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OPTIMALITY PROPERTIES OF A SPECIAL ASSIGNMENT PROBLEM†

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In this paper, it is shown that if the cost matrix of an assignment problem has the following property $c_{ij}=|j-i|$ then any basic feasible solution is optimal if and only if its unit components belong to two well-defined symmetric regions. The matrix with above mentioned property is called the 'reordering matrix,' because it arose for the first time in the reordering of nodes of a critical path and other acyclic network problems. One deals with similar matrices in some problems related to order statistics.

ASSUME we want to order the nodes of a network in such a way that for every arc (i,j), which belongs to the topology of the network, $i<j$.‡ In case of a large network, this may not be possible to do manually, as it generates a large permutation problem. An algorithm to reorder the nodes of the network is given in reference 1, where the number of steps involved is related to the magnitude of divergence of the node order of a network, where divergence is defined to be

$$S=\sum_{i=1}^{i=m} |i-a_i|,$$

where a_i is the initial order of the node having final order i.

In order to fix an upper bound to the divergence of the network, we have to determine the maximum value of S over all possible m-tuples a.

We will now show that this problem is equivalent to the classical assignment problem.

MATHEMATICAL FORMULATION

LET $t=\{a|a$ is an m-tuple chosen without repetition from the set $(1, 2, \cdots, m)$ e.g., $a=(2, m, 10, \cdots, 1, 5)\}$.

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‡ This problem is important because it reduces significantly the work involved in logical search for a critical path problem. It is impossible to perform this node ordering when loops are present in the network.

Note that t has $m!$ nonidentical components.

Then, the problem becomes

$$\max_{a\in t} S=\sum_{i=1}^{i=m} |i-a_i|. \tag{1}$$

It is now easy to see that all the components of each possible summation $\sum_{i=1}^{i=m} |i-a_i|$ are of the form $|i-j|$ where $i=1, \cdots, m$; $j=1, \cdots, m$. We can list all these possible values under matrix form where each entry is equal to the absolute difference between its row number and column number.

Define $c_{ij}=|i-j|$ and $C=\{c_{ij}\}$. Any possible S can be found by summing up m entries of the matrix selected by picking one and only one element from each row and each column. For example

i	$a_i = j$			
1	c_{11}	c_{12}	c_{13}	c_{14}
2	c_{21}	c_{22}	c_{23}	c_{24}
3	c_{31}	c_{32}	c_{33}	c_{34}
4	c_{41}	c_{42}	c_{43}	c_{44}

A possible $S=c_{12}+c_{23}+c_{31}+c_{44}$ corresponding to the 4-tuple 'a'$=(2,3,1,4)$. Thus, the problem reduces to

$$\text{maximize } S=\sum_{ij} c_{ij}\, x_{ij}$$

Subject to:

$$\begin{aligned} \sum_j x_{ij}&=1, &\quad (i&=1,\cdots,m)\\ \sum_i x_{ij}&=1, &\quad (j&=1,\cdots,m)\\ x_{ij}&\geqq 0, &\quad (x_{ij}&\text{ are integers}) \end{aligned} \tag{2}$$

which is the classical assignment problem formulation.

Our aim is to prove, in this particular set-up, that this optimum for S can be achieved by selecting any feasible solution such that all its components belong to two symmetric regions of the matrix and that no optimal solution can be found if one or more components do not belong to these two regions.

THE OPTIMAL REGION

IN THE reordering matrix let us define the two following sets of entries.

$$O_1^*=\{c_{ij}|i\leqq \tfrac{1}{2}(m+1), j\geqq \tfrac{1}{2}(m+1)\},$$

$$O_2^*=\{c_{ij}|i\geqq \tfrac{1}{2}(m+1), j\leqq \tfrac{1}{2}(m+1)\}.$$

Let $O^* = O_1^* \cup O_2^*$ then,

$$O_1^* \cap O_2^* = \begin{cases} \emptyset \text{ if } m \text{ is even,} \\ c_{m+1/2,m+1/2} \text{ if } m \text{ is odd.} \end{cases}$$

These two sets of entries are symmetric with respect to the principal diagonal:

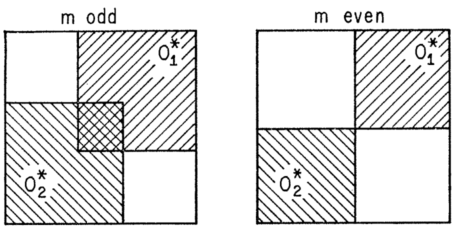

Fig. 1. Optimal region.

Let

$$O_1 = \{x_{ij} | x_{ij} \in O_1 \text{ if } c_{ij} \in O_1^*\},$$

$$O_2 = \{x_{ij} | x_{ij} \in O_2 \text{ if } c_{ij} \in O_2^*\}.$$

Then, $O = O_1 \cup O_2$, and it is called the *optimal region.*

PROPERTIES OF THE REORDERING MATRIX

1. For all $c_{rs} \notin O^*$
   either all $c_{rk} \in O^*$ are greater than or equal to c_{rs},
   or all $c_{ks} \in O^*$ are greater than or equal to c_{rs}.
2. For any submatrix of C of the form

   $$C_s = \begin{bmatrix} c_{ij} & c_{i,j+s} \\ c_{i+r,j} & c_{i+r,j+s} \end{bmatrix},$$

   such that $i \neq j$ and all its elements are either all below the principal diagonal or all above the principal diagonal, then
   $c_{ij} + c_{i+r,j+s} = c_{i+r,j} + c_{i,j+s}$.
3. For any submatrix of C of the form

   $$C_r = \begin{bmatrix} c_{ij} & c_{i,j+s} \\ c_{i+r,j} & c_{i+r,j+s} \end{bmatrix},$$

   such that $c_{i+r,j}$ is an entry below the principal diagonal of C and $c_{i,j+s}$ is an entry above the principal diagonal. Then

   $c_{ij} + c_{i+r,j+s} < c_{i+r,j} + c_{i,j+s}$.

It is possible to prove the optimality of any feasible solution in region O by using the theory of linear programming, or more specifically the assignment problem algorithm.[2] We give here an alternative proof.

OPTIMALITY THEOREM

THEOREM: *A feasible solution x^0 is optimal if and only if all its components lie in* O.

Proof: Suppose $\hat{x}$ is a feasible solution but has at least one component that does not lie in O, we will then show that we can improve S.

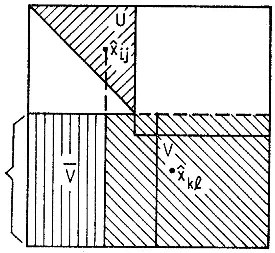

Figure 2

Let us assume that $\hat{x}_{ij}$ is such that $i<\frac{1}{2}(m+1)$, $j<\frac{1}{2}(m+1)$ and $j\geqq i$, i.e., in Fig. 2, $\hat{x}_{ij}$ lies in U. In order to be feasible x has at least a component, say $\hat{x}_{kl} \in V$, because it is impossible to 'cover' the $[\frac{1}{2}(m+1)]$† last rows with 'selections' only done in $\bar{V}$, as $\bar{V}$ has less than $[\frac{1}{2}(m+1)]$ columns (by assumption on $\hat{x}_{ij}$). We can find then a new feasible solution by replacing $\hat{x}_{ij}$ and $\hat{x}_{kl}$ by x_{il} and x_{kj} and by property 3, S will be improved.

We have proved that for any feasible solution that has one or more components outside O, it can be improved. So, we can produce an iterative procedure that will increase S, as long as x has a component which does not lie in O.

If $x^0 \in$ O, it is not possible to improve the value of S because the only acceptable substitutions are of the form

$$x^0_{ij} \quad \text{and} \quad x^0_{i+r,j+s},$$

by

$$x^0_{i+r,j} \qquad x^0_{i,j+s}\ (\in \mathrm{O}),$$

or repeated substitutions of that form. But by property 2 we know that the value of S will not change.

COROLLARY: *All the feasible solutions in region* O *are optimal.*

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1. S. PARIKH, "Project Planning by Decomposition," Research Report, Operations Research Center, University of California, Berkeley ORC, 63-2 (RR) Jan. 1963.
2. L. R. FORD AND D. R. FULKERSON, *Flows in Networks*, Princeton University Press, Princeton, New Jersey, 1962.

† $[\alpha]$ = greatest integer contained in α.