 6. The forward rates curve is rather unsettled up to the end of year 1, it actually reflects almost perfectly the 'unsettled' market situation at that time (January-February 1982).

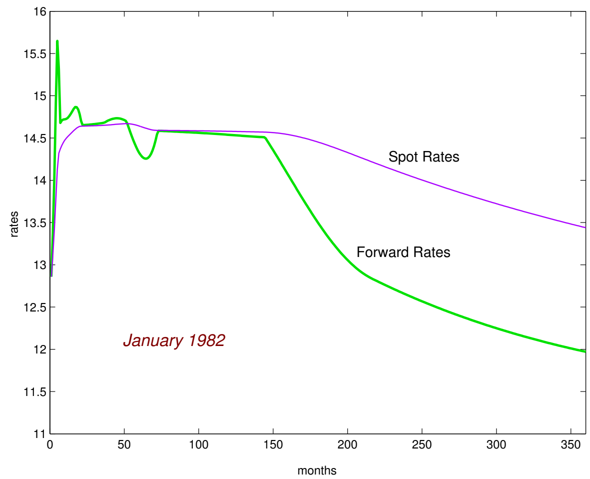


Figure 6: Spot and Forward Rates curves from spot rates.

6.2 Zero-curves from cash flow streams

We briefly review our goals and guiding principles. Given the increased complexity of the instruments being traded, it certainly is no longer sufficient to be able to build zero-curves based on just zero-coupon bonds; in fact, the LIBOR (swaps) based zero-curves seem to occupy, at present, the prominent place . The ultimate objective should be to build the spot rates curve associated with *any* collection of instruments, for example, AAA- or AA-rated corporate bonds, any mixture of swaps, futures and bonds, etc. Notwithstanding a relatively large literature devoted to zero-curves, cf. Buono, Gregory-Allen and Yaari [5], there has never been any serious attempt at dealing with the building of zero-curves at this more comprehensive level.

Of course, given an arbitrary collection of instruments, each one generating its own cash flow stream, it might be possible (assuming that maturities occur at different dates) to generate, via BootStrapping for example, any one of the zero-curves. However, as every practitioner knows all too well and as was reviewed in §2, some of the resulting curves will be, to say the least, unwieldy, and have every characteristic except 'believable.' The insistence on 'smoothness', cf. Vasicek and Fong [20], Shea [18], Adams and Van Deventer [1], is motivated by the strongly held belief, that's also supported by historical data, that zero-curves don't come with kinks, and spikes i.e., extremely abrupt changes in the rates.

Keeping this in mind, the problem of generating zero-curves could be roughly formulated as follows: Given a collection of instruments, each one generating a given cash flow stream, find smooth zero-curves so that for each instrument (in the collection), the *net present value (NPV)* of the associated cash flow matches its present price.

Although this formulation allows us to include zero-coupon bonds, coupon bonds, swaps, etc., in our collection of instruments, it does not allow for futures, future swaps, etc.

To do so, we reformulate the problem in the following more general terms: With each instrument i in our collection, we associate a

$$\text{Time Array:} \quad (t_{i1}, t_{i2}, \ldots, t_{i,L_i})$$

the dates, or time span, at which cash payments will take place, and a

$$\text{Payments Array:} \quad (p_{i1}, p_{i2}, \ldots, p_{i,L_i})$$

with cash flow p_{il} received at time t_{il}, L_i is the maturity date. One then interprets $p_{il} > 0$ as cash received and $p_{il} < 0$ as cash disbursed. For example, in the case of a coupon bond, bought today for \$100, with semi-annual \$3 coupons and a two year-maturity, one would have:

$$\text{Time Array:} \quad (0, 6, 12, 18, 24),$$

assuming the time unit is '1 month', and

$$\text{Payments Array:} \quad (-100, 3, 3, 3, 103).$$

This allows us to include almost any conceivable instrument in our collection, as long as it comes with an explicit cash flow stream. For example, in the case of the following T-bill forward: A bank will deliver in 3 months from now, a 6-month Treasury bill of face value \$100 with a 10% annual forward rate for that 6 months period. The value of such a contract would be \$95.24 that would have to be paid in 3 months. This contract would then come with the following arrays:

$$\text{Time Array:} \quad (3, 9), \qquad \text{Payments Array:} \quad (-95.24, 100).$$

In this frame of reference, the zero-curve problem could be formulated in the following terms: Given a (finite) collection of instruments that generate cash flow streams, find a *discount factor curve* such that

- the net present value (NPV) of each individual instrument (contract) turns out to be 0 when all cash payments received and all disbursements are accounted for;
- all associated zero-curves (forward, spot, discount rates) are 'smooth'.

When formulated at this level of generality, the zero-curve problem is usually not feasible. In fact, it's not difficult to fabricate an 'infeasible' problem. Simply, let the collection consist of two one-coupon bonds that have the same nominal value, the same maturity and the same price (today). Both coupons are to be collected at maturity but have different face value. Clearly, there is no discount factor curve so that the net present value (NPV) of both of these cash flows turns out to be 0! Of course, this is an unrealistic example, the financial markets wouldn't have assigned the same price to these two instruments; arbitrage would be a distinct possibility in such a situation. But since we allow for *any* collection of instruments, there is the distinct possibility that there are practical instances when one can't find a 'smooth' discount factors curve so that the NPV of *all* cash flow streams factors out to 0. So, given that we want to be able to deal with any eclectic collection of instruments, as well as the 'standard' ones, instead of asking for the NPV of all cash flow streams to be 0, we are going to ask that they be as close to 0 as possible.

Smoothness is going to be achieved by restricting the choice of the discount factors curve to $\mathcal{C}^{q,pl}$, i.e., curves whose qth derivative is continuous and piecewise linear as introduced in this section. To render our presentation more concrete, and easier to follow, we are going to proceed with $q = 2$.

The problem is now well defined mathematically:

$$\text{find a discount factors curve:} \quad df \in \mathcal{C}^{2,pl}([0,T],N) \text{ so that } \quad \|v\|_p \quad \text{is minimized .}$$

where $\|v\|$ is the ℓ^p-norm of v,

$$v = (v_1, v_2, \dots, v_I), \qquad v_i = \sum_{l=1}^{L_i} df(t_{il}) p_{il};$$

v_i is the net present value of instrument 'i' given that the cash flow is discounted using the discount factors $df(t_{il})$. The EpiSolutions Inc., implementation relies on the ℓ^∞-norm,

$$\|v\|_\infty = \max [\, |v_1|, |v_2|, \dots, |v_I|\,],$$

so let's proceed with this criterion but it should be noted that one can choose any $p \in [1, \infty)$ that might better represent the decision maker's preferences or concerns. In fact, except for extremely unusual portfolio, the differences between the solutions should be insignificant.

Since df belongs to $\mathcal{C}^{2,pl}([0,T],N$, it's of the form: for $k = 1, \dots, N$, $\delta = T/N$, $t \in (\delta(k-1), \delta k]$ and $\tau = t - \delta(k-1)$

$$\begin{aligned} df(t) &= 1 + v_0 t + \frac{1}{2} a_0 t^2 + \frac{\tau^3}{6} x_k \\ &\quad + \sum_{j=1}^{k-1} x_j \left[\frac{\delta^3}{2} \Big(\frac{1}{3} + (k-j-1)(k-j) \Big) + \delta^2 \tau (k-j-0.5) + \frac{\delta \tau^2}{2} \right]; \end{aligned}$$

where $a_0, v_0, x_1, x_2, \dots, x_N$ are parameters to be determined; note that the discount factor at time $t = 0$ is 1, so we can fix this 'constant' (z_0). But simply being of this form doesn't make df a discount factors curve. We already have that $df(0) = 1$, we need to add two conditions:

- df should remain non-negative, thus we have to introduce the constraints: $df(t) \geq 0$ for all $t \in [0, T]$;
- df should be decreasing, at least non-increasing, this means that $df'(t) \leq 0$ for all $t \in [0, T]$, a condition that translates into the constraints:

$$df'(t) = v_0 + a_0 t + \sum_{j=1}^{k-1} x_j \Big(\delta^2 (k - j - 0.5) + \delta\tau \Big) + \frac{\tau^2}{2} x_k \leq 0, \quad \forall t \in (0, T].$$

Putting this all together with df as defined above, yields the optimization problem:

$$\begin{aligned}
\min\ & \theta \\
\text{so that}\ & \theta \geq \sum_{l=1}^{L_i} df(t_{il}) p_{il}, \qquad i = 1, \ldots, I, \\
& \theta \geq - \sum_{l=1}^{L_i} df(t_{il}) p_{il}, \qquad i = 1, \ldots, I, \\
& df(t) \geq 0, \qquad t \in [0, T], \\
& v_0 + a_0 t + \sum_{j=1}^{k-1} x_j \Big(\delta^2 (k - j - 0.5) + \delta\tau \Big) + \frac{\tau^2}{2} x_k \leq 0, \quad t \in (0, T], \\
& v_0 \leq 0,\ a_0 \geq 0,\ x_k \in I\!R, k = 1, \ldots, N;
\end{aligned}$$

the restriction $v_0 \leq 0$ means that $df'(0)$ is not positive, and $a_0 \geq 0$ says that the curve should have positive curvature at $t = 0$. The constraints involving θ tell us that

$$\theta \geq \max_{i=1,\ldots,I} \left[\left| \sum_{l=1}^{L_i} df(t_{il}) p_{il} \right| \right],$$

and by minimizing θ, we minimize the max-error; this inequality is split into $2I$ constraints so that all constraints are linear.

We have a linear optimization problem with a finite number of variables $(N + 3)$, but with an infinite number of constraints $(\forall t \in [0, T])$. To solve this problem, one could consider using one of the techniques developed specifically for (linear) semi-infinite optimization problems. Because of the nature of the problem, however, one can safely replace the conditions involving 'for all $t \in [0, T]$' by for all $t \in \{1/M, 2/M, \ldots, T/M\}$ with M sufficiently large; in the EpiSolutions Inc. implementation M is usually chosen so that the mesh size $(1/M)$ is 1 month. After this time-discretization, the problem becomes a linear programming problem that can be solved using a variety of commercial packages. In addition to the constraints described earlier, the version implemented at

EpiSolutions Inc. also relies on a few additional constraints that will improve the shape the zero-curves to fit more specifically the context.

One important component of the *EpiCurves* solution is that all the zero-curves are defined at every time t, there is never any need to resort to interpolations to fill in missing time-gaps. This, of course, gives us great flexibility in choosing the right approximations when building pricing mechanisms.

6.3 More examples

Let's now consider and analyze a few more examples.

DFS-Portfolio. Let's first go back to the DFS-example of §2. Of course, by just comparing the graphs of the forward rates curves provided by Financial Providers no.1 and no.2 and corresponding EpiCurve, it's evident that there is the possibility that some financial factors/indicators might be a little bit too much off. Here is a specific example: On June 10, 1997, the June 1999 futures contract settled at \$ 93.36. After a small convexity adjustment, this implies a 90-day forward rate of approximately 6.60%. This contract was not included in the BootStrapping sub-portfolio (17 instruments) of Financial Provider No. 1. The forward rate supplied by Financial Provider No. 1 for this period is 5.10%! Not surprisingly, the forward rate for this period, supplied by the EpiCurve, that takes into account <u>all</u> 36 instruments, is the more reliable 6.45%.

Bond-Portfolio. The first one is a Bond-portfolio. This data-set includes U.S. Treasury Bill and U.S. Treasury Bond data from August 3, 2001. There are 7 instruments in all, including 3 U.S. Treasury Bills with term-to-maturities ranging from 3 to 7 months and 4 U.S. Treasury Bonds with term-to-maturities ranging from 2 to 30 years. This data was obtained from the Bloomberg U.S. Treasuries web page; details are in Table A–4 of the Appendix. As a point of comparison, we use the results of the BootStrapping technique supplied by Financial Provider no. 2; Financial Provider no. 1 BootStrapping functionality can not deal with a Bond-portfolio. The results are graphed in Figure 7.

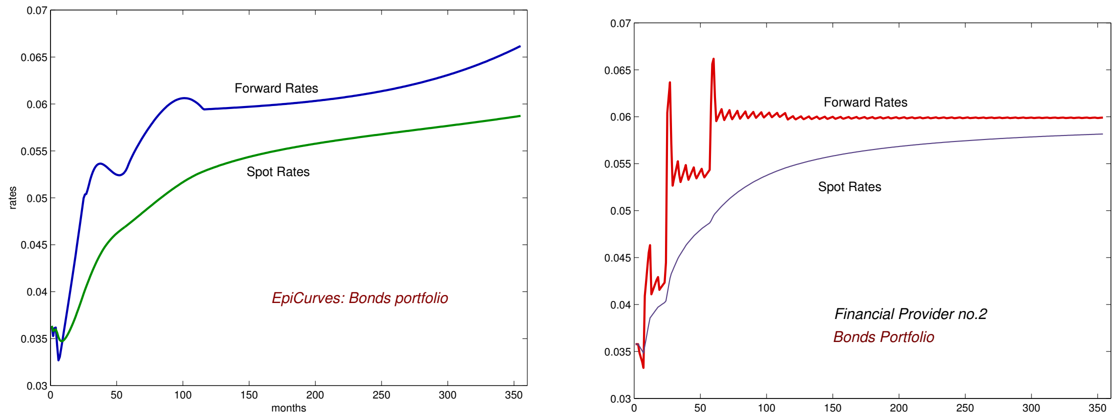


Figure 7: Spot and Forward Rates associated with a Bonds portfolio.

DFS2-Portfolio. This next example is a relatively challenging one. The portfolio includes 51 instruments: Deposits, Futures and Swaps from August 3, 2001 with quite a bit of overlap of maturity-dates. A short description of the composition of this portfolio follows here; details are in Tables A–5, A–6 of the Appendix. There are 51 instruments in all, broken down as follows: 3 Eurodollar Deposits with term-to-maturities ranging from 1 to 6 months, 40 Eurodollar Futures with 90-day deposit maturities ranging from 4 months to 10 years, and 8 Swaps with term-to-maturities ranging from 1 to 10 years. This data was obtained from the Federal Reserve (Statistical Release H.15) and the Chicago Mercantile Exchange (CME).

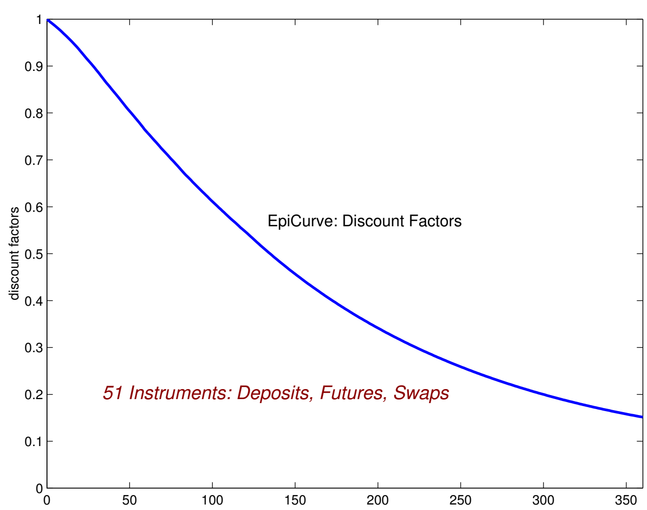


Figure 8: Discount Factors curve for the 51-instruments portfolio

In the EpiSolutions Inc. implementation of the *EpiCurve* methodology, there is an option that allows the user to fine tune the level of accuracy that will be acceptable; accuracy being defined in terms of the max-error, i.e. in terms of the objective of the optimization problem. Asking for a higher level of accuracy will usually result in a more jagged curve since one must accommodate/adjust more rapidly to even small changes in the cash flow. This is effectively illustrated by curves graphed in Figure 9. In the first one the tolerance is 5 base points, in the second one just 1 base point.

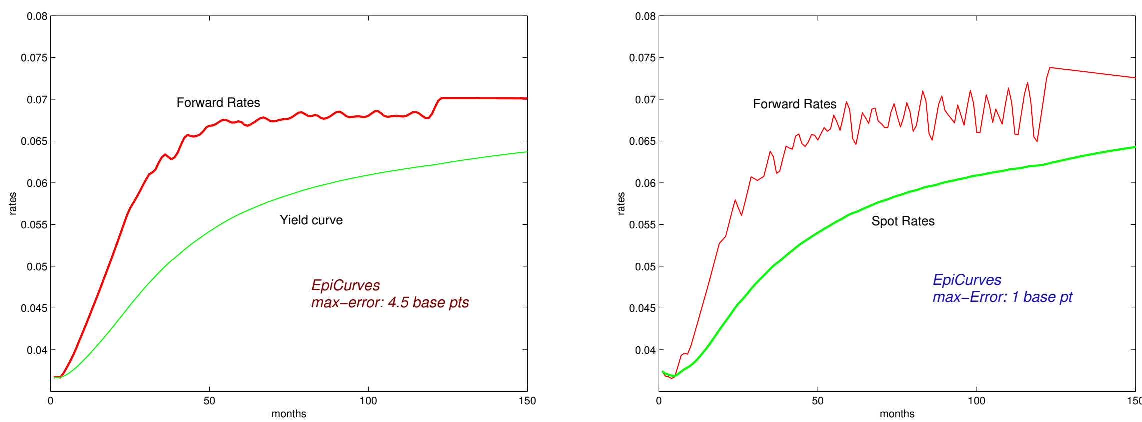


Figure 9: Variations of the zero-curves under max-error tolerance

Notwithstanding this fine tuning, *EpiCurve* is really the only methodology that provides 'serious' zero-curves associated with such a portfolio. The only BootStrapping approach that one could rely on, if one can use the word 'rely,' is the one implemented by EpiSolutions Inc. But the results are less than satisfactory, see Figure 10. Both the spot and the forward rates curves were derived for this portfolio with the forward rates curve generated by BootStrapping spiking up to 30% at one point and then immediately thereafter going negative! This suggested massaging the portfolio by introducing a *convexity adjustment* and, indeed, this improves substantially the BootStrapping results although the forward rates curve comes with some abrupt rate changes; note that this convexity adjustment has only a minor effect on the *EpiCurves* results.

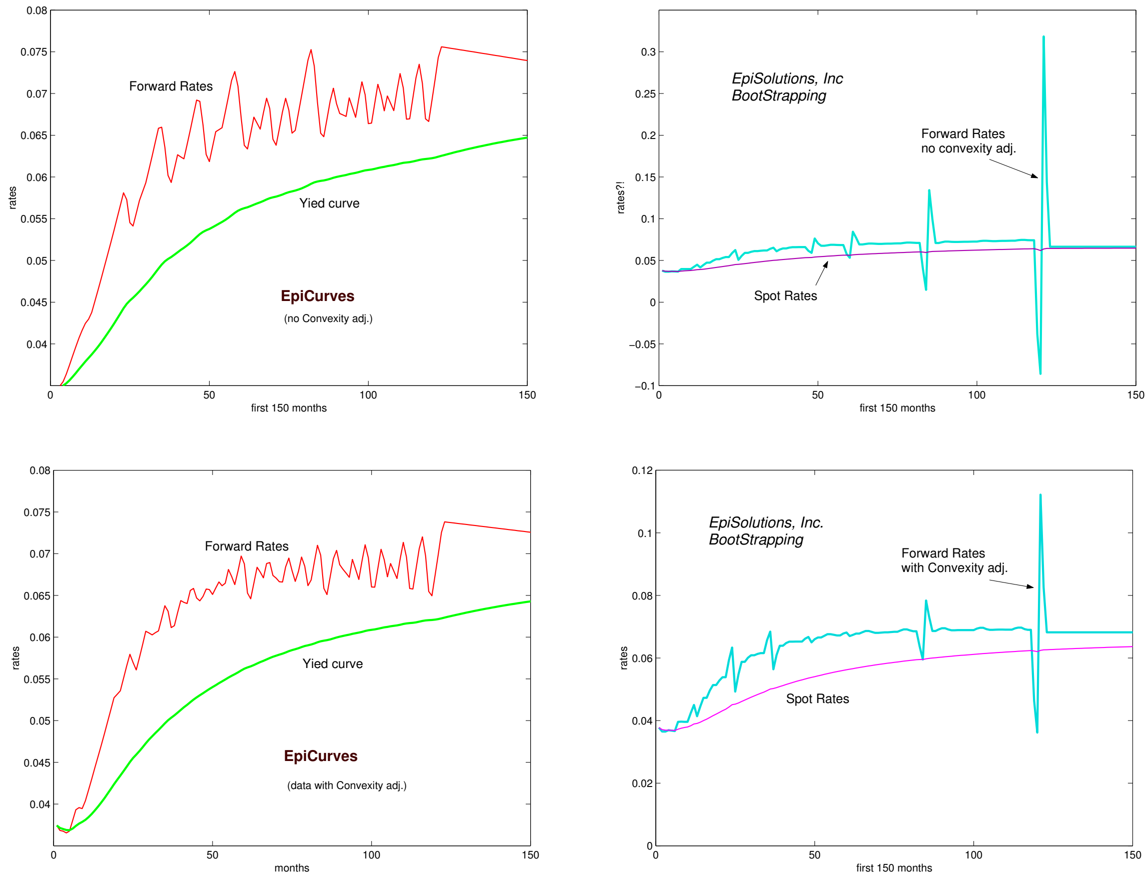


Figure 10: Spot and Forward Rates with and without Convexity adjustments

The major objective in developing the EpiCurves methodology was to overcome the inconsistent assumptions and limitations of the standard BootStrapping technique and its Maximum Smoothness variant. This is accomplished by allowing for inclusion of the *complete portfolio* of term structure instruments, while at the same time providing the *smoothness* so crucial to practitioners as a solid foundation on which to build believable valuations, forecasts and other financial analytics.

7 A comparison for U.S. Treasury curves

In this section we perform an empirical analysis of several different estimation methods using monthly U.S. Treasury bond data obtained from Mergent's Bond Record. The data set covers the period from January 31, 1999 through December 31, 2003, for a total of 60 observations (with each observation, or portfolio, containing approximately 100-110 bonds). In particular, we focus on the following methods:

- **EpiCurves, run for accuracy:** With this approach we run EpiCurves with the objective of minimizing the maximum absolute pricing error.

- **EpiCurves, run for smoothness:** With this approach we run EpiCurves with the objective of 'maximizing' the smoothness of the resulting curves. We employ Van Deventer's smoothness criterion, divided by T, to facilitate comparison of the monthly bond portfolios. We then iterate, relaxing the maximum pricing error constraint until we have achieved an acceptable level of smoothness (defined to be 0.01, for this analysis).

- **Van Deventer's maximum smoothness approach:** With this approach the primary control variable is the number of spline segments. Too many segments may result in more accurate pricing, with a resulting loss in smoothness, while too few may result in a spline that is too stiff, with a resulting loss in pricing accuracy. To address this, we set the initial number of spline segments to be approximately equal to the square root of the number of bonds in a given portfolio. We then iterated, reducing the number of spline segments by one, until the number of segments giving the best pricing results was found.

- **Svensson's extension of the Nelson-Siegel model:** With this approach we used a number of different sets of starting values, solved for each, and selected the solution with the best pricing results. β_0 and β_1 were held constant, with β_0 set to the yield-to-maturity of the bond with the longest maturity in a given portfolio, and β_1 set to the difference between this and the yield-to-maturity of the bond with the shortest maturity in the portfolio. This left us with 4 parameters $(\beta_2, \beta_3, \tau_1, \tau_2)$ each of which we allowed 5 possible values, giving us a total of 625 sets of starting values.

We examined the estimation results along several different dimensions:

- **Smoothness:** Van Deventer's smoothness criterion, divided by T.

- **In-Sample Mean Squared Error (IN MSE):** the mean squared pricing error of bonds used to perform the estimation.

- **In-Sample Weighted-Average Absolute Error (IN WAE):** the weighted-average absolute pricing error of bonds used to perform the estimation, weighted by $1/maturity$.

- **In-Sample Maximum Absolute Error (IN MAX):** the maximum absolute pricing error of bonds used to perform the estimation.

- **Out-of-Sample Mean Squared Error (OUT MSE):** the mean squared pricing error of the bonds *not* used to perform the estimation.

- **Out-of-Sample Weighted-Average Absolute Error (OUT WAE):** the weighted-average absolute pricing error of bonds *not* used to perform the estimation, weighted by $1/maturity$.

- **Out-of-Sample Maximum Absolute Error (OUT MAX):** the maximum absolute pricing error of bonds *not* used to perform the estimation.

- **Speed:** the number of minutes to complete an estimation on a 2.0 GHz machine running Windows 2000.

All pricing error results are given in basis points. *In-Sample* bonds were defined by starting with shortest maturity bond in a given portfolio, and including every other bond in ascending maturity order – with the caveat that the longest maturity bond was always *in-sample*. We did not filter the data in any other way, nor did we employ any outlier exclusion scheme during the estimations. It should be noted, however, that there were no Treasury bills or callable bonds in the data set. Some graphical estimation results can be seen in Figure 11. The numerical results are summarized in Table 1.

We can make some general observations about the results:

- The Van Deventer and Svensson results are always smoother than the EpiCurves results. This is not surprising, since they are *defined* to be so, while EpiCurves allows the user to trade-off important criteria.

- The Van Deventer method performs well in upward-sloping term structure environments (2002-2003), and poorly when the term structure is flatter (1999-2001). This is also not very surprising, since fitting splines to straight lines is difficult.

- Over the entire period studied, the Van Deventer results where more variable than the results from any of the other methods considered.

- The Svensson results were remarkably stable, while providing relatively good pricing results, in different environments.

- When run for *accuracy*, EpiCurves always gave the best pricing results.

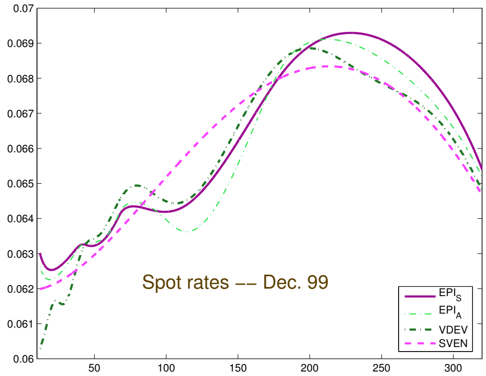
- When run for *smoothness*, EpiCurves was able to achieve an comparable level of smoothness to the other methods, while retaining excellent pricing accuracy.

- EpiCurves allowed for effective trade-off between smoothness (for strategic decision making) and pricing accuracy (for tactical decision making).

- EpiCurves was always *faster* than Van Deventer and Svensson, by a wide margin.

The last point deserves further comment. As can be seen in Table 1, even in the worst case, EpiCurves performed the estimations 4 times faster than either Van Deventer or Svensson. The original Van Deventer method was specified using a small set of zero-coupon bond prices, which requires solving a system of linear equations *once*. In the extension presented here, we are using large number of coupon-bearing bonds, making it necessary to iterate in two dimensions. In the first dimension, given a set of knot points for the spline segments (also known as term structure vertexes), we iterate to find the set of zero-coupon bond prices for the knot points that minimizes the overall pricing error. In the second dimension, we also iterated over the number of knot points to determine the optimal trade-off between curve flexibility and pricing accuracy. In the best case, this required an average of 27 minutes per estimation (2003). In the worst case, the speed deteriorated to an average of 46 minutes per estimation (1999).

One of the main criticisms of the Svensson methodology is that the parameters can be difficult to estimate. This is due to the fact that the spot and forward rate functions, though linear in the β's, are non-linear the τ's. As a result, there are multiple local minima, making it necessary to run the estimations for many different sets of starting values. To completely eliminate uncertainty in the results would require running the estimations over an unwieldy number of sets of starting values. By holding β_0 and β_1 constant, we reduced the number of free variables to 4. Then allowing these 4 variables to take on each of 5 different values, we settled on 5^4, or 625, sets of starting values. The β's are ranged from -(maximum bond yield) to +(maximum bond yield), and the τ's are ranged from minimum bond maturity to maximum bond maturity. In the best case, this required an average of 25 minutes per estimation (2003). In the worst case, the speed deteriorated to an average of 38 minutes per estimation (1999).


Spot rates –– Dec. 99
EPI_S
EPI_A
VDEV
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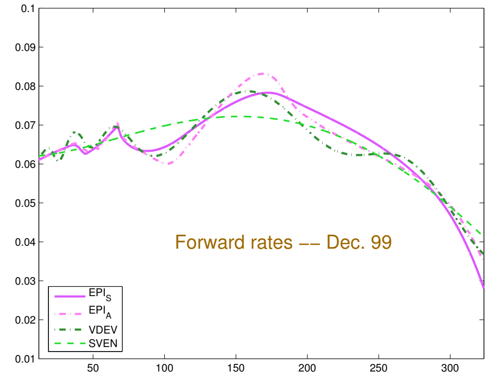
Forward rates –– Dec. 99
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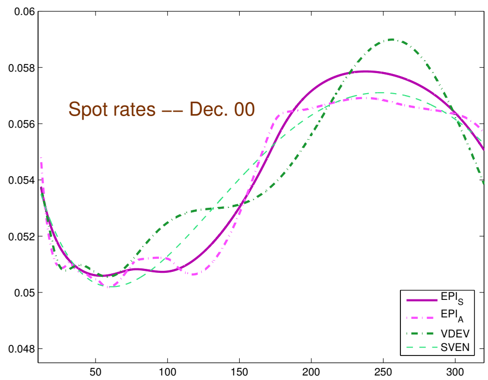
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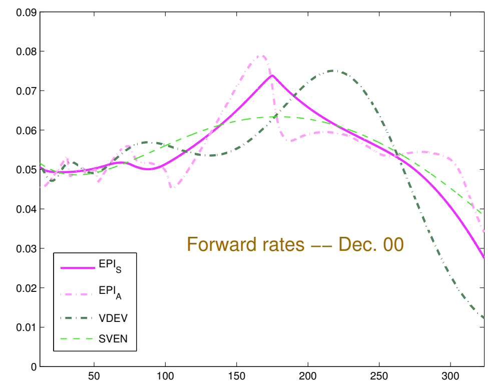
Forward rates –– Dec. 00
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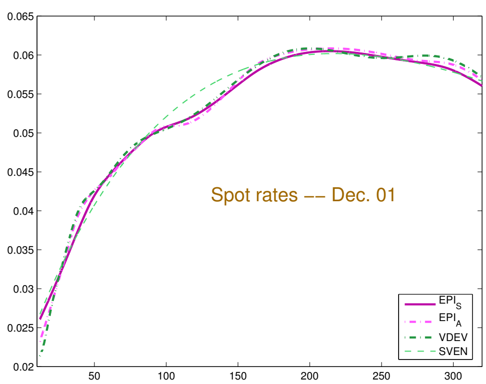
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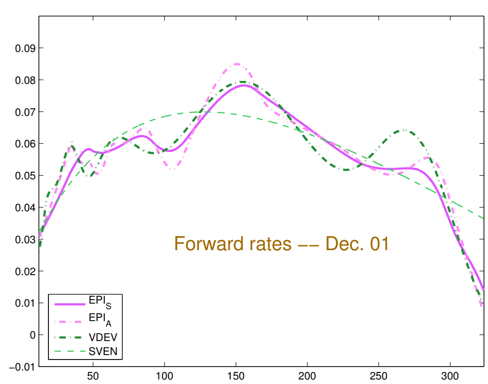
Forward rates –– Dec. 01
EPI_S
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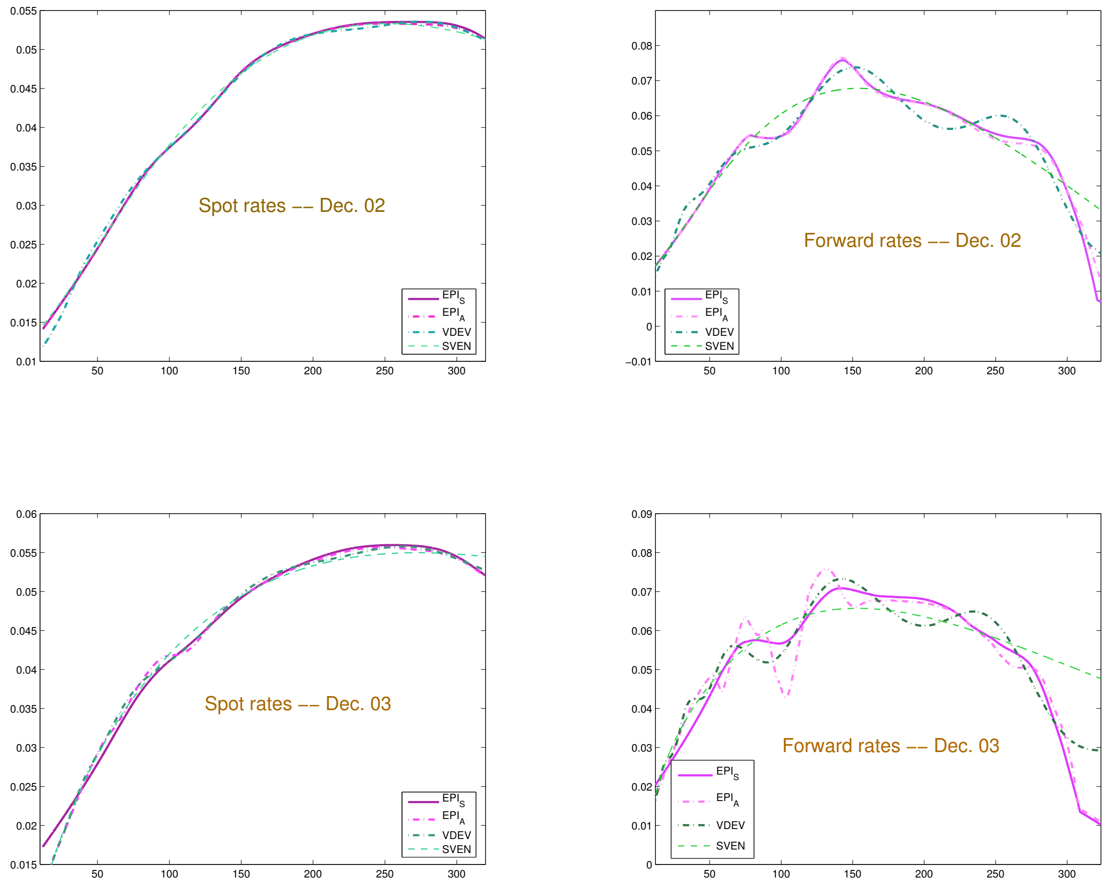

Figure 11: Comparison of Spot and Forward Rate curves.

1999-2003	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.252989	0.005163	0.000282	0.000877
IN MSE	442.03	986.59	2230.32	1214.64
IN WAE	8.95	22.42	8.66	20.74
IN MAX	58.44	89.16	145.03	133.38
OUT MSE	501.41	1032.69	2075.52	1040.42
OUT WAE	9.63	22.50	8.86	20.52
OUT MAX	79.54	96.82	151.85	122.77
Speed (min)	2	4	34	30

2003	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.731324	0.003204	0.000028	0.000559
IN MSE	298.25	1378.76	448.76	1647.35
IN WAE	4.51	34.83	5.72	22.80
IN MAX	41.86	97.72	80.70	128.69
OUT MSE	321.82	1291.56	323.74	1496.99
OUT WAE	5.87	34.55	5.89	22.53
OUT MAX	58.23	80.86	72.51	121.62
Speed (min)	4	6	27	25

2002	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.357384	0.004697	0.000112	0.003827
IN MSE	568.15	1390.76	604.60	1549.93
IN WAE	6.42	32.13	6.34	29.42
IN MAX	57.40	111.04	91.89	142.65
OUT MSE	637.45	1463.28	615.65	1391.80
OUT WAE	7.55	32.33	6.65	29.42
OUT MAX	85.89	116.09	105.80	152.67
Speed (min)	3	6	30	26

2001	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.103716	0.003786	0.000104	0.000000
IN MSE	215.98	530.63	2738.24	1072.05
IN WAE	8.18	15.11	10.36	26.68
IN MAX	39.54	61.68	151.60	110.96
OUT MSE	355.28	693.17	2376.87	973.28
OUT WAE	9.27	16.11	10.74	26.77
OUT MAX	72.58	91.73	172.03	107.12
Speed (min)	2	4	30	28

2000	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.02769	0.006766	0.000098	0.000000
IN MSE	300.23	539.01	3114.29	623.21
IN WAE	7.47	8.69	9.29	10.99
IN MAX	55.75	67.35	162.62	118.72
OUT MSE	356.57	616.53	2900.21	408.48
OUT WAE	7.63	8.72	9.55	10.46
OUT MAX	74.62	82.71	165.51	81.49
Speed (min)	1	3	34	33

1999	*EPI Accurate*	*EPI Smooth*	*Van Deventer*	*Svensson*
Smoothness	0.044831	0.007363	0.001069	0.000000
IN MSE	827.53	1093.78	4245.72	1180.67
IN WAE	18.19	21.35	11.59	13.82
IN MAX	97.65	108.00	238.32	165.90
OUT MSE	835.93	1098.92	4161.15	931.56
OUT WAE	17.82	20.81	11.49	13.43
OUT MAX	106.38	112.73	243.38	150.98
Speed (min)	2	3	46	38

Table 1: U.S. Treasury Curve Statistics (Averages)

Part II
VOLATILITY STRUCTURE

8 Setting the stage

The motivation for the *EpiVolatility* model is to provide a consistent, flexible, and *market calibrated* term structure of volatility that, in particular can serve as input to any valuation package, in particular the *EpiValuation* library of EpiSolutions Inc. No comprehensive comparisons will be made with alternative approaches because, to our knowledge, the alternative methods that have been suggested are either proprietary (Risk Metrics), are based on historical data rather than (present-day) market data, are relatively elementary (regression, linear interpolation, for example) or rely on a BootStrapping type approach that will be described below; a brief review of this literature was provided by Dupačová and Bertocchi [8, §3], see also an earlier article by these authors and their collaborators [7].

The primary option pricing models underlying a significant number of valuation packages, including the EpiValuation Library, are the Black, Derman, and Toy (BDT)(1990) binomial model and the Black (1976) model for interest-rate derivatives. The approach outlined in this contribution is in the class of popular interest rate models known as market models. In particular, our estimation of the term structure of volatility is based on the standard market model, which is also the basis for volatility estimation in the LIBOR market model (LMM). In particular, this makes *EpiVolatility* model consistent with the LIBOR market model..

The LIBOR market model, also known as the Brace, Gatarek, and Musiela (BGM)(1997) model, is an extension of the Heath, Jarrow, and Morton (HJM)(1992) model. However, where the HJM model is based on (unobservable) instantaneous forward rates, the LMM is based on observable market LIBOR rates that follow a lognormal process. This makes the LMM consistent with the Black model for pricing interest rate caps and floors, which is used by market practitioners. A similar model was developed by Jamshidian (1997) for swap rates that is consistent with the Black model for valuing European swaptions. This model is known as the Swap market model (SMM). One can refer to Hull [10] for a detailed description of these market and valuation models.

Although the LMM and the SMM are each internally consistent (neither allows opportunities for arbitrage), they are not consistent with each other, see [11]. This is because the LMM is based on a lognormal process for forward rates and the SMM is based on a lognormal process for swap rates, where swap rates can be thought of as an average of a series of forward rates. However, the difference in swaption prices between the two

models is low, see [4], and that the SMM substantially overprices caplets, see [12]. We want to be clear in stating, therefore, that for practical purposes one can reasonably *assume* that the LMM is the preferable model.
The construction of a BDT binomial interest rate tree requires three inputs: a time ruler, a yield curve, and a volatility curve. The time ruler is based on the security being priced (timing of cash flows, options, etc.). In Part I, we have dealt with the construction of the LIBOR yield curve (or zero-curve) as estimated from current market rates/prices of Eurodollar deposits, Eurodollar futures, and/or (on market) interest rate swaps. In principle, the volatility curve (or term structure of volatility) can simply be observed in the market, since interest rate cap and floor prices are *quoted* in terms of flat implied Black volatilities.

Table 2 has the cap volatility quotes from GovPx on October 24, 2002. The term of of the cap is expressed in years and the tenor of the cap is 3 months (i.e., each cap is a series of 3 month caplets). The bid and ask volatilities are expressed in percent per annum. The strike is the at-the-money strike rate of the cap. This means that the cap strike rate equals the swap rate, for a swap with the same reset dates as the cap.

Term	*Bid Vol*	*Ask Vol*	*Strike*
0.50	52.86	53.86	1.722
1.00	50.20	51.20	1.894
1.50	49.90	50.90	2.201
2.00	48.26	49.26	2.533
2.50	45.09	46.09	2.826
3.00	42.28	43.28	3.073
3.50	40.13	41.13	3.287
4.00	37.90	38.90	3.481
4.50	36.18	37.18	3.628
5.00	34.51	35.51	3.769
6.00	31.65	32.65	4.046
7.00	29.72	30.72	4.265
8.00	28.00	29.00	4.456
9.00	26.70	27.70	4.594
10.00	25.47	26.47	4.730

Table 2: Cap Volatility Quotes

Caps, however, are quoted in terms of *flat* volatilities. A flat volatility is the implied volatility of the cap, when that volatility is applied to *all* the caplets underlying the cap. As a result, caplets underlying more than one cap (for example, the 9x12 caplet is common to all caps in the table above, except the 6 month cap), will be priced with different volatilities depending on the cap being considered. An alternative approach is to use a unique volatility for each caplet in the cap series (i.e., for the 9x12 caplet, a single volatility would be used regardless of the cap being valued). These are called *spot* volatilities. Spot volatilities can be deduced from flat volatilities using a standard BootStrapping approach. With this approach, a series of cap prices is first generated using the flat volatilities, then the difference between each cap price and the previous cap price gives a forward caplet price, the Black model can then be inverted to produce the implied volatility for this caplet. This is the spot volatility of the forward rate with the same term as the caplet. Figure 12 shows the flat volatilities and spot volatilities [obtained by BootStrapping] from Table 2, based on mid-market quotes.

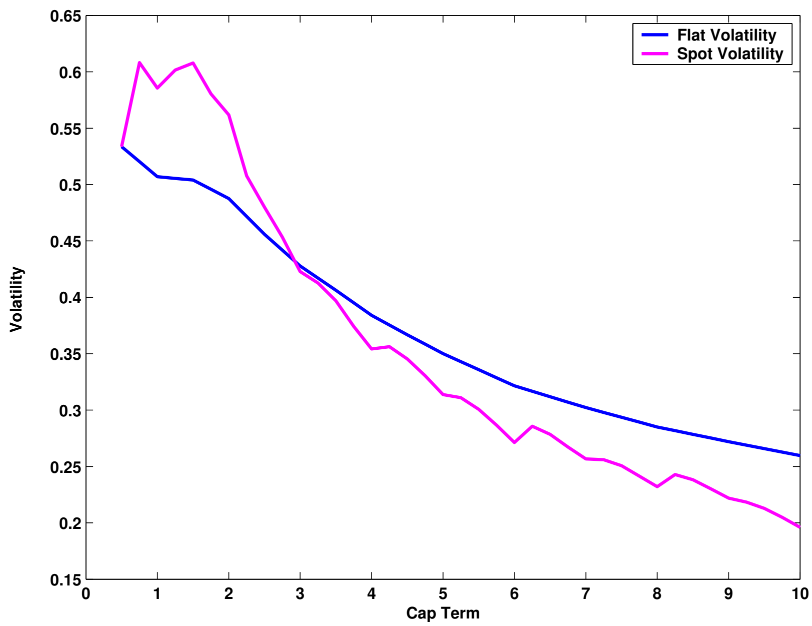

Figure 12: Flat and Spot Volatility Curves

For some applications, derivation of spot volatilities in this fashion may be acceptable. But there are some serious limitations to the way this data may be used. We will discuss these limitations and how the EpiVolatility model attempts to overcome them, but first the valuation of caps and floors with the Black model, and the Black, Derman, and Toy interest rate model are briefly discussed.

9 Some tree-based valuation models

An interest rate cap (floor) is simply a portfolio of European call (put) options on forward interest rates.

Consider a caplet with term δ_n, notional value L, and strike rate R_K. Let $F_{t_n,t_{n+1}}$ be the interest rate for the period between time t_n and time t_{n+1} (where $\delta_n = t_{n+1} - t_n$), observed at time t_n. The caplet has a payoff at time t_{n+1} of

$$L\delta_n(F_{t_n,t_{n+1}} - R_K)^+ \tag{1}$$

If the rate $F_{t_n,t_{n+1}}$ follows a lognormal process with volatility σ, the familiar Black pricing formula can be used to determine the value of the caplet at time $t_0 < t_n$ as

$$C_n = L\delta_n P(t_0, t_{n+1})(F_{t_n,t_{n+1}} N(d_1) - R_K N(d_2)) \tag{2}$$

where

$$d_1 = \frac{\ln(F_{t_n,t_{n+1}}/R_K) + \sigma^2 t_n/2}{\sigma\sqrt{t_n}}$$
$$d_2 = \frac{\ln(F_{t_n,t_{n+1}}/R_K) - \sigma^2 t_n/2}{\sigma\sqrt{t_n}}$$

and $N(\cdot)$ is the cumulative Gaussian distribution with mean 0 and variance 1. $P(t_0, t_{n+1})$ is the price, at time t_0, of a zero coupon bond maturing at time t_{n+1}.

The formula for the corresponding floorlet is

$$F_n = L\delta_n P(t_0, t_{n+1})(R_K N(-d_2) - F_{t_n,t_{n+1}} N(-d_1)) \tag{3}$$

Since a cap (floor) is a portfolio of caplets (floorlets), the price of a cap with term $T = t_{n+1}$ at time $t_0 < t_n$ is

$$C = \sum_{i=1}^{n} C_n \tag{4}$$

and

$$F = \sum_{i=1}^{n} F_n \tag{5}$$

for a floor.

In 1990, Black, Derman, and Toy proposed an algorithm for constructing a binomial interest rate tree that yields a discretized version of the model:

$$d\ln r(t) = [\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln(r)]dt + \sigma(t)dz \tag{6}$$

where $\sigma'(t)$ is the partial derivative of σ with respect to t, and $\theta(t)$ is a time-dependent parameter used to fit the model to the initial term structure [10]. Therefore, in order to use this model the volatility function must be differentiable at any time t.

10 The EpiVolatility model

10.1 Guidelines

The following are the desirable properties of a volatility curve, for use as a basis for interest rate modeling and security valuation (present and future):

- The partial derivative with respect to t, at any time t, exists and is continuous.
- The curve is defined with respect to *spot* (not flat) volatility, also known as *forward-forward* volatility.
- Spot volatility may be retrieved for a term of *arbitrary* length.
- A particular form for the volatility function need *not* be assumed.

The first three properties are requirements for the successful implementation of the BDT model in either binomial tree or Monte Carlo form. The last property is desirable from an implementation perspective, since we are not forced to assume a particular functional 'shape' that may or may not be representative of current market conditions. Further, any assumed functional form will restrict us to a few parameters for fitting the curve to the market data.

Figure 12 shows that the BootStrapped spot volatility curve does not meet these critieria:

- The partial derivative with respect to t, at any time t, is not continuous, and might even fail to exist,
- The forward volatilities retrieved from this curve are only defined for forward rates with the same start and end times as the caplets used in its construction.

Regarding this second point, if the tenor of the caps in the market data set is 3 months (for example), then the volatility curve only contains information about the forward volatilities of 3 month LIBOR forward rates that coincide with caplet dates. The curve says nothing about forward volatilities of 3 month LIBOR forward rates that do not coincide with caplet dates, nor about forward volatilities of forward rates with any other term (1 day, 1 week, 1 month, etc.). But when constructing a BDT tree for arbitrary δt, this is exactly what is needed.

10.2 EpiSolutions Inc.'s approach

Here is a brief outline of an approach to volatility estimation, that is designed to meet the *desirable* criteria listed above:

- Start with market data (broker quotes) for interest rate caps or floors.

- Fit a (smooth) curve to the flat volatility quotes (mid, bid, or ask), using *EpiCurves*, see §6.
- Fit a (smooth) curve to the strike rate curve, using *EpiCurves*.
- Construct a set of caps or floors with terms every n months (where n depends on the tenor of the securities in the market data set, typically 3 or 6 months). The price for each security is determined using the Black model, and flat volatilities and strike rates read from their respective (smooth) curves.
- Fit a (smooth) curve to the cap or floor prices determined in the previous step using *EpiCurves*.

This last curve is the end product of the *EpiVolatility* model, and is used as the input to the *EpiValuation* library. When the *EpiValuation* library needs a spot volatility for a given term, it derives a cap or floor price for that term from the price curve, and then inverts the Black model to retrieve the corresponding implied spot volatility.

Here again, the meaning of 'smooth' is bound to be mostly subjective. One could take as definition, the number of times the curve is continuously differentiable. But there are analytic curves, i.e., of class $\mathcal{C}^\infty$, that don't 'look' smooth; refer to §4, 6 for more about this issue. Practically, we shall content ourselves with curves that are less than infinitely smooth, but where we control the rate of change of the 2-nd, or higher, derivative.

In the next section, we discuss the EpiSolutions criterion and methodology for fitting 'smooth' curves to market data. Finally, we give an example of the implementation of the *EpiVolatility* model.

11 Implementation

Using the cap price data from Table 2, Table 3 shows the results of the first three steps outlined in §11:

- Fit a (smooth) curve to the flat volatility quotes (mid, bid, or ask), using *EpiCurves*, cf. §6.
- Fit a (smooth) curve to the strike rate curve, using *EpiCurves*.
- Construct a set of caps or floors with terms every n months (where n depends on the tenor of the securities in the market data set, typically 3 or 6 months). The price for each security is determined using the Black model, and flat volatilities and strike rates read from their respective (smooth) curves.

We use mid market volatility quotes and the cap tenors are assumed to be 3 months.

Term	*Flat Vol*	*Strike*	*Price*	*Term*	*Flat Vol*	*Strike*	*Price*
0.50	53.3539	1.7222	0.000595	5.50	33.4571	3.9121	0.053636
0.75	51.9915	1.7855	0.001613	5.75	32.7672	3.9811	0.056743
1.00	50.6982	1.8943	0.002945	6.00	32.1487	4.0461	0.059799
1.25	50.3926	2.0381	0.004590	6.25	31.6054	4.1060	0.062867
1.50	50.3984	2.2025	0.006613	6.50	31.1180	4.1615	0.066044
1.75	49.8694	2.3706	0.008936	6.75	30.6629	4.2141	0.069153
2.00	48.7541	2.5337	0.011497	7.00	30.2180	4.2652	0.072157
2.25	47.2321	2.6858	0.014139	7.25	29.7684	4.3160	0.075074
2.50	45.5900	2.8260	0.016878	7.50	29.3227	4.3657	0.078018
2.75	44.0749	2.9551	0.019636	7.75	28.8948	4.4129	0.080902
3.00	42.7722	3.0737	0.022441	8.00	28.4975	4.4563	0.083782
3.25	41.6809	3.1831	0.025259	8.25	28.1391	4.4948	0.086745
3.50	40.6276	3.2872	0.028185	8.50	27.8108	4.5296	0.089836
3.75	39.4827	3.3890	0.031055	8.75	27.5007	4.5622	0.092862
4.00	38.3912	3.4817	0.034002	9.00	27.1973	4.5943	0.095746
4.25	37.4882	3.5592	0.037239	9.25	26.8918	4.6271	0.098460
4.50	36.6764	3.6283	0.040725	9.50	26.5841	4.6608	0.101054
4.75	35.8452	3.6978	0.044101	9.75	26.2762	4.6952	0.103437
5.00	35.0100	3.7690	0.047326	10.00	25.9700	4.7300	0.105574
5.25	34.2084	3.8409	0.050472				

Table 3: Calibrating Cap Set

The last step is to fit a curve to the price vector from Table 3, using the *EpiCurves* technology. The result can be seen in Figure 13.

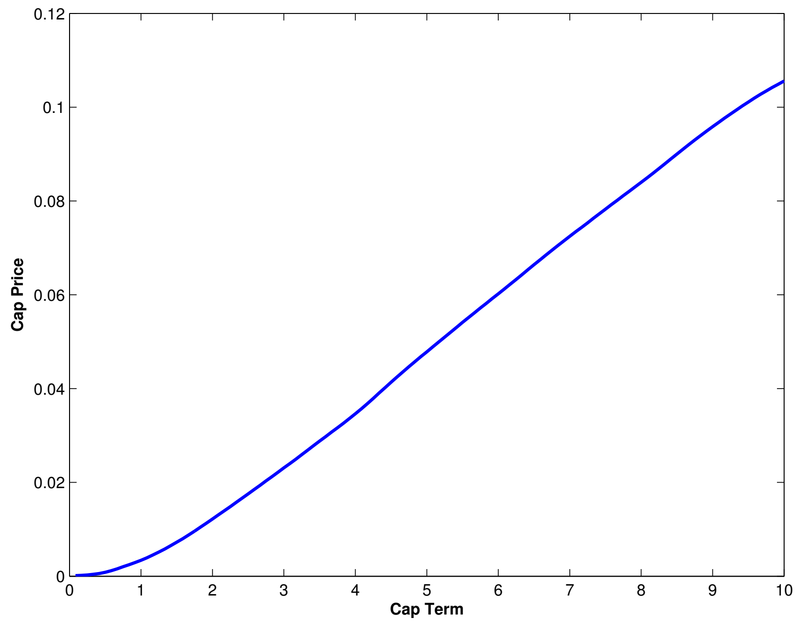

Figure 13: Fitted Cap Price Curve

This fitted cap price curve is then used as input to the *EpiValuation* library, which uses it to determine spot volatilities in interest rate modeling and option pricing. The steps the *EpiValuation* library uses to do this are straight-forward. Given a time step from time t_n to time t_{n+1},

- Read a cap price C_{t_n} from the curve with term t_n.
- Read a cap price $C_{t_{n+1}}$ from the curve with term t_{n+1}.
- The forward caplet price $C_{t_n,t_{n+1}}$ then simply equals $C_{t_{n+1}} - C_{t_n}$.
- Invert the Black model for the forward caplet (i.e., solve for volatility instead of price), assuming the caplet is at-the-money. This means the strike rate is set equal to the (simply compounded) forward rate $F_{t_n,t_{n+1}}$ for the same period.

Figure 14 shows the results for daily spot volatilities for 3 month Libor rates. This means that a volatility read from the curve is the volatility for the 3 month Libor rate starting at that time and maturing 3 months later. Figure 15 shows the results for 1 month Libor Rates.

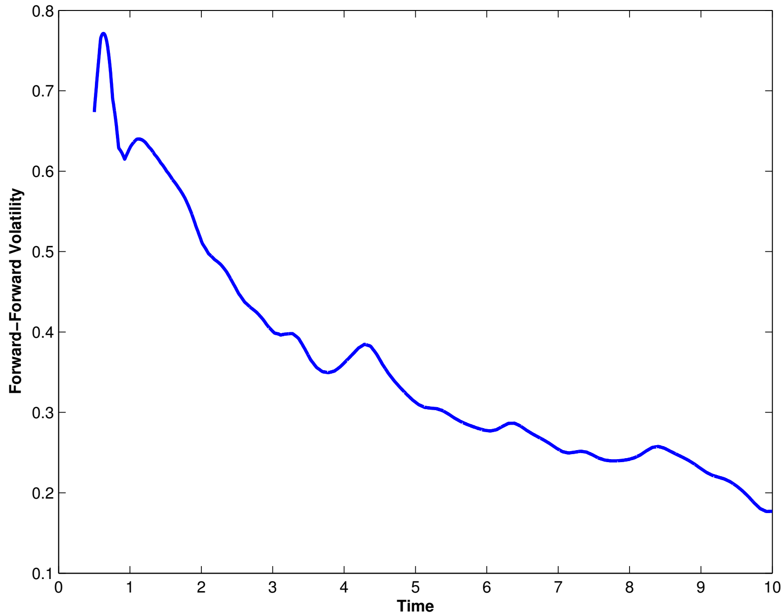

Figure 14: Daily Spot Volatilities (3 Month Rates)

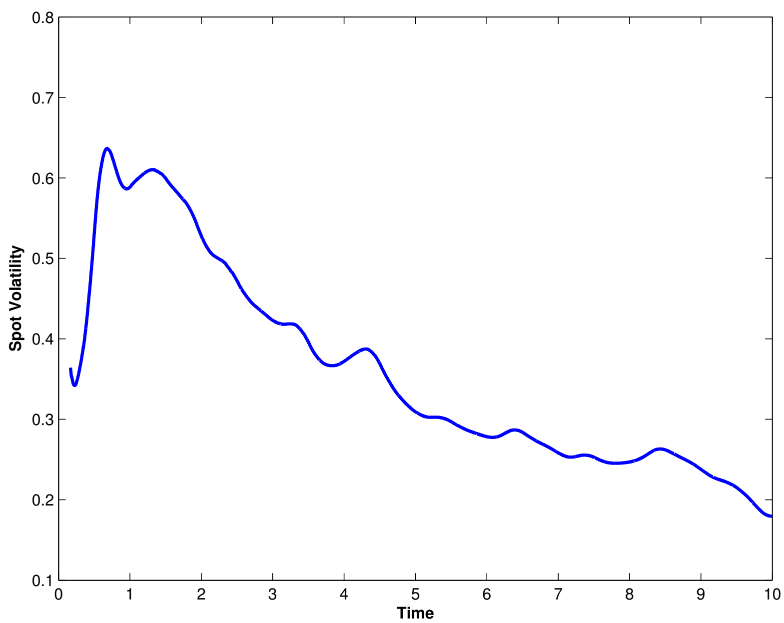

Figure 15: Daily Spot Volatilities (1 Month Rates)

12 Summary

The approach to estimating LIBOR spot volatilities outlined in the second part of this paper arose from a need for a consistent and flexible method for determining quantities required by the *EpiValuation* library. The *EpiValuation* library is used for a number of functions, including:

- Current valuation of securities and portfolios.
- Future valuation of securities and portfolios.
- Scenario based total return estimation for securities and portfolios.

And as indicated in the Introduction, scenario-based total returns are themselves input for our *EpiManager*, that deals with portfolio optimization by relying on stochastic programming techniques.

A Appendix

Settle	*Maturity*	*Rate*
06/10/97	06/11/97	0.054688
06/11/97	07/11/97	0.056250
06/11/97	08/11/97	0.057500
06/11/97	09/11/97	0.057344
06/11/97	12/11/97	0.058125
06/11/97	06/11/98	0.061875

Table A–1: Eurodollar Deposits

Settle	*Maturity*	*Rate*
06/11/97	06/11/99	0.062010
06/11/97	06/11/00	0.064320
06/11/97	06/11/01	0.065295
06/11/97	06/11/02	0.066100
06/11/97	06/11/04	0.067860
06/11/97	06/11/07	0.068575

Table A–2: Swaps .

Delivery	*Maturity*	*Price*	*Delivery*	*Maturity*	*Price*
06/16/97	09/16/97	94.195	06/19/00	09/19/00	93.220
09/15/97	12/15/97	94.040	09/18/00	12/18/00	93.190
12/15/97	03/16/98	93.820	12/18/00	03/19/01	93.190
03/16/98	06/16/98	93.725	03/19/01	06/19/01	93.120
06/15/98	09/15/98	93.610	06/18/01	09/18/01	93.080
09/14/98	12/14/98	93.510	09/17/01	12/17/01	93.050
12/14/98	03/15/99	93.410	12/17/01	03/18/02	92.980
03/15/99	06/15/99	93.390	03/18/02	06/18/02	92.980
06/14/99	09/14/99	93.360	06/17/02	09/17/02	92.940
09/13/99	12/13/99	93.330	09/16/02	12/16/02	92.900
12/13/99	03/13/00	93.260	12/16/02	03/17/03	92.830
03/13/00	06/13/00	93.250	03/17/03	06/17/03	92.830

Table A–3: Eurodollar Futures

Settle	*Maturity*	*Price*
08/03/01	11/01/01	3.44
08/03/01	01/31/02	3.36
08/03/01	02/28/02	3.33

Settle	*Maturity*	*Coupon*	*Price*
08/03/01	07/31/03	0.03875	$99 + 30/32$
08/03/01	05/15/06	0.04625	$99 + 26/32$
08/03/01	02/15/11	0.05000	$98 + 25/32$
08/03/01	02/15/31	0.05375	$99 + 00/32$

Table A–4: Bond portfolio — U.S Treasury Bills and Bonds

Delivery	*Maturity*	*Price*	*Convex.*	*Delivery*	*Maturity*	*Price*	*Convex.*
09/17/01	12/17/01	96.430	0.023	09/18/06	12/18/06	93.340	13.999
12/17/01	03/18/02	96.295	0.117	12/18/06	03/19/07	93.250	15.364
03/18/02	06/18/02	96.150	0.274	03/19/07	06/19/07	93.290	16.794
06/17/02	09/17/02	95.805	0.494	06/18/07	09/18/07	93.260	18.283
09/16/02	12/16/02	95.420	0.775	09/17/07	12/17/07	93.230	19.826
12/16/02	03/17/03	95.020	1.121	12/17/07	03/17/08	93.140	21.444
03/17/03	06/17/03	94.780	1.533	03/17/08	06/17/08	93.180	23.128
06/16/03	09/16/03	94.514	2.005	06/16/08	09/16/08	93.150	24.870
09/15/03	12/15/03	94.315	2.539	09/15/08	12/15/08	93.125	26.665
12/15/03	03/15/04	94.100	3.138	12/15/08	03/16/09	93.035	28.537
03/15/04	06/15/04	94.030	3.803	03/16/09	06/16/09	93.075	30.474
06/14/04	09/14/04	93.905	4.529	06/15/09	09/15/09	93.050	32.468
09/13/04	12/13/04	93.800	5.314	09/14/09	12/14/09	93.030	34.514
12/13/04	03/14/05	93.660	6.166	12/14/09	03/15/10	92.940	36.641
03/14/05	06/14/05	93.650	7.085	03/15/10	06/15/10	92.980	38.831
06/13/05	09/13/05	93.575	8.064	06/14/10	09/14/10	92.950	41.079
09/19/05	12/19/05	93.505	9.183	09/13/10	12/13/10	92.930	43.376
12/19/05	03/20/06	93.395	10.294	12/13/10	03/14/11	92.845	45.756
03/13/06	06/13/06	93.420	11.379	03/14/11	06/14/11	92.885	48.199
06/19/06	09/19/06	93.375	12.707	06/13/11	09/13/11	92.855	50.700

Table A–5: 51-Instruments portfolio — Eurodollar Futures

Settle	*Maturity*	*Rate*
08/03/01	09/03/01	0.0366
08/03/01	11/03/01	0.0359
08/03/01	02/03/02	0.0360

Settle	*Maturity*	*Rate*
08/03/01	08/03/02	0.0385
08/03/01	08/03/03	0.0444
08/03/01	08/03/04	0.0491
08/03/01	08/03/05	0.0523
08/03/01	08/03/06	0.0547
08/03/01	08/03/08	0.0576
08/03/01	08/03/11	0.0598
08/03/01	08/03/31	0.0632

Table A–6: 51-Instruments portfolio — Eurodollar Deposits & Swaps

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