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OUTER SEMICONTINUITY OF POSITIVE HULL MAPPINGS WITH APPLICATION TO SEMI-INFINITE AND STOCHASTIC PROGRAMMING*

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Abstract. Let W be an arbitrary subset of $\mathbb{R}^n$ and pos$W$ the positive hull of W. We are concerned with conditions under which one can guarantee continuity properties for pos$W$ as a function of W. The results are then applied in the context of semi-infinite linear programs and stochastic programs with recourse.

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1. Introduction.

Let $\{W; W^\nu,\ \nu \in \mathbb{N}\}$ be a collection of nonempty subsets of $\mathbb{R}^m$ and $\{\text{pos}W; \text{pos}W^\nu,\ \nu \in \mathbb{N}\}$ the positive hulls generated by these sets, i.e.,

$$\text{pos}W = \left\{ t = \sum_{j=1}^{q} t^j x_j | t^j \in W, x_j \geq 0, \qquad \text{with} \quad q \quad \text{finite} \right\},$$

and posW^ν is defined similarly for each W^ν. Our overall concern is with the continuity of the positive hull mapping $W \mapsto \text{pos}W$, but, more specifically, we are interested in finding conditions under which

$$\limsup_{\nu\to\infty} \text{pos}W^\nu \subset \text{pos}W$$

when the W^ν "converge" to W; here and throughout, $\limsup_\nu C^\nu$ designates the *outer limit*, in the sense of Painlevé-Kuratowski, of the sets C^ν, that is, the set of all limits of subsequences $\lim_{\nu_i\to\infty} t^{\nu_i}$, with $t^{\nu_i} \in C^{\nu_i}$; cf. [11, Chapter 4].

One can view this work as an extension, in various directions, of a result of Walkup and Wets [12, Theorem 2] where the sets W and W^ν were of constant finite cardinality, or, more simply, they consist of the points identified by the columns of constant size matrices. Questions of this type occur in a variety of variational problems. For example, in the analysis of the stability of the solutions of linear programs, semi-infinite linear programs [3], linear complementarity problems [7], [8, section 4], equilibrium and quasi-equilibrium problems [9], generalized linear programs [5, Chapter 22], and stochastic programming problems [2, 10]. Two of these applications are further analyzed in the last two sections.

2. Outer semicontinuity.

Our major objective is to obtain the inclusion $\limsup_\nu \text{pos}W^\nu \subset \text{pos}W$ that can be viewed as an outer-semicontinuity result. We

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begin with a characterization of the outer limit of a sequence of sets that could be deduced from the results in [11, Chapter 4, section H] but isn't readily available in the literature. A (closed) ball centered at x and radius ρ is denoted by $\mathbb{B}(x,\rho)$ and the unit ball simply by $\mathbb{B}$, and so, for some positive scalar η, $\eta\mathbb{B} = \mathbb{B}(0,\eta)$.

PROPOSITION 2.1 (outer limit of sets). *A closed set $C \subset \mathbb{R}^m$ is the outer limit of a sequence of sets $\{C^\nu \subset \mathbb{R}^m,\, \nu \in \mathbb{N}\}$ if and only if given any $\epsilon > 0, \rho > 0$, there is $\nu_{\epsilon,\rho}$ such that*

$$\forall\, \nu \geq \nu_{\epsilon,\rho}: \quad C^\nu \cap \rho\mathbb{B} \subset C + \epsilon\mathbb{B}.$$

Proof. We rely on the criterion provided by [11, Proposition 4.5(b)], namely, that $C \supset \limsup_\nu C^\nu$ if and only if whenever $C \cap B = \emptyset$ for a compact set B, then also $C^\nu \cap B = \emptyset$ for ν large enough. The proof, in both directions, proceeds by contradiction.

When the asserted inclusion is not satisfied for some pair $\epsilon > 0, \rho > 0$, then for a countable collection of indexes, say, for $\nu \in N^\sharp$, there is a collection

$$\left\{x^\nu \in (C^\nu \cap \rho\mathbb{B}) \setminus (C + \epsilon\mathbb{B}),\, \nu \in N^\sharp\right\}$$

converging to some $\bar{x}$; all these points belong to the compact set $\rho\mathbb{B}$. Thus, $C \cap B = \emptyset$ for the compact set $B = \{\bar{x}; x^\nu,\, \nu \in \mathbb{N}\}$, whereas there is no ν_B arbitrarily large so that $C^\nu \cap B$ is empty for all $\nu \geq \nu_B$. Hence, C can't contain the outer limit of the C^ν.

On the other hand, if there is a compact set B such that $C \cap B = \emptyset$ but for some countable collection of indexes, say, $\nu \in N^\sharp$, $C^\nu \cap B$ fails to be empty, choose $\rho > 0$ such that $B \subset \rho\mathbb{B}$ and $\epsilon > 0$ such that $(C + \epsilon\mathbb{B}) \cap B = \emptyset$. Then, for all $\nu \in N^\sharp$, $C^\nu \cap \rho\mathbb{B} \not\subset C + \epsilon\mathbb{B}$; i.e., there is no $\nu_{\epsilon,\rho}$ such that the asserted inclusion holds for all $\nu \geq \nu_{\epsilon,\rho}$. ∎

3. The core cone. To obtain the outer semicontinuity of the positive hulls, our conditions will involve two limit cones associated with a collection of sets $\{C^\nu,\, \nu \in \mathbb{N}\}$. The first one, generated by the directions at the "horizon," is the *horizon outer limit* defined as

$$\limsup{}^{\infty}_{\nu} C^\nu = \{0\} \cup \left\{t = \lim_{\nu \in N} \lambda_\nu t^\nu,\ t^\nu \in C^\nu,\ \lambda_\nu \downarrow 0\right\},$$

where $N \subset \mathbb{N}$ indicates that the limit is conceivable with respect to a subsequence. This set is a closed cone whose properties are detailed in [11, Chapter 4, section F]. The notation

$$C^\infty = \{t | \exists\, t^\nu \in C,\ \lambda_\nu \downarrow 0, \text{ with } \lambda_\nu t^\nu \to t\}$$

is reserved for the *horizon cone* associated with a set $C \neq \emptyset$ [11, Chapter 3, section B] (see also [1, Chapter 2]).

The second one, believed to be new, can be interpreted as the "inverse" of the horizon limit cone, as will be seen below. It's defined by

$$\limsup{}^{o}_{\nu} C^\nu = \left\{t = \lim_{\nu \in N} \lambda_\nu^{-1} t^\nu,\ t^\nu \in C^\nu,\ \lambda_\nu \downarrow 0\right\},$$

where, again, $N \subset \mathbb{N}$ suggests that the limit may involve only a subsequence of indexes. We refer to this limiting set as the *core outer limit* of the sequence of nonempty sets

$\{C^\nu,\, \nu \in \mathbb{N}\}$. So, rather than the direction points at the horizon, it identifies the "direction points" at the origin. It's immediate, from the definition, that this (core) limit set is also a *closed cone*. We recall that given a set C, the tangent cone of C at 0 is the cone $\limsup_{\lambda \downarrow 0} \lambda^{-1} C$. Here are some simple rules of calculus of the core outer limit.

PROPOSITION 3.1. *Let $\{C^\nu \subset \mathbb{R}^m,\, \nu \in \mathbb{N}\}$ and $\{D^\nu \subset \mathbb{R}^m,\, \nu \in \mathbb{N}\}$ be two collections of sets. Then*

(i) *the core outer limit of C^ν coincides with the intersection of the tangent cones $\limsup_{\lambda \downarrow 0} \lambda^{-1}(\cup_{\nu \geq k} C^\nu)$ for all $k \geq 1$. In particular, when the sets C^ν are constant, say, equal to C, the core outer limit is exactly the tangent cone of C at 0;*

(ii) $\limsup^o_\nu C^\nu \subseteq \limsup^o_\nu D^\nu$ *if $C^\nu \subseteq D^\nu$ for all ν;*

(iii) $\limsup^o_\nu C^\nu \cup D^\nu = \limsup^o_\nu C^\nu \cup \limsup^o_\nu D^\nu$*;*

(iv) $\limsup^o_\nu C^\nu \cap D^\nu \subseteq \limsup^o_\nu C^\nu \cap \limsup^o_\nu D^\nu$*;*

(v) $\limsup^o_\nu C^\nu + \limsup^o_\nu D^\nu \subseteq co \limsup^o_\nu (C^\nu + D^\nu)$ *provided that C^ν and D^ν contain the origin;*

(vi) $\limsup^o_\nu C^\nu + \limsup^o_\nu D^\nu \supseteq \limsup^o_\nu (C^\nu + D^\nu)$ *provided that the core outer limit of C^ν and that of $-D^\nu$ have only the zero vector in common.*

Proof. To prove the first assertion let $t = \lim_{\nu_i} \lambda_{\nu_i}^{-1} t^{\nu_i}$, with $t^{\nu_i} \in C^{\nu_i}$. Then, for each k, $t^{\nu_i} \in \cup_{\nu \geq k} C^\nu$ whenever $\nu_i \geq k$. Thus, t belongs to the tangent cone of $\cup_{\nu \geq k} C^\nu$ for every k. Conversely, let t be a vector such that, for every $k \geq 1$, there is a sequence of positive numbers $\lambda_{k,i}$ converging to 0 as i tends to ∞ and $t_k^{\nu_i} \in \cup_{\nu \geq k} C^\nu$ such that $t = \lim_i \lambda_{k,i}^{-1} t_k^{\nu_i}$. For each k, choose $i(k)$, with $\lambda_{k,i(k)} \leq \frac{1}{k}$, and $\nu(k) \geq k$, with $t_k^{\nu_{i(k)}} \in C^{\nu(k)}$. By taking a subsequence if necessary, one may assume that $\nu(k) > \nu(k-1)$ for every $k > 1$. Then t is the limit of $\lambda_{k,i(k)}^{-1} t_k^{\nu_{i(k)}}$, with $\lambda_{k,i(k)}$ tending to 0, and hence belongs to the core outer limit of C^ν.

The three assertions that follow are straightforward. The assertion (v) is obtained from (ii) and the fact that C^ν and D^ν are contained in the sum $C^\nu + D^\nu$. For the last assertion, let $v = \lim_{\lambda_i \downarrow 0} \lambda_i (t^{\nu_i} + s^{\nu_i})$, $v \neq 0$, with $t^{\nu_i} \in C^{\nu_i}$ and $s^{\nu_i} \in D^{\nu_i}$. Consider the sequences $\{\lambda_i t^{\nu_i}\}_i$ and $\{\lambda_i s^{\nu_i}\}_i$. They are both either bounded or unbounded. In the first case we may assume that they converge, respectively, to some vector v_1 of the core outer limit of C^ν and some v_2 of the core outer limit of D^ν, which yields $v = v_1 + v_2$. In the other case we divide the general terms of these sequences by $\|\lambda_i t^{\nu_i}\|$ and assume that the obtained sequences converge, respectively, to some vector u_1 of the core outer limit of C^ν and some u_2 of the core outer limit of D^ν. Then, $u_1 + u_2 = 0$, which contradicts the hypothesis. ☐

Example 3.2. The inclusion of (v) and the containment of (vi) may be strict, and, without appropriate assumptions, they may fail to hold.

Detail. In $\mathbb{R}^2$, set $C^\nu = \{(\frac{1}{\nu}, \frac{1}{\nu^2}), (0,0)\}$ and $D^\nu = \{(-\frac{1}{\nu}, \frac{1}{\nu^2}), (0,0)\}$. Then the core outer limit of C^ν consists of the vectors $(x, 0)$, with $x \geq 0$, and the core outer limit of D^ν consists of the vectors $(-x, 0)$, with $x \geq 0$. The core outer limit of the sum $C^\nu + D^\nu$ contains the above-mentioned vectors and the vectors $(0, y)$, with $y \geq 0$, as well. This shows that the inclusion of (v) is strict.

By setting $C_0^\nu = \{(\frac{1}{\nu}, \frac{1}{\nu^2})\}$ and $D_0^\nu = \{(-\frac{1}{\nu}, \frac{1}{\nu^2})\}$ we see that the hypothesis of (v) is violated for these families. The convex hull of the core outer limit of their sum is the half-space $\{(0, y) | y \geq 0\}$, which contains neither the core outer limit of C_0^ν nor the core outer limit of D_0^ν.

For the families C^ν and D^ν above, the hypothesis of (vi) does not hold, and the containment is not true. For the families $A^\nu = \{(\frac{1}{\nu}, \frac{1}{\nu})\}$ and $B^\nu = \{(-\frac{1}{\nu}, \frac{1}{\nu})\}$, direct calculation shows that the core outer limit of $A^\nu + B^\nu$ is the set of the vectors $(0, y)$, with $y \geq 0$, the core outer limit of A^ν is the ray (x, x), with $x \geq 0$, and the core outer limit of B^ν is the ray $(-x, x)$, with $x \geq 0$. Hence, the sum of the latter core outer limits contains the core outer limit of $A^\nu + B^\nu$ as a proper subset.

To see that the core outer limit set can be interpreted as an inverse of a horizon outer limit set, let's introduce the mapping

$$t \mapsto t^- = t/|t|^2 \qquad \text{for} \qquad t \in \mathbb{R}^m, t \neq 0,$$

with $0^- = 0$. For a set $C \subset \mathbb{R}^m$, by definition, $C^- = \{t^- | t \in C \subset \mathbb{R}^m\}$, and obviously $(C^-)^- = C$. The mapping $t \mapsto t^-$ has the following properties:

(a) It's a homeomorphism on $\mathbb{R}^m \setminus \{0\}$.

(b) C is bounded if and only if $\mathrm{cl}\,(C \setminus \{0\})^-$ does not contain the origin.

(c) With $C^o = \{0\} \cup \big\{t = \lim_\nu \lambda_\nu^{-1} t^\nu | t^\nu \in C,\ \lambda_\nu \downarrow 0\big\}$, the *core cone* associated with C, one has $C^o = (C^-)^\infty$ and $C^\infty = (C^-)^o$.

(d) If C is itself a cone, then $C^- = C$.

Proposition 3.3. *For a sequence of nonempty sets* $\{C^\nu \subset \mathbb{R}^m,\ \nu \in \mathbb{N}\}$*,*

$$\limsup{}^o_\nu C^\nu = \limsup{}^\infty_\nu (C^\nu)^-$$

and

$$\limsup{}^\infty_\nu C^\nu = \limsup{}^o_\nu (C^\nu)^-.$$

Proof. Indeed, let $t \neq 0$ belong to the core outer limit. Without loss of generality, one can assume that $|t| = 1$ and is the limit of some sequence $\{t^\nu/|t^\nu|\}_{\nu=1}^\infty$, with $t^\nu \in C^\nu$ and $\lim_\nu t^\nu = 0$. Then $(t^\nu)^- = t^\nu/|t^\nu|^2 \in (C^\nu)^-$, with $|(t^\nu)^-| \to \infty$. Clearly, t is the limit of the sequence $(t^\nu)^-/|(t^\nu)^-|$, and hence t belongs to the horizon outer limit of the sets C^ν. The converse is obtained by the same argument. The second equality follows from the first one via the identity $(C^-)^- = C$. ◻

4. The outer semicontinuity of pos$W$.

In addition to the limiting cones introduced in the previous section, our conditions also involve $\mathrm{lil}\,K$, the *lineality space* of a convex cone K; it's the maximal linear subspace contained in K.

Theorem 4.1 (the outer semicontinuity of positive hulls). *The taking of positive hulls is outer semicontinuous, more precisely: Given* $\{W; W^\nu,\ \nu \in \mathbb{N}\}$*, a collection of nonempty subsets of* $\mathbb{R}^m$*,*

$$\limsup{}_\nu \mathrm{pos} W^\nu \subset \mathrm{pos} W$$

under the following hypotheses:

(a) W *includes the outer limit of the sets* W^ν*,*

(b) pos$W$ *includes the horizon outer limit of the sets* W^ν*,*

(c) pos$W$ *includes the core outer limit of the sets* W^ν*,*

(d) *when* $0 \neq t \in \mathrm{lil}(\mathrm{pos}W)$ *is a cluster point of a sequence* $\{t^\nu/|t^\nu|,\ \nu \in \mathbb{N}\}$*, where* $t^\nu \in \mathrm{pos}W^\nu \setminus \{0\}$*, then* $t^\nu \in \mathrm{lil}(\mathrm{pos}W^\nu)$ *for* ν *sufficiently large and* $\mathrm{lil}(\mathrm{pos}W) \supset \limsup_\nu \mathrm{lil}(\mathrm{pos}W^\nu)$.

Proof. Let $\{t^\nu \in \mathrm{pos}W^\nu\}_{\nu=1}^\infty$ converge to $t \in \mathbb{R}^m$. One needs to show that $t \in \mathrm{pos}W$. According to Carathéodory's theorem [11, Theorem 2.29], one can always express t^ν as

$$t^\nu = \sum_{i=1}^m \lambda_{\nu,i} w^{\nu,i}, \text{ with } \lambda_{\nu,i} \geq 0,\ w^{\nu,i} \in W^\nu,$$

i.e., as a nonnegative linear combination of no more than m vectors in W^ν. Fix an index $i \in \{1,\dots,m\}$, and consider the sequence $\{\lambda_{\nu,i} w^{\nu,i}\}_{\nu=1}^\infty$.

CLAIM 1. *Any cluster point, say, $\hat{t}^i$, of $\{\lambda_{\nu,i} w^{\nu,i},\ \nu \in \mathbb{N}\}$ belongs to* pos$W$.

In all of the arguments that follow, it's taken for granted that one passes to a subsequence whenever that's required or appropriate. Certainly, if $\hat{t}^i = 0$, it belongs to pos$W$. When $\hat{t}^i \neq 0$, one has to consider three possibilities: (i) $\lim_\nu \lambda_{\nu,i} = 0$ (i.e., the limit of some subsequence is 0), (ii) $\lim_\nu \lambda_{\nu,i} = \lambda_i > 0$ is finite, or (iii) $\lim_\nu \lambda_{\nu,i} = \infty$. In case (i), the sequence $\{w^{\nu,i}\}$ must be unbounded, and all of its "cluster points" belong to the horizon outer limit $\limsup_\nu^\infty W^\nu$. From (b) it follows that they also belong to pos$W$. In case (ii), the (sub)sequence $\{w^{\nu,i},\ \nu \in \mathbb{N}\}$ must be bounded, and hence $\hat{t}^i = \lambda_i w^i$ for some $w^i \in \limsup_\nu W^\nu$. From (a), it then follows that every such cluster point $\hat{t}^i$ also belongs to pos$W$. In the third case (iii), i.e., $\lim_\nu \lambda_{\nu,i} = \infty$, $\hat{t}^i$ then belongs to the core outer limit of the sets W^ν, in which case $\hat{t}^i \in$ pos$W$ by (c). This completes the proof of the assertion.

Let i_0 be an index such that

$$|\lambda_{\nu,i_0} w^{\nu,i_0}| = \max_{i=1,\dots,m} |\lambda_{\nu,i} w^{\nu,i}|,$$

and consider the sequence $\{\lambda_{\nu,i_0} w^{\nu,i_0}\}$; i_0 is assumed to be common for all ν, again passing to a subsequence if required. If it is bounded, then all of the sequences $\{\lambda_{\nu,i} w^{\nu,i}\}$, $i = 1,\dots,m$, are bounded, and one can assume that, for each i, they converge to (cluster at) some $\hat{t}^i \in \mathbb{R}^m$. In view of the earlier claim, these limits belong to pos$W$. Thus, also $t = \hat{t}^1 + \cdots + \hat{t}^m$ belongs to pos$W$. There remains only to consider the case when the sequence $\{\lambda_{\nu,i_0} w^{\nu,i_0}\}$ is unbounded, say, $\lim_\nu |\lambda_{\nu,i_0} w^{\nu,i_0}| = \infty$. For all $i = 1,\dots,m$, the sequences $\{\lambda_{\nu,i} w^{\nu,i}/|\lambda_{\nu,i_0} w^{\nu,i_0}|\}$ are bounded, and one can assume that, for each i, they converge to (equivalently, cluster at) some $u^1,\dots,u^m$. By Claim 1, these u^i belong to pos$W$.

CLAIM 2. *These limit points $u^1,\dots,u^m$ belong to* lil(pos$W$)*. Consequently, if u^i is nonzero, then, for ν sufficiently large, $w^{\nu,i}$ belong to* lilposW^ν.

Since $|\lambda_{\nu,i_0} w^{\nu,i_0}| \to \infty$, dividing t^ν by $|\lambda_{\nu,i_0} w^{\nu,i_0}|$ and passing to the limit when ν tends to ∞, one obtains $u^1 + \cdots + u^m = 0$. Since pos$W$ is a convex cone, we conclude that $u^1,\dots,u^m$ must belong to the lineality space of pos$W$. The second part of the assertion now follows directly from condition (d), and thus Claim 2 is verified.

Let I_0 denote the set of all indexes i such that $u^i = 0$ and J_0 the set of remaining indexes. Note that i_0 belongs to J_0 since $|u^{i_0}| = 1$. According to Claim 2, if the index set I_0 is empty, then t^ν belongs to lilposW^ν for ν large enough and, consequently, so does $t \in$ pos$W$. Thus, one has only to consider the case when I_0 is nonempty. Let's write t^ν as the sum of two following terms:

$$t^\nu = z^\nu + y^\nu, \quad \text{where} \quad z^\nu = \sum_{j\in J_0} \lambda_{\nu,j} w^{\nu,j},\ y^\nu = \sum_{i \in I_0} \lambda_{\nu,i} w^{\nu,i}.$$

Now, consider the sequence $\{z^\nu\}_{\nu=1}^\infty$. It's bounded or not.

Bounded case: The sequence $\{z^\nu\}$ is bounded. Let z be a cluster point of this sequence that necessarily belongs to pos$W$. Then the corresponding (sub)sequence $\{y^\nu\}$ converges to $t - z$. There are two possibilities as far as this latter sequence is concerned. The first one is when all sequences $\{\lambda_{\nu,i} w^{\nu,i}\}$ are bounded; hence, one may assume that they converge to some w^i, with $i \in I_0$. In view of Claim 1, these limits w^i belong to pos$W$, the limit $t - z = \sum_{i\in I_0} w^i$ also belongs to pos$W$, and one may conclude that $t \in$ pos$W$. When all of the sequences $\{\lambda_{\nu,i} w^{\nu,i}\}$ are not necessarily bounded, there is a sequence, again possibly passing to a subsequence, $\{\lambda_{\nu,i_1} w^{\nu,i_1}\}$ such that

$$|\lambda_{\nu,i_1} w^{\nu,i_1}| = \max_{i \in I_0} |\lambda_{\nu,i} w^{\nu,i}| \text{ and } \lim_{\nu} |\lambda_{\nu,i_1} w^{\nu,i_1}| = \infty.$$

One can appeal to the same argument as that used earlier for the sequence $\{\lambda_{\nu,i_0} w^{\nu,i_0},\ \nu \in \mathbb{N}\}$, and one can find subsets $I_1 \subset I_0$ and $J_1 = I_0 \setminus I_1$ such that

$$y^\nu = v^\nu + \sum_{i \in I_1} \lambda_{\nu,i} w^{\nu,i},$$

where $v^\nu = \sum_{j \in J_1} \lambda_{\nu,j} w^{\nu,j}$ belongs to $\text{lilpos}W^\nu$ for ν sufficiently large. One can rewrite t^ν as follows:

$$t^\nu = (z^\nu + v^\nu) + \sum_{i \in I_1} \lambda_{\nu,i} w^{\nu,i}.$$

Note that the first term $(z^\nu + v^\nu)$ belongs to $\text{lilpos}W^\nu$ for ν sufficiently large and that the index set I_1 has cardinality strictly smaller than that of I_0. One can proceed in this manner, one eventually exhausts all possible indexes, and one is led to conclude that $t \in \text{pos}W$.

Unbounded case: The sequence $\{z^\nu\}$ is unbounded, say, $\lim_\nu |z^\nu| = \infty$. Assume that $\lim_\nu z^\nu/|z^\nu| = v \in \text{pos}W, v \neq 0$, and

$$0 = \lim_{\nu} \frac{t^\nu}{|z^\nu|} = v + \lim \frac{y^\nu}{|z^\nu|}.$$

Set $\mu_{\nu,i} = \lambda_{\nu,i}/|z^\nu|$, and consider the sequences $\{\mu_{\nu,i} w^{\nu,i}\}_{\nu=1}^\infty$, with $i \in I_0$. By the same argument as in the "bounded case" for the sequences $\{\lambda_{\nu,i} w^{\nu,i}\}_{\nu=1}^\infty$, we come to the conclusion that either $w^{\nu,i}$ belong to $\text{lilpos}W^\nu$ for ν sufficiently large and $i \in I_0$ or there is a nonempty subset $J_{1,ubdd}$ of I_0 such that

$$t^\nu = \sum_{j \in J_0 \cup J_{1,ubdd}} \lambda_{\nu,j} w^{\nu,j} + \sum_{i \in I_{1,ubdd}} \lambda_{\nu,i} w^{\nu,i},$$

where the elements of the first sum belong to $\text{lilpos}W^\nu$ for ν sufficiently large and $I_{1,ubdd} := I_0 \setminus J_{1,ubdd}$ is of cardinality strictly smaller than that of I_0. Continuing this procedure and remembering that the number of indexes is finite (m), we arrive at the final step in which either all terms t^ν belong to $\text{lilpos}W^\nu$ for ν sufficiently large or t is a sum of the limit points that belong to $\text{pos}W$. In both cases, t is an element of $\text{pos}W$ because the latter set is a convex cone. This completes the proof of the theorem. ◻

Remark 4.2. Let's record the following observations about the hypotheses of this theorem:

- (a) When Theorem 4.1(a) holds, so does Theorem 4.1(c), trivially, when either of the following conditions is satisfied:
  - (a_1) The closure of W does not contain the origin;
  - (a_2) $[\limsup^o_\nu W^\nu] \setminus \{0\} \subset \text{int}(\text{pos}W)$.
- (b) If $\text{pos}W$ is pointed, then Theorem 4.1(d) is trivially satisfied.
- (c) In Theorem 4.1(d), the inclusion $\text{lil}(\text{pos}W) \supset \limsup_\nu \text{lil}(\text{pos}W^\nu)$ when
  - (c_1) $\text{pos}W \supset \liminf_\nu W^\nu$;
  - (c_2) $\dim \text{lil}(\text{pos}W^\nu) \leq \dim \text{lil}(\text{pos}W)$.

Clearly, condition 4.1(a) is essential. Next, we give three examples to show that each of the remaining conditions can't be neglected either.

Example 4.3 (necessity of condition 4.1(b)). Let $W = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}$, and let $W^\nu = W \cup \{(\nu, \nu)\}$. Then $\text{pos}W = \{(0, y) \in \mathbb{R}^2 : y \geq 0\}$, while $\text{pos}W^\nu = \text{pos}W \cup \{(x, x) : x \geq 0\}$. All of the conditions of the theorem are fulfilled except for the second one.

Example 4.4 (necessity of condition 4.1(c)). Let $W = \{(0, y) : y \geq 0\}$, and let $W^\nu = W \cup \{(1/\nu, 1/\nu^2)\}$. Then $\text{pos}W$ coincides with W, but $\limsup_\nu \text{pos}W^\nu$ is the positive orthant of $\mathbb{R}^2$. In this example only the third condition is violated.

Example 4.5 (necessity of condition 4.1(d)). Let $W = \{(-1, 0), (1, 0)\}$, and let $W^\nu = \{(-1, 1/\nu), (1, 1/\nu)\}$. Then $\text{pos}W = \{(x, 0) : x \in \mathbb{R}\}$, while $\limsup_\nu \text{pos}W^\nu = \{(x, y) : y \geq 0\}$. In this example all of the conditions of the theorem are satisfied except for the fourth one.

When W^ν consists of the columns $a^{\nu,1}, \ldots, a^{\nu,k}$ of a constant size $m \times k$-matrix, Theorem 4.1 yields the following improvement of [12, Theorem 2].

COROLLARY 4.6 (positive hull of converging matrices). *Assume that the vectors* $a^{\nu,1}, \ldots, a^{\nu,k}$ *converge, respectively, to* $a^i, i = 1 \ldots, k$. *Set* $W = \{a^1, \ldots, a^k\}$, *and assume further that*

(a) *for all* ν, $\dim \text{lil}(\text{pos}W^\nu) = \dim \text{lil}(\text{pos}W)$*;*

(b) $\text{pos}W$ *includes all of the cluster points of* $\{a^{\nu,i}/|a^{\nu,i}|\}$ *when* $a^i = 0$*;*

(c) *if* $0 \neq a^i \in \text{lil}(\text{pos}W)$, *then* $a^{\nu,i} \in \text{lil}(\text{pos}W^\nu)$ *for all* ν.

Then $\limsup_\nu \text{pos}W^\nu \subset \text{pos}W$.

Proof. The proof combines the observations in Remark 4.2(c) with the assertion of Theorem 4.1. ∎

The condition (b) of [12, Theorem 2] requires that Corollary 4.6(c) holds even when $a^i = 0$. It is clear that this condition then implies both conditions 4.6(b) and 4.6(c), but the converse is not the case, as seen by the next example.

Example 4.7 (relaxed outer semicontinuity). Let $a^{\nu,1} = (0, 1)$ and $a^{\nu,2} = (\frac{1}{\nu}, \frac{1}{\sqrt{\nu}})$.

Detail. Then $a^1 = (0, 1)$ and $a^2 = (0, 0)$. The lineality spaces of $\text{pos}W^\nu$ and $\text{pos}W$ are the null space, and therefore conditions 4.6(a) and 4.6(c) clearly hold. Since $a^{\nu,2} \neq 0$, condition (b) of [12, Theorem 2] is not satisfied. However, one still has $\limsup_\nu \text{pos}W^\nu \subset \text{pos}W$ according to the previous corollary.

The outer semicontinuity of the positive hulls can also be characterized in terms of the inner limits of their positive polar cones. Given a nonempty subset W of $\mathbb{R}^m$ the polar cone of W consists of linear functions that are positive on W, that is,

$$W^* := \{u \in \mathbb{R}^m : \langle u, t\rangle \geq 0, t \in W\}.$$

The *inner limit* of a collection of sets $\{C^\nu, \nu \in \mathbb{N}\}$ is denoted by $\liminf_\nu C^\nu$, which consists of those vectors v for which there exist $v_\nu \in C^\nu$ for every ν such that v is the limit of the sequence $\{v_\nu\}_\nu$.

PROPOSITION 4.8 (limits under polarity). *Let* $C \subset \mathbb{R}^n$ *be a closed convex cone. Then the following conditions are equivalent:*

(a) $\limsup_\nu \text{pos}W^\nu \subset C$,

(b) $\liminf_\nu (\text{pos}W^\nu)^* \supset C^*$.

Hence, under the assumptions of Theorem 4.1,

$$\liminf_\nu (W^\nu)^* \supset W^*.$$

Proof. Observe that the outer limit of posW^ν coincides with the outer limit of their closures. Now, apply [11, Corollary 11.35] to the closed convex cones $\text{cl}\,\text{posW}^\nu$

to obtain the first assertion. The second assertion is deduced from (a) and Theorem 4.1. ☐

Let us close up this section by some remarks on lower semicontinuity and upper semicontinuity of the positive hull mappings. It is clear that the inclusion

$$\liminf_{\nu} \operatorname{pos}W^\nu \supset \operatorname{pos}W,$$

which characterizes the lower semicontinuity of the positive hull, is true under a quite standard assumption that $\liminf_\nu W^\nu \supset W$. The upper semicontinuity means that, for any open set A containing pos$W$, one can find an index ν_0 such that A contains all posW^ν for $\nu > \nu_0$. Since posW^ν are cones, the above condition implies that pos$W$ contains all of the cones posW^ν whenever $\nu \geq \nu_0$. This is the reason why we focus our attention to outer semicontinuity only.

5. Application: Semi-infinite programs. Consider the following semi-infinite linear program,

$$\text{(siLP)} \qquad \begin{array}{rl} & \min \langle c, x\rangle \\ \text{so that} & \langle a_t, x\rangle + \beta_t \leq 0,\ t \in T, \end{array}$$

where $a_t \in \mathbb{R}^n, \beta_t \in \mathbb{R}$, and T is supposed to be a compact metric space. The system of constraints $\langle a_t, x\rangle + \beta_t \leq 0,\ t \in T$, is denoted by σ and its solution set (the feasible set of (siLP)) is denoted by S. Set

$$W = \{a_t | t \in T\} \qquad \text{and} \qquad \hat{W} = \{(a_t, \beta_t) | t \in T\}.$$

The convex cones generated by W and $\hat{W}$ are, respectively, called the first-moment and the second-moment cone of the system σ (see [6]). We recall also that a sequence $\{x^\nu\}_\nu$ of vectors in $\mathbb{R}^n$ is said to be an asymptotic solution of the system σ if for every $t \in T$ one has $\liminf_{\nu\to\infty}\langle a_t, x^\nu\rangle + \beta_t \leq 0$. It is clear that the system σ is consistent (that is, S is nonempty) if and only if it has a bounded asymptotic solution. A system that has no asymptotic solutions is called strongly inconsistent. Of course, an inconsistent system may have asymptotic solutions as well.

Now we consider a collection of perturbed problems of the same form: For $\nu \in \mathbb{N}$,

$$\text{(siLP}^\nu) \qquad \begin{array}{rl} & \min \langle c^\nu, x\rangle \\ \text{so that} & \langle a_t^\nu, x\rangle + \beta_t^\nu \leq 0,\ t \in T^\nu. \end{array}$$

As functions of t, a_t, a_t^ν, β_t, and β_t^ν are continuous. When the spaces $C(T)$ and $C(T^\nu)$ of the continuous functions on T and T^ν are equipped with the max-norm topology, their (topological) duals are the spaces of measures $M(T)$ and $M(T^\nu)$, respectively. The cone of positive measures is denoted by $M_+(T^\nu)$, and $M_+^F(T^\nu)$ is the cone of positive measures with finite support. The system σ^ν and the sets S^ν, W^ν, and $\hat{W}^\nu$ are defined accordingly as σ, S, W, and $\hat{W}$ above.

The first result that can be derived from Theorem 4.1 is on the stability of consistency and strong inconsistency of the system of constraints of (siLP).

PROPOSITION 5.1. *Assume that the collection* $\{\hat{W}; \hat{W}^\nu, \nu \in \mathbb{N}\}$ *satisfies the hypotheses of Theorem* 4.1. *Then the following assertions hold:*

(a) *If the system σ is consistent, then, for ν sufficiently large, the systems σ^ν are consistent.*

(b) *If the systems σ^ν are strongly inconsistent, then the system σ is strongly inconsistent.*

Proof. According to the consistency tests (Theorem 4.4 of [6]) the system σ is consistent if and only if the cone $\mathrm{clpos}\hat{W}$ does not contain the vector $e := (0, \dots, 0, 1)$ of the space $\mathbb{R}^{n+1}$. Since the outer limit of $\mathrm{pos}\hat{W}^\nu$ coincides with the outer limit of $\mathrm{clpos}\hat{W}^\nu$, in view of Theorem 4.1, when ν is sufficiently large, the cone $\mathrm{clpos}\hat{W}^\nu$ does not contain that vector either, and hence the system σ^ν is consistent.

Now, if the systems σ^ν are strongly inconsistent, then again, in view of the consistency tests, the cones $\mathrm{pos}\hat{W}^\nu$ contain the vector e. By Theorem 4.1, the vector e belongs to the cone $\mathrm{pos}\hat{W}$, and hence the strong inconsistency of the system σ follows. ∎

It is known that the horizon cone of the feasible set S (when it is nonempty) is the solution set to the homogeneous system $\langle a_t, x\rangle \le 0, t \in T$. It is exactly the negative polar cone of the set W. We derive the outer and inner continuity of the horizon cone of S as follows.

PROPOSITION 5.2. *The following assertions hold:*

(a) *If the collection $\{W; W^\nu, \nu \in \mathbb{N}\}$ satisfies the hypotheses of Theorem* 4.1 *and if the problems* (siLP$^\nu$) *have feasible solutions, then*

$$S_\infty \subseteq \liminf_\nu S^\nu_\infty.$$

(b) *If $W \subseteq \liminf_\nu W^\nu$ and the problem* (siLP) *has feasible solutions, then*

$$S_\infty \supseteq \limsup_\nu S^\nu_\infty.$$

Proof. If S is empty, then the inclusion of (a) holds trivially. If S is not empty, as we have already mentioned, the horizon cone of S is the negative polar cone of the set W. Then (a) follows from Theorem 4.1 and Proposition 4.8. The second assertion is obtained from Proposition 4.8 and the remark at the end of section 4 on the inner semicontinuity of the positive hull mappings. ∎

An immediate consequence of the above proposition is on the boundedness of the feasible set.

COROLLARY 5.3. *If nonempty, the feasible set S of* (siLP) *is bounded provided that the hypotheses of Theorem* 4.1 *are satisfied and that the feasible sets S^ν are nonempty and bounded. For ν sufficiently large, if nonempty, the sets S^ν are bounded provided that S is nonempty and bounded and that $W \subseteq \liminf_\nu W^\nu$.*

Proof. It is known that, being nonempty, the set S is bounded if and only if its horizon cone is trivial. The corollary now follows from Proposition 5.2. ∎

We notice that the hypotheses of Theorem 4.1 do not guarantee that problem (siLP) has feasible solutions even if (siLP$^\nu$) have. Next we turn to the stability the of existence of optimal solutions for (siLP).

PROPOSITION 5.4 (existence of solutions). *Assume the following:*

(a) (siLP) *and* (siLP$^\nu$) *satisfy the Slater condition;*

(b) *the collection $\{W; W^\nu, \nu \in \mathbb{N}\}$ satisfies the hypotheses of Theorem* 4.1 *and $c^\nu \to c$;*

(c) *problems* (siLP$^\nu$) *have finite optimal values.*

Then the optimal value of (siLP) *is finite.*

Proof. Direct calculation, either via the Lagrangian function or conjugate calculus (see also [3]), yields the dual problem of (siLP$^\nu$):

$$(siD^\nu) \qquad \begin{aligned} &\max \int_{T^\nu} \beta_t^\nu \, d\mu(t) \\ \text{so that} \quad &\mu \in M_+(T^\nu), \\ &c^\nu + \int_{T^\nu} a_t^\nu d\mu(t) = 0. \end{aligned}$$

The *finite support dual*, whose (dual) variables are restricted to be measures with finite support, is

$$(\text{siFD}^\nu) \qquad \begin{aligned} &\max \sum_{t \in \text{supp}(\mu)} \beta_t^\nu \mu(t) \\ &\mu \in M_+^F(T^\nu), \\ &c^\nu + \sum_{t \in \text{supp}(\mu)} a_t^\nu \mu(t) = 0, \end{aligned}$$

where supp(μ) designates the (finite) support of the measure μ. With $v(\cdot)$ denoting the optimal value, one has the following weak duality inequalities [3]:

$$v(\text{siLP}^\nu) \geq v(\text{siD}^\nu) \geq v(\text{siFD}^\nu).$$

By assumption $v(\text{siLP}^\nu)$ is finite, and hence the set of feasible solutions of (siFD$^\nu$) is nonempty [3, Theorem 5.27], which means that

$$0 \in c^\nu + \text{pos} W^\nu.$$

Passing to the limit and applying Theorem 4.1, one obtains

$$0 \in c + \limsup_\nu \text{pos} W^\nu \subset c + \text{pos} W.$$

Hence, (siFD) is feasible, and in turn this implies that v(siLP) is finite. ∎

Remark 5.5 (convergence of the solutions). The propositions that we have established in this section are direct applications of Theorem 4.1. Some more convergence properties which are less direct from the above-said theorem can also be said about (siLP). For instance, the convergence of the feasible solutions is stated as follows.

(i) If $\hat{W} \subseteq \liminf_\nu \hat{W}^\nu$, then

$$S \supseteq \limsup_\nu S^\nu;$$

(ii) if the family $\hat{W}^\nu$ is relaxed upper semicontinuous in the sense that, for every $\epsilon > 0$, there is some integer ν_0 such that all $\hat{W}^\nu$, $\nu \geq \nu_0$, are within an ϵ-neighborhood of $\hat{W}$ in the product space $\mathbb{R}^m \times \mathbb{R}$, and if S admits a strong Slater point, that is, a point $\overline{x} \in S$, with $\langle a_t, \overline{x} \rangle + \beta_t \leq -\delta$ for all $t \in T$ and some $\delta > 0$, then

$$S \subseteq \liminf_\nu S^\nu.$$

The proof of these assertions presents no difficulty, so we omit it. The convergence of the optimal solutions and the optimal value of (siLP) can also be obtained by using the methods of [4, 6], in which a rather complete analysis of convergence of (siLP) has been exposed in the case when the index sets T^ν are common for all ν.

6. Application: Stochastic programs. Here, we consider an extension of the *"linear" stochastic program recourse model* [2, 10] to one where the recourse problem takes on the form of a generalized linear program. This brings us into the realm of problems with nonlinear recourse, and, for this particular class, we derive rather explicit expressions for the induced constraints as well as a lower semicontinuity result for the optimal value function. Let $\Im$ stand for the random components of the problem that takes its values, denoted by ξ, in $\Xi \subset \mathbb{R}^d$, the (closed) support of the associated probability measure P. The *recourse cost function* is

$$Q(\xi, x) = \inf \left\{ \sum_{j=1}^{J} q_j y_j | y_j \geq 0, \text{ with } \xi - Tx = \sum_{j=1}^{J} t^j y_j, t^j \in W, j = 1, \ldots, J, \text{ and } J \in \mathbb{N} \right\},$$

where, for $j = 1, \ldots, J$,

$$\begin{pmatrix} q_j \\ t^j \end{pmatrix} \in C \subset \mathbb{R}^{m+1},$$

with C convex; it could be the epigraph of (nonlinear) convex functions, for example. The *expected recourse function* is

$$EQ(x) = E\{Q(\Im, x)\} = \int_{\Xi} Q(\xi, x)\, P(d\xi),$$

with the stochastic program

$$\begin{aligned} \min\ & f_0(x) + EQ(x) \\ \text{so that} \quad & Ax = b, x \geq 0, \end{aligned}$$

with $f_0 : \mathbb{R}^n \to \mathbb{R}$. Let

$$K_1 = \left\{ x \in \mathbb{R}^n_+ | Ax = b \right\} \quad \text{and} \quad K_2 = \text{dom} EQ.$$

The problem is said to have *relatively complete recourse* when $K_1 \subset K_2$. That's a desirable property, but, unfortunately, this is not universally the case. When $E\{\Im\}$ is finite [13], as we now assume, for the set of *induced constraints*, one has

$$K_2 = \text{dom} EQ = \bigcap_{\xi \in \Xi} \text{dom} Q(\xi, \cdot) = \bigcap_{\xi \in \Xi} \{x | \xi - Tx \in \text{pos} W\}.$$

Our emphasis here will be on the dependence of the set K_2 on perturbations affecting T, W, and Ξ, in particular when the set of feasible solutions of the stochastic program is not necessarily bounded. Let's denote these perturbed versions by T^ν, W^ν, and Ξ^ν.

LEMMA 6.1 (outer semicontinuity of the induced constraints). *When*

(a) $\liminf_\nu \Xi^\nu \supset \Xi$ *and* $T^\nu \to T$,

(b) *the collection* $\{W; W^\nu, \nu \in \mathbb{N}\}$ *satisfies the hypotheses of Theorem* 4.1,

then $\limsup_\nu K_2^\nu \subset K_2$.

Proof. Let $\{x^\nu \in K_2^\nu\}_{\nu=1}^\infty$ converge to x and $\xi \in \Xi$. In view of (a), one can find $\xi^\nu \in \Xi^\nu$ such that $\lim_\nu \xi^\nu = \xi$. Then $\lim_\nu (\xi^\nu - T^\nu x^\nu) = \xi - Tx$ that belongs to pos$W$ according to Theorem 4.1. It follows that $x \in K_2$. □

Generally, the inclusion $\limsup_\nu D^\nu \subset D$ does not imply $\limsup_\nu^\infty D^\nu \subset D^\infty$. A remarkable exception is the case of induced constraints.

LEMMA 6.2 (the horizon outer limit of the induced constraints). *Under the hypothesis of Lemma* 6.1*, one has*

$$\limsup_\nu^\infty K_2^\nu \subset K_2^\infty.$$

Moreover, when $K_1 \cap K_2 \neq \emptyset$*, also* $\limsup_\nu^\infty (K_1 \cap K_2^\nu) \subset (K_1 \cap K_2)^\infty$.

Proof. Observe first that a vector x belongs to the horizon cone K_2^∞ if and only if $-Tx$ belongs to clpos$W$. Let $z \in \limsup_\nu^\infty K_2^\nu$, say, $z = \lim_k \lambda_k z^{\nu_k}$, where $z^{\nu_k} \in K_2^{\nu_k}$ and $\lambda_k \downarrow 0$. For $\xi \in \Xi$, in view of Lemma 6.1(a), one can find $\xi^k \in \Xi^{\nu_k}$ such that $\xi^k \to \xi$. Thus, $\xi^k - T^{\nu_k} z^{\nu_k} \in \text{pos}W^{\nu_k}$ and $\lambda_k \xi^k - T^{\nu_k}(\lambda_k z^{\nu_k}) \in \text{pos}W^{\nu_k}$, as well. Passing to the limit when k goes to ∞ yields $-Tz \in \text{pos}W$ via Lemma 6.1. Thus, $z \in K_2^\infty$. The second assertion is obtained from the first one by relying on [11, Proposition 3.9]. □

This leads us to a lower-semicontinuity result for the optimal value. Here, we also allow for perturbations f_0^ν of the objective function f_0 as well as for perturbations P^ν of the probability measure P. The expected recourse functions $E^\nu Q$ of the perturbed problems are then defined by

$$EQ^\nu(x) = \int_{\Xi^\nu} \left[\inf \left\{ \sum_{j=1}^J q_j y_j | y_j \geq 0 \text{ such that } \xi - T^\nu x = \sum_{j=1}^J t^j y_j, t^j \in W^\nu, \right.\right.$$
$$\left.\left. j = 1, \ldots, J, \text{with } J \text{ finite} \right\} \right] P^\nu(d\xi).$$

Let v and v^ν denote the optimal values of the given stochastic program and of its perturbations, respectively.

PROPOSITION 6.3 (lower semicontinuity of the optimal value). *When*
(a) $\liminf_\nu \Xi^\nu \supset \Xi$ *and* $T^\nu \to T$,
(b) *the collection* $\{W; W^\nu, \nu \in \mathbb{N}\}$ *satisfies the hypotheses of Theorem* 4.1*,*
(c) *given* (b)*,* $E^\nu Q$ *epiconverges to* EQ*,*[1]
(d) *the functions* $f^\nu = f_0^\nu + E^\nu Q$ *are quasi-convex,*
(e) *the functions* f_0^ν *converge continuously to* f_0*,*
(f) *the set of solutions of our (given) stochastic program is nonempty and bounded,*
then $\liminf_\nu v^\nu \geq v$.

Proof. A standard argument about the inf-projection and the summation, or integration, of convex functions yields the convexity of EQ and $E^\nu Q$. Assumptions (c) and (e) imply that the functions f^ν epiconverge to f [11, Theorem 7.46(b)]. We now proceed by contradiction. Suppose that one can find x^ν such that $\liminf_\nu f^\nu(x^\nu) < v$. Let $x^0 \in \text{argmin}_{K_1 \cap K_2} f$. If the sequence $\{x^\nu, \nu \in \mathbb{N}\}$ is bounded, in view of Lemma 6.1, one may assume that it converges to some $x \in K_1 \cap K_2$. This means, by epiconvergence, that $\liminf_\nu f^\nu(x^\nu) \geq f(x) \geq v$, and that's a contradiction. If the sequence $\{x^\nu\}$ is unbounded, we may assume that $\{x^\nu/|x^\nu|\}$ converges to some nonzero vector $z \in \limsup_\nu^\infty (K_1 \cap K_2^\nu)$. By Lemma 6.2, $z \in (K_1 \cap K_2)^\infty$; i.e., given $\lambda > 0$, one can find $\lambda_\nu > 0$ converging to 0 such that $x^0 + \lambda_\nu(x^\nu - x^0)$ converges to $x^0 + \lambda z$. Since the f^ν are quasi-convex, one must have

$$f^\nu(x^0 + \lambda_\nu(x^\nu - x^0)) \leq \max\{f^\nu(x^0), f^\nu(x^\nu)\} \leq \max\{f^\nu(x^0), v\}.$$

[1]This is not a very demanding condition, and it actually occurs under rather minimal assumptions; cf. [14, sections 6 and 8] for a brief survey.

This implies that

$$f(x^0 + \lambda z) \leq \max\{f(x^0), v\} = v.$$

Thus, for all $\lambda \geq 0$, $x^0 + \lambda z$ minimizes f on $K_1 \cap K_2$, in contradiction with assumption (f). □

Example 6.4 (without quasi-convexity). If the functions f^ν are not quasi-convex, then the inequality $\liminf_\nu v^\nu \geq v$ is no longer valid.

Detail. To see this, let us define the objective function $f_0 : \mathbb{R} \to \mathbb{R}_+$ by

$$f(x) = \begin{cases} -x-1 & \text{if } x \leq 0, \\ x-1 & \text{if } 0 < x < 3, \\ 2 & \text{if } x \geq 3 \end{cases}$$

and choose the constraints so that the feasible set $K_1 \cap K_2 = \mathbb{R}_+$. (The set W consists of one element 1, T is the matrix (-1), and the random variable Ξ takes the values $0, 1, 2, \ldots$.) The perturbed problems are given with the same feasibility set, and the objective functions are

$$f^\nu(x) = \begin{cases} f(x) & \text{if } x \leq \nu, \\ -x+\nu+1 & \text{if } \nu < x < \nu+3, \\ -f(x) & \text{if } x \geq \nu+3. \end{cases}$$

Then all of the hypotheses of the proposition are fulfilled except for the quasi convexity of the f^ν. The optimal value v^ν is -2 while the optimal value $v = -1$.

Example 6.5 (without boundedness of argmin$f$). The conclusion of Proposition 6.3 is no longer true if the set of minimizers of f on $K_1 \cap K_2$ is unbounded as demonstrated by the following example.

Detail. Again, let $K_1 \cap K_2 = \mathbb{R}_+$. The objective function is

$$f(x) = \begin{cases} -x-1 & \text{if } x \leq 0. \\ -1 & \text{if } x > 0, \end{cases} \quad \text{and } f^\nu(x) = \begin{cases} f(x) & \text{if } x \leq \nu, \\ -x+\nu-1 & \text{if } \nu < x < \nu+1, \\ x-\nu-3 & \text{if } x \geq \nu+1. \end{cases}$$

Then the set of minimizers of f is unbounded, and the optimal value of the perturbed problems is $v^\nu = -2$ while $v = -1$.

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