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Towards an Algebraic Characterization of Convex Polyhedral Cones

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Summary. It is shown that the theory of positive linear independence and the properties of Jordan-equivalent matrices can be utilized effectively in order to obtain an algebraic characterization of a face structure of convex polyhedral cones.

1. Introduction

To each subset $S \subset \mathfrak{R}^m$ of cardinality n $(n \geqq 1)$ corresponds in a natural way an $m \times n$ matrix A. We assume that $\{0\} \nsubseteq S$, i.e. A does not contain zero columns. The linear hull and the positive hull of S will be denoted respectively by

$$\operatorname{lin} A = \{t \mid t = A\,x,\ x \in \mathfrak{R}^n\} \quad \text{and} \quad \operatorname{pos} A = \{t \mid \mathrm{t} = A\,x,\ x \in \mathfrak{R}^n_{\oplus}\},$$

where $\mathfrak{R}^n_{\oplus}$ stands for the nonnegative orthant of $\mathfrak{R}^n$. In matters pertaining to the theory of positive linear independence, we follow the terminology and notations of REAY [6]. In relation to convex cones and polytopes, we follow the terminology of KLEE [4]. In particular, a *k-face* of a convex cone C is a face of dimension k. A *simplicial k-face* is a k-face which contains exactly k extreme rays. An $(m-1)$-face of an m-cone C is called a *facet*. Furthermore we denote by A_j and ${}_iA$ the j^{th} column and the i^{th} row, respectively, of a matrix A.

If $\bar{A}$ and A are *Jordan-equivalent*, i.e. the rows of $\bar{A}$ and A generate the same linear subspace[1], then $C = \operatorname{pos} A$ and $\bar{C} = \operatorname{pos} \bar{A}$ have identical face structure. In particular, if C is *pointed*, i.e. of *lineality dimension* [3] zero, then so is $\bar{C}$, and if $\operatorname{pos}(A_{j_1}, \ldots, A_{j_l})$ is a k-face of C, then $\operatorname{pos}(\bar{A}_{j_1}, \ldots, \bar{A}_{j_l})$ is a k-face of $\bar{C}$. A matrix A is in *canonical form* if some of its columns form an identity submatrix, the *basis*; all the other columns are *nonbasic columns*. By $\mathscr{A}$ we denote the class of all matrices in canonical form which are Jordan-equivalent to A, and all row and column permutations of such matrices. By $\bar{A}$ we always denote a member of $\mathscr{A}$. A matrix is *lexico-positive* (*-negative*) if all its columns are lexico-positive (-negative).

A minimal set of generators A_j of $C = \operatorname{pos}(A_1, \ldots, A_n)$ is called a *frame* [1]. A frame of a face F of C *determines* F. A linearly independent subset T of a k-face F *subdetermines* F if $C \cap \operatorname{lin} T = F$. If, for instance, $A = (A_1, A_2, A_3)$ with

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[1] Note that A and $\bar{A}$ need not have the same number of rows. For instance, the deletion of rows which are identically zero will produce a Jordan-equivalent matrix.

$A_1=\begin{pmatrix}1\\0\end{pmatrix}$, $A_2=\begin{pmatrix}0\\1\end{pmatrix}$, $A_3=\begin{pmatrix}-1\\0\end{pmatrix}$, then A_1, A_2 subdetermine (but do not determine) $C=\operatorname{pos} A$.

For C a pointed m-cone, an $(m-k)$-face G is said to be *complementary* to a k-face F if $F\cap G=\{0\}$ and the linear hull of F and G has the same dimension as C.

(1) Lemma. If C is a pointed cone, then (i) every extreme ray of C has a complementary facet, and (ii) every facet of C has a complementary extreme ray.

Proof. (i) Suppose R is an extreme ray of C without a complementary facet. Then R would be contained in every facet, whence in the lineality space of C. But this contradicts the fact that C is pointed and therefore of lineality dimension zero. (ii) Suppose F is a facet of C without a complementary extreme ray. Then every extreme ray of C lies in F, contradicting $C\neq F$.

Note that this lemma is a special case of a theorem by EGGLESTON, GRÜNBAUM and KLEE [2] which, when adapted to cones, states that every k-face of a pointed cone has a complementary $(m-k)$-face.

2. Convex Cones and Equivalent Matrices

(2) Proposition. $C=\operatorname{pos} A\neq\{0\}$ is a pointed cone if and only if there is a lexico-positive matrix $\bar{A}$ in $\mathscr{A}$.

Proof. Let $\bar{A}$ be lexico-positive. The lineality dimension of $\bar{C}$, and therefore of C, is different from zero if and only if there exists $\bar{A}_j$ such that $-\bar{A}_j\in\bar{C}$ [3 and 5]. But $\bar{A}$ is lexico-positive; thus $\operatorname{pos}(\bar{A}_1,\ldots,\bar{A}_n)\nsupseteq\{-\bar{A}_j\}$ for $j=1,\ldots,n$ since every $-\bar{A}_j$ is lexico-negative. This completes the proof in one direction. If C is a pointed m-cone ($m>0$), then at least one of its generators, say A_1, is extreme. By lemma (1), $\operatorname{pos} A_1$ possesses a complementary facet F, say $F=\operatorname{pos}(A_2,\ldots,A_m,A_{m+1},\ldots,A_{m+t})$ with $A_2,\ldots,A_m$ subdetermining F. Construct A' such that the columns $A'_1,\ldots,A'_m$ (in that order) constitute the basis. Since $\operatorname{lin}(A'_2,\ldots,A'_m)=\{x\mid x_1=0\}$ is a supporting hyperplane of C', all entries in row ${}_1A'$ must be of the same sign. Therefore ${}_1A'\geqq 0$, since $a'_{11}=1$. All nonbasic columns with positive entry in the first row are trivially lexico-positive, while those with zero entry belong to $\operatorname{lin}(A'_2,\ldots,A'_m)$ and thus to F', which is itself a pointed $(m-1)$-cone. We use the same construction to make $a''_{2j}\geqq 0$ for $\operatorname{pos} A''_j\subseteq F''$, viz. let $\operatorname{pos} A'_2$ be an extreme ray of F', select a complementary face in F' and a set of $(m-2)$ columns subdetermining this subfacet of C'; construct A'' such that A''_1, A''_2, and these $(m-2)$ columns constitute a basis. This keeps ${}_1A'$ unchanged. The proof is then easily completed by recursion.

It is possible to strengthen somewhat the "only if" direction of Proposition (2). In fact, the proof of the proposition yields:

(3) Remark. If $C=\operatorname{pos} A\neq\{0\}$ is a pointed cone, then there is a lexico-positive matrix $\bar{A}$ in $\mathscr{A}$ such that its basic columns correspond to extreme rays of C.

(4) Theorem. F is a k-face of a pointed m-cone $C=\operatorname{pos} A$ $(0<k<m)$ if and only if there exists in $\mathscr{A}$ a matrix $\bar{A}$ of the form

$$\bar{A}=\begin{pmatrix} I^{(m-k)} & 0 & 0 & V\\ 0 & I^{(k)} & U & W\end{pmatrix},$$

where $I^{(k)}$ is an identity matrix of order k, U and V are lexico-positive, and those columns in A which correspond to $\begin{pmatrix} 0 \\ I^{(k)} \end{pmatrix}$ subdetermine F. (The columns of $\begin{pmatrix} 0 \\ U \end{pmatrix}$ characterize the remaining generators of F.)

Proof. F is contained in some facet $F_{(m-1)}$. By Lemma (1) there exists an extreme ray complementary to $F_{(m-1)}$, say $\operatorname{pos} A_1$. We obtain A' in $\mathscr{A}$ by selecting a basis formed by A_1, and $(m-1)$ extreme generators of $F_{(m-1)}$. After an appropriate arrangement of columns and rows, we can write

$$A' = \left[\begin{array}{c|c|c|c} 1 & & 0 \ldots 0 & + \cdots + \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & * & * \end{array}\right],$$

where A_1' corresponds to A_1, and the columns $A_2', \ldots, A_m'$ subdetermine $F'_{(m-1)}$. The arguments proving that ${}_1A' \geqq 0$ are identical to those given in Proposition (2). If ${}_1A_j' = 0$, then $A_j' \in \operatorname{lin}(A_2', \ldots, A_m')$ and therefore $A_j' \in F'_{(m-1)}$. If $F' = F'_{(m-1)}$, i.e. if $k = m-1$, the construction terminates. Otherwise there exists a facet $F'_{(m-2)}$ of $F'_{(m-1)}$ containing the k-face F'. By Jordan equivalence, the cone C' spanned by the columns of A' is pointed since C is pointed. The facet $F'_{(m-1)}$ is pointed since it is a subcone of the pointed cone C'. Hence Lemma (1) applies, yielding the existence of an extreme ray, say $\operatorname{pos} A_2'$, complementary to $F'_{(m-2)}$ in $F'_{(m-1)}$. By selecting a basis consisting of A_1', A_2', and $(m-2)$ extreme generators of $F'_{(m-2)}$, we obtain

$$A'' = \left[\begin{array}{c|c|c|c|c} 1 & & 0 \ldots 0 & 0 \ldots 0 & + \cdots + \\ \hline \;\; 1 & & 0 \ldots 0 & + \cdots + & \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & * & * & * \end{array}\right]$$

where A_1'', A_2'' correspond to A_1', A_2', and the columns $A_3'', \ldots, A_m''$ subdetermine $F''_{(m-2)}$. We note that ${}_1A'' = {}_1A'$, and that ${}_2A_j'' \geqq 0$ if $A_j'' \in F''_{(m-1)}$. The columns whose two first components vanish belong to $F''_{(m-2)}$. If $k < m-2$, the construction continues until it reaches

$$A^{(m-k)} = \left[\begin{array}{c|c|c|c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & & 0 & \begin{matrix} & & + \cdots + \\ & + \cdots + & \\ + \cdots + & & \end{matrix} \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & * & * \end{array}\right] = \begin{pmatrix} I^{(m-k)} & 0 & 0 & V \\ 0 & \tilde{I}^{(k)} & \tilde{U} & \tilde{W} \end{pmatrix},$$

where V is lexico-positive, the columns corresponding to $\begin{pmatrix} 0 \\ \tilde{I}^{(k)} \end{pmatrix}$ subdetermine F and all columns whose $(m-k)$ first components are zero characterize generators in F.

Any change of basis involving only columns in $\begin{pmatrix} 0 & 0 \\ \tilde{I}^{(k)} & \tilde{U} \end{pmatrix}$ will not affect the first $(m-k)$ rows of $A^{(m-k)}$. We proceed to use this fact to transform $A^{(m-k)}$ into a lexico-positive $\bar{A}$ with the desired properties. The columns in $\begin{pmatrix} 0 & 0 \\ \tilde{I}^{(k)} & \tilde{U} \end{pmatrix}$ determine a subcone of a pointed cone, which is therefore pointed itself. The same is true, because of Jordan-equivalence, for the cone spanned in $\mathfrak{R}^k$ by the columns of $(\tilde{I}^{(k)} \tilde{U})$. By Proposition (2), there exists a lexico-positive matrix $(\tilde{I}^{(k)} \tilde{U})$ which — after a possible rearrangement of columns — is Jordan-equivalent to $(I^{(k)}\ U)$. The corresponding basis change yields A, completing the proof in one direction.

If there exists $\bar{A}$ in $\mathscr{A}$ as above, $C = \text{pos}\, A$ is a pointed cone by Proposition (2). Moreover, the columns of $\begin{pmatrix} 0 & 0 \\ I^{(k)} & U \end{pmatrix}$ determine a pointed subcone $\bar{F}$ of $\bar{C} = \text{pos}\, \bar{A}$. $\bar{F}$ is a k-face of $\bar{C}$, and therefore F is a k-face of C, since there exists $\varepsilon > 0$ such that the vector $(\varepsilon, \varepsilon^2, \ldots, \varepsilon^{m-k}, 0, \ldots, 0)$ is the normal of a supporting hyperplane $\bar{P}$ of $\bar{C}$ with $\bar{P} \cap \bar{C} = F$.

In view of Remark (3) and the particular use of Proposition (2) made in the proof of Theorem (4), it is possible to strengthen somewhat the "only if" direction of this theorem.

(5) **Remark.** If F is a k-face of a pointed m-cone $C = \text{pos}\, A$, then there exists $\bar{A}$ in $\mathscr{A}$ as in (4) and such that the basic columns in $\begin{pmatrix} 0 \\ I^{(k)} \end{pmatrix}$ correspond to columns of A which determine extreme rays of F.

(6) **Corollary.** Let $C = \text{pos}\, A$ be a pointed m-cone. Then $A_1, \ldots, A_k$ $(k < m)$ determine a simplicial k-face of C if and only if there exists $\bar{A}$ in $\mathscr{A}$ as in (4) where $I^{(k)}$ is an identity matrix of order k which corresponds to the columns $\bar{A}_1, \ldots, \bar{A}_k$, V is lexico-positive, and U is a nonnegative matrix.

(7) **Corollary.** Let $C = \text{pos}\, A$ be a pointed cone. Then $\text{pos}\, A_1$ $(\text{pos}\,(A_1, A_2))$ is an extreme ray of C (is a 2-face of C) if and only if there exists $\bar{A}$ as in (6) with $k = 1$ $(k = 2)$.

Proof. It suffices to observe that every 1- or 2-face of a pointed cone is a simplicial face.

3. Reay Matrices and Positive Bases

(8) **Definition.** We call an $m \times s$ matrix $R \leqq 0$ a *Reay matrix* if it is upper block triangular, $R = \begin{bmatrix} D_1 & & * \\ & D_2 & \\ & & \ddots \\ 0 & & D_s \end{bmatrix}$, such that each D_i is a $t_i \times 1$ negative matrix with $t_i \geqq t_{i+1} \geqq 0$ for $i = 1, \ldots, s-1$ and $\sum t_i = m$.

(9) **Proposition** (REAY [6]). If there exists $\bar{A}$ in $\mathscr{A}$ such that $\bar{A} = (I, R)$ where R is an $m \times s$ Reay matrix with $1 \leqq s \leqq m$, then the matrix A determines a positive basis [5] for $\mathfrak{R}^m$.

Proof. Let $u_j = \sum_1^j t_i$ with $u_0 = 0$. If A determines a positive basis for $\mathfrak{R}^m$, by [7, Theorem 2], there exists a decomposition of the columns of A into s subsets $A^{(1)}, \ldots, A^{(s)}$ of decreasing cardinality where card $A^{(i)} = t_i + 1$, $i = 1, \ldots, s$, and $A^{(i)} = A_{u(i-1)+1}, \ldots, A_{u_i}, A_{m+i}$, such that $\bigcup_{k=1}^{s} A^{(k)} = A$ and $\operatorname{pos}(A^{(1)} \cup \ldots \cup A^{(j)})$ is a linear subspace of $\mathfrak{R}^m$ of dimension u_j. We can choose $v_j \leqq u_j$ and renumber such that

$$\dim \operatorname{lin}\left(A^{(1)} \cup \ldots \cup A^{(j-1)} \cup \{A_{u(j-1)+1}, \ldots, A_{v_j}\}\right) = u_j$$

and hence so that

$$\dim \operatorname{lin}\left(\{A_1, \ldots, A_{v_1}\} \cup \{A_{u_1+1}, \ldots, A_{v_2}\} \cup \ldots \cup \{A_{u(j-1)+1}, \ldots, A_{v_j}\}\right) = u_j.$$

The proof is completed by renumbering so that for $j = s$ the vectors in the above union become $A_1, \ldots, A_m$ and by letting $(\bar{A}_1, \ldots, \bar{A}_m)$ be the basis of A.

Note that for the "if" direction, the fact that the size of the matrices D_i decreases is not necessary. From Propositions (1) and (6) we obtain:

(10) Corollary. Let $C = \operatorname{pos} A$ be an m-cone. Then $\operatorname{lin}(A_1, \ldots, A_k)$ is the lineality space of C if and only if there exists $\bar{A}$ in $\mathscr{A}$ such that $\bar{A} = \begin{pmatrix} I^{(k)} & 0 & U & W \\ 0 & I^{(m-k)} & 0 & V \end{pmatrix}$ where $I^{(k)}$ is an identity matrix of order k which corresponds to the columns $A_1, \ldots, A_k$, U contains a $k \times s$ Reay matrix where $1 \leqq s \leqq k$ and V is a lexico-positive matrix. If A corresponds to a frame for $\operatorname{pos} A$, then U is a Reay matrix.

Algorithms for finding matrices of the form referred to by Theorem (4) and Reay matrices are described in [8].

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