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Algorithms for Frames and Lineality Spaces of Cones*

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A frame of a cone C is a minimal set of generators, and the lineality space L of C is the greatest linear subspace contained in C. Algorithms are described for determining a frame and the lineality space of a cone $C(S)$ spanned by a finite set S. These algorithms can be used for determining the vertices, edges, and other faces of low dimension of the convex hull of a finite set $H(S)$. All algorithms are based on the simplex method of linear programming. The problem of finding the lineality space can be successively reduced to problems in spaces of lower dimensions.

Key Words: Algorithm, cone, convex hull, face, frame, lineality space, linear programming.

Introduction

Let

$$S = \{A_1, \ldots, A_n\}$$

be a family of points in R^m. We denote by $L(S)$ its *linear hull*, by $H(S)$ its *convex hull*, and by $C(S)$ its *conical* or *positive hull*, i.e., the convex polyhedral cone expressible as the set of all nonnegative weighted sums of elements of S. We use the conventions $L(\phi) = C(\phi) = \{0\}$.

A subfamily $T \subseteq S$ is called a *frame* [4][1] of $C(S)$ if $C(T) = C(S)$ but $C(T - \{A_j\}) \neq C(T)$ for each $A_j \in T$. The greatest linear subspace contained in $C(S)$ is called its *lineality space* [5]. Two main problems are, given S, to find a frame of $C(S)$ and to determine the lineality space of $C(S)$. These two problems are closely related.

Several important problems are equivalent to or included in these two main problems. Consider, for instance, the system of linear inequalities

$$A_j^T X \leq b_j, \qquad j = 1, \ldots, n,$$

for $X \in R^m$. Removing redundant constraints amounts to finding a frame for the cone spanned by the vectors $\hat{A}_j = \begin{pmatrix} A_j \\ b_j \end{pmatrix}$, $j = 1, \ldots, n$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Determining the dimension of a polyhedron given by linear inequalities, can be reduced to determining the dimension of a polyhedral cone $C = \{U | A^T U \leq 0\}$ (see [10]). As to the latter problem, it suffices to note that the linear hull $L(C)$ is the orthogonal complement of the lineality space of the polar cone $C^p = C\{A_1, \ldots, A_n\}$, where $A_1, \ldots, A_n$ are the columns of A. This problem also arises if the uniqueness of an optimal solution to a linear program is to be established in the presence of degeneracy.

Clearly, the problem of finding the vertices of a convex hull $H(S)$ can be solved by finding the frame of a suitable cone. It should not be confused with the problem of finding the vertices of a polyhedron defined by linear inequalities. The latter problem corresponds to finding the *facets*, that is, the proper faces of highest dimension, of some $H(S)$. Even a moderate number of inequalities is apt to generate a huge number of vertices [6, 9], and by the same token, the number of facets of $H(S)$ may be extremely large compared to the cardinality of S. Thus finding the facets of $H(S)$ is inherently more difficult than finding the vertices of $H(S)$.

The problem of finding the *edges* of $H(S)$, and other facets of low dimension, may still be expected to be essentially easier than to find the facets of $H(S)$. Indeed, one is tempted to conjecture that an efficient determination of the facets requires prior determination of the lower dimensional faces.

Note that finding the edges of $H(S)$ could be accomplished by finding the vertices of $H(S_2)$, where

$$S_2 := \{A_{(ij)} = \tfrac{1}{2}(A_i + A_j) | i < j\}.^2$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

[2] The symbol ":=" stands for "is defined by."