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# ON THE CONTINUITY OF THE VALUE OF A LINEAR PROGRAM AND OF RELATED POLYHEDRAL-VALUED MULTIFUNCTIONS

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Dedicated to Professor George B. Dantzig on the occasion of his seventieth birthday.

Results about the continuity of the optimal value of a linear program and of related polyhedral-valued multifunctions (determined by the constraints) are reviewed. A framework is provided for studying their interconnections.

*Key words*: Linear Programming, Polyhedral-Valued Multifunctions, Infimal Value Function, Marginal Functions.

We study the continuity properties of the following function:

$$Q(t) := \inf_{x \in \mathbb{R}^n} \{cx \mid Ax \geqslant b, x \geqslant 0\}$$

where

$$t := (c; A_1; A_2; \ldots; A_m; b^{\mathrm{T}})$$
$$= (c_1, \ldots, c_n; a_{11}, \ldots, a_{1n}; \ldots; a_{m1}, \ldots, a_{mn}; b_1, \ldots, b_m).$$

Thus

$$t \mapsto Q(t) : \mathbb{R}^N \to \bar{\mathbb{R}} = [-\infty, +\infty] \quad \text{with } N = n + m \cdot n + m.$$

Let

$$\mathcal{T} := \{t \in \mathcal{R}^N \mid -\infty < Q(t) < +\infty\}$$

denote the set on which $Q$ is finite. We deal exclusively with sufficient conditions for the continuity of $Q$; necessary conditions are much too involved to be verifiable, and consequently are of rather limited interest in applications. The stability—which means continuity of some type—of the optimal value (and optimal solutions) of an optimization problem under data perturbations is usually the key issue in the validation of the modeling process. For example, the work of Dantzig Folkman and Shapiro [5] was motivated by stability questions in chemical equilibria; for more

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examples one could refer to [18], and for a general discussion to [12, Section 1]. My own interest in this area stems from a variety of questions that arise in stochastic programming [13, 14, 17]. For example, some of the parameters $t = (c, A, b)$ of the linear program may only be known in a statistical sense. The data available suggest a distribution function $F$ for the random vector $\boldsymbol{t}$, but there is no absolute guarantee that it actually is the (true) distribution function. We are interested in the continuity (stability) of

$$E[Q(t)] = \int Q(t)\, \mathrm{d}F(t)$$

under perturbations of $F$. Assuming that the range $T$ of possible values for $\boldsymbol{t}$ is compact, then the continuity of $\int Q(t)\, \mathrm{d}F(t)$ with respect to $F$ would follow directly from the continuity of $Q$ on $T$. Recall that a sequence $\{F^\nu, \nu = 1, \ldots\}$ of distribution functions defined on $\mathbb{R}^N$ converges to $F$ if

$$\lim_{\nu \to \infty} F^\nu(t) = F(t) \quad \text{for all } t \in \{z \mid F \text{ is continuous at } z\}$$

and then for every bounded continuous function $g$

$$\lim_{\nu \to \infty} \int g(t)\, \mathrm{d}F^\nu(t) = \int g(t)\, \mathrm{d}F(t).$$

Similar continuity questions would arise if $F$ were known to be the actual distribution function, but it became necessary to replace $F$ by an approximation in order to simplify the computation of $E[Q(t)]$.

Dantzig, Folkman and Shapiro [5], Bereanu [2], Wets [17], Martin [9], Robinson [10] and Klatte [8] have formulated a number of sufficient conditions for the continuity of $Q$, and some others can be derived from general results for infimal functions of optimization problems, such as found in Berge [3], Dinkelbach [6], Hogan [7], Bank, Guddat, Klatte, Kummer and Tammer [1], and Rockafellar and Wets [12], for example. One of the main purposes of this note is to exhibit the relationship between these various conditions. As a by-product, or more exactly as a prerequisite, we prove a number of results about the continuity of polyhedral-valued multifunctions. The two following (convex) *polyhedral-valued multifunctions*

$$t \mapsto K(t) := \{x \mid Ax \leqslant b, x \geqslant 0\} \quad \text{and} \quad t \mapsto D(t) := \{y \mid yA \leqslant c, y \geqslant 0\},$$

that correspond respectively to the set of primal and dual feasible solutions of the linear program that defines $Q$, play an important role in what follows. The function $Q$ is finite precisely on

$$\{t \in \mathbb{R}^n \mid K(t) \neq \emptyset, D(t) \neq \emptyset\} = \mathcal{T}.$$

If $K(t) \neq \emptyset$ but $D(t) = \emptyset$, then $Q(t) = -\infty$, otherwise, i.e. when $K(t) = \emptyset$, then $Q(T) = \infty$. Recall that a multifunction $t \mapsto \Gamma(t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is *upper semicontinuous at* $t$, if

$$t = \lim_{\nu \to \infty} t^\nu, \quad x^\nu \in \Gamma(t^\nu) \quad \text{and} \quad x = \lim_{\nu \to \infty} x^\nu$$

implies that $x \in \Gamma(t)$; it is *lower semicontinuous at t*, if

$$t = \lim_{\nu\to\infty} t^\nu \quad \text{and} \quad x \in \Gamma(t)$$

implies the existence of $x^\nu \in \Gamma(t^\nu)$ such that $x = \lim_{\nu\to\infty} x^\nu$; and $\Gamma$ is *continuous at t* if it is both upper and lower semicontinuous at $t$. For the polyhedral-valued multifunctions $K$ and $D$ we have

**1. Proposition.** *The multifunctions K and D are upper semicontinuous on $\mathcal{T}$.*

**Proof.** It suffices to prove the assertion for either $K$ or $D$. Suppose that $\{t^\nu, \nu = 1, \ldots\} \subset \mathcal{T}$ and $t = \lim_{\nu\to\infty} t^\nu$. Let $\{x^\nu, \nu = 1, \ldots\}$ be a sequence such that for all $\nu = 1, \ldots, x^\nu \in K(t^\nu)$—which implies that $A^\nu x^\nu \geq b^\nu$—and $\lim_{\nu\to\infty} x^\nu = x$. Since

$$\|A^\nu - A\| \to 0, \quad \|x^\nu - x\| \to 0 \quad \text{and} \quad \|b^\nu - b\| \to 0$$

it follows that $Ax \geq b$ and $x \geq 0$ which yields $x \in K(t) \neq \emptyset$. Note that any norm for matrices and vectors will do, the most natural one to use here is

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|. \quad \square$$

In general $K$ and $D$ are not continuous, or equivalently in view of Proposition 1, they are not lower semicontinuous. For example, consider $t^\nu = (c^\nu = \nu^{-1}; A = \nu^{-1}; b = \nu^{-1})$ and $t = (0; 0; 0)$. Then $K(t) = \mathbb{R}_+$, but for all $\nu$, $K(t^\nu) = [1, \infty)$; the point $\frac{1}{2} \in K(t)$ cannot be reached by any sequence $\{x^\nu, \nu = 1, \ldots\}$ with $x^\nu \in [1, \infty)$. Also, $D(t) = \mathbb{R}_+$ but for all $\nu$, $D(t^\nu) = [0, 1]$, again any point $p \in (1, \infty)$ cannot be reached by any sequence $\{x^\nu, \nu = 1 \ldots\}$ with $x^\nu \in [0, 1]$. Note that $Q(t) = \lim_{\nu\to\infty} Q(t^\nu)$. The importance of having $K$ and $D$ (lower semi)continuous is underscored by the following theorem.

**2. Theorem.** *Let $t \in T \subset \mathcal{T}$. Suppose the multifunction K is continuous at t relative to T. Then Q is upper semicontinuous at t relative to T. Similarly, if the multifunction D is continuous at t relative to T, then Q is lower semicontinuous at t relative to T. Finally, continuity of K and D at t (relative to T) implies the continuity of Q at t (relative to T).*

**Proof.** If $t = (c; A; b) \in T \subset \mathcal{T}$, then both $K(t)$ and $D(t)$ are nonempty, and there exist $x \in T(t)$ and $y \in D(t)$ such that

$$yb = Q(t) = cx$$

as follows from the Duality Theorem for linear programs. The lower semicontinuity of $K$ (or $D$) means that for any sequence $\{t^\nu = (c^\nu; A^\nu; b^\nu), \nu = 1, \ldots\} \subset T$ converging to $t$, there exist $\{x^\nu \in K(t^\nu), \nu = 1, \ldots\}$ (or $\{y^\nu \in D(t), \nu = 1, \ldots\}$ resp.) such that $x = \lim_{\nu\to\infty} x^\nu$ (or $y = \lim_{\nu\to\infty} y_1$ resp.). Moreover, we have that for all $\nu$,

$$Q(t^\nu) \leq c^\nu x^\nu$$



from which it follows that

$$Q(t) = cx = \lim_{\nu\to\infty} c^\nu x^\nu \geq \limsup_{\nu\to\infty} Q(t^\nu)$$

giving us the upper semicontinuity of $Q$ at $t$. For similar reasons, the lower semicontinuity of $D$ at $t$ yields:

$$Q(t) = yb = \lim_{\nu\to\infty} y^\nu b^\nu \leq \liminf_{\nu\to\infty} Q(t^\nu)$$

The assertion about continuity of $Q$ at $t$ follows from the two preceding inequalities. $\square$

The remainder of this note is devoted to obtaining sufficient conditions for the (lower semi) continuity of the polyhedral-valued multifunctions $K$ and $D$. The conditions that we shall exhibit can never be more than sufficient conditions for the continuity of $Q$, as is clearly demonstrated by the simple example that precedes Theorem 2. There, neither $K$ nor $D$ is lower semicontinuous at $t = (0; 0; 0)$ but $Q$ itself is continuous. We go from the weakest conditions to the strongest. To simplify the formulation of the results we shall make the assertions in terms of the global continuity of functions and multifunctions on a set $T \subset \mathcal{T}$, although in every case we could make a somewhat stronger statement in terms of *local* continuity at a point $t$ relative to a set $T \subset \mathcal{T}$, exactly as done in Theorem 2.

**3. Proposition.** *The multifunction $t \mapsto D(t)$ is continuous on $T \subset \mathcal{T}$ if and only if the polyhedral convex-cone valued multifunction*

$$t \mapsto \text{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix} = \left\{ \begin{matrix} u \\ \eta \end{matrix} \,\middle|\, \begin{matrix} u = Ax - Is, \\ \eta = cx + \beta, \end{matrix} \quad x \geq 0, s \geq 0, \beta \geq 0 \right\}$$
$$= \{(u, \eta) \in \mathbb{R}^{m+1} \mid u \leq Ax, \eta \geq cx, x \geq 0\}$$

*is upper semicontinuous on T. Similarly $t \mapsto K(t)$ is continuous on $T \subset \mathcal{T}$ if and only if the convex-cone-valued multifunction*

$$t \mapsto \text{pos}\begin{pmatrix} A^{\mathrm{T}} & I & 0 \\ b^{\mathrm{T}} & 0 & -1 \end{pmatrix} = \left\{ \begin{matrix} v \\ \theta \end{matrix} \,\middle|\, \begin{matrix} v^{\mathrm{T}} = yA + rI, \\ \theta = yb - \alpha, \end{matrix} \quad y \geq 0, r \geq 0, \alpha \geq 0 \right\}$$
$$= \{(v, \theta) \in \mathbb{R}^{n+1} \mid v^{\mathrm{T}} \geq yA, \theta \leq yb, y \geq 0\}$$

*is upper semicontinuous on T.*

**Proof.** For reason of symmetry, it really suffices to prove the assertions involving $D$. We first note that

$$t \mapsto C(t) := \text{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$$

is upper semicontinuous if and only if the polar multifunction

$$t \mapsto \mathrm{pol}\, C(t) = \{(y, \beta) \mid yA \leqslant \beta c, y \geqslant 0, \beta \geqslant 0\}$$

is lower semicontinuous as follows in a straightforward fashion from the definitions of lower semicontinuity and of the map pol [13, Proposition 1]. In turn this multifunction pol $C$ is lower semicontinuous if and only if $D$ is lower semicontinuous as follows from the identity

$$\text{(4)} \qquad \mathrm{pol}\, C(t) = \mathrm{cl}\{\lambda(y, 1) \mid y \in D(t), \lambda \in \mathbb{R}_+\}$$

where cl denotes closure. The inclusion $\supset$ follows directly from the fact that pol $C(t)$ is a closed cone that contains $(D(t) \times \{1\})$. For the converse, let $(y, \beta) \in \mathrm{pol}\, C(t)$. If $\beta > 0$, then $\beta^{-1}y \in D(t)$ and $(y, \beta) = \lambda(\beta^{-1}y, 1)$ with $\lambda = \beta$. If $\beta = 0$ then

$$yA \leqslant 0 \quad \text{and} \quad y \geqslant 0.$$

Take any $\bar{y} \in D(t)$; recall that $D(t) \neq \emptyset$ since $t \in \mathcal{T}$. For any $\nu = 1, 2, \ldots,$ we have

$$(\bar{y} + \nu y)A \leqslant c, \qquad (\bar{y} + \nu y) \geqslant 0,$$

and thus $(\bar{y} + \nu y) \in D(t)$ for all $\nu = 1, \ldots,$ and hence the sequence of points

$$\{\nu^{-1}(\bar{y} + \nu y, 1), \nu = 1, \ldots\}$$

is in the set $\{\lambda(y', 1) \mid y' \in D(t), \lambda \in \mathbb{R}_+\}$ which implies that $(y, 0)$ belongs to its closure. This completes the proof of (4).

Now suppose that $D$ is lower semicontinuous at $t \in T \subset \mathcal{T}$. To show that pol $C(t)$ is also lower semicontinuous at $t$, for any $(y, \beta) \in \mathrm{pol}\, C(t)$ and $\{t^\nu, \nu = 1, \ldots\}$ any sequence in $T$ we have to exhibit a sequence $\{(y^\nu, \beta^\nu) \in \mathrm{pol}\, C(t^\nu), \nu = 1, \ldots\}$ converging to $(y, \beta)$. First assume that $\beta > 0$. Then $\beta^{-1}y \in D(t)$ and by lower semicontinuity of $D$ at $t$ there exist $\{\bar{y}^\nu \in D(t), \nu = 1, \ldots$ converging to $\beta^{-1}y$. The desired sequence is obtained by setting $y^\nu = \beta\bar{y}^\nu$ and $\beta^\nu = \beta$ for all $\nu$. Next if $\beta = 0$, the argument above has shown that then there exist $y^k \in D(t)$ such that

$$(y, 0) = \lim_{k \to \infty} k^{-1}(y^k, 1).$$

Again by lower semicontinuity of $D$ at $t$, we know that for all $k$

$$y^k = \lim_{\nu \to \infty} y^{k\nu} \quad \text{with } y^{k\nu} \in D(t^\nu), \quad \nu = 1, \ldots.$$

The desired sequence is obtained by a standard diagonalization selection procedure.

If pol $C$ is lower semicontinuous at $t \in T \subset \mathcal{T}$, let $y \in D(t)$ and $\{t^\nu, \nu = 1, \ldots\}$ be any sequence of points in $T$. From (4) we know that $(y, 1) \in \mathrm{pol}\, C(t)$ and thus there exist a sequence $\{(y^\nu, \beta^\nu) \in \mathrm{pol}\, C(t^\nu), \nu = 1, \ldots\}$ converging to $(y, 1)$. For $\nu$ sufficiently large $\beta^\nu > 0$, in which case $((1/\beta^\nu)y^\nu, 1) \in \mathrm{pol}\, C(t^\nu)$, i.e., $(\beta^\nu)^{-1}y \in D(t^\nu)$ and $y = \lim_{\nu \to \infty} (\beta^\nu)^{-1}y^\nu$. □

To pass from Proposition 3 to our next characterization of lower semicontinuity of $K$ and $D$ we rely on a Theorem of Walkup and Wets that gives sufficient conditions



for the continuity of polyhedral convex-cone valued multifunctions, it is reproduced here for the convenience of the reader.

**5. Theorem** [13, Theorem 2]. *Suppose $Z \subset \mathbb{R}^{m \cdot n}$ and for every matrix $\hat{A} \in Z$, with* pos $A = \{u \mid u = Ax, x \geqslant 0\}$,

(*a*) dim(pos $\hat{A} \cap (-\mathrm{pos}\, \hat{A})$) *is constant, i.e., the dimension of the lineality space of* pos $\hat{A}$ *is constant on $Z$,*

(*b*) *there exists a neighborhood $V$ of $\hat{A}$ such that if any column $\hat{A}^j$ of $\hat{A}$ lies in the lineality space of* pos $\hat{A}$, *then the corresponding column $A^j$ of any matrix $A$ in $V \cap Z$ lies in the lineality space of* pos $A$.

*Then the restriction of $A \mapsto$* pos $A$ *to $Z$ is continuous.*

**6. Proposition.** *Suppose $T \subset \mathcal{T}$ and for all $t \in T$*

($\mathrm{i_a}$) *the dimension of $K(t)$ is constant on $T$,*

($\mathrm{i_b}$) *there exists a neighbourhood $V$ of $t$ such that whenever*

$$K(t) \subset \{x \mid A_i x = b_i, i \in I\} \cap \{x \mid x_j = 0, j \in J\}$$

*for index subsets $I$ and $J$ of $\{i = 1, \ldots, m\}$ and $\{j = 1, \ldots, n\}$ respectively, then for all $t' \in T \cap V$*

$$K(t') \subset \{x \mid A'_i x = b'_i, i \in I\} \cap \{x \mid x_j = 0, j \in J\}$$

*Then $K$ is continuous on $T$.*

*Similarly if, for all $t \in T \subset \mathcal{T}$,*

($\mathrm{ii_a}$) *the dimension of $D(t)$ is constant on $T$,*

($\mathrm{ii_b}$) *there exist a neighborhood $W$ of $t$ such that whenever*

$$D(t) \subset \{y \mid yA^j = c_j, j \in J\} \cap \{y \mid y_i = 0, i \in I\}$$

*for $J$ and $I$ index subsets of $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, then, for all $t' \in T \cap W$,*

$$D(t') \subset \{y \mid y(A')^j = c'_j, j \in J\} \cap \{y \mid y_i = 0, i \in I\}$$

*Then $D$ is continuous on $T$.*

**Proof.** Again let $C(t) := \mathrm{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$. If dim $D$ is constant on $T$, then the dimension of pol $C$ is also constant on $T$, cf. (4), which in turn implies that the dimension of the lineality space of $C$ is constant on $T$. This is condition (a) of Theorem 5. Condition (b) of this Theorem 5 requires that there exist a neighbourhood $W$ of $t$, such that whenever the linear systems

$$-A^j \leqslant Ax, \quad -c_j \geqslant cx, \qquad x \geqslant 0$$

for some indices $j \in \{1, \ldots, n\}$, and for fixed $k \in \{1, \ldots, m\}$

$$1 \leqslant A_k x, \quad 0 \leqslant A_i x \ \text{ for } i \neq k, \quad 0 \geqslant cx, \ \ x \geqslant 0,$$

are consistent, then they remain consistent for all $t' \in W \cap T$. From these relations we obtain condition ($ii_b$) through a straightforward application of Farkas Lemma (Theorem of the Alternatives for Linear Inequalities) using the fact that $D$ is nonempty on $T \subset \mathcal{T}$. The assertions involving $K$ are proved similarly. □

In the particular case when $K$ and $D$ are of full dimension on $T$, we have the following version of Proposition 6.

**7. Corollary.** *Suppose that $K$ has nonempty interior on $T \subset \mathcal{T}$, i.e. for all $t \in T$, $\operatorname{int} K(t) \neq \emptyset$, and that for all $t \in T$ no row of $(A, b)$ is identically 0. Then $K$ is continuous on $T$. Similarly, if for all $t \in T$, $\operatorname{int} D(t) \neq \emptyset$, and no column of $\binom{A}{c}$ is identically 0, then $D$ is continuous on $T$.*

**Proof.** Let us proceed with $K$ restricted to $T$. Clearly condition ($i_a$) of Proposition 6 is satisfied since $\dim K(t) = n$ for all $t \in T$. Moreover if $\operatorname{int} K(T) \neq \emptyset$, $K(t)$ is not contained in any hyperplane, consequently condition ($ii_b$) could only fail if for some $t \in T$, there was a row $(A_i, b_i) = (0, 0)$. □

Another way of proving Corollary 7 is to show that the assumptions imply that for all $t \in T$, $C(t)$: $\operatorname{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix}$ is pointed, i.e., $C(T) \cap (-C(t)) = \{0\}$. Indeed suppose otherwise, then there exists $0 \neq v \in C(t)$ such that for all $u \in \operatorname{pol} C(t)$

$$v \cdot u \leqslant 0 \quad \text{and} \quad -v \cdot u \leqslant 0.$$

This would mean that $\operatorname{pol} C(t) \subset \{u \mid v \cdot u = 0\}$ and this in turn implies that $\operatorname{int} \operatorname{pol} C(t)$ would be empty. But that cannot be since $\operatorname{int} D(t) \neq \emptyset$ yields $\operatorname{int} \operatorname{pol} C(t) \neq \emptyset$, as follows from (4). We now appeal to Theorem 5 knowing that

(a′) $\dim[C(t) \cap (-C(t))] = 0$, and

(b′) no column of $\binom{A}{c}$ is identically 0,

which in view of (a′) is enough to guarantee that Condition (b) of Theorem 5 is satisfied.

The sufficient conditions provided by the subsequent results allow us to ascertain whether the assumptions of Corollary 7 are satisfied.

**8. Proposition.** *Suppose that, for all $t \in T \subset \mathcal{T}$,*

$$R(t) := \{x \mid Ax \geqslant 0, cx \leqslant 0, x \geqslant 0\} = \{0\},$$

*then $D$ is continuous on $T$. Similarly, if for all $t \in T \subset \mathcal{T}$,*

$$S(t) := \{y \mid yA \leqslant 0, yb \geqslant 0, y \geqslant 0\} = \{0\}$$

*then $K$ is continuous on $T$.*

**Proof.** Again for reasons of symmetry it really suffices to prove the first part of the Proposition. Again let

$$C(t) := \operatorname{pos}\begin{pmatrix} A & -I & 0 \\ c & 0 & 1 \end{pmatrix} = \{(u, \eta) \mid u \geqslant Ax, \eta \leqslant cx, x \geqslant 0\}.$$



We show that if $R(t) = \{0\}$ on $T$, then $C(t)$ is pointed and no column of $\binom{A}{c}$ can be identically 0 on $T$. Suppose $C(t)$ is not pointed, i.e., there exists $(u, \eta) \neq \emptyset$ such that

$$u \leqslant Ax^1, \quad \eta \geqslant cx^1 \quad \text{for some } x^1 \geqslant 0,$$

and

$$-u \leqslant Ax^2, \quad -\eta \geqslant cx^2 \quad \text{for some } x^2 \geqslant 0.$$

This implies that for $(x_1 + x_2) \geqslant 0$,

$$0 \leqslant A(x^1 + x^2) \quad \text{and} \quad 0 \geqslant c(x^1 + x^2).$$

But then $x^1 + x^2 = 0 = x^1 = x^2$ if $t = (c, A, b) \in T$ since $R(t) = \{0\}$. This in turn yields $(u, \eta) = 0$, which contradicts the working assumption that $C(t)$ is not pointed. Also, if some column $\binom{A^j}{c_j}$ is identically 0, then $R(t) \neq \{0\}$ since then any nonnegative multiple of the $j$th unit vector $u$ (with $u_1 = 0$ if $1 \neq j$ and $u_j = 1$) satisfies the inequalities

$$Ax \geqslant 0, \quad cx \leqslant 0, \quad x \geqslant 0.$$

This means that the assumption of Theorem 5 are satisfied—see above the alternate proof of Corollary 7—which yields the upper semicontinuity of $C$ on $T$ and we then appeal to Proposition 3 to obtain the continuity of $D$. □

There are a number of ways, all equivalent, to express the conditions of Proposition 8. For example: $R(t) = \{0\}$ if and only if

$$0 \neq \hat{x} \in \{x \geqslant 0 \mid Ax \geqslant 0\} \quad \text{implies} \quad c\hat{x} > 0, \tag{9}$$

or still

$$c \in \operatorname{int} \operatorname{pos}(A^{\mathrm{T}}, I) \tag{9′}$$

Similarly $S(t) = \{0\}$ if and only if

$$0 \neq \hat{y} \in \{y \geqslant 0 \mid yA \leqslant 0\} \quad \text{implies} \quad \hat{y}b > 0, \tag{10}$$

or still

$$b \in \operatorname{int} \operatorname{pos}(A, -I) \tag{10′}$$

**11. Corollary.** *Suppose that for all $t \in T \subset \mathcal{T}$, $K(t)$ is bounded, then $D$ is continuous on $T$. Similarly if all $t \in T \subset \mathcal{T}$, $D(t)$ is bounded then $K$ is continuous on $T$.*

**Proof.** The convex polyhedron $K(t)$ is bounded if and only if $\{x \mid Ax \geqslant b, x \geqslant 0\} = \{0\}$. This implies that $R(t) = \{0\}$ with $R(t)$ as defined in Proposition 8. The lower semicontinuity of $D$ now follows from Proposition 8. One argues similarly for $K$ using this time the boundedness of $D$ to conclude that $S(t) = \{0\}$. □

**12. Corollary.** *Suppose that for all $t \in T \subset \mathcal{T}$, either $c > 0$ or all columns $A^j$ of $A$ are nonpositive and nonzero. Then $D$ is continuous on $T$. Similarly if for all $t \in T$, either*

*$b<0$ or all rows $A_i$ of $A$ are nonnegative and nonzero, then $K$ is continuous on $T$. Hence, if for all $t\in T$, $A<0$ and $b<0$ or $A>0$ and $c>0$, then $Q$ is continuous on $T$.*

**Proof.** If $A^j\leq 0$ and $A^j\neq 0$ then $\{x\geq 0\mid Ax\geq 0\}=\{0\}$ and thus $K(t)$ is bounded for all $t\in T$. The lower semicontinuity of $D$ then follows from Corollary 11. If $c>0$ then for every $0\neq x\in\mathbb{R}^n_+$, $cx>0$ and from (9) it follows that $R(t)=\{0\}$ and in turn the lower semicontinuity of $D$ follows from Proposition 8. Again, the lower semicontinuity of $K$ is obtained by arguing similarly using $A_i\geq 0$ and $b<0$. The assertions about $Q$ now follow from the above using, naturally, Theorem 2. □

There is another way to prove Corollary 12, which also shows how to generalize it. The alternative proof of Corollary 7 and the proof of Proposition 8 show that many of the sufficient conditions for the lower semicontinuity of $D$ boil down to checking if

$$C(t):=\text{pos}\begin{pmatrix} A & -I & 0\\ c & 0 & 1\end{pmatrix}$$

is pointed. The last $m+1$ columns $\left(\begin{smallmatrix}-I & 0\\ 0 & 1\end{smallmatrix}\right)$ of the matrix that generate $C(t)$ determine an orthant and thus $C(t)$ will certainly be pointed if the remaining columns $\{\left(\begin{smallmatrix}A^j\\ c_j\end{smallmatrix}\right), j=1,\ldots,n\}$ belong to this orthant or are such that when added to $\left(\begin{smallmatrix}-I & 0\\ 0 & 1\end{smallmatrix}\right)$ they keep the cone pointed. Sufficient conditions of this type are provided by Corollary 12, but they clearly do not exhaust the realm of possibilities. For example, if there exists a vector $\pi\in\mathbb{R}^m$ with $\pi_i>0$ for all $i=1,\ldots,m$ such that $\pi A<c$ then $C(t)$ is pointed, since then all the columns of $\left(\begin{smallmatrix}A & -I & 0\\ c & 0 & 1\end{smallmatrix}\right)$ have strictly positive inner product with the vector $(-\pi,1)\in\mathbb{R}^{m+1}$. Here we are naturally very close to the conditions of Corollary 7 and Proposition 8.

The diagram of Table 1 summarizes the results about the continuity of $D$. There is of course a mirror image diagram for the multifunction $K$.

As indicated in the beginning of this short note one of our goals was to exhibit the connections between the various conditions that have appeared in the literature. We conclude with the pertinent references as well as some comments about the general continuity results for infimal functions of optimization problems depending on parameters.

Theorem 2 and Proposition 3 are taken from Wets [17]. Proposition 6 and Corollary 7 are due to Dantzig, Folkman and Shapiro [5], here they are obtained as application of a theorem in Walkup and Wets [13], see the Appendix for further comments. Klatte [8] rewords this result by replacing the conditions $t\in\mathcal{T}$, i.e., $K(t)\neq\emptyset$, and

$$K(t)\subset\{x\mid A_ix=b_i,\ i\in I\}\cap\{x\mid x_j=0,\ j\in J\}$$

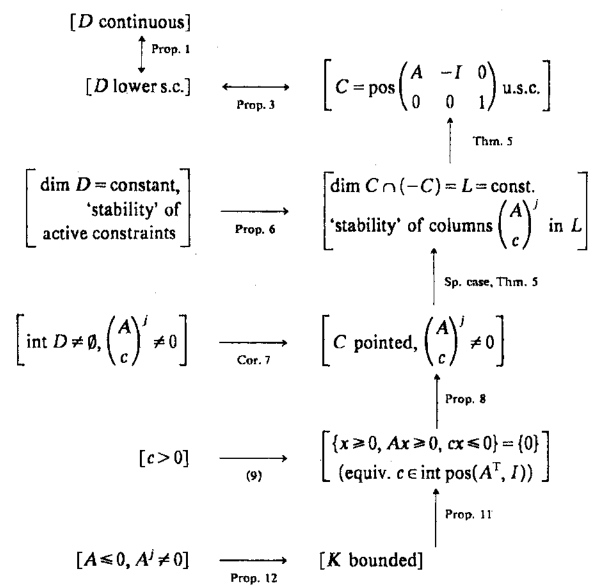
with

$$\text{rint } K(t)\subset\{x\mid A_ix=b_i,\ i\in I\}\cap\{x\mid x_j=0,\ j\in J\}\cap\{x\mid A_ix>b_i,\ i\notin I\}$$
$$\cap\{x\mid x_j>0,\ j\notin J\},$$



Table 1

Summary of results




where rint stands for relative interior. Klatte's condition clearly implies the other one since cl rint $K(t)=K(t)$ and for convex sets, $\text{cl}(C\cap D)=\text{cl } C\cap\text{cl } D$ whenever $\text{rint } D\cap\text{rint } C\neq\emptyset$. Bereanu [2, Theorem 2.2] proves the continuity of $Q$ under conditions (9) and (10). Version (9′) and (10′) of these conditions are those of Robinson [10], when applied to linear programs of the type considered here. He also shows that they are equivalent to having the sets of optimal solutions of the primal and dual programs bounded and these are the conditions used by Martin [9] to obtain the continuity of $Q$. This can be argued as follows. Since $t\in\mathcal{T}$, $Q(t)$ is finite. Thus the set of optimal solutions is determined by the inequalities

$$\text{argmin } Q(t)=\{x\mid Ax\geq b,\ x\geq 0,\ cx\leq Q(t)\},$$

and this polyhedral set is bounded if and only if $R(t)=\{0\}$, with $R(t)$ as defined in Proposition 8, or equivalently if (9) or (9′) holds.

As far as general continuity results for the infimal functions of parameterized optimization problems, the theory of epi-convergence [12, Theorem 3.37] when applied to this case, provides us with the next result that sharpens results of Hogan [7] and Bank, Guddat, Klatte, Kummer and Tammer [1, Theorem 4.3.4].

**13. Theorem.** *Let $T \subset \mathcal{T}$. Suppose $K$ is lower semicontinuous on $T$, then $Q$ is upper semicontinuous on $T$. If there exists a compact set $C \subset \mathbb{R}^n$ such that for all $t \in T$*

$$C \cap \operatorname{argmin} Q(t) \neq \emptyset,$$

*then $Q$ is lower semicontinuous.*

The first hypothesis is exactly that used in Theorem 2. The second one means that we could, for theoretical purposes, replace the definition of $Q$ by

$$Q(t) = \inf\{cx \mid Ax \geq b,\ x \geq 0,\ x \in P\}$$

where $P$ is a bounded polyhedron containing $C$. Thus the set of feasible solutions of this modified problem

$$K'(t) = \{x \mid\ Ax \geq b,\ x \geq 0,\ x \in P\}$$

is (uniformly) bounded on $T$. And this is stronger than necessary since to prove lower semicontinuity of $D$, which in turn yields the lower semicontinuity of $Q$ (cf. Theorem 2), all that is needed is to have $K$ bounded on $T$, see Corollary 11. Note that Theorem 3.37 of [12] actually makes a stronger assertion involving $\varepsilon$-optimal solutions, that are of limited interest in linear programming.

We could also rephrase our results in terms of variational systems [12] that provide a general framework for the study of parametric optimization problems. We would work with the variational system

$$f_t(x) := \begin{cases} cx & \text{if } Ax \geq b,\ x \geq 0, \\ +\infty & \text{otherwise,} \end{cases}$$

and we would be concerned with the epi-semicontinuity of this variational system and its dual

$$g_t(y) := \begin{cases} yb & \text{if } yA \leq c,\ y \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$$

By *upper epi-semicontinuity* of the variational system $\{f_t,\ t \in \mathbb{R}^N\}$, one means that the multifunction

$$t \mapsto \operatorname{epi} f_t = \{(x, \alpha) \mid \alpha \geq cx,\ Ax \geq b,\ x \geq 0\}$$

from $\mathbb{R}^N$ into $\mathbb{R}^{n+1}$ is upper semicontinuous; *lower episemicontinuity* is defined similarly. We could then use the results of [12, Section 3] to obtain conditions for the continuity of $Q$, since

$$Q(t) = \inf f_t = \sup g_t.$$

In general, there is much to be gained from such an approach because perturbations of the constraints and the objective are blended together. However, here we would reproduce the earlier results since upper and lower epi-semicontinuity of $f_t$ corresponds to upper and lower semicontinuity of $K$ as can easily be checked.



If the entries of $A$ do not vary, then $Q$ is always continuous. In particular we obtain:

**14. Theorem.** *Suppose that for all $t \in T \subset \mathcal{T}$ the matrix $A$ is constant. Then $Q$ is continuous on $T$.*

**Proof.** In this case, the multifunctions $K$ and $D$ are not only continuous on $T$ but in fact Lipschitz continuous on $T$. Indeed, with $A$ fixed (constant), let

$$P := \{(x, b) \in \mathbb{R}^n \times \mathbb{R}^m \mid Ax - Ib \geq 0,\ x \geq 0\}.$$

This is a polyhedral cone. Then

$$K(t) = \{x \mid (x, b) \in P\} = \pi_1[P \cap \pi_2^{-1}(b)]$$

with $\pi_1$ and $\pi_2$ the canonical projections from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. We now appeal to [15, Theorem 1] that guarantees the Lipschitz continuity of

$$b \mapsto P \cap \pi_2^{-1}(b).$$

(A similar argument was used in [11, Proposition 1] to obtain the local upper Lipschitz continuity of polyhedral multifunctions.) Of course the same applies to $D$ (with $A$ fixed). The continuity of $Q$ results again from Theorem 2. □

In fact in this case $Q$ possesses stronger continuity properties. It is well-known [16, Basis Decomposition Theorem] that $Q$ is a piecewise linear convex function of $b$ for all fixed $(c, A)$, and that $Q$ is a piecewise concave function of $c$ for fixed $(A, b^{\mathrm{T}})$. Thus in each one of these two cases $Q$ is actually Lipschitz. This implies that with $A$ constant but $c$ and $b$ varying, $Q$ is Lipschitzian on compact subsets of $\mathcal{T}$. In general, however, it is not Lipschitz on $\mathcal{T}$. For example, with $A = I$ (constant) consider the function

$$Q(t) = \inf\{cx \mid Ix \geq b,\ x \geq 0\}.$$

Then, with $m = n$,

$$\mathcal{T} = \{(c, A, b) \mid c \in \mathbb{R}^n_+,\ A = I,\ b \in \mathbb{R}^m\},$$

and for $t \in \mathcal{T}$,

$$Q(t) = \sum_{j=1}^{n} c_j \max(0, b_j),$$

which of course is not Lipschitz since for $b \in \mathbb{R}^m_+$, $Q$ takes the form

$$Q(t) = cb.$$

To conclude these paranthetical remarks, let us also record that if only $b$ varies, then not only is $Q$ Lipschitz continuous but there exists a continuous function

$$t \mapsto x^*(t) : \mathcal{T} \to \mathbb{R}^n \ldots$$

such that for all $t$, $x^*(t) \in K(t)$ and $cx^*(t) = Q(T)$ [14, Theorem], [4]. If only $c$ varies a similar statement can be made, viz., there exists $y^*(\cdot): \mathcal{T} \to \mathbb{R}^m$, continuous such that $y^*(t) \in D(t)$ and $y^*(t)b = Q(t)$.

**Remarks. 1.** Robinson [10] formulates his pair of dual linear programs to take into account problems involving both equalities and constraints. For such cases there are also appropriate versions of Theorem 2 and Propositions 1 and 3. For example, if

$$Q(t) = \inf_{x \in \mathbb{R}^n} [cx \mid Ax = b, x \geq 0]$$

then we should study the continuity of the maps

$$t \mapsto \operatorname{pos}\begin{pmatrix} A & 0 \\ c & 1 \end{pmatrix} \quad \text{and} \quad t \mapsto \operatorname{pos}\begin{pmatrix} A^{\mathrm{T}} & -A^{\mathrm{T}} & I & 0 \\ b^{\mathrm{T}} & -b^{\mathrm{T}} & 0 & -1 \end{pmatrix}$$

2. The proof of Theorem 5, and by implication that of Proposition 6, relies on long and subtle arguments from the theory of linear inequalities. An alternate proof, that relies on arguments from the theory of multifunctions (using the concepts of strong lower semicontinuity [1, Lemma 2.2.5.] and of index stability sets for linear systems [8]), appears in D. Klatte thesis: "Untersuchungen zut lokalen Stabilität konvexer parametrischer Optimierungsprobleme", Humbold-Universität zu Berlin, 1977. In [8] this proof has been adopted to polyhedral-valued multifunctions.

## Appendix

Dantzig, Folkman and Shapiro actually prove a sharper result than Proposition 6 that does not fit neatly in the pattern laid out in the summary of the results. For the sake of completeness we state and prove this result. They key step for this new (and simple) proof is due to Dr. Duncan Martin, CSIR National Research Institute for Mathematical Sciences, Pretoria, South Africa.

**Theorem** [5, Theorem II.2.2]. *Let $\{t^\nu = (A^\nu, b^\nu)\}_{\nu=1}^{\infty}$ be a sequence converging to $t = (A, b)$ with*

$$K(t^\nu) = \{x \mid A^\nu x \geq b^\nu\}$$

*and*

$$K(t) = \{x \mid Ax \geq b\}$$

*and let*

$$I := \{i \mid A_i x = b_i \text{ for all } x \in K(T)\}.$$

*Suppose that*

$$\limsup_{\nu \to \infty} \operatorname{rank}(A_i^\nu, i \in I) \leq \operatorname{rank}(A_i, i \in I).$$



*Then either*

$$\lim_{\nu \to \infty} K(t^\nu) = K(t),$$

*i.e., $K$ is lower semicontinuous on $T = \{t^1, t^2, \ldots, t\}$ at $t$, or $\liminf_{\nu \to \infty} K(t^\nu)$ is empty.*

**Proof.** It really suffices to prove the theorem in the case when

A.1. $\liminf_{\nu \to \infty} K(t^\nu)$ is nonempty.

A.2. $I = \{1, \ldots, m\}$.

To see that it suffices to work with A.2, observe that

$$K(t) = K_I(t) \cap K_{NI}(t)$$

where

$$K_I(t) = \{x \mid A_i x = b_i, i \in I\} \quad \text{and} \quad K_{NI}(t) = \{x \mid A_i x \geq b_i, i \notin I\}.$$

Then $\operatorname{int} K_{NI}(t) \neq \emptyset$ and, from the definition of $I$, it follows that $((A_i, b_i) \neq 0, i \notin I)$. This implies that for $\nu$ sufficiently large, $\operatorname{int} K_{NI}(t^\nu)$ is nonempty; note that $\bar{x} \in \operatorname{int} K_{NI}(t)$ implies that $\bar{x} \in \operatorname{int} K_{NI}(t^\nu)$ for $t^\nu$ sufficiently close to $t$. Hence by Corollary 7, $K_{NI}$ is lower semicontinuous at $t$ on $T = \{t^1, t^2, \ldots, t\}$, i.e., $\liminf_{\nu \to \infty} K_{NI}(t^\nu) = K_{NI}(t)$. Moreover since

$$K_I(t) \cap \operatorname{int} K_{NI}(t) \neq \emptyset,$$

and the sets $K_{NI}^\nu(t^\nu)$ and $K_I(t^\nu)$ are convex, we have

$$\liminf_{\nu \to \infty}(K_{NI}(t^\nu) \cap K_I(t^\nu)) = \liminf_{\nu \to \infty} K_{NI}(t^\nu) \cap \liminf_{\nu \to \infty} K_I(t^\nu)$$

$$= K_{NI}(t) \cap K_I(t),$$

provided we can show that

$$K_I(t) = \liminf_{\nu \to \infty} K_I(t^\nu)$$

when $\liminf_{\nu \to \infty} K_I(t^\nu)$ is nonempty.

So, we now also accept A.2. From the hypotheses, in particular: $\limsup_{\nu \to \infty} \operatorname{rank} A^\nu \leq \rho := \operatorname{rank} A$, it follows for $\nu$ sufficiently large rank that $A^\nu = \rho$. We may as well assume that the first $\rho$ rows of $A$—and thus also of the $A^\nu$ for $\nu$ sufficiently large—are linearly independent. Without loss of generality assume that this holds for all $\nu$, and that the columns of $A$ and of the $A^\nu$ have been reordered so that the matrices can be partitioned in the form:

$$A = \begin{bmatrix} L & C \\ M & D \end{bmatrix}, \quad A^\nu = \begin{bmatrix} L_\nu & C_\nu \\ M_\nu & D_\nu \end{bmatrix}$$

with $L$ and $L_\nu$ invertible $\rho \times \rho$-matrices. Since $A = \lim A^\nu$, it follows that

$$L = \lim_{\nu \to \infty} L_\nu \quad \text{and} \quad C = \lim_{\nu \to \infty} C_\nu,$$

which implies that

$$L^{-1}C = \lim (L_\nu)^{-1} C_\nu.$$

Now consider any

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{such that } z \in \ker A,\ z_1 \in \mathbb{R}^p,\ z_2 \in \mathbb{R}^{n-p}.$$

This means that

$$z_1 = -L^{-1}Cz_2.$$

Then, with

$$z^\nu = \begin{bmatrix} -L_\nu^{-1}C_\nu z_2 \\ z_2 \end{bmatrix},$$

we have that $A^\nu z^\nu = 0$ and $\lim_{\nu\to\infty} z^\nu = z$. Hence $z \in \liminf_{\nu\to\infty} \ker A^\nu$. Thus so far we have shown that

A.3. $\ker A \subset \liminf_{\nu\to\infty} \ker A^\nu$.

To complete the proof note that by A.1 there exists a convergent sequence $\{z^\nu\}_{\nu=1}^\infty$ to some $z$, with $z^\nu$ in $K(t^\nu)$. Then by Proposition 1, $z$ is in $K(t)$ and by A.2 we have $Az = b$. Consider any point $x$ in $K(t)$, again by A.2; it follows that

$$v = x - z \in \ker A.$$

By A.3, there exists a sequence $\{v^\nu\}_{\nu=1}^\infty$ converging to $v$ with $v^\nu \in \ker A^\nu$. Hence with

$$x^\nu = z^\nu + v^\nu$$

we have $A^\nu x^\nu = A^\nu z^\nu \geqslant b^\nu$, so that $x^\nu \in K(t^\nu)$ while

$$\lim x^\nu = z + v = x$$

which shows that $x \in \liminf_{\nu\to\infty} K(t^\nu)$. □

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