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PROGRAMMING UNDER UNCERTAINTY: THE EQUIVALENT CONVEX PROGRAM*

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Abstract. This paper is an attempt to describe and characterize the equivalent convex program of a two-stage linear program under uncertainty. The study has been divided into two parts. In the first one, we examine the properties of the solution set of the problem and derive explicit expressions for some particular cases. The second section is devoted to the derivation of the objective function of the equivalent convex program. We show that it is convex and continuous. We also give a necessary condition for its differentiability and establish necessary and sufficient conditions for the solvability of the problem. Finally, we give the equivalent convex program of certain classes of programming under uncertainty problems, i.e., when the constraints and the probability space have particular structures.

1. Introduction. The *standard form* of the problem to be considered in this paper is:

$$
\begin{array}{llll}
 & \text{minimize} & z(x) = cx + E_\xi\{qy\}, & \\
 & \text{subject to} & Ax \qquad\qquad = b, & \\
(1) & & Tx + \quad My = \xi, & \xi \text{ on } (\Xi, \mathfrak{F}, F), \\
 & & x \geqq 0, \quad y \geqq 0, &
\end{array}
$$

where A is an $m \times n$ matrix, T is an $\bar{m} \times n$ matrix, M is an $\bar{m} \times \bar{n}$ matrix, ξ is a random vector defined on the probability space $(\Xi, \mathfrak{F}, F)$. We shall assume that (1) is solvable.

This problem belongs to the class of stochastic linear programming problems for which one seeks a *here-and-now solution*. Problem (1) is known in the literature as the *two-stage linear program under uncertainty*. One interprets it as follows: the decision maker must select the activity levels for x, say $x = \hat{x}$, he then observes the random event $\xi = \hat{\xi}$, and he is finally allowed to take a corrective action y, such that $y \geqq 0$, $My = \hat{\xi} - T\hat{x}$, and qy is minimum. This second stage decision y is taken when no uncertainties are left in the problem.

It is clear that we could also write the objective function of (1) as

$$(1') \qquad z(x) = cx + E_\xi\{\min qy \mid x\}.$$

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The interpretation given above indicates that (1) as well as (1′) are conventional ways to express the same concept. Many practical problems can be formulated to fit the standard form, e.g., inventory problems, planning problems, transportation problems with uncertain demand, etc.

All quantities considered here belong to the reals, denoted $\mathfrak{R}$. Vectors will belong to finite-dimensional spaces $\mathfrak{R}^n$ and whether they are to be regarded as row vectors or column vectors will always be clear from the context in which they appear. Thus, for example, the expressions,

$$x = (x_1, x_2, \cdots, x_i, \cdots, x_n),$$
$$Tx = \chi,$$
$$y^+y^- = \sum_{i=1}^{\bar{m}} y_i^+y_i^-,$$

are easily understood. No special provisions have been made for transposing vectors.

For the sake of simplicity, we shall assume that $(\Xi, \mathfrak{F}, F)$ is the probability space induced in $\mathfrak{R}^{\bar{m}}$, F determines a Lebesgue-Stieltjes measure and $\mathfrak{F}$ is the completion for F of the Borel algebra in $\mathfrak{R}^{\bar{m}}$. We also assume that $\bar{\xi} = E\{\xi\}$ exists. Also, note that our notation ξ on $(\Xi, \mathfrak{F}, F)$ is meant to imply that the first stage decision has no effect on the probability space on which ξ is defined. In other words, ξ is independent of x.

The marginal probability space for $i = 1, \cdots, \bar{m}$ will be denoted by $(\Xi_i, \mathfrak{F}_i, F_i)$. If it exists, we denote the density function of ξ_i by $f_i(\xi_i)$. If ξ_i is a discrete random variable, we denote its probability mass function also by $f_i(\xi_i)$. No confusion should arise from this abuse of notation. Moreover, let α_i and β_i be respectively the greatest lower bound and the least upper bound of Ξ_i. If Ξ_i is not bounded below, we set $\alpha_i = -\infty$; if Ξ_i is not bounded above, we set $\beta_i = +\infty$.

We usually think of Ξ as the convex hull of all elements of $\mathfrak{F}$ with positive measure. The probability measure may be discrete, continuous, or a mixture of both. Only in one particular case (§2A) shall we use another characterization of Ξ, namely, $\Xi = \{\xi \mid f(\xi) \neq 0\}$.

The first part of this paper characterizes the solution set of (1), and it points out some of its properties. In the second part, we derive a programming problem whose set of optimal solutions is identical to the set of optimal solutions to problem (1).

2. The solution set. We are only interested in the here-and-now decision to be taken. Thus, a solution to (1) is not a pair (x^0, y^0). To see this, it suffices to remark that once x is selected and ξ is observed, the set of optimal second stage decisions y is uniquely determined by solving the linear program

$$\text{(2)} \qquad \begin{aligned} &\text{minimize} && qy, \\ &\text{subject to} && My = \xi - Tx, \\ & && y \geqq 0. \end{aligned}$$

It is thus obvious that the only decision variable of problem (1) is x.

Nevertheless, the second stage affects our decision on x in two ways. First, we need to limit our set of acceptable first stage decisions to those for which there exists a feasible second stage decision, i.e., problem (2) is feasible. Also, for each selection of a vector x, we must take into account the expected costs of the second stage decisions such an x may generate: $E_\xi\{\min qy \mid x\}$.

2A. The set of feasible solutions. *A feasible solution to* (1) is a vector x such that it satisfies the first stage constraints and such that it is always possible to find a feasible solution to the second stage problem (2), whatever be the value assumed by ξ on Ξ. Dantzig and Madansky [2] call such a solution a permanently feasible solution. The word "permanently" was introduced to reinforce this notion of feasibility of the second stage problem for all values of ξ. We have rejected this terminology because it sometimes leads to confusion in the understanding of problem (1).

The following example shows how the Dantzig-Madansky definition of permanent feasibility differs from what one may believe to be meant by permanent feasibility. We reserve the terms "permanently feasible" for the following concept: select a vector x such that the constraints are satisfied with probability one. Consider the following problem:

$$\text{(3)} \qquad \begin{aligned} &\text{minimize} && z(x) = cx + Q(Tx - \xi), \\ & && x \in \Omega, \end{aligned}$$

where ξ is an $\bar{m}$-dimensional random vector on $(\Xi, \mathfrak{F}, F)$, T is an $\bar{m} \times n$ matrix, $\Omega = \{x \mid Ax = b, x \geqq 0\} \subset \mathfrak{R}^n$, and Q is a real-valued function. If Q is defined as follows:

$$\begin{aligned} Q(Tx - \xi) &= 0 && \text{if } Tx \geqq \xi, \\ Q(Tx - \xi) &= +\infty, && \text{otherwise,} \end{aligned}$$

then, for each fixed ξ, (3) is a linear programming problem. Such a function $Q(Tx - \xi)$ requires permanent feasibility, i.e., if there exists a solution $(\bar{z}(x) \neq +\infty)$ to problem (3) it must satisfy the condition

$$\text{(4)} \qquad Tx \geqq \xi, \qquad \text{for every } \xi \in \Xi.$$

To see that problem (1) is not as restrictive, e.g., let

$$Q(Tx - \xi) = E_\xi\left\{\sum_{i=1}^{\bar{m}} Q_i(T_i x - \xi_i)\right\},$$

where

$$Q_i(T_i x - \xi_i) = 0 \qquad \text{if } T_i x \geqq \xi_i,$$

$$Q_i(T_i x - \xi_i) = q_i^-(\xi_i - T_i x) \qquad \text{if } T_i x \leqq \xi_i.$$

Such a function $Q(Tx - \xi)$ no longer imposes permanent feasibility, i.e., $z(x)$ is no longer identically equal to $+\infty$ for all x which do not satisfy condition (4). We can then rewrite (3) as follows:

$$(5) \quad \begin{array}{lrl} \text{minimize} & z(x) = cx + E_\xi\{0 \cdot y^+ + q^- y^-\}, & \\ \text{subject to} & Ax & = b, \\ & Tx + Iy^+ - Iy^- = \xi, & \xi \text{ on } (\Xi, \mathfrak{F}, F), \\ & x \geqq 0, \quad y^+ \geqq 0, \quad y^- \geqq 0. & \end{array}$$

Problem (5) is a special case of problem (1), known as the complete problem [6].

From our definition of feasible solution, it is clear that the decision maker is limited in its decision by a double set of constraints. Let

$$K_1 = \{x \mid Ax = b, x \geqq 0\}.$$

We say that K_1 is the set determined by the *fixed constraints.*

(6) PROPOSITION. *K_1 is a convex polyhedron.*

A set C *is convex* if $x_1, x_2 \in C$ implies $[x_1, x_2] \subset C$. By *convex polyhedron* we mean that K_1 can be written as the sum of a convex polytope (convex hull of a finite number of points in $\mathfrak{R}^n$) and a convex polyhedral cone.

Let

$$K_2 = \{x \mid \text{for every } \xi \in \Xi, \text{ there exists } y \geqq 0 \text{ such that } My = \xi - Tx\}.$$

We say that K_2 is the set representing the constraints imposed on our vector x by the *induced constraints.* The word "induced" means that these constraints are the restrictions imposed on x by the condition: the second stage problem (2) must be feasible for all $\xi \in \Xi$. This is the meaning of the equality sign found in the constraints of the standard form:

$$Tx + My = \xi, \qquad \xi \text{ on } (\Xi, \mathfrak{F}, F).$$

Let

$$K_{2\xi} = \{x \mid Tx = \xi - My \text{ for some } y \geqq 0\}.$$

It is easy to see that $K_{2\xi}$ is a convex polyhedron.

(7) PROPOSITION. *K_2 is convex.*

We have $K_2 = \bigcap_{\xi \in \Xi} K_{2\xi}$; then K_2 is either empty, a singleton or for all pairs of points $x_1, x_2 \in K_2$ we have $x_1, x_2 \in K_{2\xi}$ for all $\xi \in \Xi$. Then for every $\xi \in \Xi$, $[x_1, x_2] \subset K_{2\xi}$; hence $[x_1, x_2] \subset \bigcap_{\xi \in \Xi} K_{2\xi} = K_2$.

The next result is an immediate consequence of (6) and (7).

(8) PROPOSITION. *$K = K_1 \cap K_2$ is a convex set, where K is the set of feasible solutions.*

Remark. We have expressed the set of feasible solutions in terms of x alone, rather than x and y.

In what follows, we assume that K has full dimension. If this were not the case, one would need to appeal to the relative topology. Most of our proofs do not require this assumption, but it simplifies our treatment and terminology.

The set K_1 is immediately available in terms of linear equations and inequalities involving x only. The set K_2 presents much more difficulty. In general, say when Ξ is a continuum, i.e., when $\bigcap_{\xi \in \Xi} K_{2\xi}$ is an infinite intersection of convex polyhedrons, then the characterization of K_2 in terms of x alone is a much more complex problem. One main difficulty one encounters in trying to solve a program under uncertainty (no assumptions on the probability space or on the structure of the constraints of (1)) lies in determining whether or not a given x belongs to K.

We now examine some special cases where the assumptions made either on the constraints structure of problem (2) or on the probability space $(\Xi, \mathfrak{F}, F)$ allow us to obtain fairly easily an explicit expression for the set K_2 (and so for K).

Case 1. *Ξ has a finite number of points* (card $|\Xi| < \infty$). The intersection $\bigcap_{\xi \in \Xi}$ is finite and since $K_{2\xi}$ is a convex polyhedron, so is K_2, and so is K. Let $\xi^1, \xi^2, \cdots, \xi^k$ be the values of ξ for which $f(\xi) \neq 0$. Then,

$$K_2 = \{x \mid Tx + My^l = \xi^l, \quad l = 1, \cdots, k\}.$$

Case 2. *The matrix $M = I$ (identity) and Ξ is compact.* Then

$$K_2 = \{x \mid \text{for every } \xi \in \Xi, \text{ there exists } y \geqq 0 \text{ and } y = \xi - Tx\},$$

which implies

$$x \in K_2 \quad \text{if and only if for every} \quad \xi \in \Xi, \qquad \xi - Tx \geqq 0.$$

Since Ξ is bounded, there exists a smallest closed interval, say $\Xi^* \subset \mathfrak{R}^{\bar{m}}$, with lower bound α, such that $\Xi \subset \Xi^*$. The α_i's correspond to the lower bounds for the random variables ξ_i, $i = 1, \cdots, \bar{m}$.

(9) PROPOSITION. *$\xi - Tx \geqq 0$ for every $\xi \in \Xi$ if and only if $Tx \leqq \alpha$.*

The proof of this proposition is trivial. We have

$$K_2 = \{x \mid Tx \leqq \alpha\}. \tag{10}$$

Case 3. *$M = (I, -I)$. The problem is complete.* One says that problem (1) is *complete* [6] when the matrix M (after an appropriate rearrangement of rows and columns) can be partitioned in two parts, where the first part is the identity matrix and the second part is the negative of an identity matrix, $M = (I, -I)$. This case seems to represent a very important class of applications of programming under uncertainty. It is thus an encouraging fact that the set K can be expressed immediately in terms of linear constraints in x. No assumption at all is necessary on the probability space $(\Xi, \mathfrak{F}, F)$.

Let us partition the vector y as follows:

$$y = (y^+, y^-),$$

where y^+ corresponds to I and y^- to $-I$; then

$$K_2 = \{x \mid \text{for every } \xi \in \Xi, \text{ there exists } y^+ \geqq 0, y^- \geqq 0 \text{ such that } y^+ - y^- = \xi - Tx\}.$$

(11) PROPOSITION. $K = K_1$.

Since $K_2 = \mathfrak{R}^n$ (it is always possible to express any number as the difference of two nonnegative numbers), we have $K = K_1 \cap \mathfrak{R}^n = K_1$.

This property, $K = K_1$, gives an intuitive justification for the use of the word "complete". Nevertheless, we should remark that $K = K_1$ does not imply that $M = (I, -I)$.

2B. A feasibility test. We now fix x and ξ and concentrate our attention on the feasibility of problem (2). From Farkas' lemma we get:

(12) Either the equations $My = \xi - Tx$ have a nonnegative solution or the inequalities $uM \geqq 0$, $u(\xi - Tx) < 0$, have a solution.

(13) PROPOSITION. $x \in K_2$ *if and only if for every* $\xi \in \Xi$ *we have* $U(x, \xi) \geqq 0$, *where*

$$U(x, \xi) = \{\min u(\xi - Tx) \mid uM \geqq 0\}.$$

If for a given $\bar{x}$ and for every $\xi \in \Xi$ we have $U(\bar{x}, \xi) \geqq 0$, then the system of inequalities, $uM \geqq 0$ and $u(\xi - Tx) < 0$, has no solution. By (12), the system $My = \xi - T\bar{x}$ has then a nonnegative solution, for all $\xi \in \Xi$. This means that $\bar{x} \in K_2$.

Proposition (13) yields a test which allows us to determine if a given $x \in K_1$ is or is not a feasible solution to (1). Nonetheless, such a procedure would be completely inefficient if we had to perform this test for all ξ in Ξ. If Ξ does not have finite cardinality, this test for any given x would involve solving an infinite number of linear programs of the form,

$$\begin{aligned} &\text{minimize} && u(\xi - Tx), \\ &\text{subject to} && uM \geqq 0. \end{aligned}$$

If problem (1) is stated in a slightly different form (it is very often possible to reduce problem (1) to (14)), viz.,

$$
(14)\qquad
\begin{aligned}
&\text{minimize} && z(x) = cx + E_\xi\{qy\},\\
&\text{subject to} && Ax \qquad\qquad = b,\\
& && Tx + \quad My \geqq \xi, \qquad \xi \text{ on } (\Xi, \mathfrak{F}, F),\\
& && x \geqq 0, \quad y \geqq 0,
\end{aligned}
$$

it is possible to obtain a more efficient test. We then apply the following form of Farkas' lemma: exactly one of the two alternatives hold: either the inequality, $My \geqq \xi - Tx$, has a nonnegative solution, or the inequalities

$$uM \geqq 0, \qquad u(\xi - Tx) < 0,$$

have a nonnegative solution.

(15) PROPOSITION. *$x \in K$ if and only if $x \in K_1$ and for every $\xi \in \Xi$, $U(x, \xi) \geqq 0$, where*

$$U(x, \xi) = \{\min u(\xi - Tx) \,|uM \geqq 0, u \geqq 0\}.$$

If Ξ has a lower bound—from a practical point of view this is a very mild condition—then let $\boldsymbol{\alpha}$ be such that $\boldsymbol{\alpha} \in \Xi$ and $\alpha_i \leqq \xi_i$ for all $\xi_i \in \Xi_i$, $i = 1, \cdots, \bar{m}$. Since u is restricted to be nonnegative, we have

$$U(x, \alpha) \leqq U(x, \xi), \qquad \text{for every } \xi \in \Xi.$$

Moreover, $\alpha \in \Xi$ and $U(x, \alpha) < 0$ imply that there exists at least one point of Ξ for which the condition $U(x, \xi) \geqq 0$ does not hold. By (10) this x is not a feasible solution. We have proved:

(16) PROPOSITION. *$x \in K$ if and only if $x \in K_1$ and $U(x, \alpha) \geqq 0$.*

For this case, it is thus sufficient to solve one linear program to test the feasibility of a given x which belongs to K_1. Proposition (11) is not true if $\alpha_i \leqq \xi_i$ for all $\xi_i \in \Xi_i$, $i = 1, \cdots, \bar{m}$, but $\alpha \notin \Xi$. For instance, consider the following example. Let

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

$$\Xi = \{\xi \,|\, -1 \leqq \xi_1 \leqq 0, \quad \xi_2 \leqq 2, \quad \xi_1 + \xi_2 \geqq 0\},$$

and let $\bar{x} = (\bar{x}_1, \bar{x}_2) = (0, 0)$ belong to K_1. By definition of Ξ, $\alpha = (\alpha_1, \alpha_2) = (-1, 0)$. It is easy to see that $\Xi \subset \{\zeta \mid \zeta = My, y \geqq 0\}$ and that the affine transformation obtained by translating Ξ by Tx, maps Ξ into itself ($T\bar{x} = 0$), i.e., $\bar{x} = 0 \in K$. But $U(\bar{x}, \alpha)$ is not bounded below.

Suppose now that we have at hand $\hat{x}$ such that $\hat{x} \in K_1$ and $U(\hat{x}, \alpha) < 0$, where $U(x, \alpha)$ is as defined in (15). Let $\hat{u}$ be an optimal solution to

$$\begin{array}{ll} \text{minimize} & u(\alpha - T\hat{x}), \\ \text{subject to} & uM \geqq 0, \\ & u \geqq 0. \end{array}$$

Since $U(\hat{x}, \alpha) < 0$, we have $\hat{u}\alpha < \hat{u}T\hat{x}$ and by (16), $\hat{x} \notin K$. Thus, every $x \in K$ must satisfy the inequality

$$(17) \qquad (\hat{u}T)x \leqq \hat{u}\alpha.$$

We can add this condition (17) to the fixed constraints, $Ax = b$, $x \geqq 0$. It has the effect of *cutting off* part of the set K_1.

3. The equivalent convex programming problem. We now show that a linear program under uncertainty can be expressed in terms of the first stage decision variable x, as a convex program that we shall call the *equivalent convex programming problem.* We derive the properties of the objective function of the equivalent convex program and construct the equivalent convex program when the constraints and the probability space satisfy the assumptions made in §2.

3A. The equivalent convex program.

(18) DEFINITION. A programming problem, minimize $f(x)$, $x \in K$, is an *equivalent programming problem* to (1), if $f(x)$ is given explicitly for each x (not just as a function of x, y, and ξ as in (1′)), if K is the set of feasible solutions to (1), and if an optimal solution to the equivalent programming problem is an optimal solution to (1).

In §2, we have already characterized the set of feasible solutions to (1). To exhibit an equivalent *convex* program to (1), it suffices to show that (1′) is convex in x. Let us consider the second stage problem (2) for a fixed ξ in Ξ, as a function of x. Then, by (A3) of the Appendix,

$$(19) \qquad P(x, \xi) = \{\min qy \mid My = \xi - Tx, y \geqq 0\}$$

is convex in x on $\{x \mid Tx = \xi - My, y \geqq 0\}$ and in particular on K_2.

By the duality theorem for linear programs, we have

$$(20) \qquad P(x, \xi) = Q(x, \xi),$$

where

$$Q(x, \xi) = \{\max \pi(\xi - Tx) \mid \pi M \leqq q\}$$

for fixed ξ in Ξ. Let

$$(21) \qquad \begin{aligned} Q(x) &= E_\xi\{\min qy \mid My = \xi - Tx, y \geqq 0\} \\ &= E_\xi\{Q(x, \xi)\} = E_\xi\{P(x, \xi)\} \end{aligned}$$

be the expected value of the second stage problem (2) for a given x in K_2.

(22) PROPOSITION. *$Q(x)$ is convex on K_2* (see [2]).

Since by (A3) of the Appendix, $Q(x, \xi)$ is convex in x on K_2, it suffices to remark that applying the operator E_ξ to $Q(x, \xi)$ is equivalent to performing a positive weighted linear combination of convex functions, i.e., $Q(x)$ is convex on K_2.

Thus the equivalent convex program to (1) is,

$$\text{(23)} \qquad \begin{aligned} &\text{minimize} && z(x) = cx + Q(x), \\ &\text{subject to} && x \in K. \end{aligned}$$

(24) PROPOSITION. *$Q(x)$ is continuous on K_2.*

Since $Q(x)$ is convex on K_2, the result is immediate if K_2 is open. To see that K_2 could be open, consider the following example. Let $M = 1$, $T = 1$ and $\Xi = (0, 1)$; then $K_2 = (-\infty, 0)$. In general, by (A12) of the Appendix, $Q(x, \xi)$ is uniformly continuous in x and ξ; thus

$$Q(x) = \int_{\xi \in \Xi} Q(x, \xi)\, dF(\xi)$$

is continuous in x on K_2.

Consider the dual to the second stage problem (2),

$$\text{(25)} \qquad \begin{aligned} &\text{maximize} && \pi(\xi - Tx), \\ &\text{subject to} && \pi M \leqq q, \end{aligned}$$

and let $\pi(x, \xi)$ be the optimal solution to (25) for fixed x and ξ. In what follows, we assume that $\pi(x, \xi)$ and $Q(x, \xi)$ are defined for all x in K and all ξ in Ξ. Define

$$\text{(26)} \qquad \pi(x) = E_\xi\{\pi(x, \xi)\} = \int_{\xi \in \Xi} \pi(x, \xi)\, dF(\xi)$$

as the expected optimal solution to problem (25) for a given x. Also, let

$$\psi(x) = E_\xi\{\pi(x, \xi)\xi\} = \int_{\xi \in \Xi} \pi(x, \xi)\xi\, dF(\xi).$$

Note that $\pi(x)$ is an $\bar{m}$-dimensional vector and that $\psi(x)$ is a scalar.

(27) PROPOSITION. *$[c - \pi(\bar{x})T]x = -\psi(\bar{x})$ is a supporting hyperplane of $z(x)$ at $x = \bar{x}$, where $\bar{x} \in K$* (see [2]).

Since $[c - \pi(\bar{x})T]\bar{x} + \psi(\bar{x}) = z(\bar{x})$, it suffices to show that, for every $x \in K$, $z(x) \geqq [c - \pi(\bar{x})T]x + \psi(\bar{x})$. But this is true, since for all $x \in K$ and for all $\xi \in \Xi$,

$$\pi(x, \xi)(\xi - Tx) \geqq \pi(\bar{x}, \xi)(\xi - Tx).$$

Integrating both sides with respect to $dF(\xi)$ and adding cx on both sides, we get

$$z(x) = [c - \pi(x)T]x + \psi(x) \geqq [c - \pi(\bar{x})T]x + \psi(\bar{x}).$$

(28) COROLLARY. *$[c - \pi(\bar{x})T]$ is a gradient of $z(x)$ at $\bar{x}$.*

We give to the term "gradient" the same meaning as Minty gives to "generalized gradient" in [5].

(29) PROPOSITION. *If $F(\xi)$ is continuous, then $z(x)$ is differentiable on K.*

By (A15) of the Appendix, $\pi(x, \xi)$ is piecewise constant. Moreover, since the set of points where $\pi(x, \xi)$ is multivalued has measure zero, $\pi(x)$ and $\psi(x)$ are unique for all $x \in K$. This implies that $z(x)$ has a unique supporting hyperplane for all x in. K By (22), $z(x)$ is convex, i.e., $z(x)$ is differentiable on K.

The condition that $F(\xi)$ is continuous is sufficient but not necessary; e.g., let

$$T = M = I, \qquad \Xi = \left\{\xi^1 = \begin{pmatrix}1\\2\end{pmatrix}, \xi^2 = \begin{pmatrix}2\\1\end{pmatrix}\right\},$$

$$f(\xi^1) = f(\xi^2) = \frac{1}{2},$$

$$c = [2, 2] \qquad q = [1, 1], \qquad x = [x_1, x_2];$$

then $z(x) = x_1 + x_2$.

(30) PROPOSITION. *Let $x^0 \in K$; then x^0 is optimal if and only if there exists $\pi(x^0)$ such that for every $x \in K$,*

$$[c - \pi(x^0)T]x^0 \leqq [c - \pi(x^0)T]x.$$

The proof is a direct application of (28) and the monotonicity properties of the "gradient" of a convex function [5].

(31) COROLLARY. *If $z(x)$ is differentiable, then x^0 is optimal if and only if for all x in K,*

$$[c - \pi(x^0)T]x^0 \leqq [c - \pi(x^0)T]x.$$

One could regard (30) and (31) as statements related to the solvability of problem (1). If we disregard the inconsistent case (K is empty), we can write: (1) is solvable if and only if there exists a pair $(x^0, \pi(x^0))$ such that

$$[c - \pi(x^0)T]x^0 \leqq [c - \pi(x^0)T]x$$

for all x in K. Note that (1) can have an infinite or a finite infimum. Moreover, since $z(x)$ is continuous, $z(x)$ may fail to achieve a minimum on K only if K is not bounded.

3B. Special cases. When the constraints of problem (1) and the probability space satisfy the assumptions considered in §1, we show that the equivalent convex programs are programming problems for which satisfactory algorithms exist.

Case 1. *Ξ is finite.* Let $\xi^1, \xi^2, \cdots, \xi^k$ be the values assumed by the random vector ξ with probabilities $f^1, f^2, \cdots, f^k$, respectively. We have seen in §1 that the induced constraints can be expressed explicitly in terms of linear equations and linear inequalities. The equivalent convex program is a linear programming problem which can be expressed as follows:

$$
\begin{array}{lllllll}
\text{minimize} & z(x) = cx + f^1qy^1 + f^2qy^2 + \cdots + f^kqy^k, & & & & & \\
\text{subject to} & Ax & & & & = b, \\
 & Tx + My^1 & & & & = \xi^1, \\
(32) & Tx + & My^2 & & & = \xi^2, \\
 & \vdots & & \ddots & & \vdots \\
 & Tx + & & & My^k & = \xi^k, \\
 & x \geqq 0,\ y^1 \geqq 0,\ y^2 \geqq 0, \cdots, y^k \geqq 0. & & & &
\end{array}
$$

Dantzig and Madansky [2] have shown that there exists a dual of this problem which is in the standard form for the application of the decomposition algorithm of Dantzig and Wolfe [4]. To find this dual problem, we use a more direct approach than the one found in [2].

Let $(\sigma, \bar{\pi}^1, \bar{\pi}^2, \cdots, \bar{\pi}^k)$ be the variable appearing in the usual dual formulation of (32). Define

$$\pi^l = \frac{1}{f^l}\bar{\pi}^l, \qquad l = 1, \cdots, k;$$

then the dual reads

$$
\begin{array}{llllll}
\text{maximize} & \sigma b + f^1\pi^1\xi^1 + f^2\pi^2\xi^2 + \cdots + f^k\pi^k\xi^k, & & & & \\
 & \sigma A + f^1\pi^1T + f^2\pi^2T + \cdots + f^k\pi^kT & & & & \leqq c, \\
 & \pi^1M & & & & \leqq q, \\
 & & \pi^2M & & & \leqq q, \\
 & & & \ddots & & \vdots \\
 & & & & \pi^kM & \leqq q.
\end{array}
$$

This problem has an "angular" structure. The first n inequalities can be used to generate the master program. The last $k \times \bar{n}$ inequalities constitute the subproblem. Depending on T and M, it may be advantageous to use variants of the decomposition algorithm, e.g., see Abadie [1].

Another simple transformation gives the problem (32) the structure of a multi-stage system (so-called "staircase" system) where the linear constraints for all stages but one are identical. This last feature may simplify considerably the computation. To obtain this form, subtract from each row of $Tx + My^{l+1} = \xi^{l+1}$ the corresponding row of $Tx + My^{l} = \xi^{l}$ for $l = 1, \cdots, k-1$. Problem (32) becomes

$$\begin{array}{ll} \text{minimize } z(x) = cx + f^1qy^1 + f^2qy^2 + \cdots + f^{k-1}qy^{k-1} + f^kqy^k, & \\ \text{subject to} \quad Ax & = b, \\ \quad Tx + My^1 & = \xi^1, \\ \quad\quad - My^1 + My^2 & = \xi^2 - \xi^1, \\ \quad\quad\quad \ddots & \vdots \\ \quad\quad\quad\quad -My^{k-1} + My^k & = \xi^k - \xi^{k-1}, \\ x \geqq 0, y^1 \geqq 0, y^2 \geqq 0, \cdots, \quad y^{k-1} \geqq 0, y^k \geqq 0. & \end{array}$$

Case 2. *M is square and nonsingular, and Ξ is bounded.* We show that under these assumptions there exists a linear programming problem whose set of optimal solutions is the set of optimal solutions of the linear program under uncertainty.

(a) *M is the identity* ($M = I$). The problem under consideration is

$$\begin{array}{lll} \text{minimize} & z(x) = cx + E_\xi\{qy\}, & \\ \text{subject to} & Ax = b, & \\ & Tx + Iy = \xi, & \xi \text{ on } (\Xi, \mathfrak{F}, F), \\ & x \geqq 0, \quad y \geqq 0. & \end{array}$$

For fixed x and ξ, the second stage problem (2) is

$$\begin{array}{ll} \text{minimize} & qy, \\ \text{subject to} & Iy = \xi - Tx, \\ & y \geqq 0. \end{array} \tag{33}$$

If (33) is feasible, then $\min qy = q(\xi - Tx)$. Moreover, if $x \in K$, then (33) is feasible for all ξ in Ξ, i.e., $\xi - Tx \geqq 0$ for all ξ in Ξ. We have

$$E_\xi\{\min qy \mid x \in K\} = E_\xi\{q(\xi - Tx)\} = q\bar{\xi} - qTx.$$

By (9) and (10) there exists a vector α such that

$$K = \{x \mid Ax = b, Tx \leqq \alpha, x \geqq 0\}.$$

Thus the linear program,

$$\begin{aligned} \text{minimize} \quad & z(x) = (c - qT)x, \\ \text{subject to} \quad & Ax = b, \\ & Tx \leqq \alpha, \\ & x \geqq 0, \end{aligned}$$

yields the set of optimal solutions to our problem. If Ξ is compact, then each neighborhood of α_i has positive measure. If the random variables ξ_i, $i = 1, \cdots, \bar{m}$, are independent, then Ξ is an interval (in $\mathfrak{R}^{\bar{m}}$) and $\Xi^* = \Xi$ (9).

(b) *M is square and nonsingular*. The problem reads:

$$\begin{aligned} \text{minimize} \quad & z(x) = cx + E_\xi\{qy\}, \\ \text{subject to} \quad & Ax \qquad\quad = b, \\ & Tx + \quad My = \xi, \qquad \xi \text{ on } (\Xi, \mathfrak{F}, F), \\ & x \geqq 0, \quad y \geqq 0. \end{aligned}$$

If one multiplies both sides of $Tx + My = \xi$ on the left by $M^{-1} = [\mu_{ij}]$, one obtains

$$\tilde{T}x + Iy = \tilde{\xi}, \qquad \tilde{\xi} \text{ on } (\tilde{\Xi}, \tilde{\mathfrak{F}}, \tilde{F}),$$

where

$$\begin{aligned} \tilde{T} &= M^{-1}T, \\ M^{-1}&: \Xi \to \tilde{\Xi}. \end{aligned}$$

Since Ξ is bounded by assumption and M^{-1} is a nonsingular linear mapping, $\tilde{\Xi}$ is also bounded. Hence, our new problem is similar to the previous case ($M = I$). Let Ξ^* be the smallest interval containing $\tilde{\Xi}$ and let α^* be the lower bound of Ξ^*. The equivalent convex program then reads:

$$\begin{aligned} \text{minimize} \quad & z(x) = (c - qM^{-1}T)x, \\ \text{subject to} \quad & Ax = b, \\ & M^{-1}Tx \leqq \alpha^*, \\ & x \geqq 0. \end{aligned}$$

The components of the vector α^* can be computed as follows. Let α_i and β_i be respectively the greatest lower bound and the least upper bound for ξ_i; then

$$\alpha_i^* = \min \sum_{j=1}^{\bar{m}} \mu_{ij}\xi_j, \quad \text{where} \quad \alpha_j \leqq \xi_j \leqq \beta_j;$$

$$\alpha_i^* = \sum_{j=1}^{\bar{m}} \mu_{ij}\xi_j^*, \quad \text{where} \quad \xi_j^* = \begin{cases} \alpha_j & \text{if } \mu_{ij} \geqq 0, \\ \beta_j & \text{if } \mu_{ij} < 0. \end{cases}$$

From this computation procedure for α^*, it is easy to see that the condition that Ξ is bounded is too strong; all we need is that α^* exists.

Some generalizations are possible. For example, let M be a Leontief matrix with substitutions and let $\xi - Tx \geqq 0$ for all ξ in Ξ and all x in K. One then shows that such a problem can be reduced to the case where M is square and nonsingular [3]. In this case, the condition $\xi - Tx \geqq 0$, for all ξ in Ξ and all x in K, is not restrictive, since $\xi_i - T_i x < 0$, for some i, is meaningless if the second stage problem (2) is a Leontief system with substitution.

Case 3. *The problem is complete.* $M = (I, -I)$. By (11), the equivalent convex program has the form,

$$\text{(34)} \qquad \begin{aligned} \text{minimize} \quad & z(x) = cx + Q(x), \\ \text{subject to} \quad & Ax = b, \\ & x \geqq 0. \end{aligned}$$

This problem was studied in detail in [6]. For completeness, we list the particular forms of this convex program for some specific distribution functions $F(\xi)$.

Assumptions on $(\Xi, \mathfrak{F}, F)$	*Equivalent convex program*
Ξ is finite (ξ discrete)	Linear program with upper bounds
$F(\xi)$ uniform	Quadratic program
F(ξ) continuous	
If one approximates ξ by a sum of uniformly distributed random variables, then	Quadratic program
F(ξ) exponential	
If one approximates the objective function, then	Quadratic program
In general	Separable convex program

Moreover, many generalizations of the complete problem lead to an identical class of equivalent convex programs. Let us, for instance, consider the following problem:

$$\begin{aligned} \text{minimize} \quad & z(x) = cx + E_\xi\{q^+y^+ + q^-y^-\}, \\ \text{subject to} \quad & Ax = b, \\ & Tx + Iy^+ - Iy^- = \xi, \quad \xi \text{ on } (\Xi, \mathfrak{F}, F), \\ & x \geqq 0, \quad y^+ \in H, \quad y^- \geqq 0, \end{aligned}$$

where

$$H = \{y^+ \mid y^+ = Lz, z \geqq 0\}.$$

If L is a Leontief matrix with substitution such that H contains some $y^+ > 0$, and if $q^+ + q^- \geqq 0$, then one can show [3] that such a problem has also an equivalent convex program of the form (34).

Appendix. A linear program can be considered as a function of its parameters

$$f(c, A, b) = \{\min cx \mid Ax = b, x \geqq 0\}.$$

We study the properties of this function where b is variable. Let

$$f(t) = \{\min cx \mid Ax = t, x \geqq 0\},$$

$$\mathfrak{T} = \{t \mid t = Ax, x \geqq 0\}.$$

(A1) LEMMA. *$\mathfrak{T}$ is a convex polyhedral cone containing the origin* **0**.

For the sake of simplicity, we shall assume that the matrix A has full row rank, so in particular $m \leqq n$. The case $f(t) = -\infty$ for all t in $\mathfrak{T}$ is without interest; moreover we have the following.

(A2) LEMMA. *$f(t) = -\infty$ for some $t \in \mathfrak{T}$ if and only if for every $t \in \mathfrak{T}$, $f(t) \equiv -\infty$.*

Thus we shall assume in what follows that $f(t) > -\infty$ for all $t \in \mathfrak{T}$. Note that $f(t)$ is defined only for t in $\mathfrak{T}$.

(A3) PROPOSITION. *$f(t)$ is convex on $\mathfrak{T}$.*

Consider any $t_0, t_1 \in \mathfrak{T}$ and $\lambda \in [0, 1]$. Let $t_\lambda = \lambda t_0 + (1 - \lambda)t_1$; by (A1) we have $t_\lambda \in \mathfrak{T}$. Let x_i be such that

$$f(t_i) = cx_i = \{\min cx \mid Ax = t_i, x \geqq 0\},$$

for $i = 0, \lambda, 1$; then $\bar{x} = \lambda x_0 + (1 - \lambda)x_1$ is a feasible but not necessarily optimal solution to: min cx such that $Ax = t_\lambda$, $x \geqq 0$. Consequently, $f(t)$ satisfies the basic inequality,

$$\lambda f(t_0) + (1 - \lambda)f(t_1) = \lambda cx_0 + (1 - \lambda)cx_1 = c\bar{x} \geqq cx_\lambda = f(t_\lambda),$$

for all t_0, t_1 in $\mathfrak{T}$ and $0 \leqq \lambda \leqq 1$. Loosely speaking, we can rephrase (A3) as follows. A linear program is a convex function of its right-hand side.

(A4) COROLLARY. *Let*

$$f^*(t) = \{\min tx \mid Ax = b, x \geqq 0\},$$

and let $\mathfrak{T}^* = \{t \mid f^*(t) > -\infty\}$. Then $f^*(t)$ is concave on $\mathfrak{T}^*$.

(A5) PROPOSITION. *If A is square and nonsingular, then $\mathfrak{T}$ is a simplicial cone and $f(t)$ is linear on $\mathfrak{T}$.*

It suffices to remark that $f(t) = cA^{-1}t$ on $\mathfrak{T} = \{t \mid A^{-1}t \geqq 0\}$.

(A6) PROPOSITION. *Let B be a submatrix of A such that B is an optimal* *basis for some t. Then B is an optimal basis for all t in* $\mathfrak{T}_B = \{t \mid B^{-1}t \geqq 0\}$, *B is a simplicial cone, and* $\mathfrak{T}_B \subset \mathfrak{T}$.

(A7) Corollary. *If B is an optimal basis for some t, then* $\mathfrak{T}_B$ *is the unique subset of* $\mathfrak{T}$ *for which B constitutes an optimal basis.*

By (A5) and (A6), we have the following.

(A8) Corollary. $f(t)$ *is linear on* $\mathfrak{T}_B$.

(A9) Proposition. *There exists a decomposition of* $\mathfrak{T}$ *into simplicial cones* $\mathfrak{T}_1, \cdots, \mathfrak{T}_k$ *such that*

(i) $\mathfrak{T}_i = \{t \mid B_i t \geqq 0\}$, $i = 1, \cdots, k$, *where* B_i *is a square, nonsingular submatrix of A of rank m,*

(ii) B_i *is an optimal basis for some t,*

(iii) $\bigcup_{i=1}^k \mathfrak{T}_i = \mathfrak{T}$,

(iv) $\operatorname{int} \mathfrak{T}_i \cap \operatorname{int} \mathfrak{T}_j = \emptyset$ *for* $i \neq j$.

This proposition can be proved using (A5), (A6) and (A7). It is easy to see that this decomposition may not be unique. By (A8) and (A9) we get the next results.

(A10) Proposition. $f(t)$ *is piecewise linear on* $\mathfrak{T}$.

(A11) Proposition. $f(t)$ *is continuous on* $\mathfrak{T}$.

Since $f(t)$ is convex it is continuous on int $\mathfrak{T}$. Moreover, (A8), (A9) and (A10) imply that $f(t)$ is linear on, and in the neighborhood of, the boundary.

(A12) Corollary. $f(t)$ *is uniformly continuous on* $\mathfrak{T}$.

This is immediate by (A10) and (A11).

Consider the following problem:

$$\text{(A13)} \qquad \text{maximize } \pi t, \qquad \text{subject to } \pi A \leqq c,$$

and let $\pi(t)$ be an optimal solution to (A13) for a given t in $\mathfrak{T}$.

(A14) Proposition. *If A is square and nonsingular, then* $\pi(t)$ *is constant on* $\mathfrak{T}$.

It suffices to remark that $\pi(t) = cA^{-1}$ on $\mathfrak{T} = \{t \mid A^{-1}t \geqq 0\}$.

(A15) Proposition. $\pi(t)$ *is a piecewise constant function on* $\mathfrak{T}$.

This proposition can be proved using (A9) and (A14). Let us remark that $\pi(t)$ may be multivalued on the boundaries of the simplicial cones determining the decomposition of $\mathfrak{T}$, but it is single valued on their interior.

(A16) Proposition. $\pi(\bar{t}) \cdot t$ *is a supporting hyperplane to* $f(t)$ *at* $t = \bar{t}$, $\bar{t} \in \mathfrak{T}$.

Since at $t = \bar{t}$, the hyperplane $\pi(\bar{t}) \cdot t$ intersects $f(t)$, it suffices to show that

$$\pi(\bar{t}) \cdot t \leqq f(t), \qquad \text{for every } t \in \mathfrak{T}.$$

But this is true, since by the definition of $\pi(t)$,

$$\pi(\bar{t})\cdot t \leqq \pi(t)\cdot t = f(t).$$

This last proposition, (A10), and (A11) imply the following.

(A17) PROPOSITION. *The graph of* $f(t)$, $\{(z, t) \mid z \geqq f(t),\ t \in \mathfrak{T}\}$, *is a convex polyhedral cone with vertex* 0.

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