OPTIMIZATION TECHNOLOGY IN STATISTICAL ESTIMATION: FUSION OF HARD AND SOFT INFORMATION

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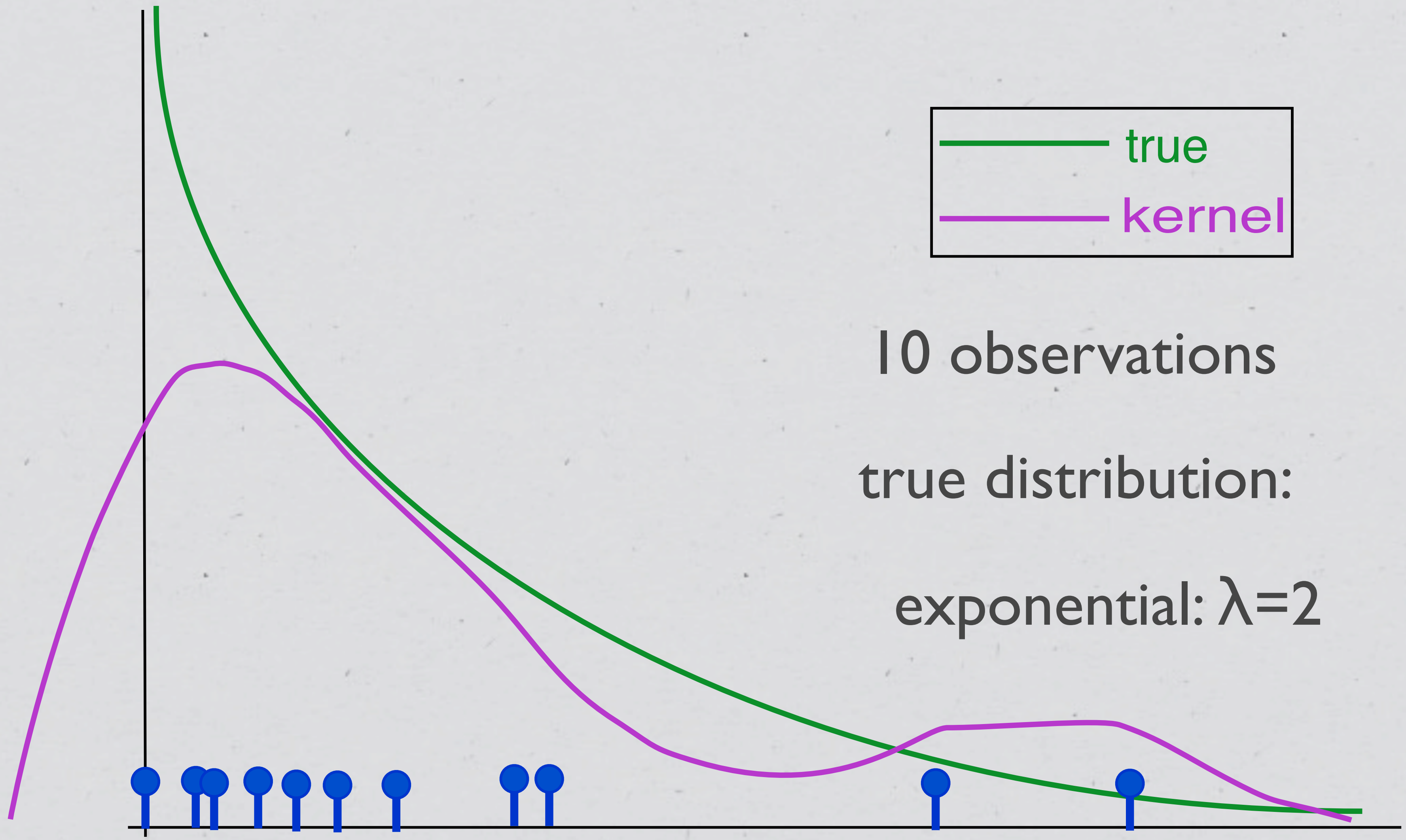
Praha, Summer 2011

R-STAT: Kernel Estimate



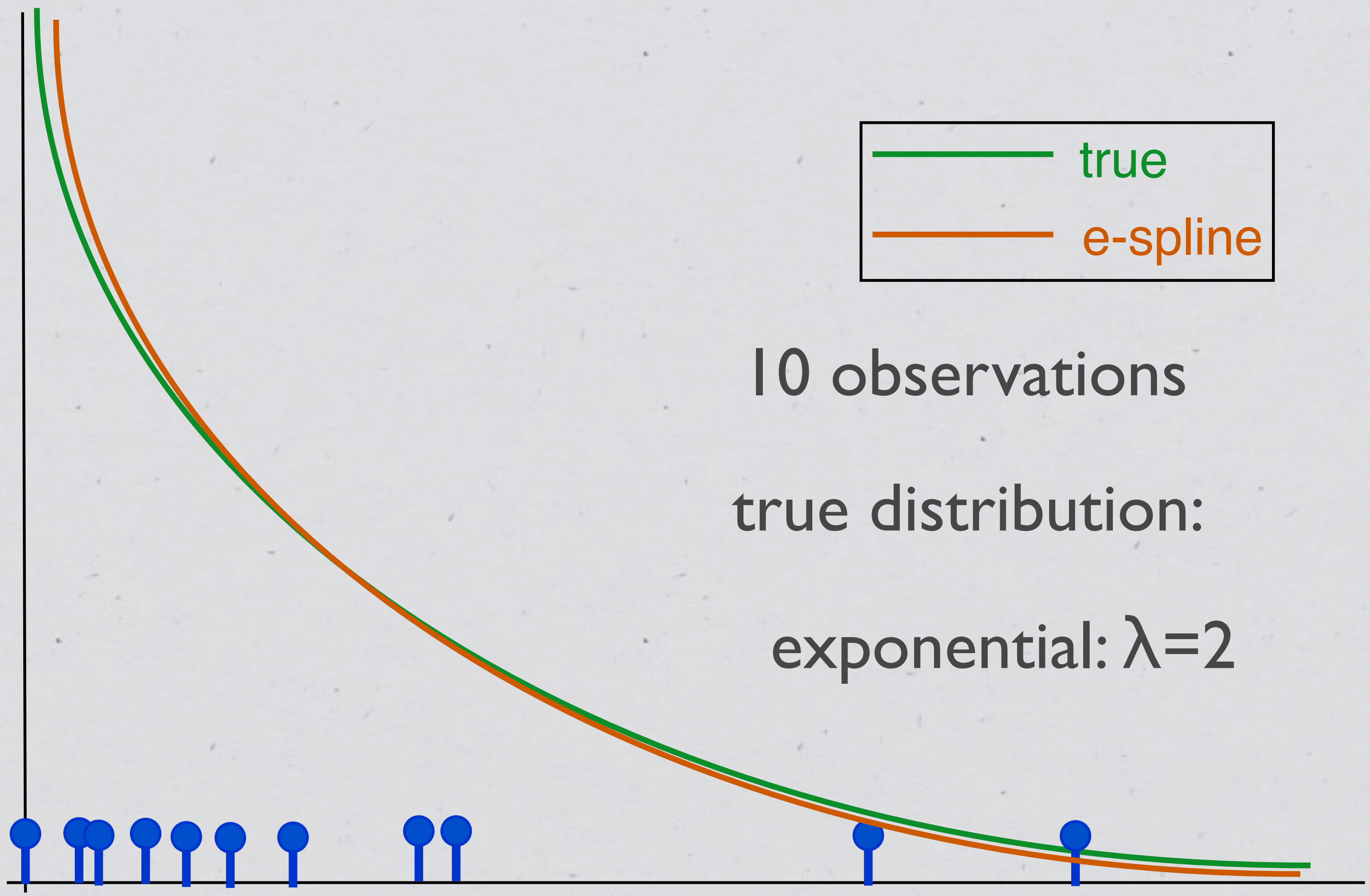
10 observations

true distribution:

exponential: λ=2

Information used: "continuous distribution" + ...

“our” Estimate



10 observations

true distribution:

exponential: $\lambda=2$

exploiting hard and soft information

density + support & decreasing

“all” information

- Observations (hard data): $\xi^1, \xi^2, \ldots, \xi^\nu$
- Non-data facts (soft information)
  - Support: (un)bounded , density or discrete distribution,
  - bounds on expectation, moments,
  - heavy tails
  - shape: unimodal, decreasing, parametric class
- still `softer' information (modeling assumptions):
  - see above + ... level of smoothness, `Bayesian' neighborhood, ..

An optimization viewpoint

- Find *h in H=fcns-class*($\mathbb{R}^n$) & density (≥ 0, ∫=1)
- that maximizes the probability of observing $\xi^1, \xi^2, \ldots, \xi^\nu$

$$\max \frac{1}{\nu}\sum_{l=1}^{\nu} \ln h(\xi^l) \quad \text{(likelihood)}$$

- $\max E^\nu\left\{\ln h(\xi)\right\} = \max \int \ln h(\xi) P^\nu(d\xi)$

"soft" information

support: $S = [\alpha, \beta], \ S = [\alpha, \infty), \ldots$

bounds on moments: $a \le \int \xi h(\xi) d\xi \le b$

unimodal: $h(x) = e^{-Q(x)}$, Q convex (Hessian SDP)

decreasing: $h'(x) \le 0, \ \forall x \in S$

'smoothness': $h \in H_0^1, \ \int \frac{h'(\xi)^2}{h(\xi)} d\xi \le \kappa$ (Fisher info.)

'Bayesian': $\| h - h^0 \| \le \beta$, objective: $\alpha \cdot \text{bayes}(h, h^0)$

Estimation: "Opt"-version

$$\max E^{\nu}\left\{\ln h(x)\right\} = \frac{1}{\nu}\sum_{l=1}^{\nu} \ln h(x_l)$$

$$\text{such that } \int h(x)dx = 1,$$

$$h(x) \geq 0,\ \forall x \in \mathbb{R}$$

$$h \in A^{\nu} \subset H$$

A^{ν}: soft (non-data) information constraints

$H = C^2(S), L^p(S), H^1(S), \ldots \quad S$ subset $\mathbb{R}$

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H= (*lsc-fcns*(*S*), *aw-(epi)-topo*), Polish space

Consistency Theorem

Suppose $\nu \to \infty$ (more data is acquired) and

$A^\nu \to A$ (valid information is acquired) then

estimates $h^\nu \to h^{true}$ $a.s.$ (with probability 1).

1949 A. Wald: consistency of parametric estimates (MLH-SAA)

1982-1983 Klonias & Prakasa Rao: consistency of nonparametric estimatesr

1971 Good & Gaskins: nonparametric roughness penalties (proposed)

1982 B. Silverman & 1990 J. Thompson/R.Tapia: consistency with penalization

1985 P. Groeneboom: Estimating a monotone density

==

1979 R. Wets: statistical approach to solution of stochastic program (Tech. Note)

1988 (with J. Dupacova): asymptotics of constrained estimators (parametric)

1991 (with H. Attouch): Law of Large Numbers for random lsc functions

2000 (with X. Dong): consitency of constrained estimators (non-parametric)

2006-10 (with M. Casey): rates of convergence (RKHS, space of lsc-functions)

RATES OF CONVERGENCE

via Consistency proof

Estimation problem: solution h^ν

$$\max F^\nu(h) = E^\nu\left\{f(\xi,h)\right\},\ h \in \text{lsc-fcns}(\mathbb{R}^n)$$

$$f(\xi,h) = \ln h(\xi) \text{ when } \int h = 1,\ h \geq 0,\ h \in A^\nu$$

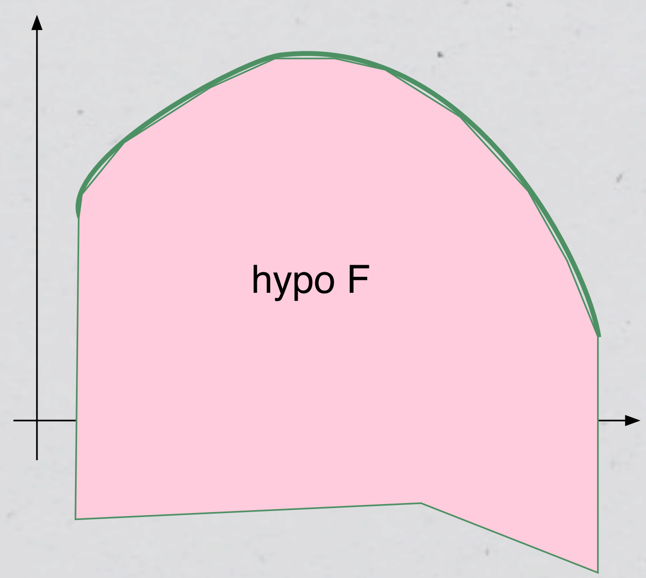
Limit problem: solution h^{true}

$$\max F^\infty(h) = E\left\{f(\xi,h)\right\},\ h \in \text{lsc-fcns}(\mathbb{R}^n)$$

hypo $F^\nu \to_{a.s}$ hypo F^∞ (LLN for random usc-fcns)

with repect to the aw-distance =hypo-topology

$\Rightarrow \text{dist}(h^\nu, h^{true}) \leq \kappa \text{ hypo-dist}(F^\nu, F^\infty) \to 0$



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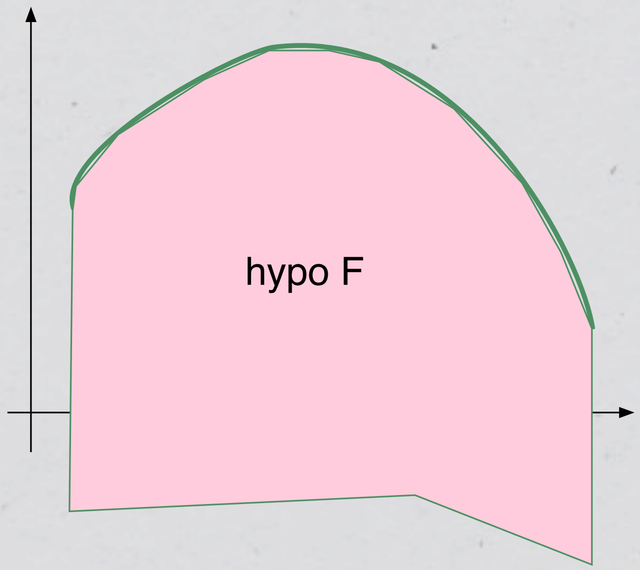


$$\text{hypo-dist}(F^{\nu}, F^{\infty}) \leq \kappa_1 \mu\text{-dist}(P^{\nu}, P)$$

$\Rightarrow$ convergence rate of empirical processes (van der Vaart & Wellner)

Under investigation
(large Deviations)

- Hoeffding's inequalities
- Nemirovski's inequalities
- & Nemirovski's inequalities revisited
  - (L. Dümbgen, S. van de Geer, M. Veraar & J. Wellner)
- Sanov's Theorem
- CLT for (unbounded) random sets (A. Pichler & Co.)

NUMERICAL STRATEGIES

from infinite to finite dimensional Optimization

$$h \approx \sum_{k=1}^{q} u_k \phi_k(\cdot)$$

Fourier coefficients, wavelets, kernel-dictionary, …

$$h = \exp\big(s(\bullet)\big) \quad \text{exponential epi-spline}$$

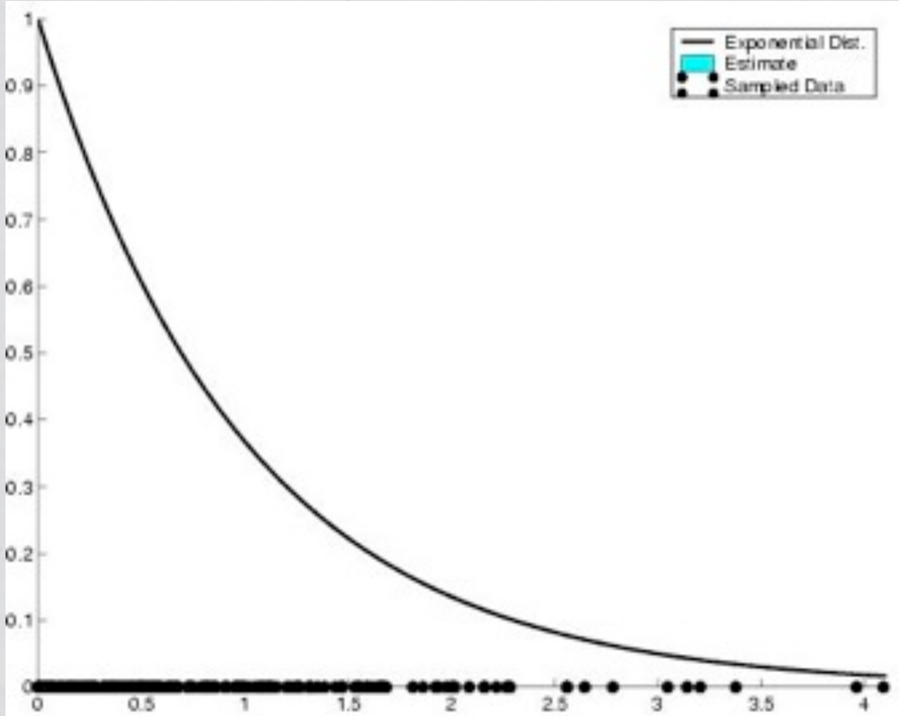
$s(\bullet)$ cubic (or quadratic) epi-spline, spline-like

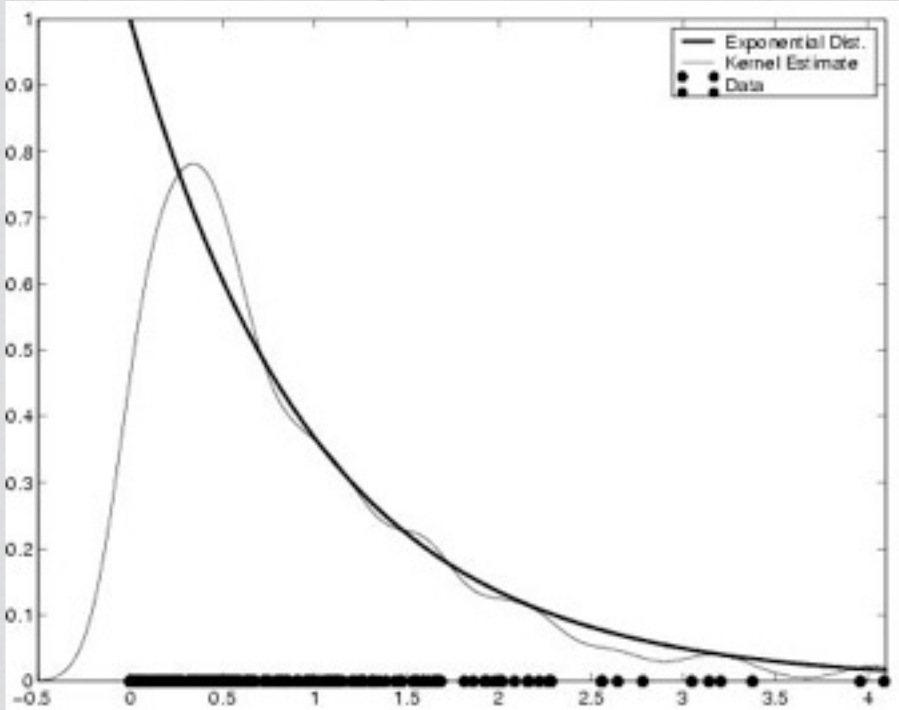
$\Rightarrow$ n-dimensional theory of epi-splines

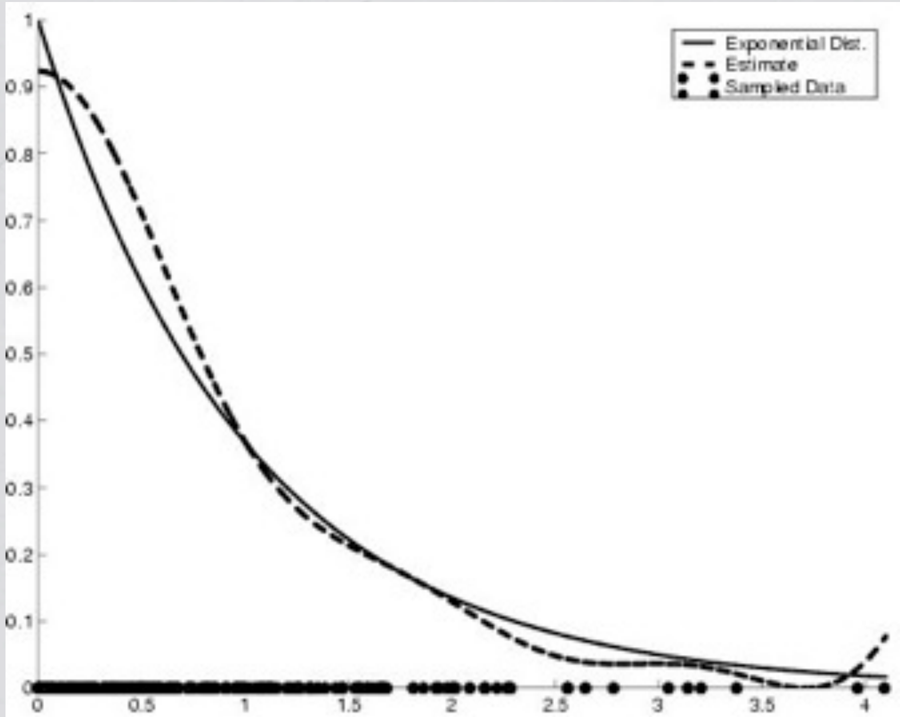
Test Case: exponential

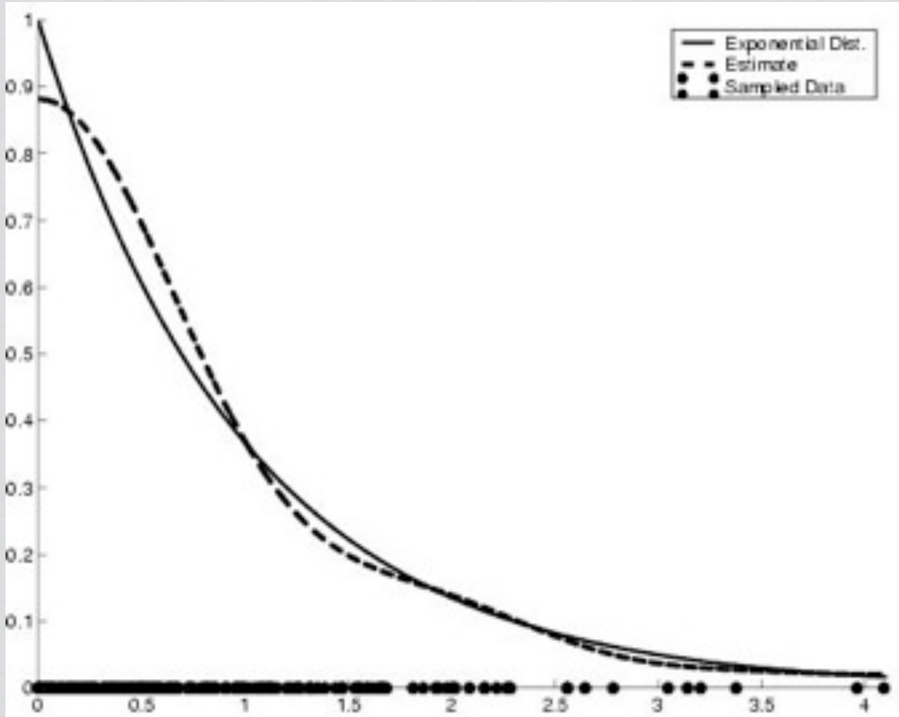
- $h^{true}(x) = \lambda e^{-\lambda x}$ if $x \geq 0$; $= 0$ if $x < 0$ $(\lambda = 1)$
- "empirical" estimate
- kernel estimate from **R-stat**
- unconstrained with support (non-negative)
- constrained (h decreasing)
- parametric, i.e., $h \in \exp\text{-class}$

200-observations



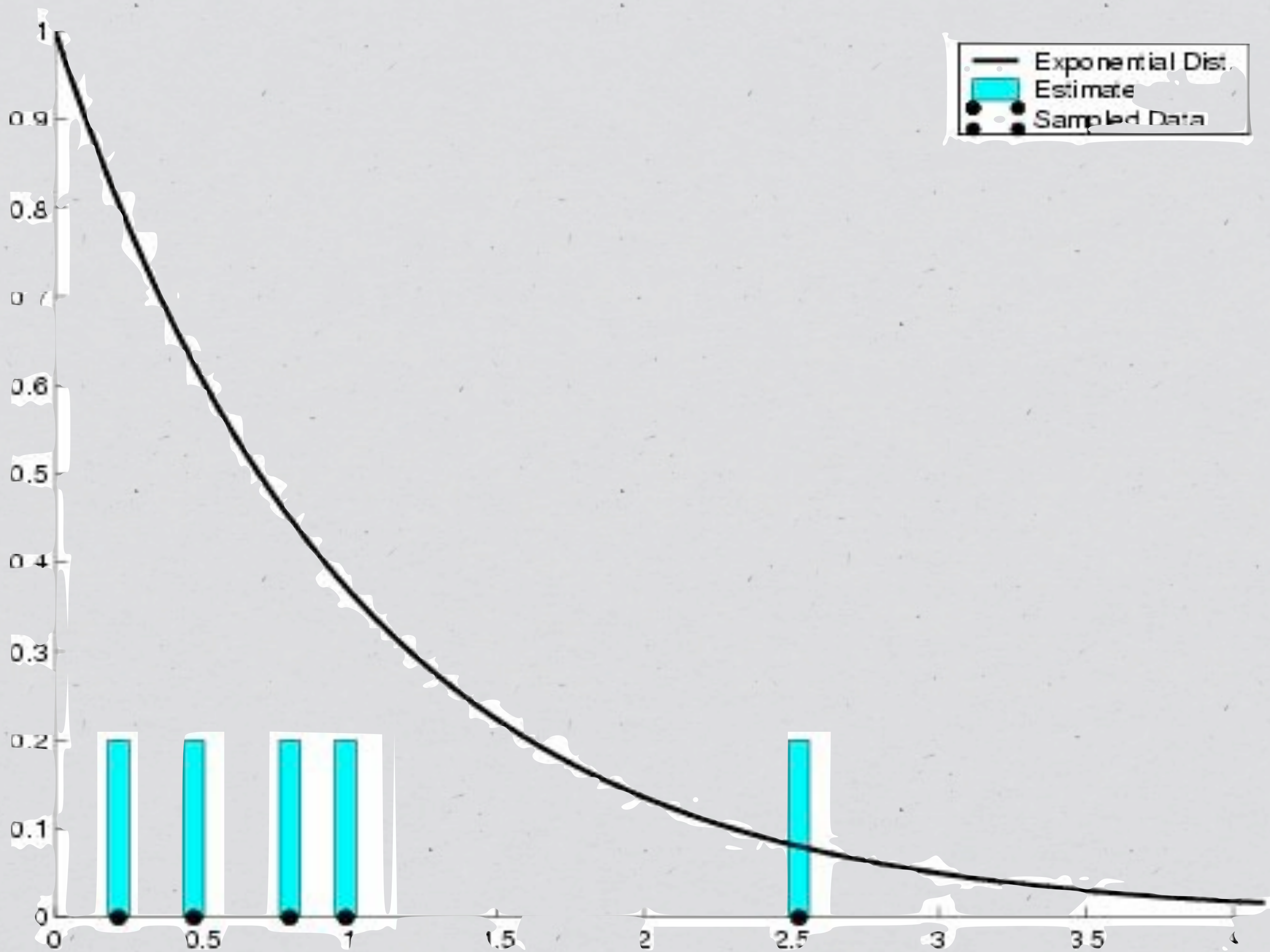




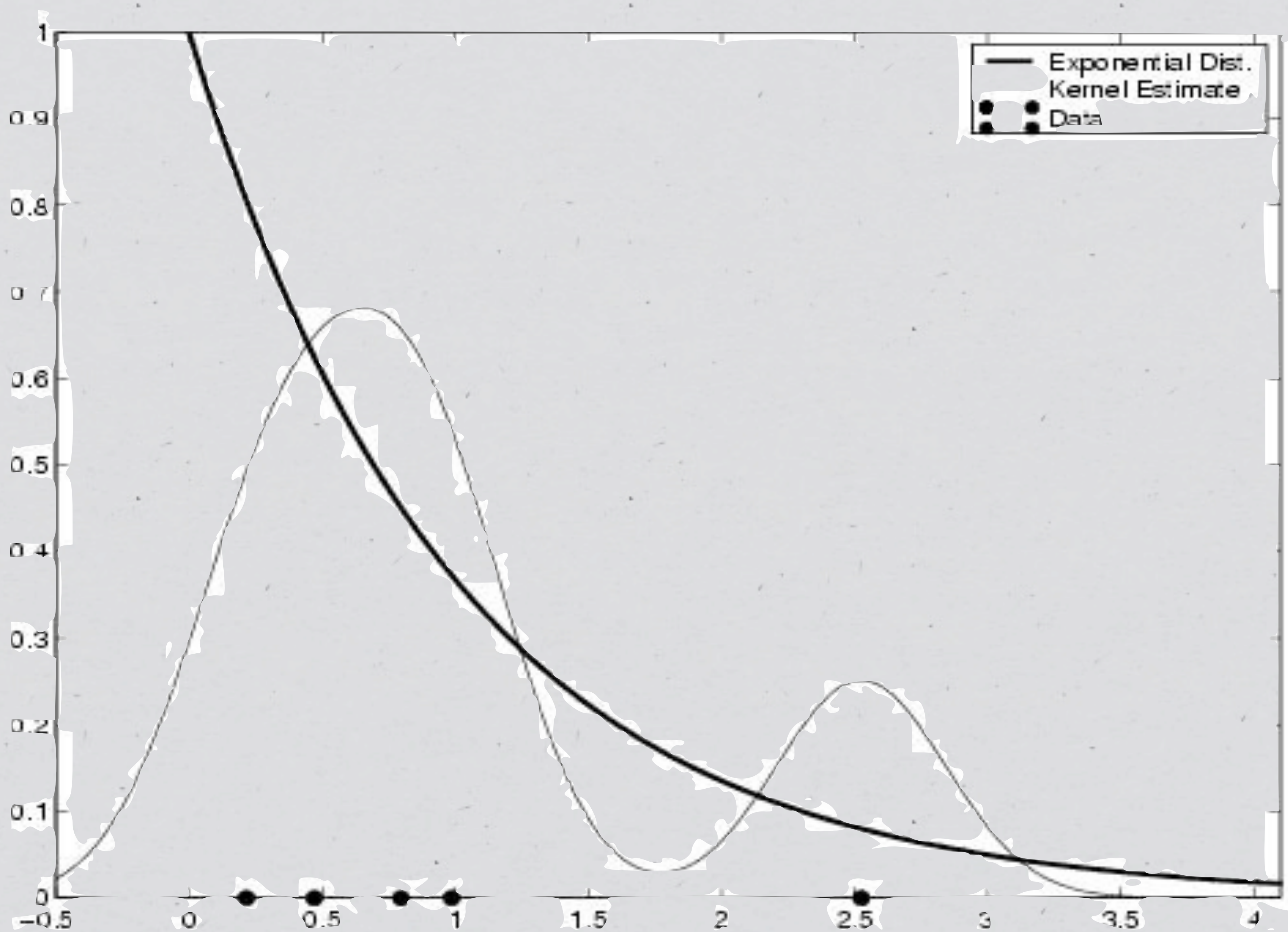


But what about 5 sample points?

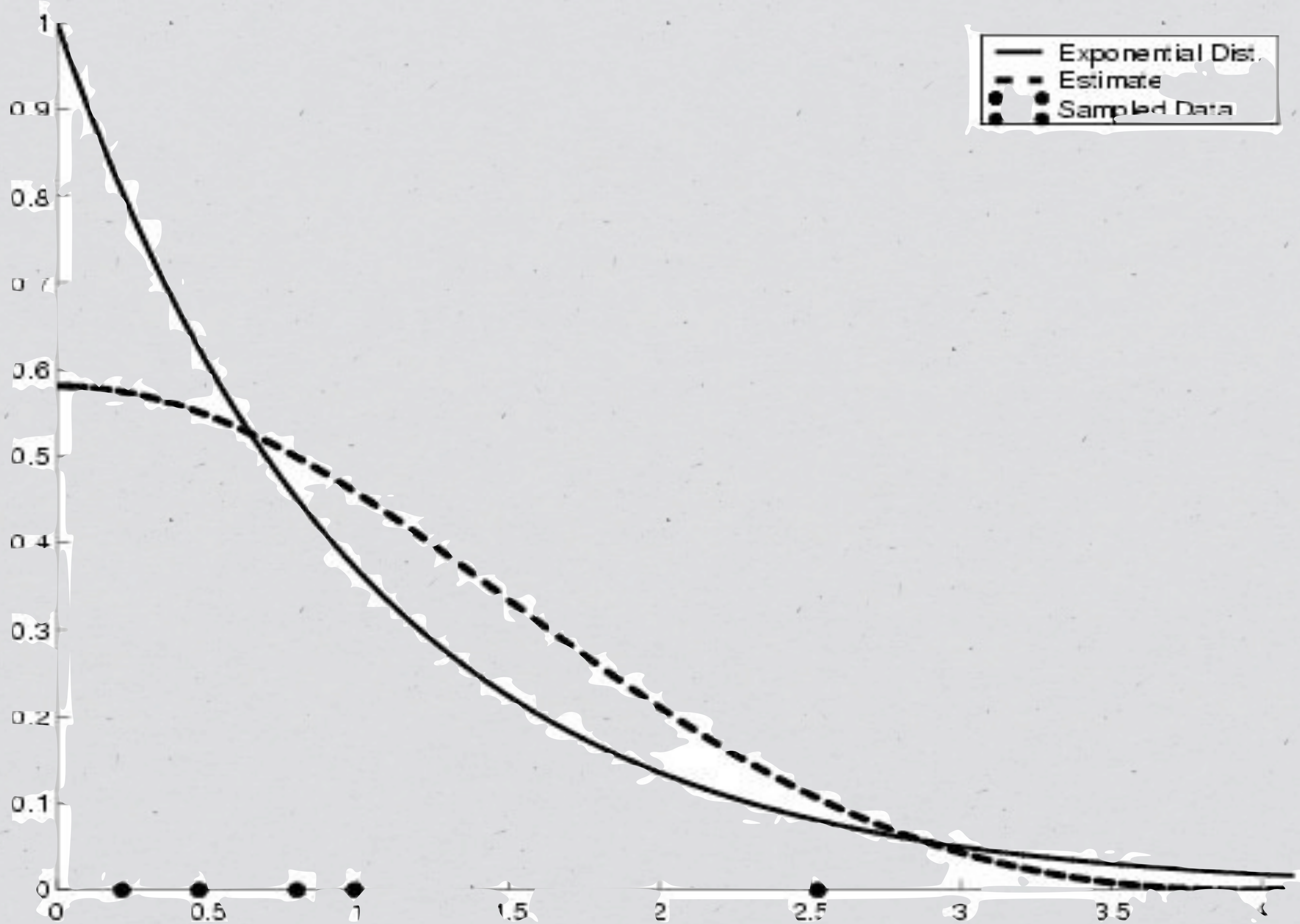
h^{est} : empirical estimate



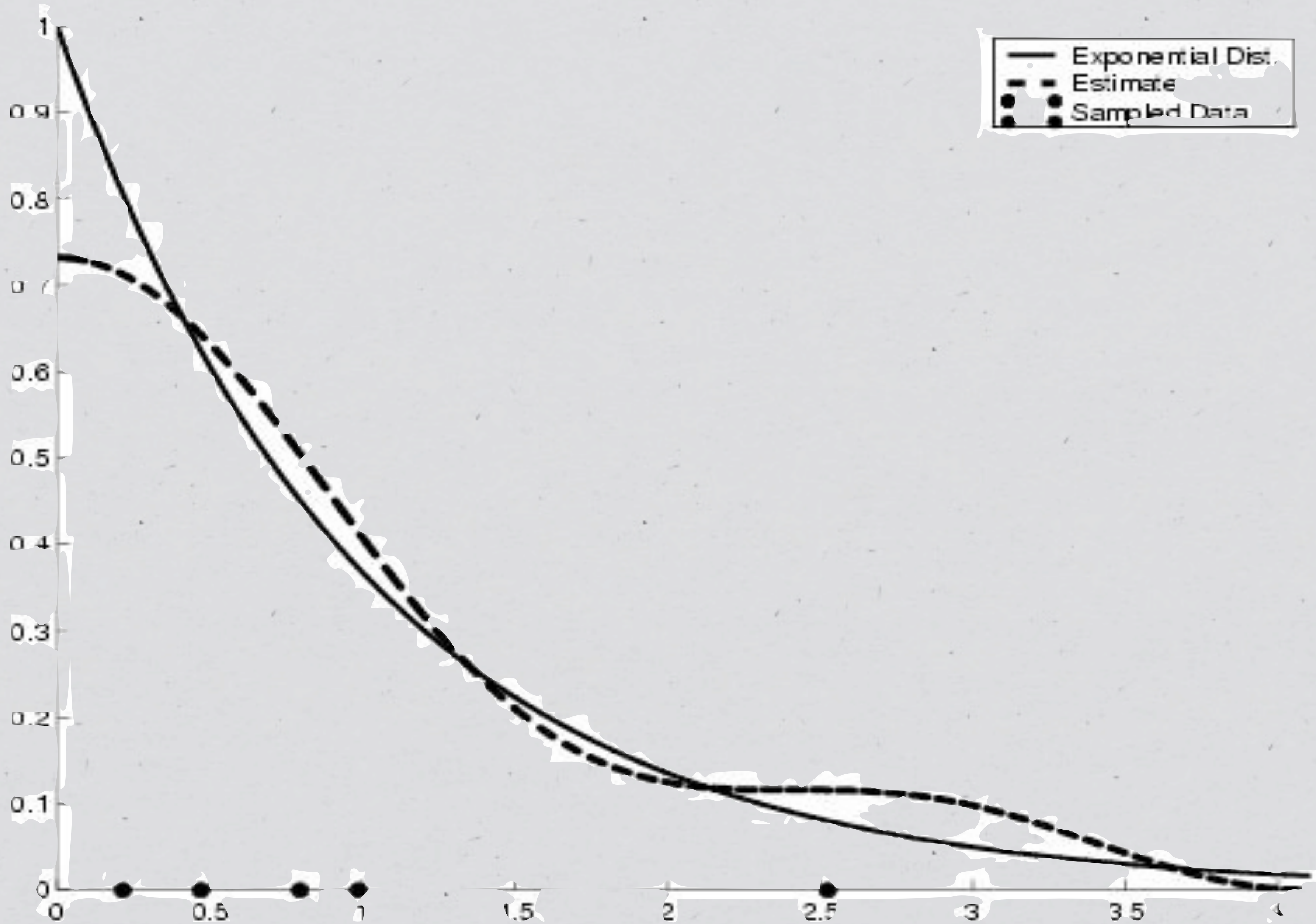
h^{est} : kernel estimate (R-stat)



h^{est} : ..with known support



h^{est} : decreasing density



ERROR ‘ANALYSIS’

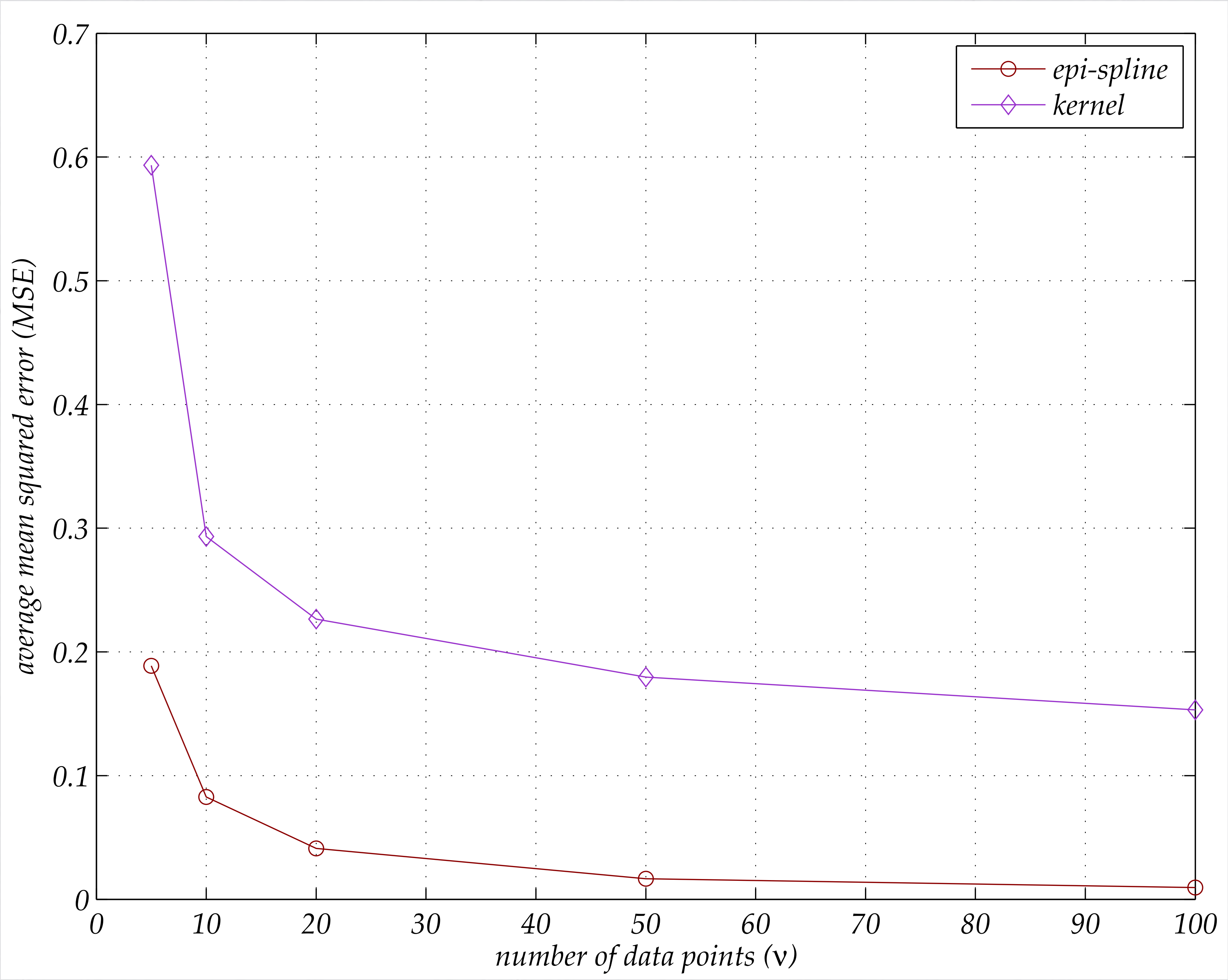
NUMERICAL EXPERIMENTATION

#runs: 10,000

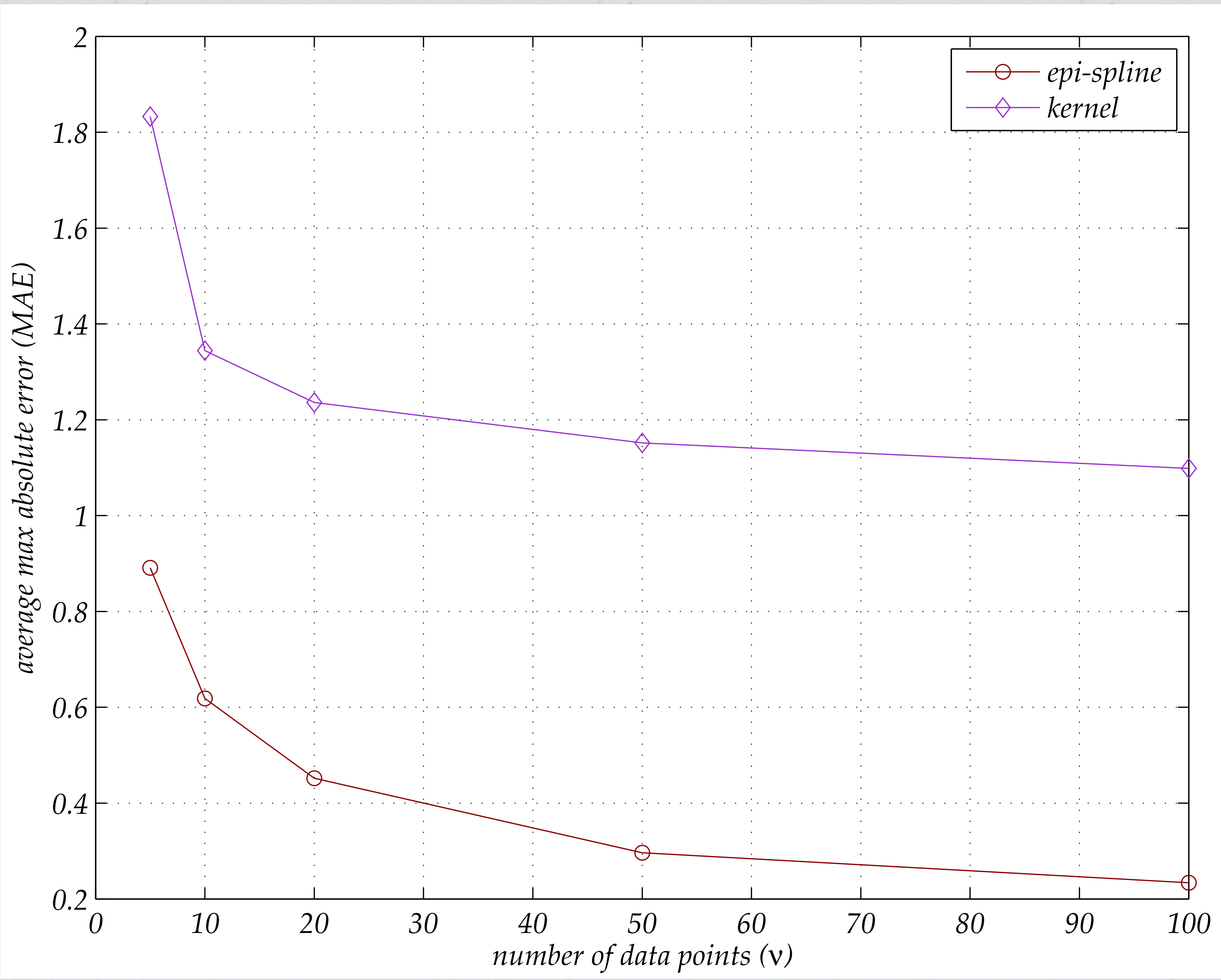
mean square error

mean absolute error

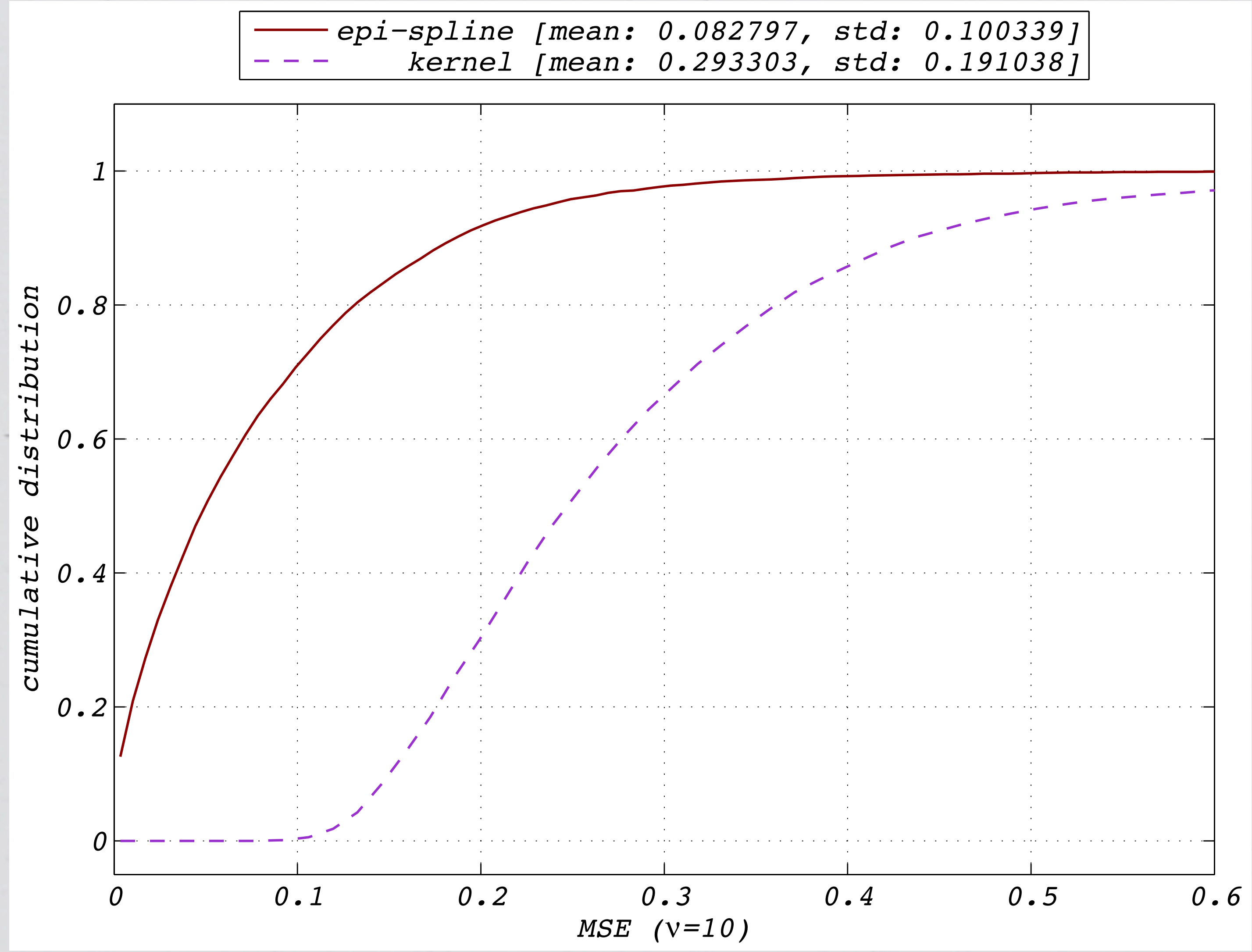
Exponential DistributioN



Exponential DistributioN



Does it pay off?
Error Analysis



Mean square error

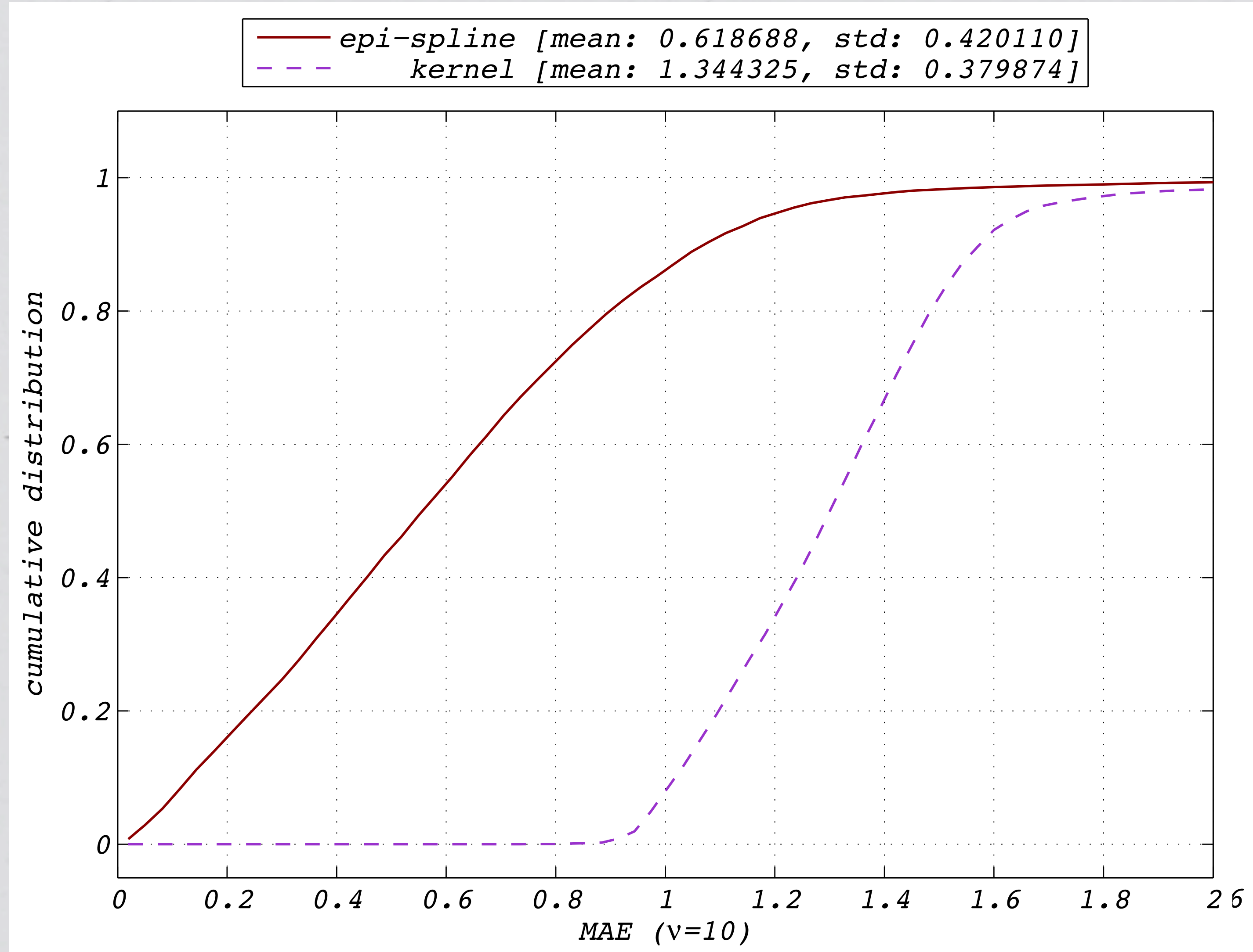
samples: 10

runs: 10,000

kernel estim: 0.293

epi-spline est.: 0.083

Does it pay off? Error Analysis



Max. absolute error

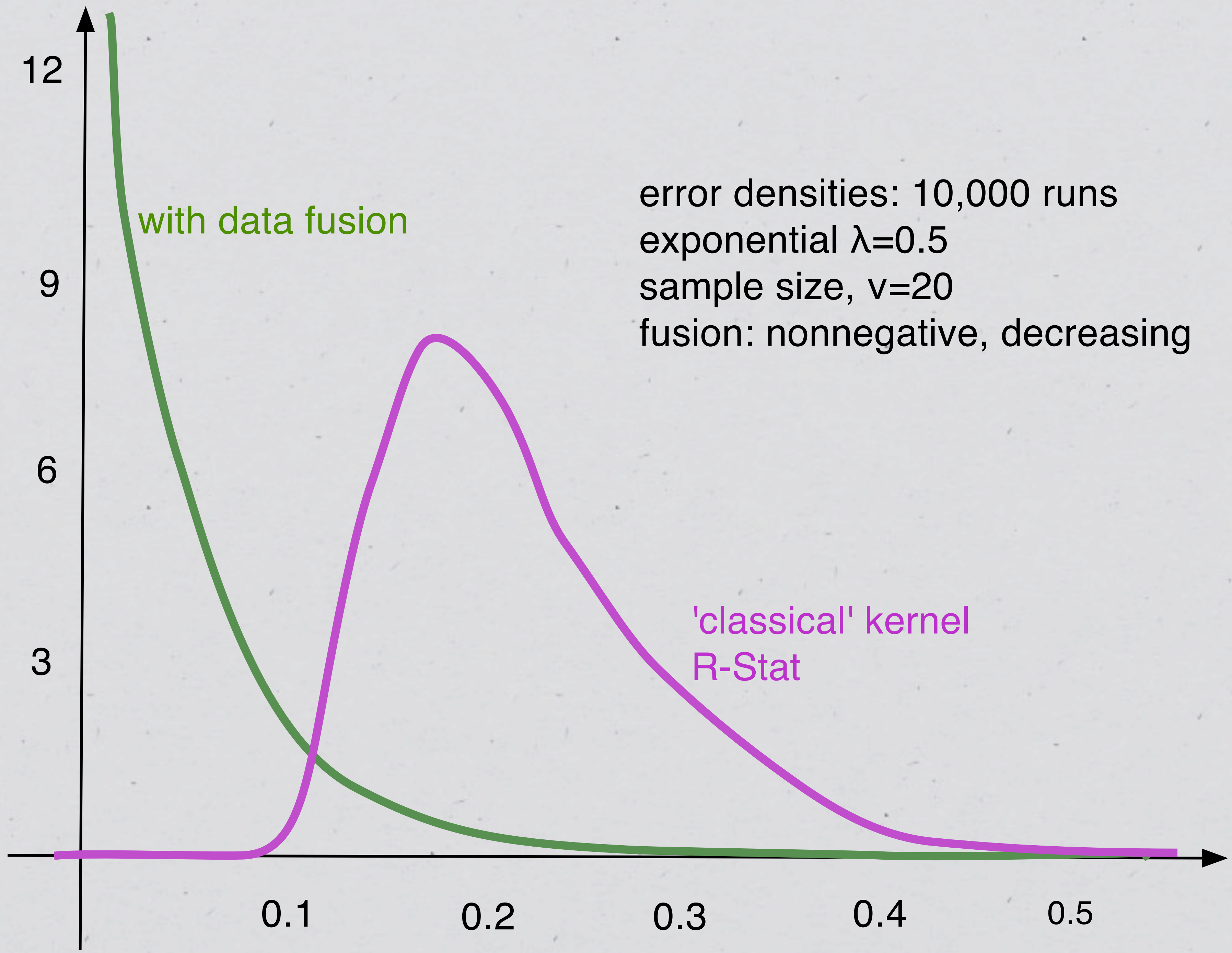
samples: 10

runs: 10,000

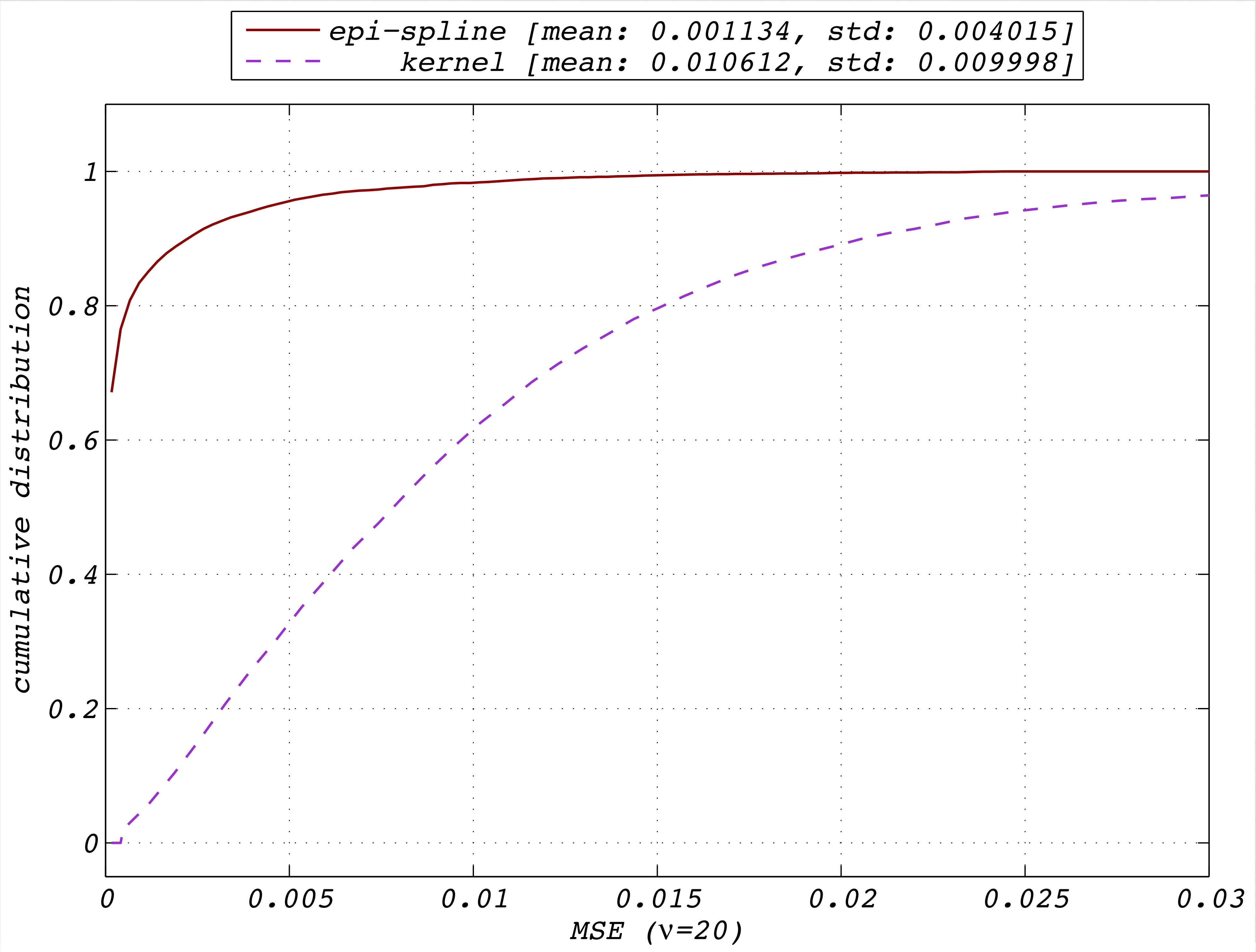
kernel estim: 1.344

epi-spline est.: 0.619

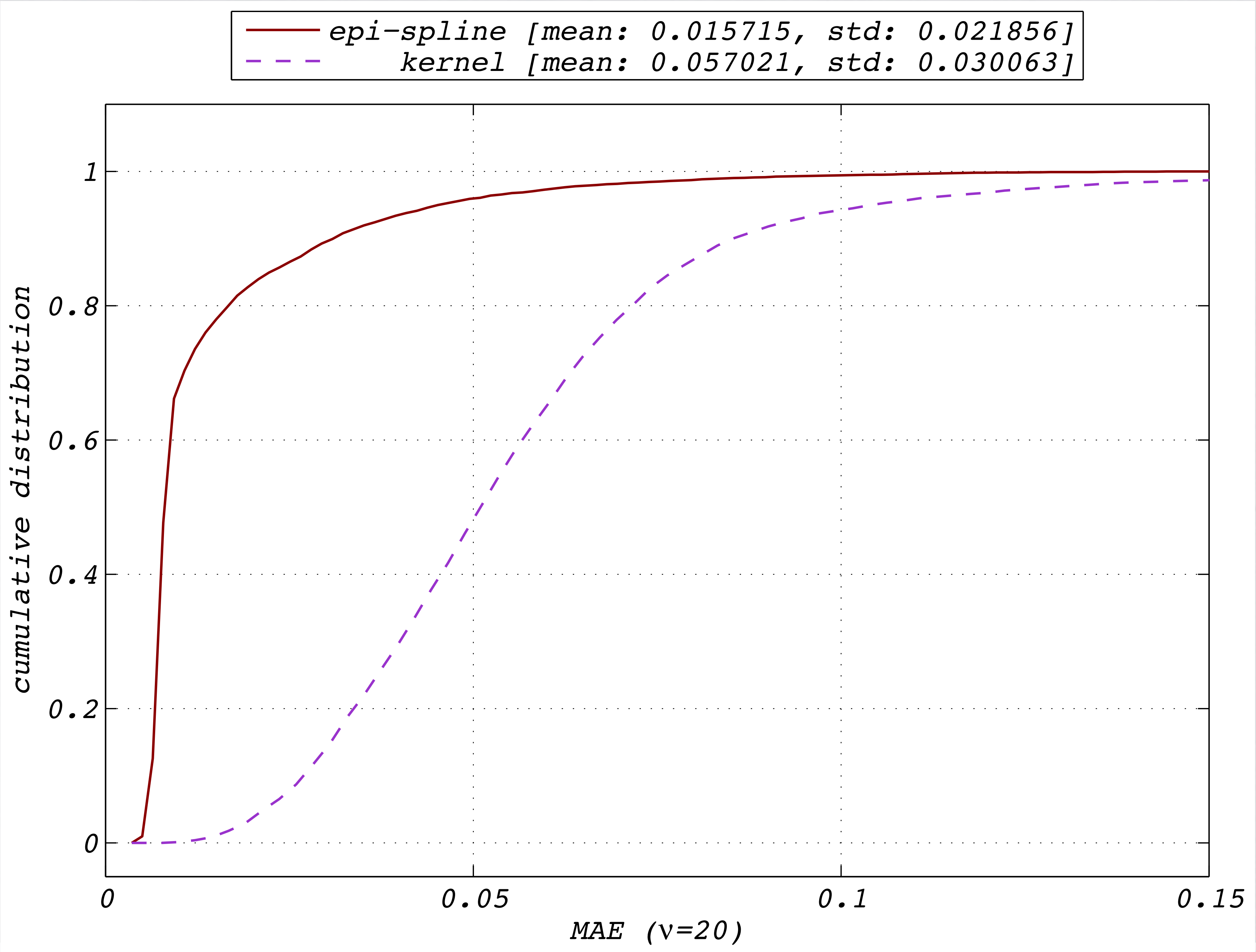
Densities: Error distributions exponential case



Gaussian Distribution



Gaussian Distribution



EXTENSIVE EXPERIMENTATION

www.math.ucdavis.edu/~prop01

EPI-SPLINES
An Approximation Tool

epi-spline: 2nd Order

$c:(a,b]\to\mathbb{R}$ twice differentiable (not C^2)

$c'':(a,b]\to\mathbb{R}$ $2nd$ derivative approzimated by $z:\mathbb{R}\to\mathbb{R}$

split (a,b]: $\{(x_{k-1},x_k],k=1,\dots,N\}$, N relatively large

fix $z(t)=z_k$ (constant) for $t\in(x_{k-1},x_k]$

$2nd$ order epi-spline (1-dim.) +++ mesh = $\max_{k=1,\dots,N}|x_k - x_{k-1}| = m$

$$s_m(x)=s_0+v_0x+\int_0^x dr\int_0^r dt\,z(t),\quad z(t)\equiv z_k \text{ on } (x_k,x_{k+1}]$$

$$=s_0+v_0x+\sum_{j=1}^{k}a_{kj}z_j \quad \text{when } x\in(x_k,x_{k+1}]$$

$s_m\in C^{1,+pl}$, linear w.r.t. $s_0,v_0,z_1,\dots,z_N$ (finite # parameters)

as mesh $m\searrow 0$, $\max\|s_m-c\|^2\to 0$ and s_m epi-converges to c

Epi-splines

originally (Wets, Bianchi & Yang, 2002):

- derive financial curves,
- later, also stochastic volatility

Epi-splines of k*th* order:

- piece-constant k*th* derivative
- still linear w.r.t. its parameters
- and epi-convergence to c

Exponential epi-spline 1-dimensional, 2nd order

$h(x) = e^{-s(x)}$

$s(x) = s_0 + v_0 x + \int_0^x dr \int_0^r dt\, z(t), \quad z(t) \equiv z_k \text{ on } (x_k, x_{k+1}]$

$= s_0 + v_0 x + \sum_{j=1}^{k} a_{kj} z_j \quad \text{when } x \in (x_k, x_{k+1}]$

$\max E^\nu \left[\ln h(\xi) \right] = \frac{1}{\nu} \sum_{l=1}^{\nu} \ln h(\xi^l) = \min \frac{1}{\nu} \sum_{l=1}^{\nu} s(\xi^l)$

such that $\int e^{-s(\xi)} d\xi \leq 1. \qquad (h \geq 0)$

$z_k \in \left[-\kappa_l, \kappa_u \right]$ 'constrained' curvature

unimodal: $\kappa_l = 0 \Rightarrow s(\cdot)$ convex

Higher-dimensional epi-splines

2-DIMENSIONAL - FIRST VERSION

$$s(x,y) = z_0 + v_1 x + v_2 y + \int_0^x d\tau \int_0^\tau d\theta\, a(\theta)$$

$$+ \int_0^y d\tau \int_0^\tau d\theta\, b(\theta) + \int_0^x d\tau \int_0^y d\theta\, c(\tau,\theta)$$

on $(x_{k-1}, x_k] : a(x) = a_k$,

on $(y_{k-1}, y_k] : b(y) = b_k$,

on $(x_{k-1}, x_k] \times (y_{l-1}, y_l) : c(x,y) = c_{kl}$

requires bouundary continuity properties

estimation : a_k and $b_k \Rightarrow$ marginal distributions

c_{kl} correlation coefficients (locally)

Higher-dimensional epi-splines
2-DIMENSIONAL - FIRST VERSION

$$\nabla^2 s(x) = \begin{pmatrix} a_{k-1,l-1} & a_{k-1,l} \\ a_{k,l-1} & a_{k,l} \end{pmatrix} \text{ Hessian}$$

on open rectangle $(x_{k-1}, x_k) \times (y_{l-1}, y_l)$

unimodal $h(x) = e^{-s(x)} \Rightarrow s$ convex (globally)

$\nabla^2 s(x)$ positive semidefinite, symmetric

(3 parameters)

rectangle boundary values: (at a mesh)

via monotonicity of $\nabla s(x)$

all conditions included in the optimization problem

example: Normal density

mean = (0,0) ... data samples correlated

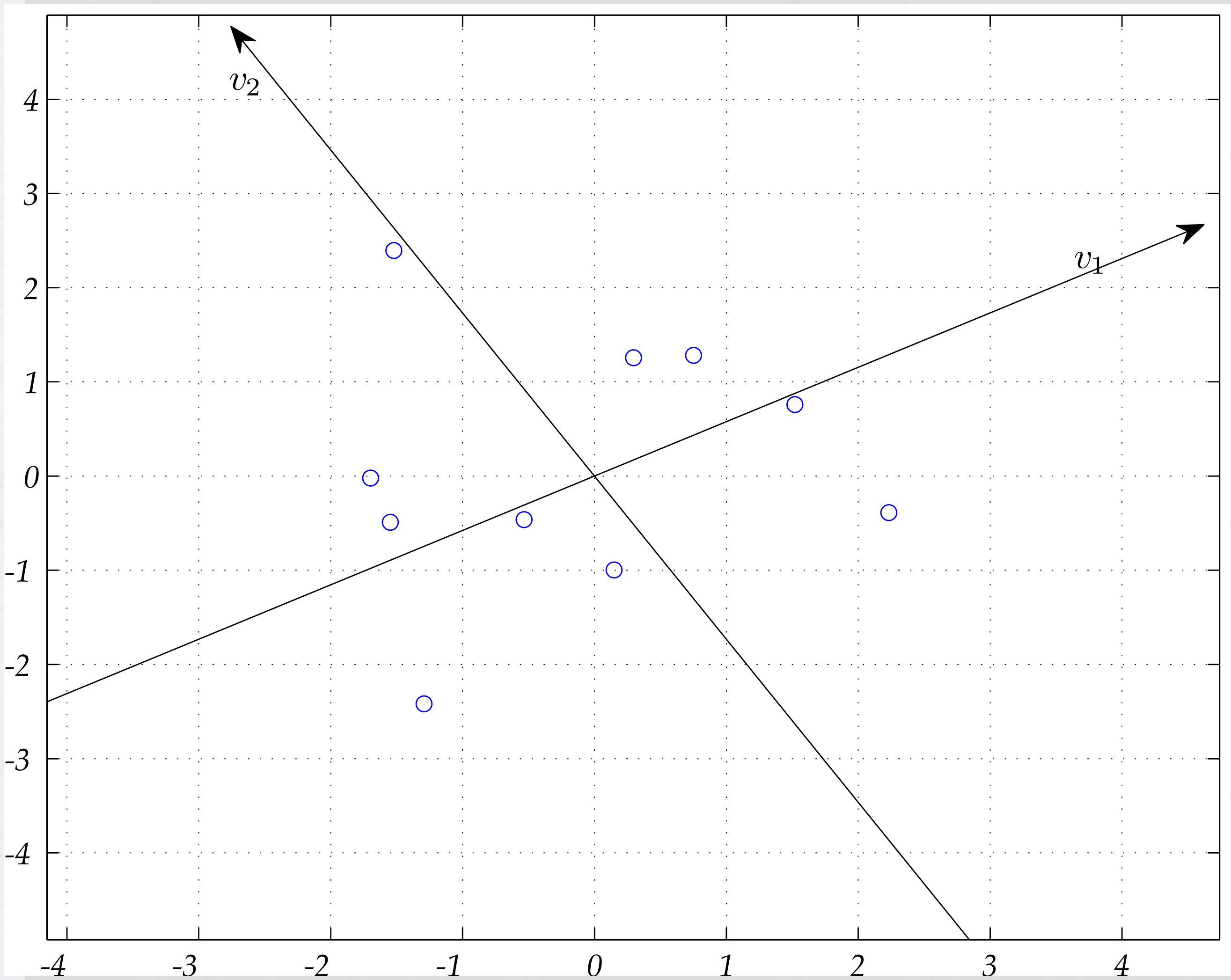
covariance: MDM^T, $D =$ diag(4,1), $M = \begin{pmatrix} \cos(\pi/6) & \cos(2\pi/3) \\ \sin(\pi/6) & \sin(2\pi/3) \end{pmatrix}$

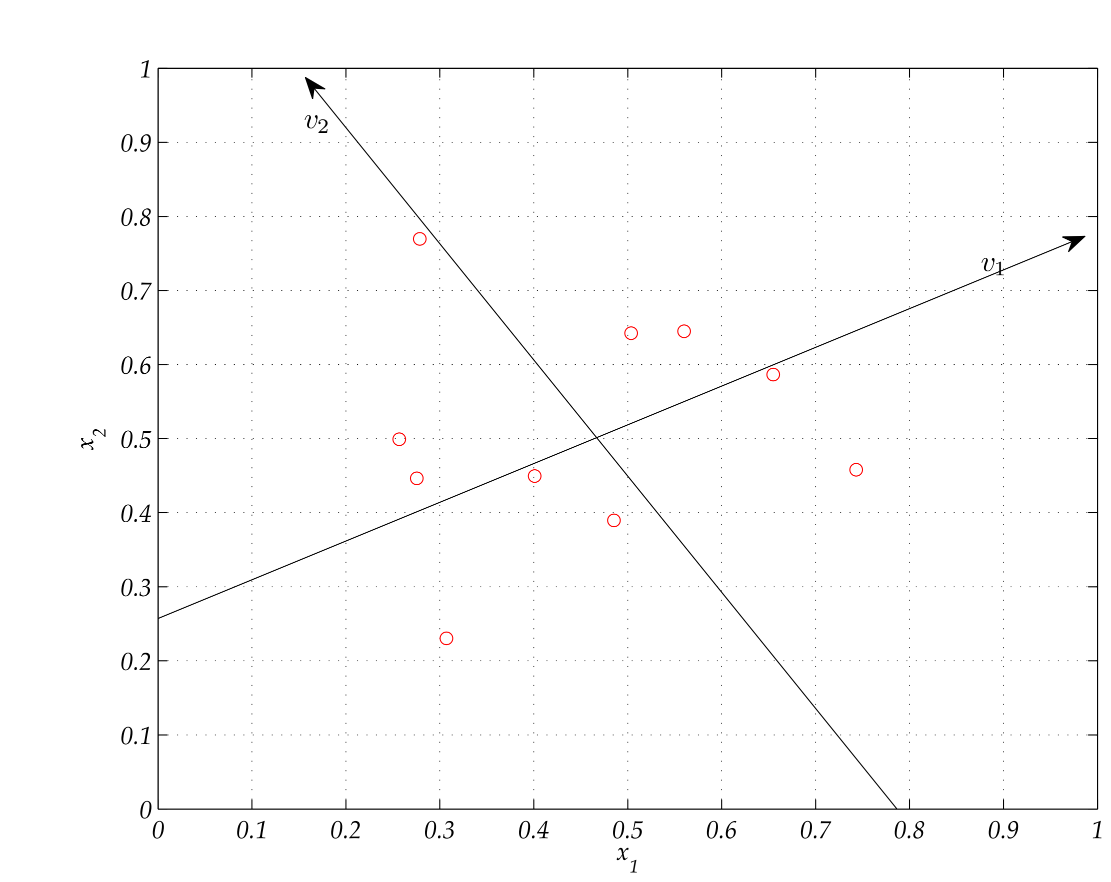
samples: $\nu = 10$, "soft" information: h unimodal

Results:

$$\left\| h^{true} - h^{est} \right\|_2^2 = 0.028, \quad \left\| h^{true} - h^{est} \right\|_\infty = 0.006$$

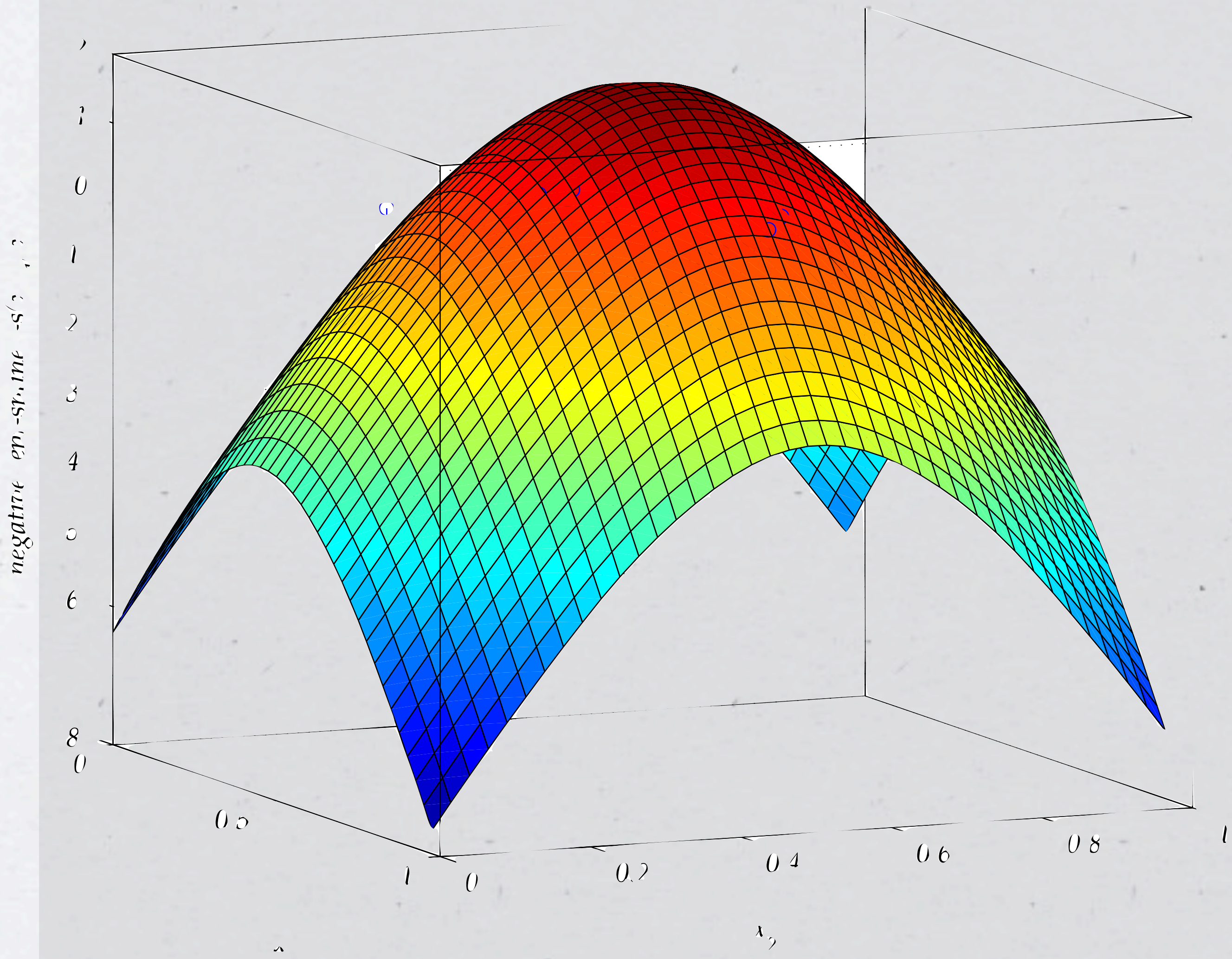
Sampled data

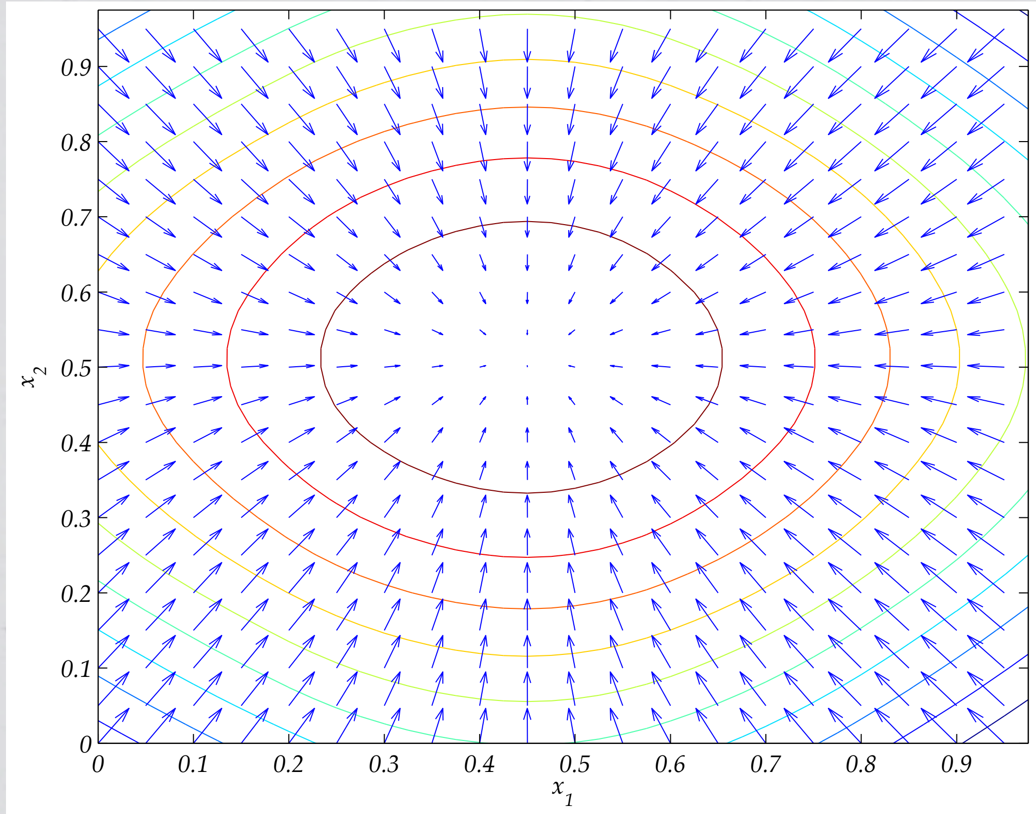




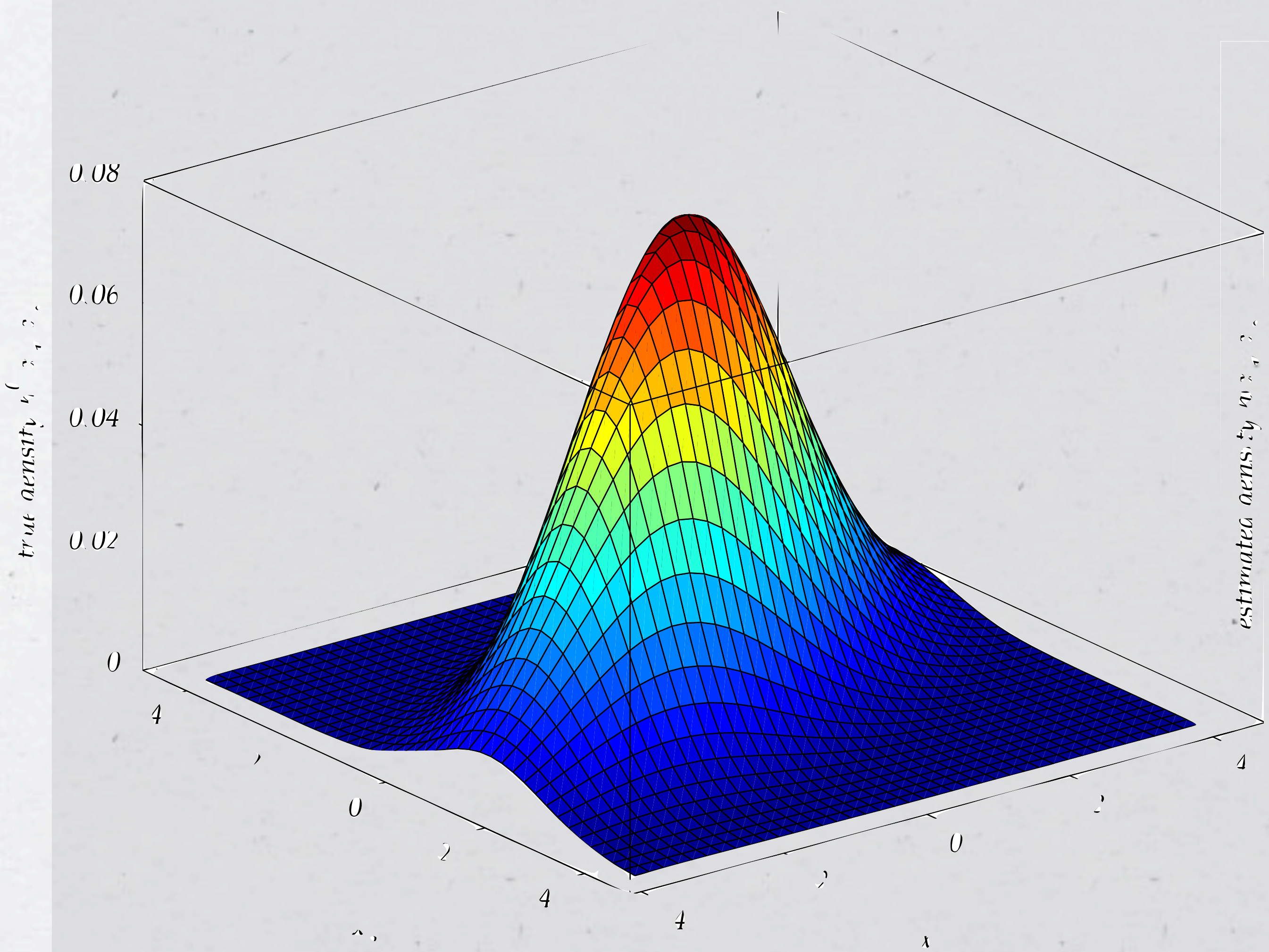
normalized

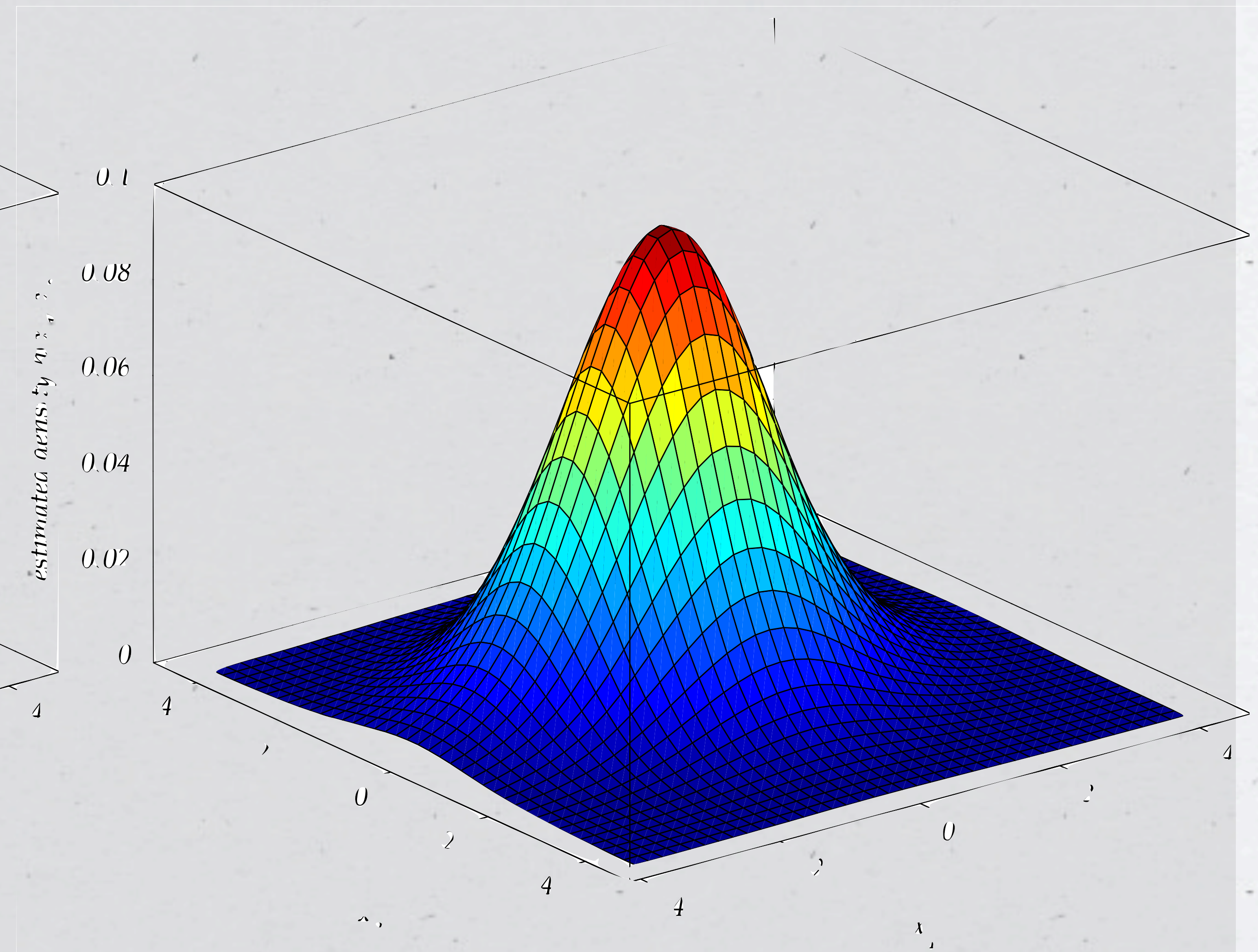
Epi-SpLine & vector field



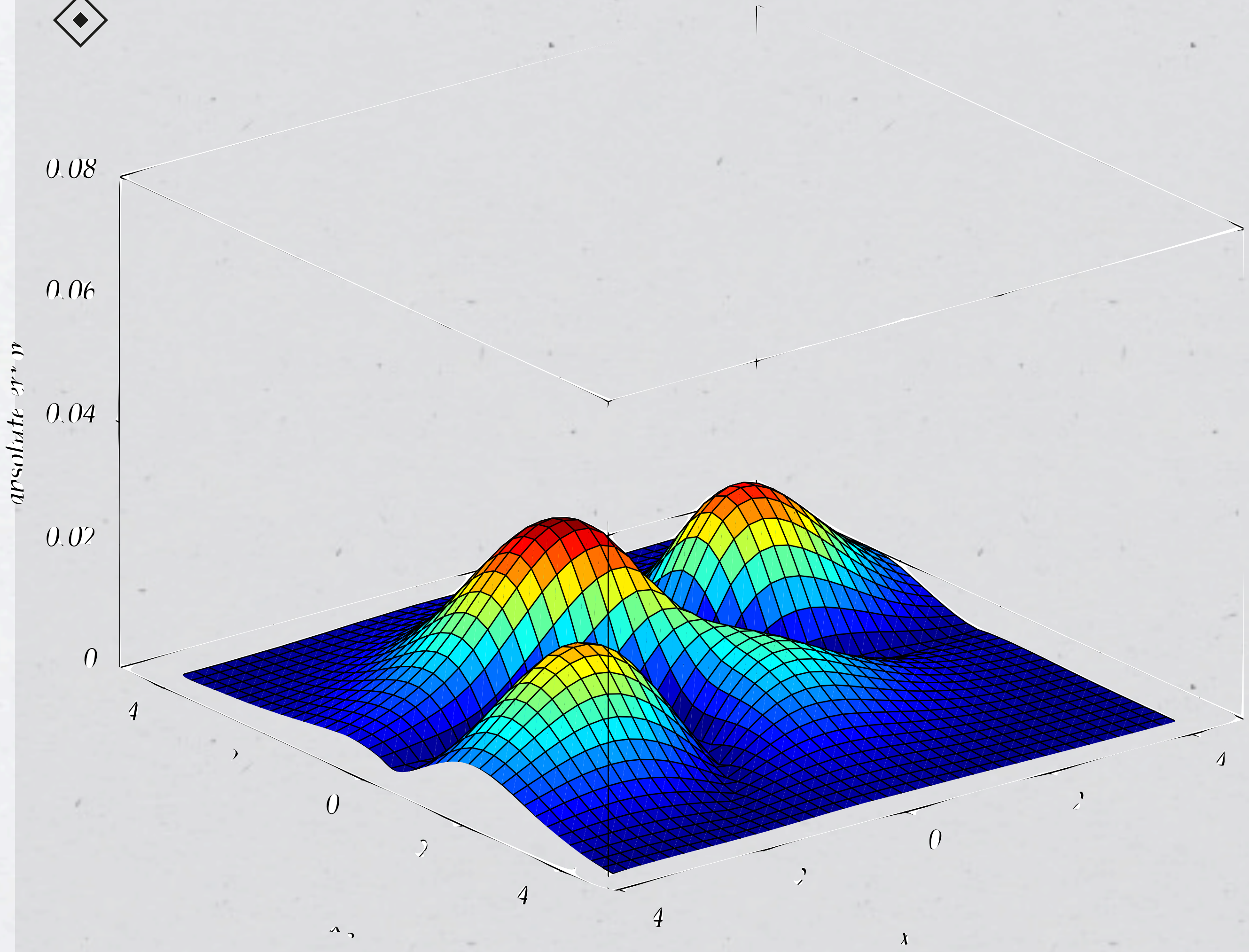


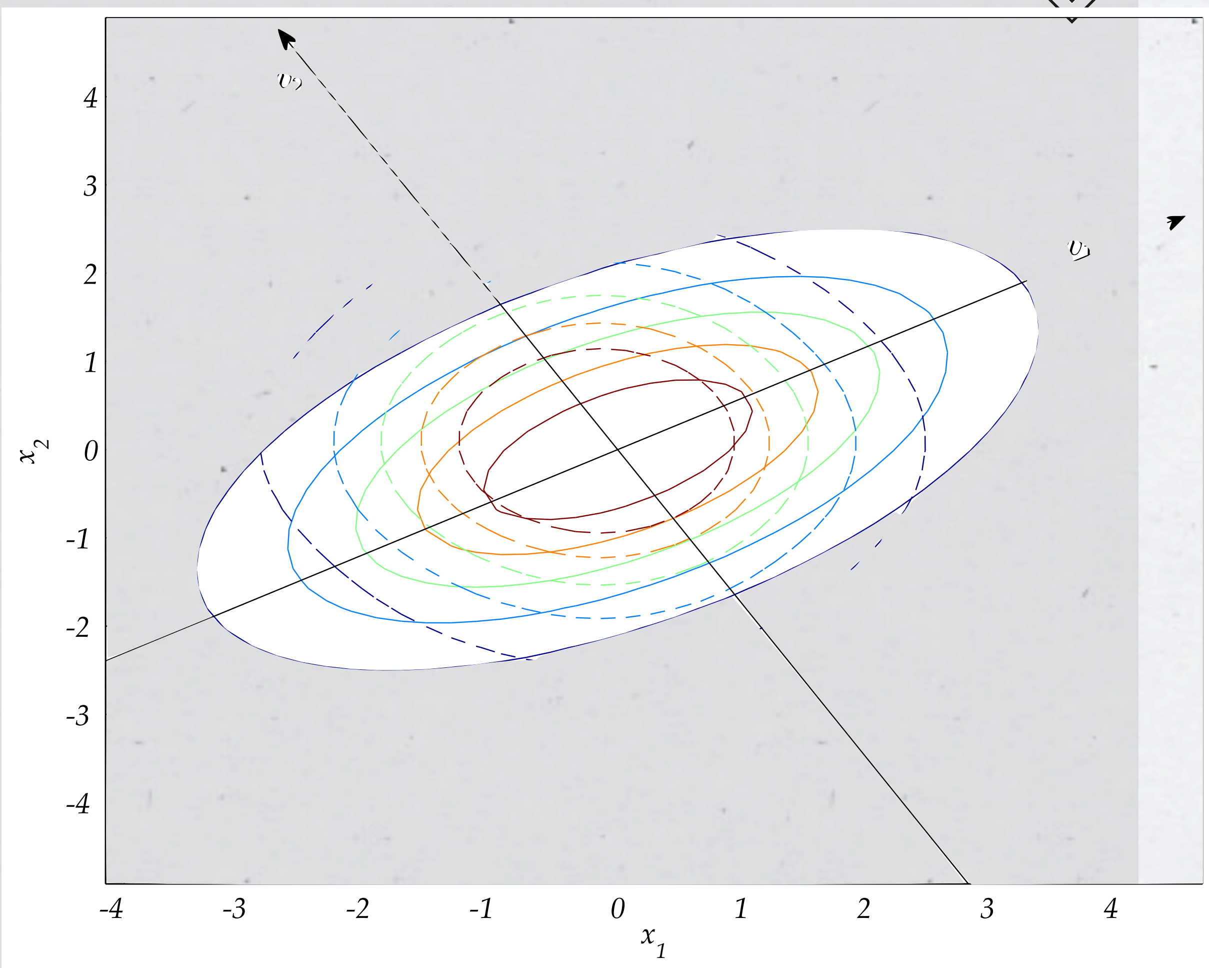
True & Estimated density





Measurement Errors



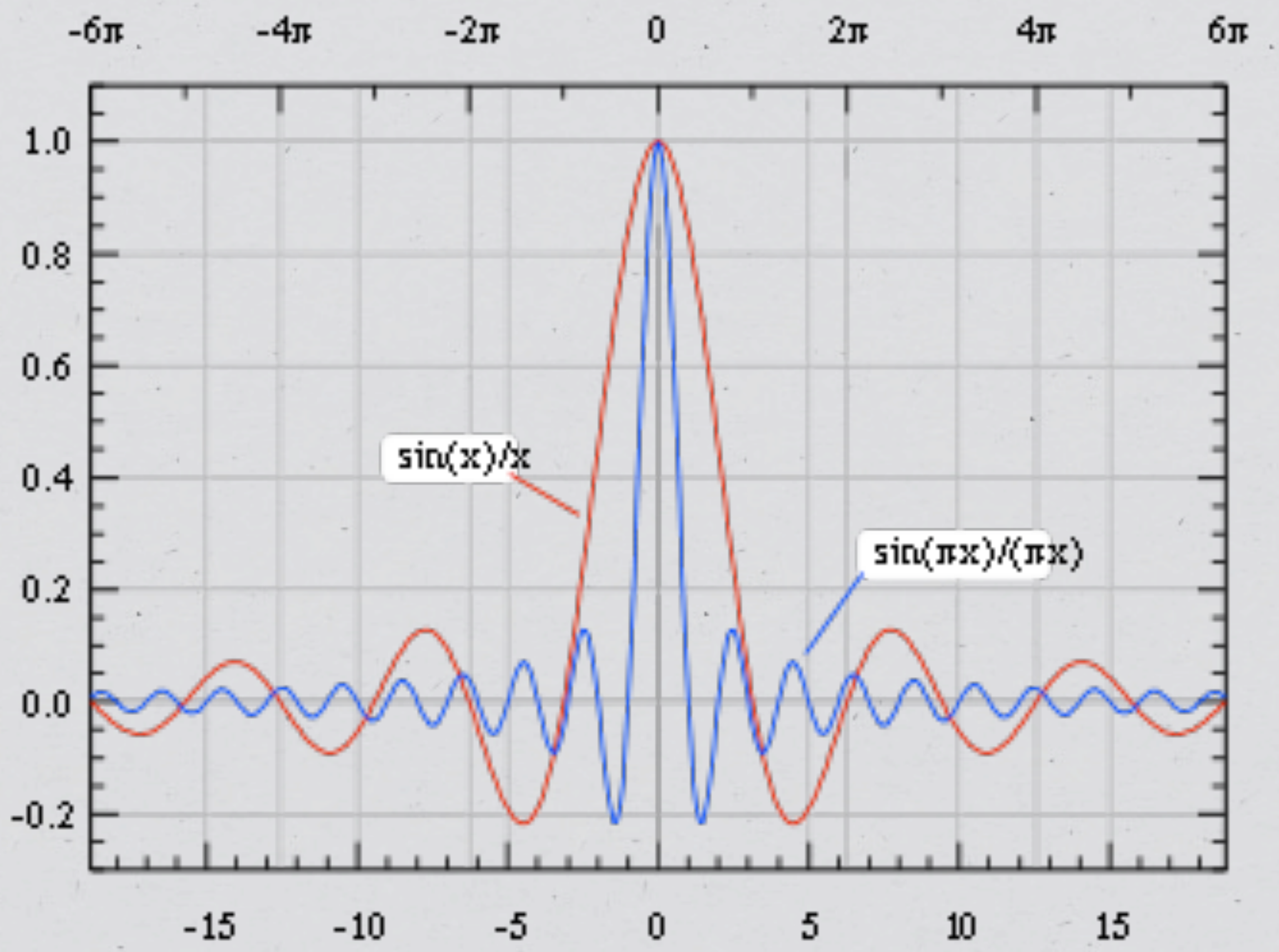


Absolute Error

Level curves: true & estimate

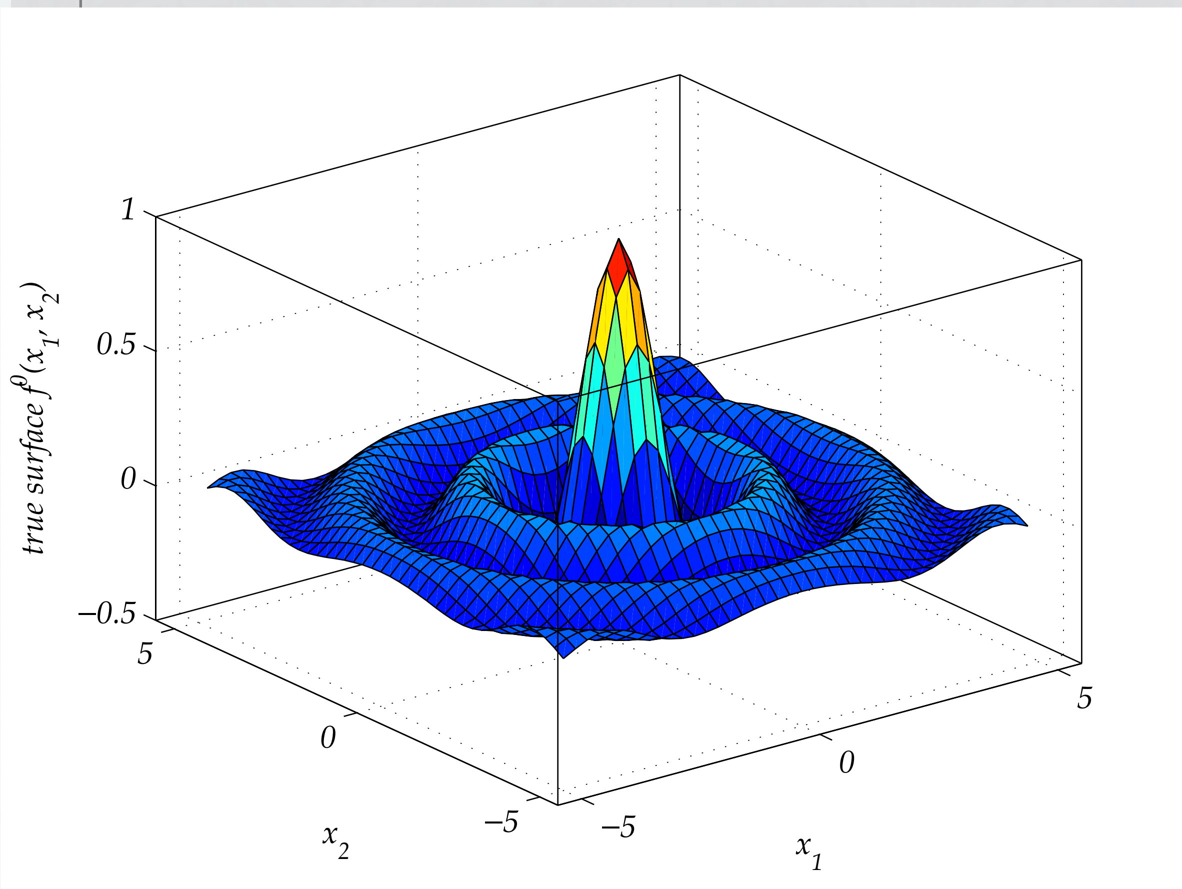
SPLINES & EPI-SPLINES

Dirichlet (sinc-)function

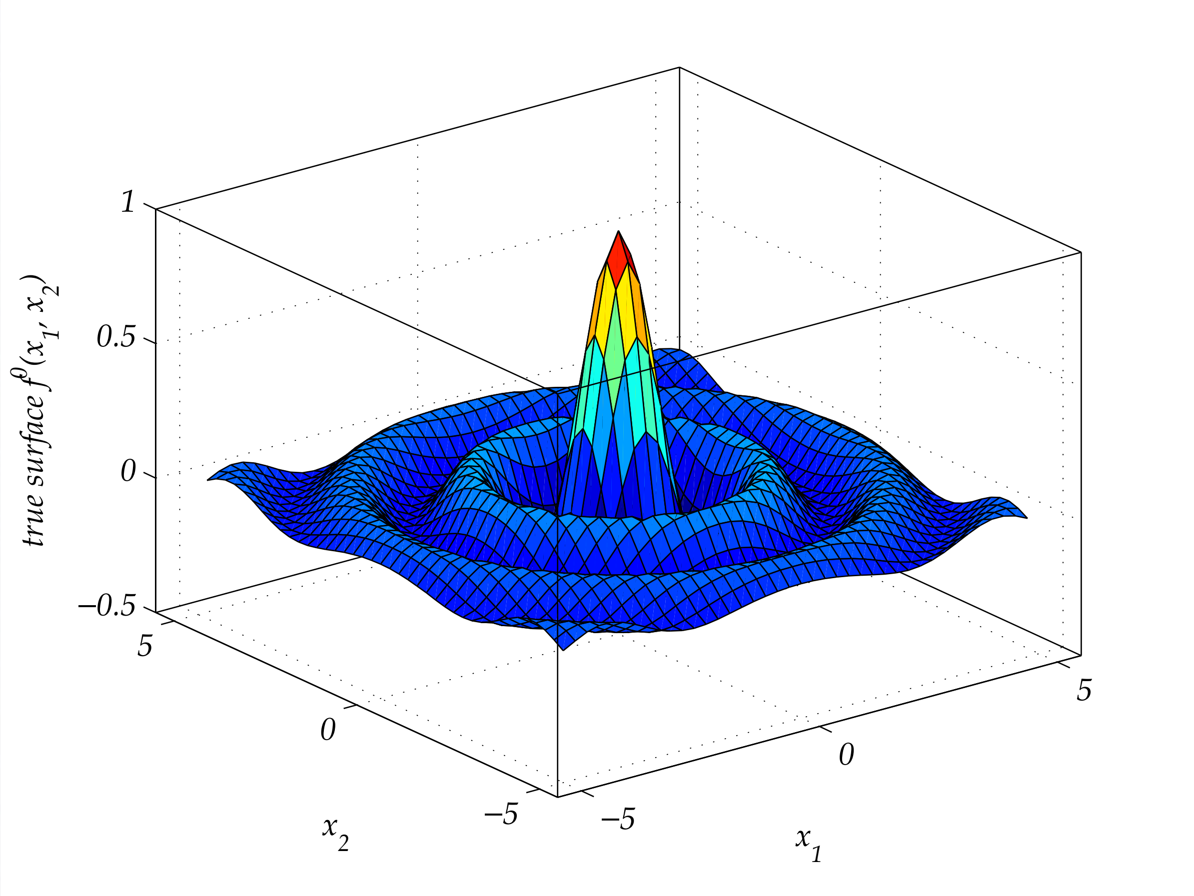


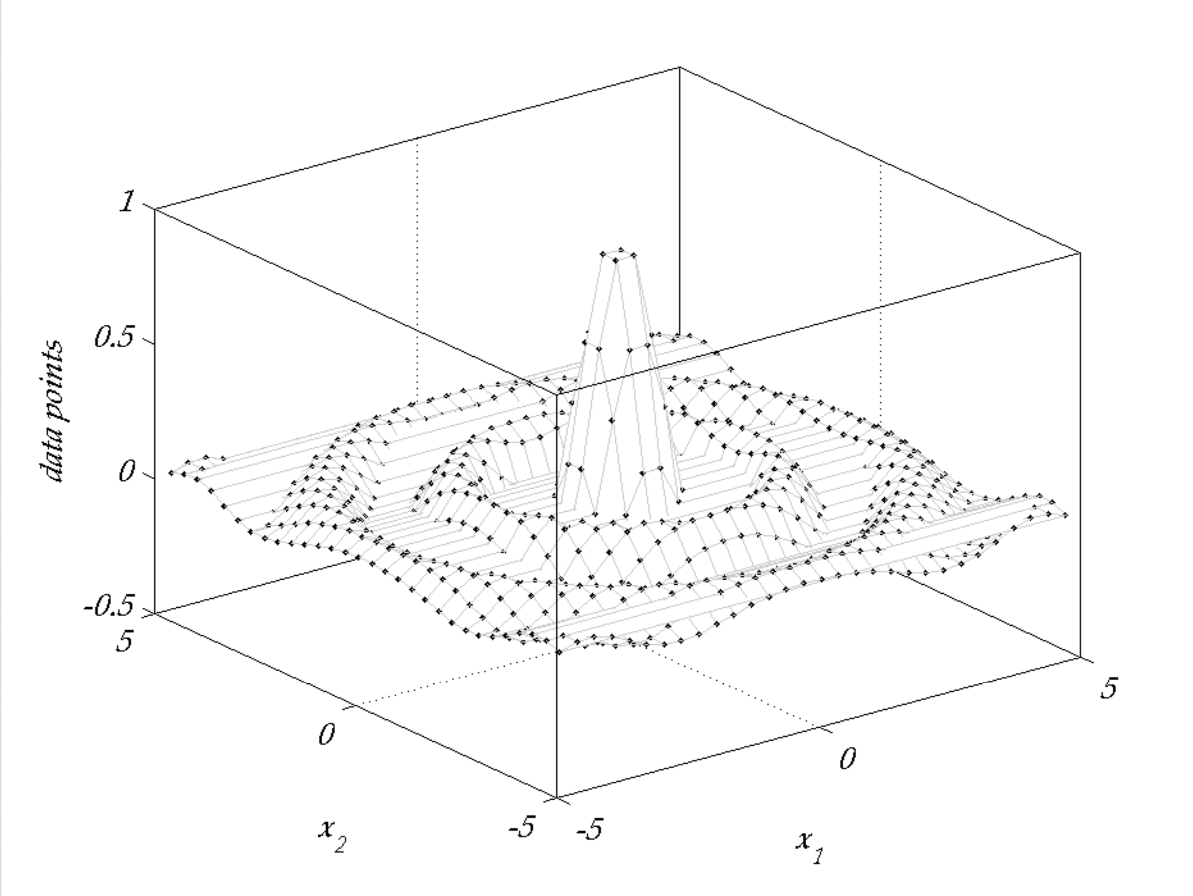
$$f(x) = \sin(\pi x / 2) / \pi x \text{ for } x \neq 0, \quad = 1 \text{ for } x = 0$$

SPLINES & EPI-splines

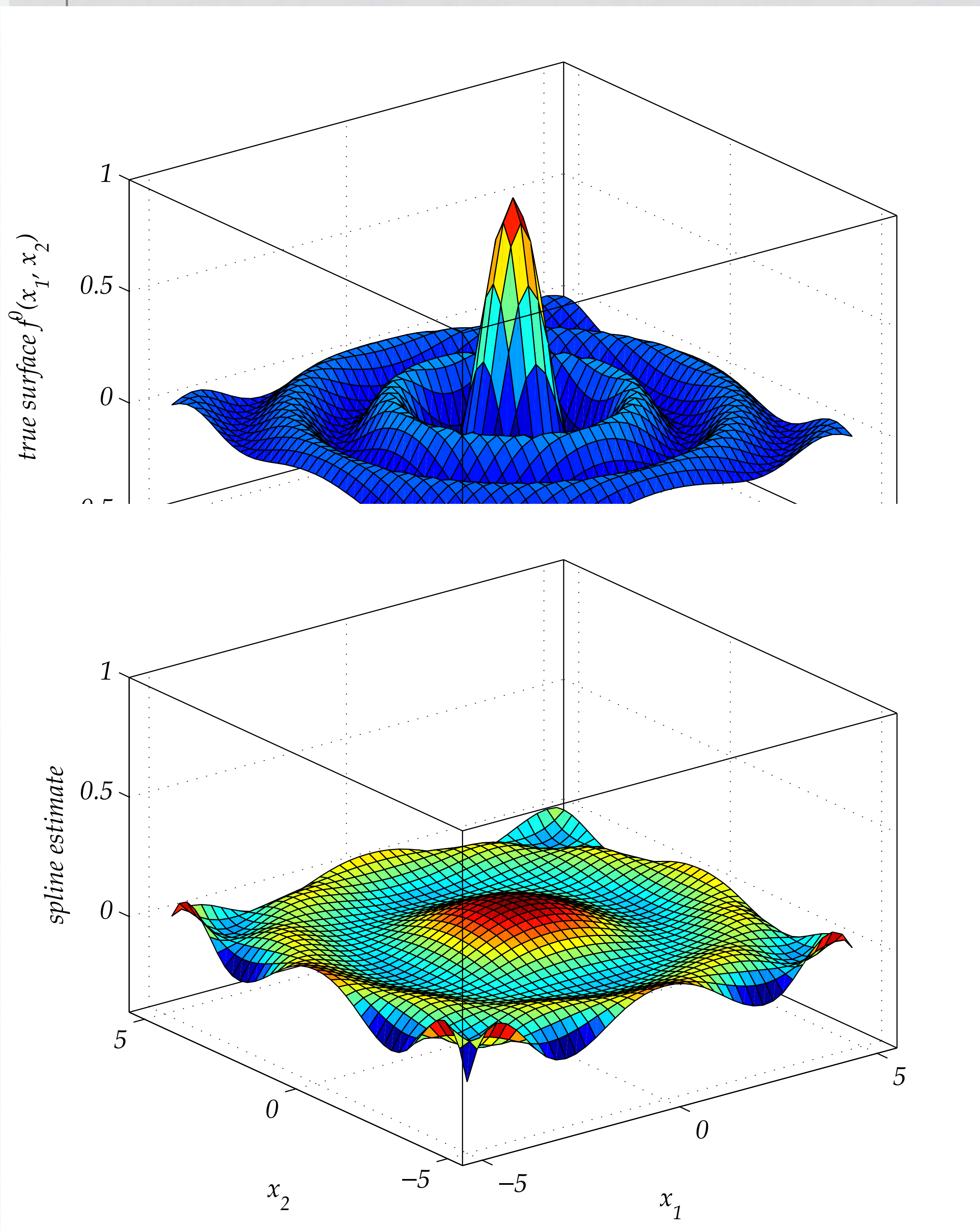


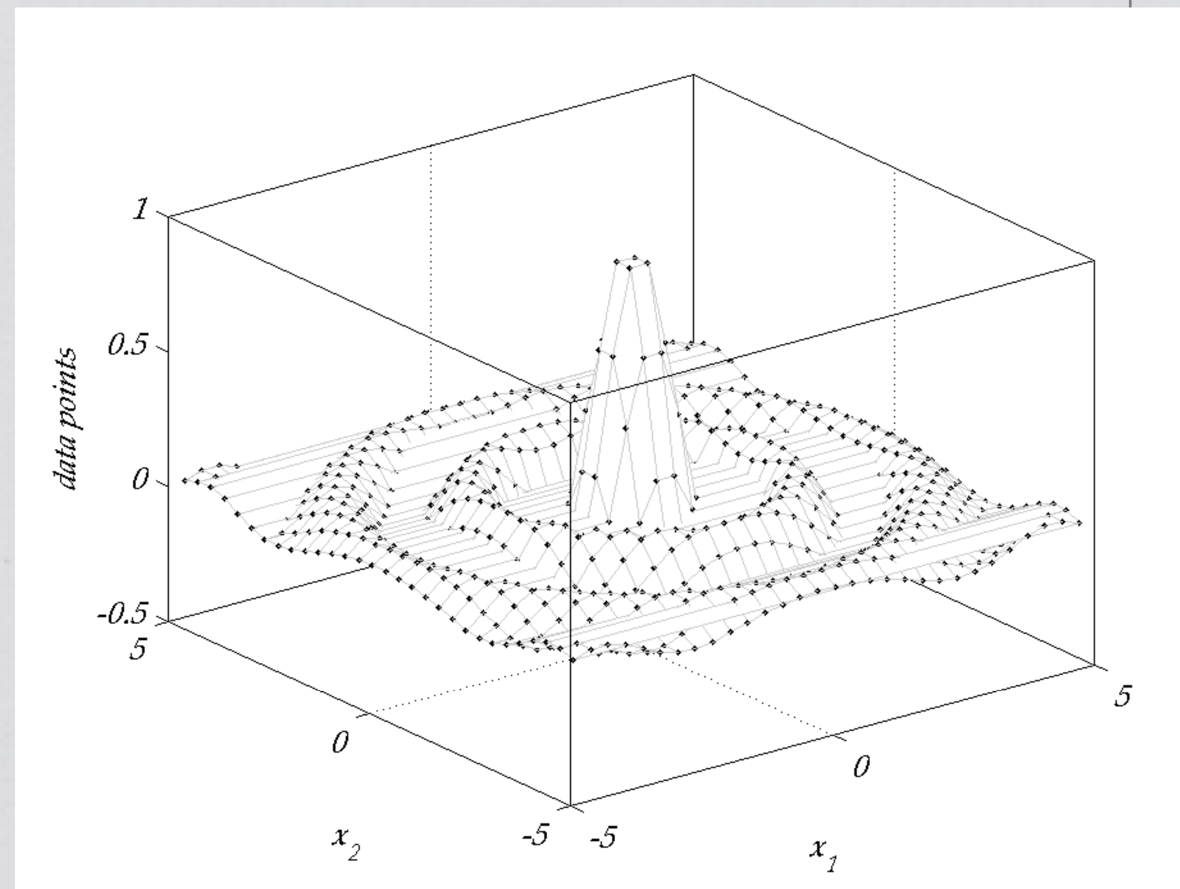
SPLINES & EPI-splines



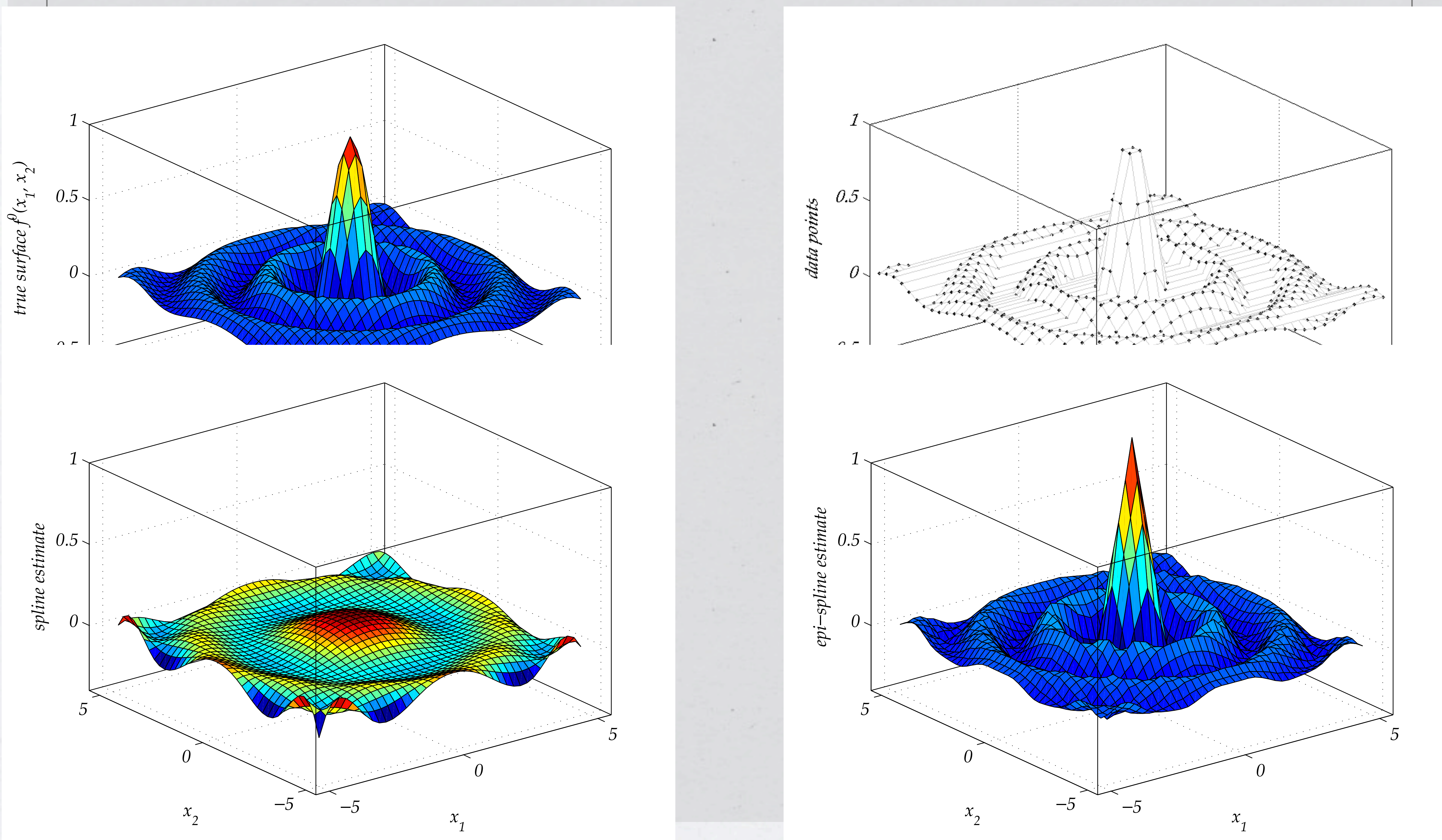


SPLINES & EPI-splines

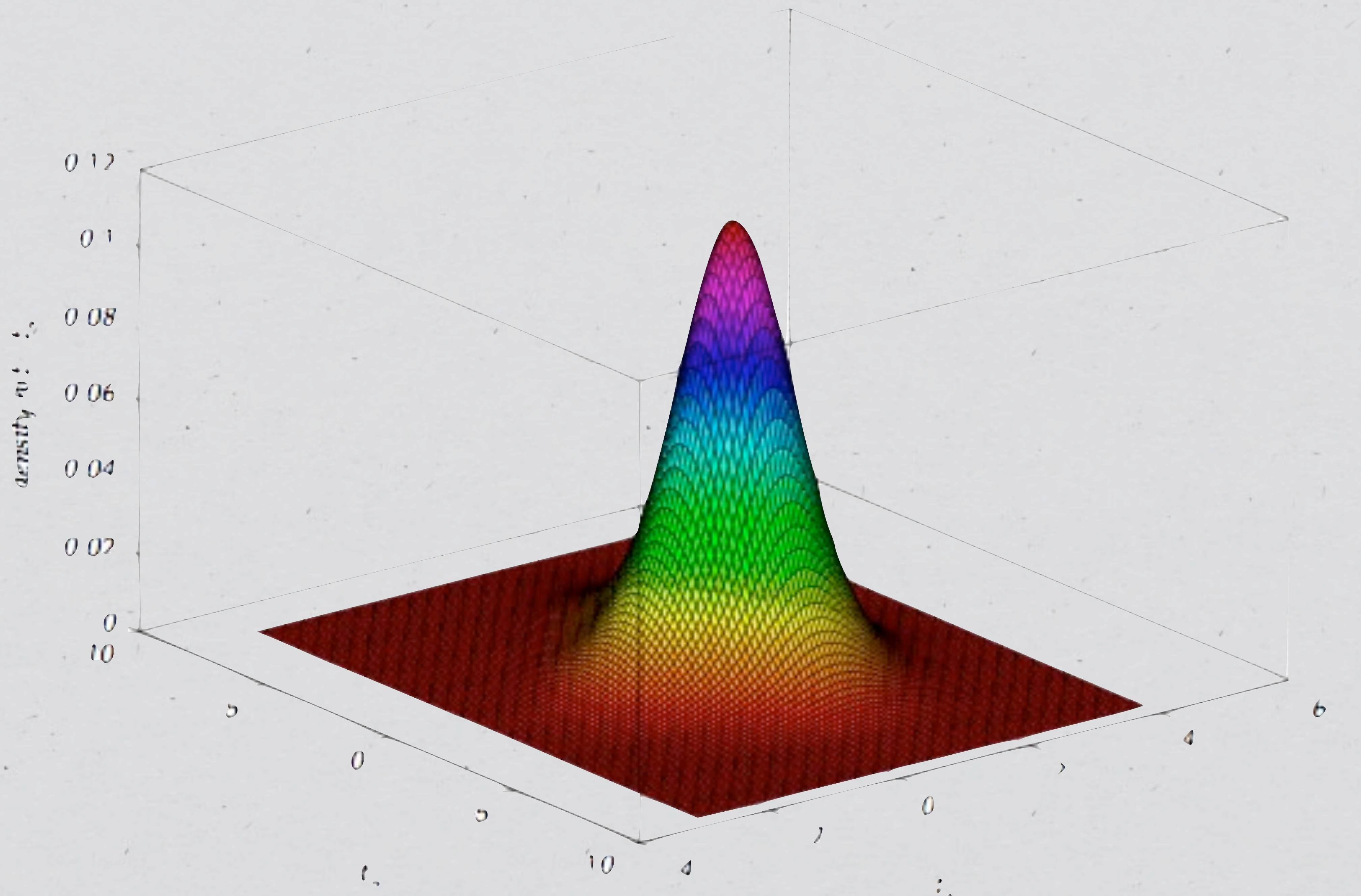




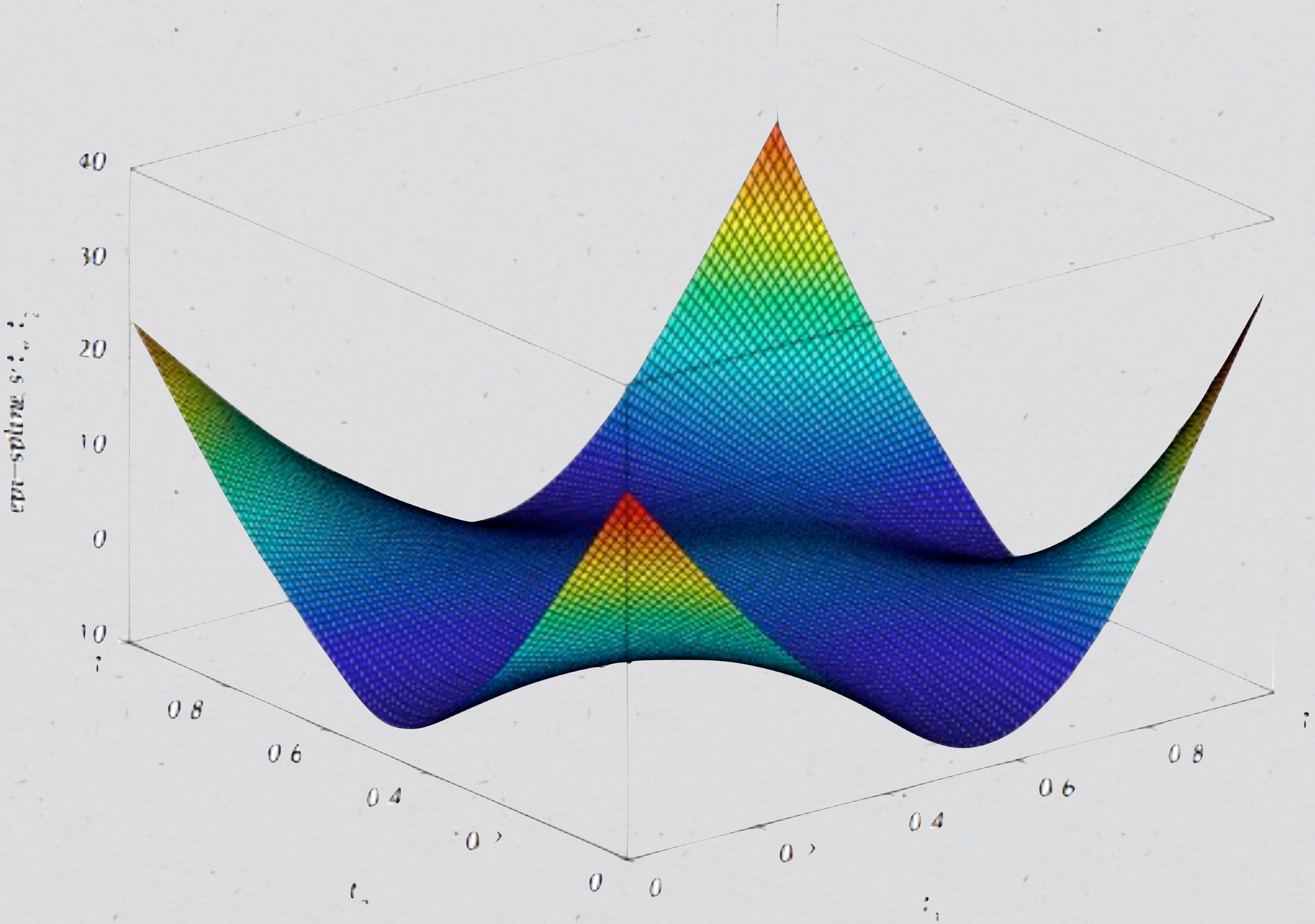
SPLINES & EPI-splines



Normal, 20 samples
Diagonal Covariance



… the -(epi-spline function)



Approximation Theory

$\min f_0(x)$ such that $x \in X \subset H$ (for our use: H Polish space)

$\min f(x)$ on $x \in X$ with $f(x) = \begin{cases} f_0(x) & \text{when } x \in X \\ \infty & \text{otherwise} \end{cases}$ lsc function: $H \to \overline{\mathbb{R}}$

$(f_0^\nu, X_0^\nu) \to (f_0, X)$ sequence of optimization problems converging(?) to, f

$\arg\min f^\nu \to \arg\min f \quad (\inf f^\nu \to \inf f)$

$\arg\min (f^\nu + g) \to \arg\min(f + g)$, g continuous perturbation

$\Rightarrow f^\nu$ epi-converges to f $(\text{epi } f^\nu \to \text{epi } f)$: for all $x \in H$,

(a) $\forall x^\nu \to x$, $\liminf_\nu f^\nu(x^\nu) \geq f(x)$,

(b) $\exists x^\nu \to x$, $\limsup_\nu f^\nu(x^\nu) \leq f(x)$.

pointwise?, uniform?,

LLN: Random lsc funtionS

$f : \Xi \times H \to \overline{\mathbb{R}}$ a random lsc function, ξ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in h, $(\forall \xi \in \Xi)$

(b) (ξ, h)-measurable $\quad (\mathcal{A} \times B_X)$-measurable

recall: $f(\xi,h) = f_0(\xi,h)$ when $h \in X(\xi)$ -- stochastic constraints

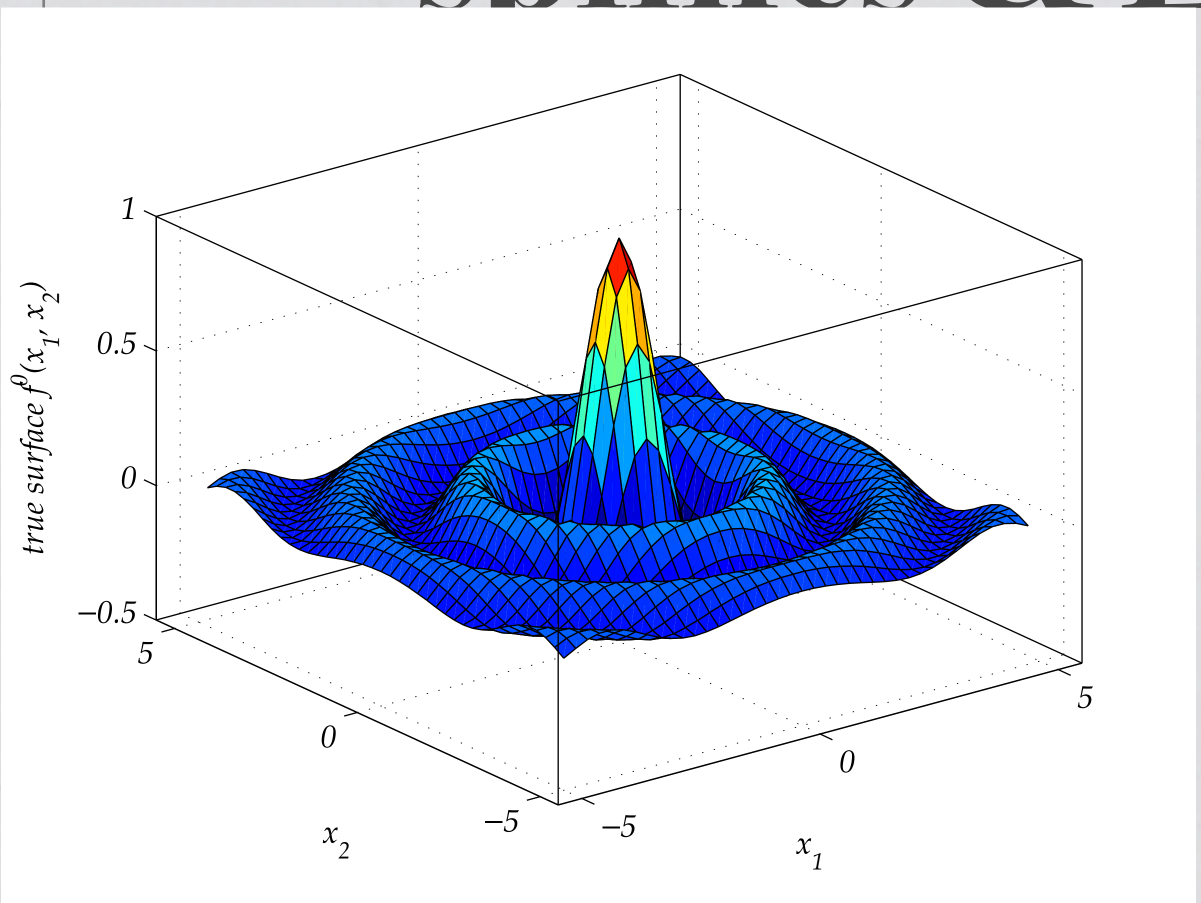
$$f^\nu(\xi,h) = \begin{cases} \frac{1}{\nu}\sum_{l=1}^{\nu} \ln h(\xi^l) \text{ if } h \geq 0, \int_\Xi h(\xi)d\xi = 1, \; h \in A^\nu \\ \infty \text{ otherwise} \quad (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi,\cdot)$ epi-converge to $\mathbb{E}\{f(\xi,h)\}$ P-a.s.?

$$h^{true} \in \arg\min \mathbb{E}\{f(\xi,h)\}$$

where $$f(\xi,h) = \begin{cases} \ln h(\xi) \text{ if } h \geq 0, \int_\Xi h(\xi)d\xi = 1, h \in A \\ \infty \text{ otherwise} \end{cases}$$

splines & Epi-Splines 2



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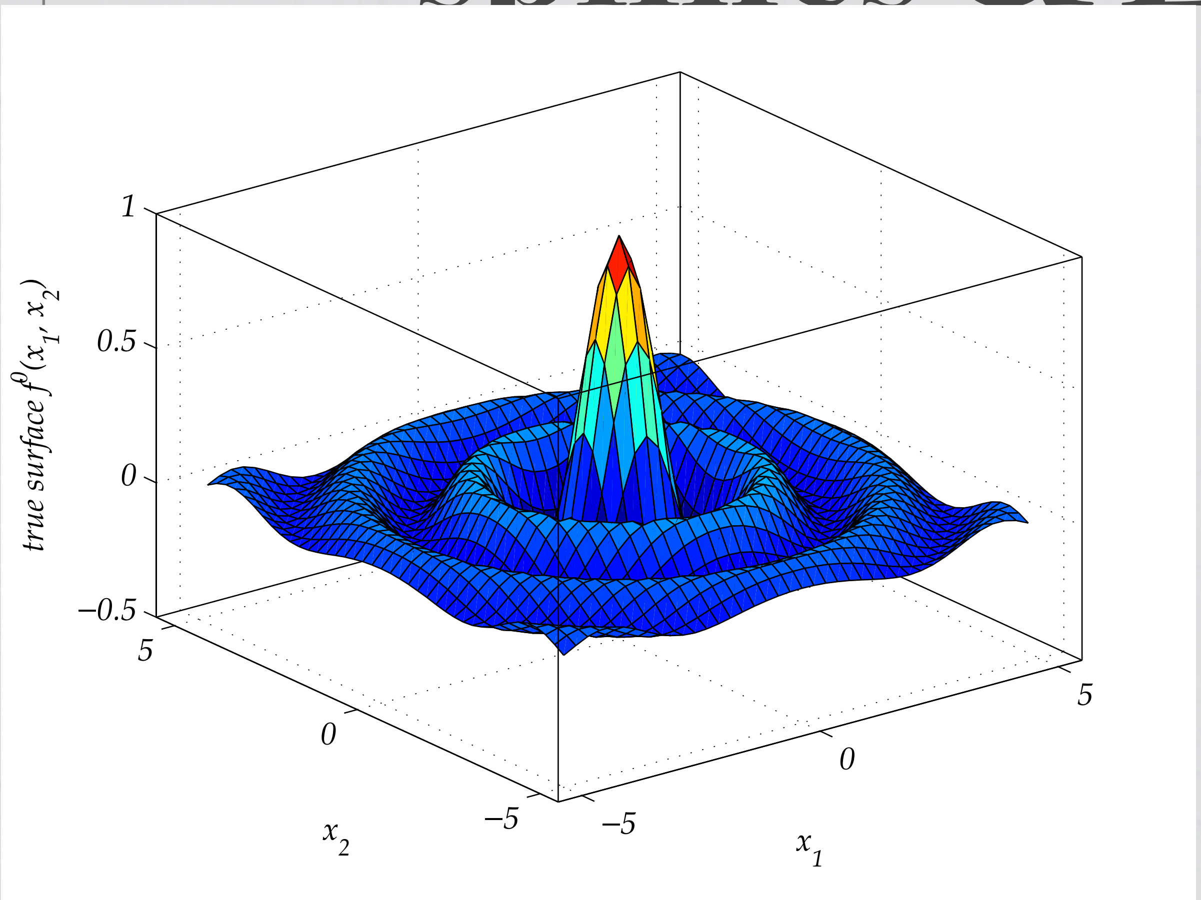

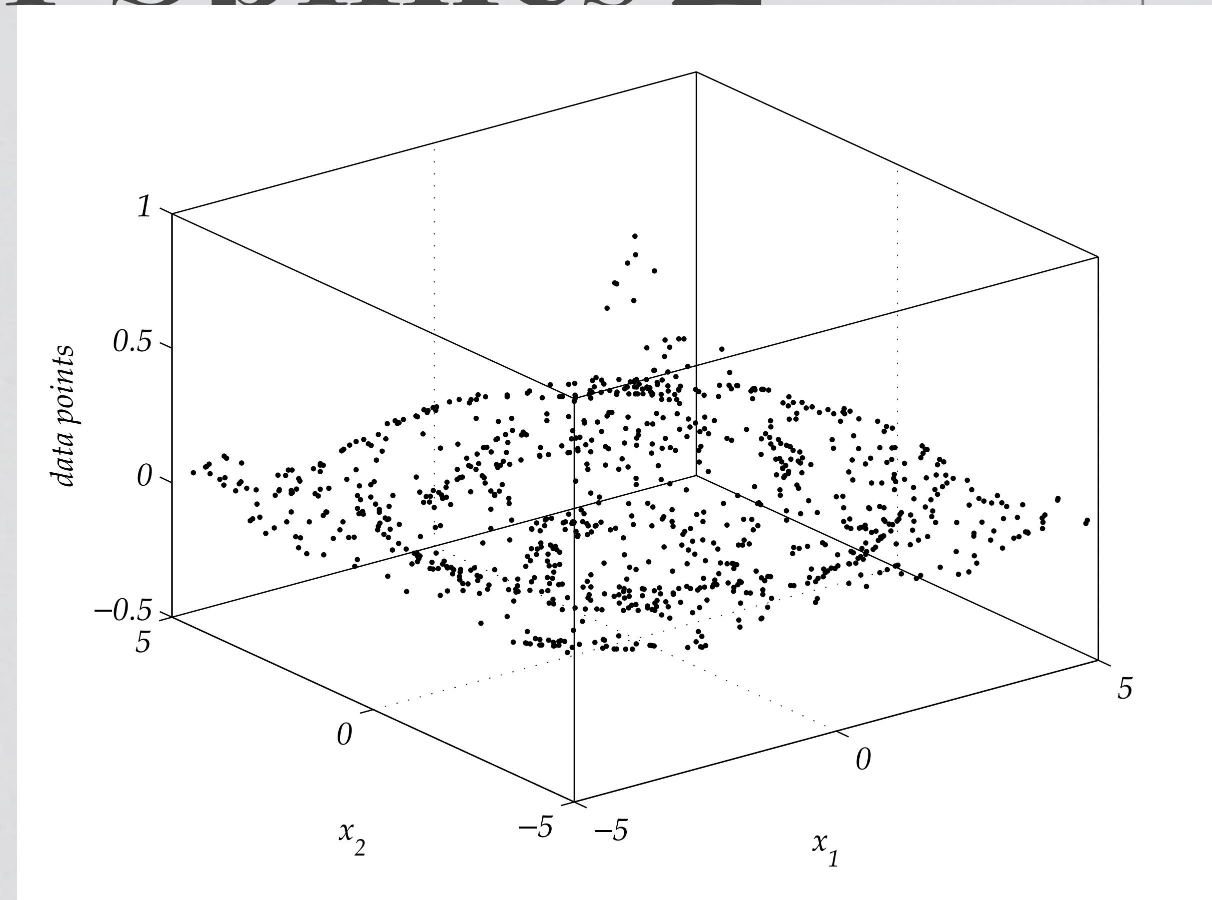

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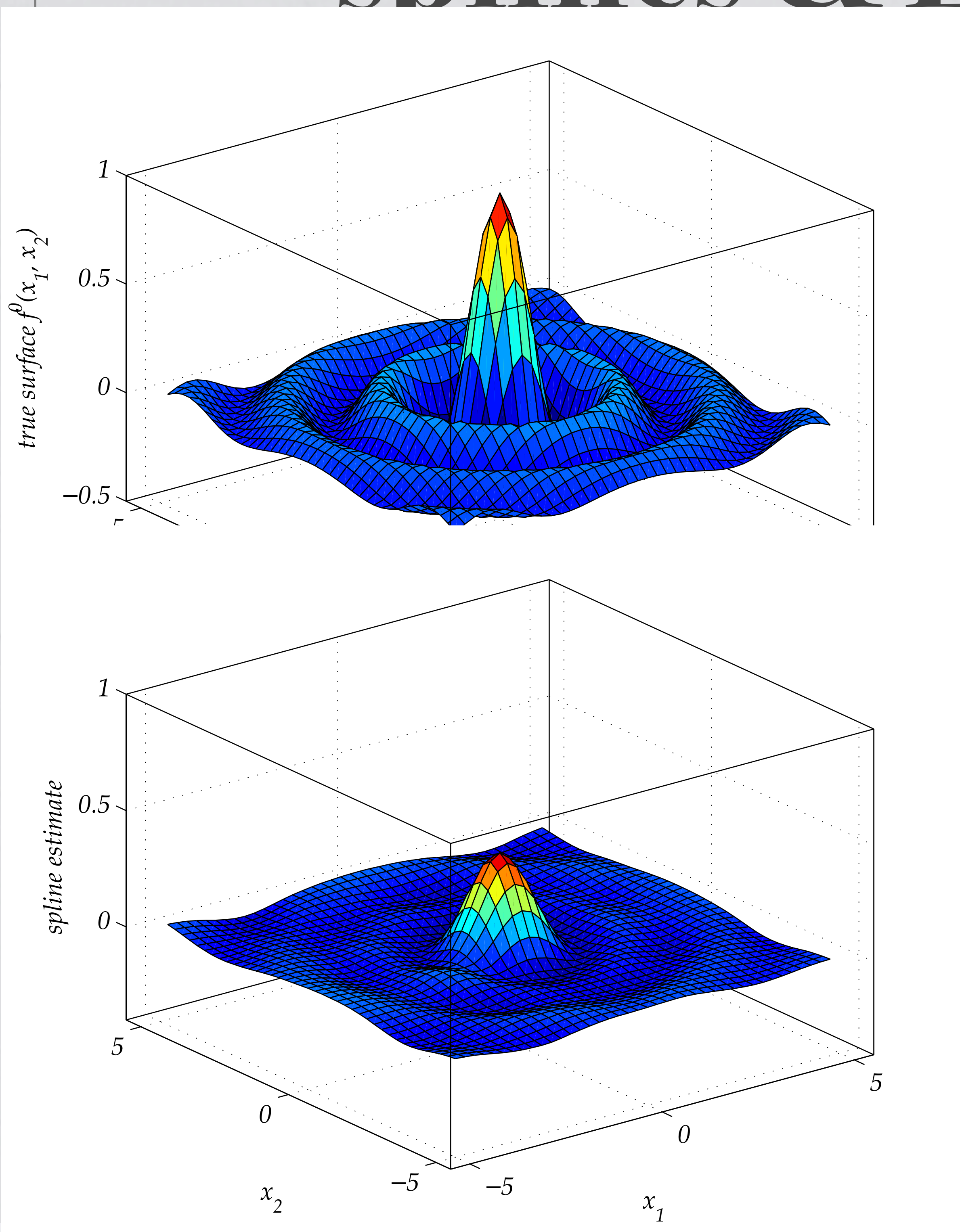

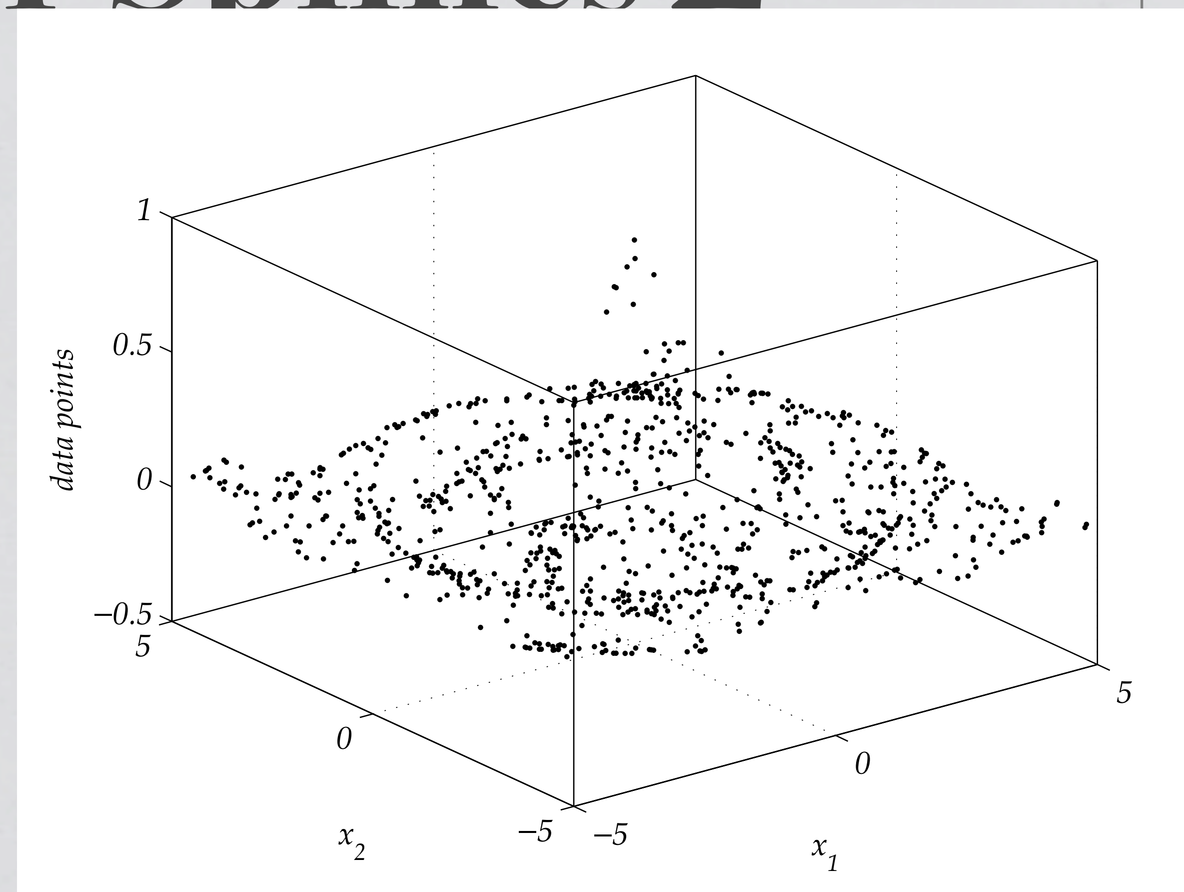

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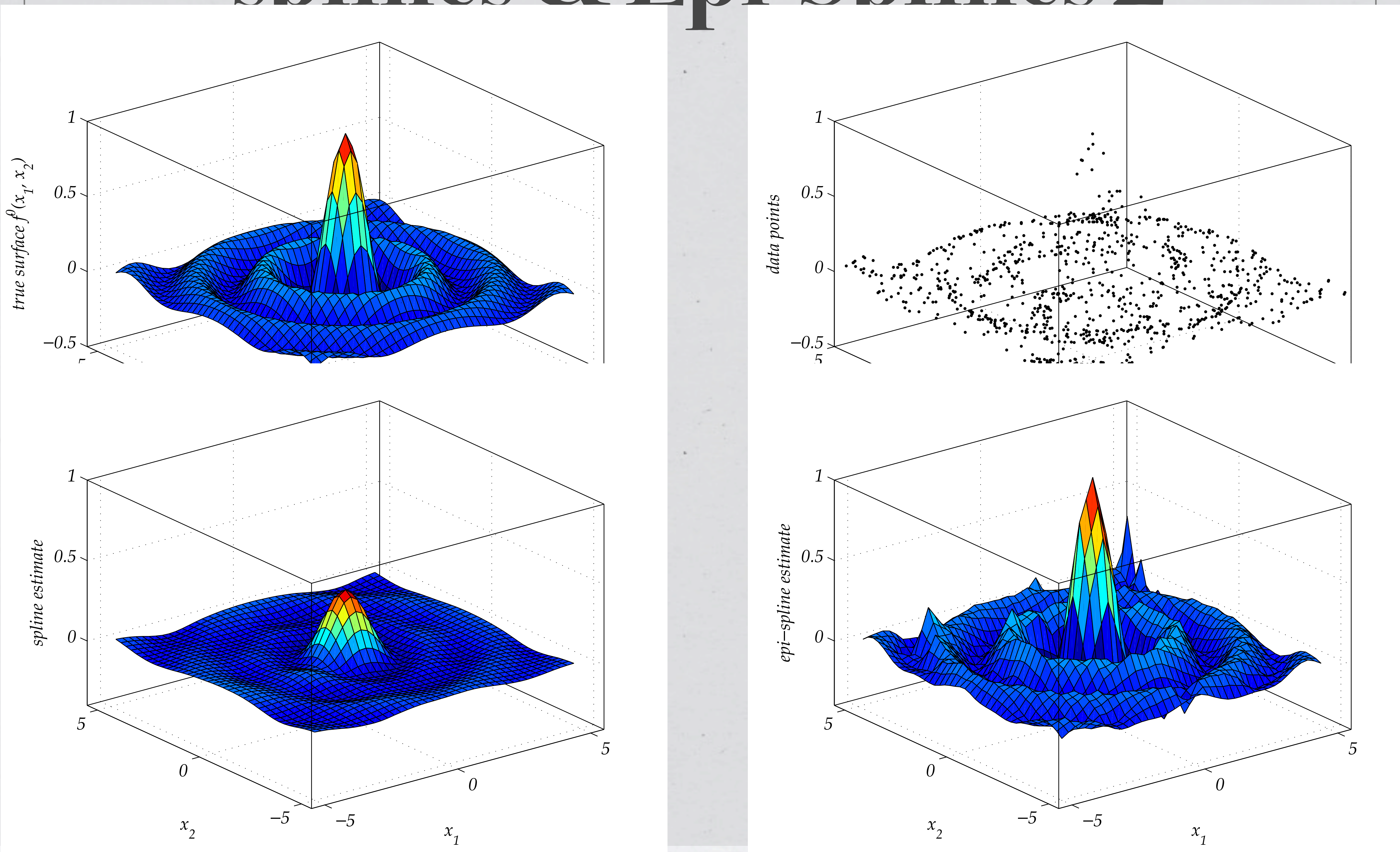

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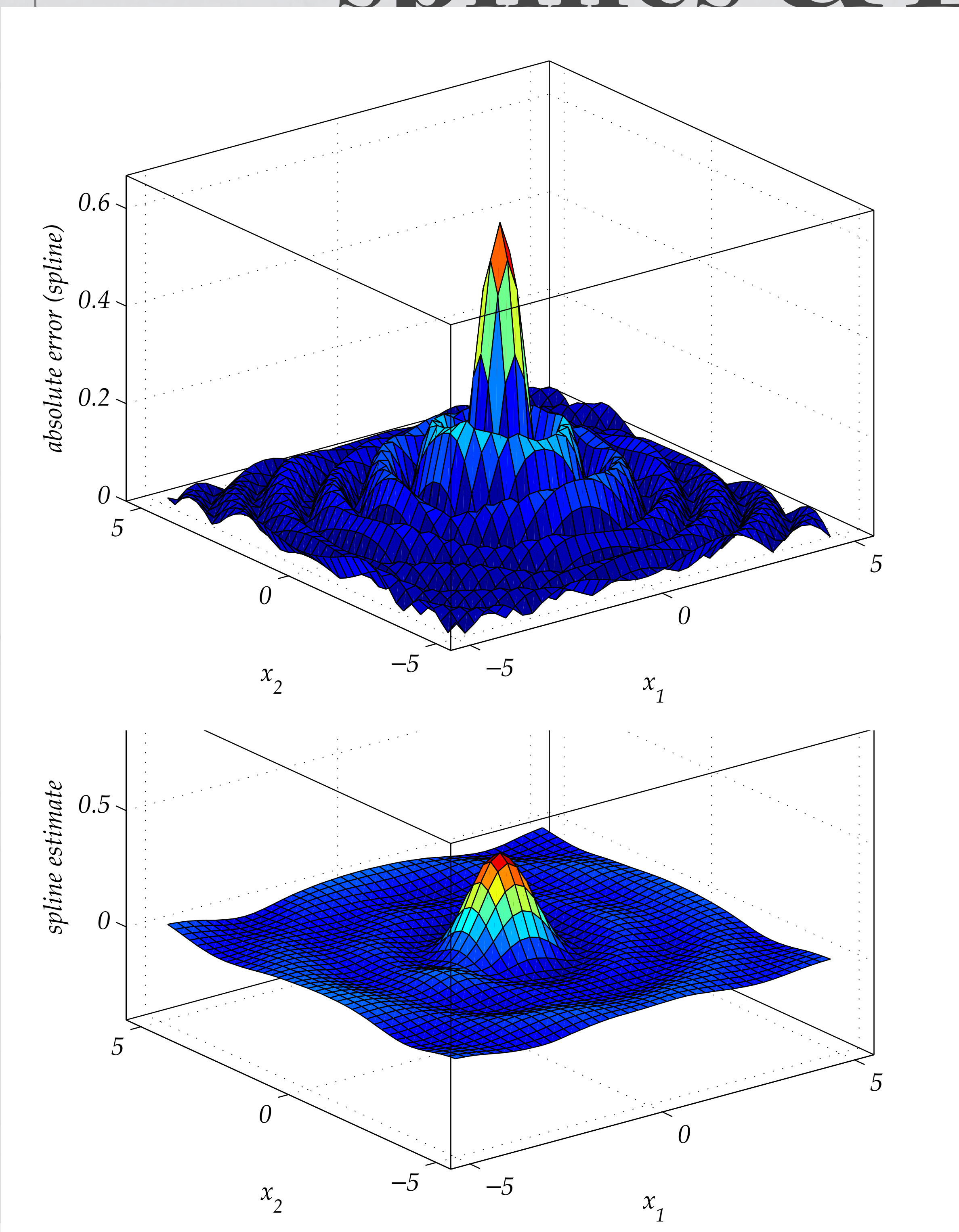

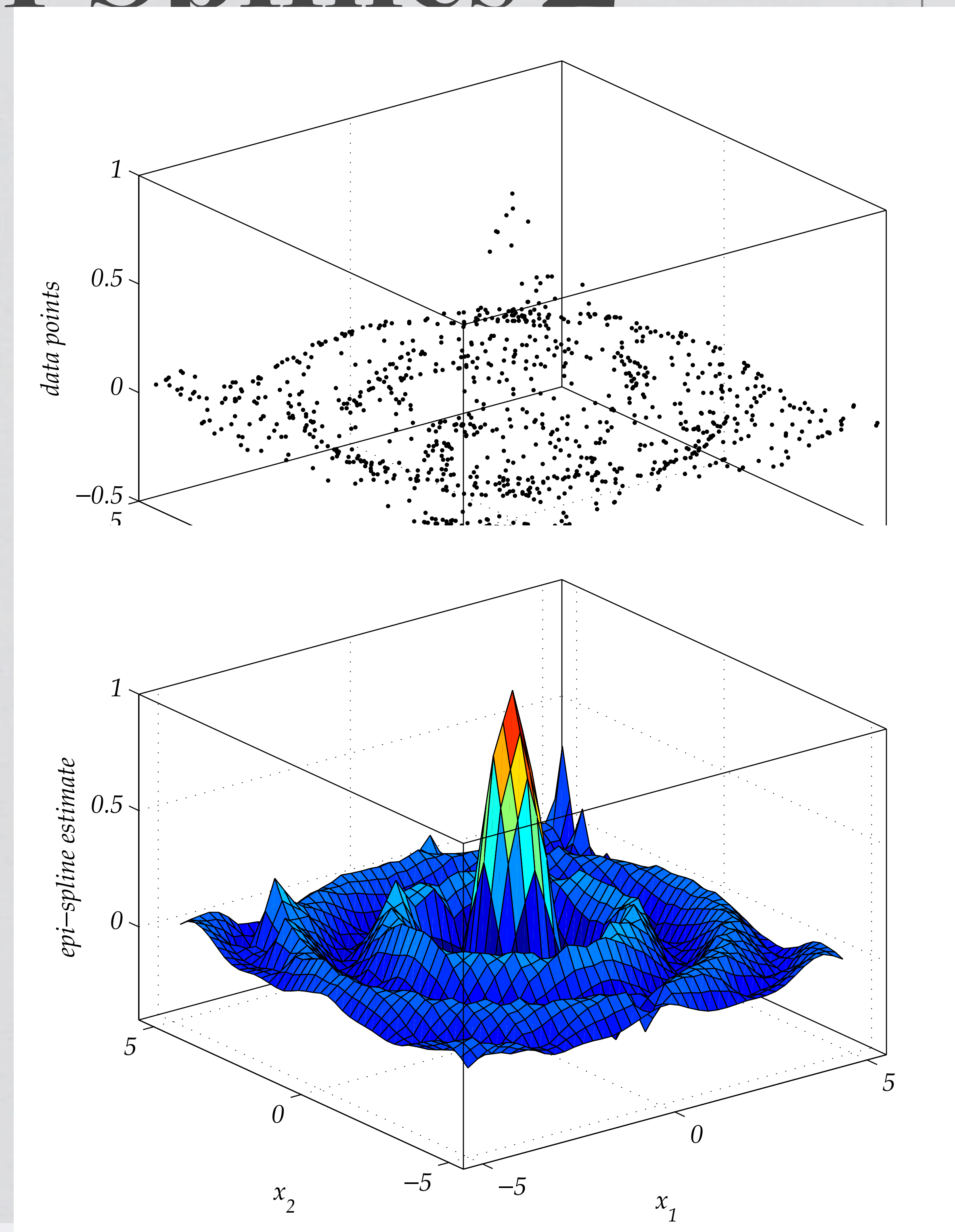

splines & Epi-Splines 2