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PROGRAMMING UNDER UNCERTAINTY: THE SOLUTION SET*

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Abstract. In a previous paper [8], we described and characterized the equivalent convex program of a two-stage linear program under uncertainty. We proved that the solution set of a linear program under uncertainty is convex and derived explicit expressions for this set for some particular cases. The main result of this paper is to show that the solution set is not only *convex* but also *polyhedral*. It is also shown that the equivalent convex program of a multi-stage programming under uncertainty problem is of the form: Minimize a convex function subject to *linear* constraints.

1. Introduction.

The *standard form* of the problem to be considered in this paper is:

$$\text{(1)} \quad \text{Minimize } z(x) = cx + E_\xi \{\min qy\} \text{ subject to}$$

$$Ax = b,$$

$$Tx + Wy = \xi, \qquad \xi \text{ on } (\Xi, \mathfrak{F}, F),$$

$$x \geqq 0, \qquad y \geqq 0,$$

where A is a $m \times n$ matrix, T is $\bar{m} \times n$, W is $\bar{m} \times \bar{n}$, and ξ is a random vector defined on a probability space $(\Xi, \mathfrak{F}, F)$.

We will assume that problem (1) is solvable. One interprets it as follows: The decision-maker must select the activity levels for x, say $x = \hat{x}$, he then observes the random event $\xi = \hat{\xi}$, and he is finally allowed to take a corrective action y, such that $y \geqq 0$, $Wy = \hat{\xi} - T\hat{x}$ and qy is minimum. This corrective action y can be thought of as a *recourse* the decision-maker possesses to "fix-up" the discrepancies between his first decision and the observed value of the random variable. This recourse decision y is taken when no uncertainties are left in the problem.

All quantities considered here belong to the reals, denoted by $\mathfrak{R}$. Vectors will belong to finite-dimensional spaces $\mathfrak{R}^n$ and whether they are to be regarded as row vectors or column vectors will always be clear from the context in which they appear. No special provision has been made for transposing vectors.

We assume that $(\Xi, \mathfrak{F}, F)$ is the probability space induced in $\mathfrak{R}^{\bar{m}}$, F determines a Lebesgue-Stieltjes measure and $\mathfrak{F}$ is the completion for F of the Borel algebra in $\mathfrak{R}^{\bar{m}}$. Ξ, the set of all *possible* outcomes of the random variables, is assumed convex. If not, we replace it by its convex hull and fill up $\mathfrak{F}$ with the appropriate sets of measure zero. We use the notation

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ξ to denote a random vector of dimension $\bar{m}$, as well as the specific values assumed by this random variable, i.e., points of Ξ. No confusion should arise from this abuse of notation.

The marginal probability space for $i = 1, \cdots, \bar{m}$ will be denoted by $(\Xi_i, \mathfrak{F}_i, F_i)$ where Ξ_i is a subset of the real line. If they exist, let α_i and β_i be respectively the greatest lower bound and least upper bound of Ξ_i.

In the first part of this paper, we characterize the solution set of problem (1); in the second part, we generalize our results to multi-stage problems.

2. The solution set. A solution of (1) is a decision to be made (here and now), thus a selection of activity levels for the vector x. The value to be assigned to the vector y can be determined by solving the deterministic linear program:

(2) Minimize qy subject to

$$Wy = \xi - Tx,$$

$$y \geqq 0,$$

i.e., after x is selected and ξ is observed.

In this paper, we are only interested in some of the properties of a solution, i.e., the decision variable x. Nevertheless, the recourse problem (2) affects our selection of x in two ways. For each selection of a vector x, we must take into account the expected costs of the recourses such an x may generate. But also, we need to limit our selection of a vector x to those for which there exists a feasible recourse, i.e., problem (2) is feasible. This latter restriction and the conditions $Ax = b$, $x \geqq 0$ determine the set of feasible solutions of problem (1).

DEFINITION. A vector x is a *feasible solution* to (1) if it satisfies the first stage constraints and if problem (2) is feasible for all ξ in Ξ.

We do not call such a solution a permanently feasible solution. We rejected these terms since they led to certain confusions, see [6] and [8].

2.1. Definition and notation. A set C is *convex* if $x_1, x_2 \in C$ implies that $[x_1, x_2] \subset C$. C is a *cone* with vertex zero if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geqq 0$. C is a *convex cone* if $x_1, x_2 \in C$ implies that $x_1 + x_2 \in C$. (a) is a *ray*, i.e., $(a) = \{z \mid z = \lambda a$ for λ in $[0, +\infty)$ and $a \in \mathfrak{R}^n\}$. The rays (a) and (b) are distinct if $b \notin (a)$ or $a \notin (b)$. C^* is the *polar cone* of C if $C^* = \{y \mid yx \geqq 0, \forall x \in C\}$. The polar cone of a ray (a) is a half-space that we denote by $(a)^*$. For further reference, see [4].

The theory of positive linear dependence was developed by Chandler Davis [3]. We review some of the definitions. A set of rays $\{(a^1), (a^2), \cdots\}$

spans positively a cone C if $(b) \in C$ implies that $b = \sum_{j=1}^{n} \lambda_j a^{i_j}$ for some selection of n rays a^{i_j} and some $\lambda_j \geqq 0$, $j = 1, \cdots, n$. A set of rays $\{(a), \cdots\}$ is *positively independent* if none of the a^i is a positive combination of the others. Otherwise, the set is positively dependent. A set of vectors $\{a^1, a^2, \cdots\}$ determines a *frame* for the cone C if $\{(a^1), (a^2), \cdots\}$ are positively independent and span positively the cone C.

C is a *convex polyhedral cone* if it is the sum of a finite number of rays, $C = \{(b) \mid (b) = \sum_{i=1}^{r} (a^i)\}$. Equivalently C is a *convex polyhedral cone* if it is the intersection of a finite number of half-spaces whose supporting hyperplanes pass through the origin. A set K is a *convex polyhedron* if it is the intersection of a finite number of half-spaces. A bounded convex polyhedron P is a *polytope.* It is easy to verify the following lemmas.

LEMMA 1 [5]. *Let C be a convex polyhedral cone, then C^* is also a convex polyhedral cone.*

LEMMA 2. *If C is a convex polyhedral cone, then every frame of C is finite.*

LEMMA 3. *Every convex polyhedron K can be obtained as the sum of a polytope P and a convex polyhedral cone C.*

2.2. The polar matrix. Let A be a $m \times n$ matrix, and let C be the cone spanned positively by the columns of A, i.e.,

$$C = \{y \mid y = Ax, x \geqq 0\},$$

then C is a convex polyhedral cone. Moreover, a subset of A determines a frame for C. The same cone C can also be defined as the intersection of a finite number of half-spaces. Moreover, there exists a minimal set of hyperplanes which support C. This concept led to the following definition.

DEFINITION. A^* is the *polar matrix* of A if

$$\{y \mid y = Ax, x \geqq 0\} = C = \{y \mid A^*y \geqq 0\}$$

and the matrix A^* has minimal row cardinality. It is easy to see that if A has full rank, i.e., rank $A = m$, then the matrix A^* has m nonzero columns. If A^* has m rows, then C is a simplicial cone; and if A^* has one row, then C is a half-space. If $C = \mathfrak{R}^m$, then no hyperplane supports C, i.e., the number of rows of A^* is zero. An algebraic characterization of the faces of convex polyhedrals is given in [9].

2.3. Fixed constraints. Let

$$K_1 = \{x \mid Ax = b, x \geqq 0\}.$$

Since K_1 does not involve any constraints involving later stages conditions, and since the constraints of K_1 are well-determined, we say that the set K_1 is the set representing the fixed constraints.

PROPOSITION 1. *K_1 is a convex polyhedron. If $m = 0$, then $K_1 = \mathfrak{R}_+^n$.*

2.4. Induced constraints. Let

$$K_2 = \{x \mid \forall\xi \in \Xi, \exists y \geqq 0 \quad \text{such that} \quad Wy = \xi - Tx\}.$$

We say that K_2 is the set representing the constraints induced on x by the following condition: No matter what value is assumed by the random variable ξ, there exists a feasible recourse y. If we define

$$K_{2\xi} = \{x \mid Wy = \xi - Tx \quad \text{for some} \quad y \geqq 0\},$$

then

$$K_2 = \bigcap_{\xi \in \Xi} K_{2\xi}.$$

In [8] it was shown that K_2 is convex. Without loss of generality, we can assume that the matrix W has full dimension ($\bar{m}$); otherwise there exists an equivalent system of linear equations to the system $Tx + Wy = \xi$ with at least one equation of the form: $T_{i}x + 0 \cdot y = \xi_i$, where 0 is a row vector of dimension $\bar{n}$. If ξ_i is a constant, we can add that equation to the system of equations $Ax = b$. If ξ_i is not a constant, then problem (1) is not solvable and is thus without interest.

In order to be able to appeal to some intuitive geometric concepts, we define the sets:

$$L = \{\chi \mid \forall\xi \in \Xi, \exists y \geqq 0 \quad \text{such that} \quad Wy = \xi - \chi\}$$

and

$$L_\xi = \{\chi \mid \chi = \xi - Wy \quad \text{for some} \quad y \geqq 0\}.$$

We also get

$$L = \bigcap_{\xi \in \Xi} L_\xi.$$

LEMMA 4. $K_{2\xi} = \{x \mid Tx = \chi, \chi \in L_\xi\}$ *and* $K_2 = \{x \mid Tx = \chi, \chi \in L\}$.

PROPOSITION 2. *If* L_ξ *is a convex polyhedron so is* $K_{2\xi}$. *Similarly, if* L *is a convex polyhderon so is* K_2.

Proof. If L_ξ is a convex polyhedron, then by the definition of a convex polyhedron there exists a matrix, say V, and a vector d such that $\chi \in L_\xi$ if and only if $\chi \in \{\chi \mid V\chi \geqq d\}$. Then, by Lemma 4, $K_{2\xi} = \{x \mid VTx \geqq d\}$. Similarly for L and K_2.

Thus, in order to show that K_2 is a convex polyhedron, it suffices to show that L is a convex polyhedron. For each ξ, L_ξ is the translate by ξ of the cone spanned positively by the columns of the matrix $-W$. Thus, L is the result of the intersection of "parallel" cones, each one being the translate of the same cone and having for vertex the point ξ.

Let W^* be the polar matrix of the matrix W, then $-W^*$ is the polar matrix of $-W$. W^* is of dimension $\bar{m}$ by l, where $l = 0$ if $L_\xi = \mathfrak{R}^{\bar{m}}$, $l = 1$ if L_ξ is a half-space, and so on.

LEMMA 5. $L_\xi = \{\chi \mid W^*\chi \leqq W^*\xi\}$.

Proof. By definition of the polar matrix, we have that $\{t \mid t = -Wy$ for some $y \geqq 0\} = \{t \mid W^*y \leqq 0\}$. Then, by translation of the cone so defined, we obtain the desired result.

PROPOSITION 3. $L = \{\chi \mid W^*\chi \leqq \alpha$, *where* $\alpha_i^* = \inf W_i^*\xi$ *for* $i = 1, \cdots, l\}$.

Proof. The result is immediate if $L = \mathfrak{R}^{\bar{m}}$, i.e., $l = 0$. Let us assume that $l \geqq 1$. By definition $L = \bigcap_{\xi \in \Xi} L_\xi$. For ξ in Ξ all the hyperplanes $W_i^*\chi = W_i^*\xi$, $i = 1, \cdots, l$, are parallel. Thus $\chi \in L$ if $W_i^*\chi \leqq \inf_{\xi\in\Xi} W_i^*\xi = \alpha_i^*$. If for each i the linear form $W_i^*\xi$ attains its infimum on the convex set Ξ, the set L is well-defined. If for some i, no infimum exists, the set of feasible solutions is empty and problem (1) is not solvable.

We have thus proved that L is a convex polyhedron. L is a cone if and only if there exists ξ in $\mathfrak{R}^{\bar{m}}$ such that $W^*\xi = \alpha^*$. From Propositions 2 and 3, we derive the following theorem.

THEOREM 1. $K_2 = \{x \mid W^*Tx \leqq \alpha^*\}$ *is a convex polyhedron.*

2.5. The solution set. It now is easy to conclude that Proposition 4 is true.

PROPOSITION 4. K, *the set of feasible solutions, is a convex polyhedron.*

Proof. By Proposition 1 and Theorem 1, K_1 and K_2 are convex polyhedrons; and since $K = K_1 \cap K_2$, the proof is immediate.

3. The equivalent convex program. In [8] it was shown that to each problem of the form (1), there exists an equivalent deterministic problem, in terms of the decision variables x, which is a convex program, i.e., the minimization of a convex function $(cx + Q(x))$ on a convex set K. We have shown here that this set K is polyhedral and consequently that this equivalent convex program has the general form:

(3) Minimize $cx + Q(x)$ subject to

$$Ax = b,$$
$$W^*Tx \leqq \alpha^*,$$
$$x \geqq 0,$$

where $Q(x)$ is defined in [8] and W^* and α^* are as above.

4. Multi-stage linear program under uncertainty. In this last section, we will show that the equivalent convex program (in terms of the decision variable x) of a multi-stage linear program under uncertainty is also of the form (3), i.e., the minimization of a convex function subject to linear constraints. We first consider the following generalization of problem (1):

$$\text{Minimize } cx + E_\xi\{\min q(y)\} \text{ subject to}$$

$$(4) \qquad \begin{aligned} Ax &= b, \\ Tx + Wy &= \xi, \qquad \xi \text{ on } (\Xi, \mathfrak{F}, F), \\ x \geqq 0, \qquad & y \in D, \end{aligned}$$

where $q(y)$ is a convex functional in y on $\mathfrak{R}^{\bar{n}}$, D is a convex polyhedron, and A, T, W and ξ are as above. In [7], it was shown that problem (4) possesses also an equivalent convex program, whose objective reads:

$$\text{Minimize} \quad cx + Q(x),$$

where

$$(5) \qquad Q(x) = E_\xi\{\min q(y) \mid Wy = \xi - Tx, y \geqq 0\}.$$

We now show that the solution set of (4) is a convex polyhedron.

Intuitively, we could argue as follows: Since D is a convex polyhedron, so is $W(D) = \{t \mid t = Wy, y \in D\}$, and so is $\xi - W(D)$ for each ξ. Since all polyhedrons $\xi - W(D)$ are the translate by ξ of a given polyhedron, these polyhedrons are "parallel", and their intersection is a convex polyhedron of the same "form", with the possible exclusion of some faces. Then by Proposition 2, the set of induced constraints, K_2, of problem (4) is also a convex polyhedron. Since K_1 is the same as for problem (1), we have that K, the set of feasible solutions, is also polyhedral.

LEMMA 6. *$W(D)$ is a convex polyhedron.*

Proof. By definition of $W(D)$.

In order to point out some of the properties of this set $W(D)$, we give more detailed construction of the set $W(D)$. By Lemma 3, every convex polyhedron D can be expressed as the sum of a polytope D_p and a convex polyhedral cone D_c. Let

$$D_c = \{r \mid r = V_c z, z \geqq 0\}$$

and

$$D_p = \{s \mid V_p s \geqq f\},$$

i.e., $y \in D$ if and only if $y = p + q$ where $p \in D_p$ and $q \in D_c$. Similarly

$$W(D) = W(D)_p + W(D)_c,$$

and it is easy to see that

$$W(D)_p = W(D_p) \quad \text{and} \quad W(D)_c = W(D_c).$$

Also

$$\xi - W(D) = \xi - [W(D_p) + W(D_c)].$$

If we construct the polar matrix of (WV_c), we can find an explicit expression for the supporting hyperplanes of $W(D_c)$ [9]. Similarly, the extreme points of D_p can be found by examining the basis of V_p, see [1]. The linear operator W maps the extreme points of D_p into the extreme points of $W(D_p)$. From these we can find the bounding hyperplanes of $W(D_p)$. It is thus theoretically possible to find an expression for $W(D)$ of the form:

$$W(D) = \{t \mid \tilde{W}t \leqq \tilde{d}\} \tag{6}$$

for some matrix $\tilde{W}$ of dimension $m \times l$ and some vector $\tilde{d}$. If $l = 0$, then $W(D) = \Re^{\bar{m}}$; if $l = 1$, then $W(D)$ is a half-space, and so on.

PROPOSITION 5. $L = \bigcap_{\xi \in \Xi} \{\chi \mid \chi = \xi - W(D)\}$ *is a convex polyhedron.*

Proof. By (6),

$$L = \{\chi \mid \tilde{W}\chi \leqq \tilde{\alpha}, \quad \text{where} \quad \tilde{\alpha}_i = \tilde{d}_i + \inf_{\xi \in \Xi} \tilde{W}_i \xi,\ i = 1, \cdots, l\}.$$

If the linear form $\tilde{W}_i\xi$ has no infimum on Ξ, then $L = \phi$.

COROLLARY 1. $K_2 = \{x \mid Tx = \chi, \chi \in L\}$ *is a convex polyhedron.*

COROLLARY 2. K, *the set of feasible solutions of* (4), *is a convex polyhedron.*

We now apply these results to the following multi-stage programming under uncertainty problem:

$$\begin{aligned} \text{Minimize} \quad & c^1x^1 + E_{\xi^2}\{\min c^2(x^2) \\ & + E_{\xi^3}[\min c^3(x^3) + (\cdots + E_{\xi^m}(\min c^m(x^m))\cdots)]\} \end{aligned} \tag{7}$$

subject to

$$\begin{aligned} &A_{11}x^1 && = b, \\ &A_{21}x^1 + A_{22}x^2 && = \xi^2, \\ &\qquad A_{32}x^2 + A_{33}x^3 && = \xi^3, \\ &\qquad\qquad \cdot \qquad \cdot && \cdot \\ &\qquad\qquad\quad \cdot \qquad \cdot && \cdot \\ &\qquad\qquad\qquad \cdot \qquad \cdot && \cdot \\ &\qquad\qquad\qquad A_{m,m-1}x^{m-1} + A_{m,m}x^m && = \xi^m, \\ &x^1 \geqq 0, x^2 \geqq 0, x^3 \geqq 0, \cdots, x^m \geqq 0, \end{aligned}$$

where the ξ^i are independent random vectors for $i = 2, \cdots, m$ and the $c^i(x^i)$ are convex functions for $i = 1, \cdots, m$. This problem is readily seen to be equivalent to the general structure of the multi-stage problem found in [2, Chap. 25]. By Theorem 1, we obtain the equivalent $m - 1$ stage programming under uncertainty problem:

$$\text{(8)} \quad \begin{aligned} &\text{Minimize } c^1(x)^1 + E_\xi\{\min c^2(x^2) + E_\xi[\min c^3(x^3) \\ &\qquad + \cdots + E_\xi(\min c^{m-1}(x^{m-1}) + Q_{m-1}(x^{m-1}))\cdots]\} \end{aligned}$$

subject to

$$\begin{array}{lr} A_{11}x^1 & = b, \\ A_{21}x^1 + A_{22}x^2 & = \xi^2, \\ \qquad A_{32}x^2 + A_{33}x^3 & = \xi^3, \\ \qquad \cdots & \cdot \\ \qquad\qquad A_{m-1,m-2}x^{m-2} + A_{m-1,m-1}x^{m-1} & = \xi^{m-1}, \end{array}$$

$$x^1 \geqq 0, x^2 \geqq 0, x^3 \geqq 0, \cdots, x^{m-2} \geqq 0, \qquad x^{m-1} \in D^{m-1},$$

where $Q_{m-1}(x^{m-1})$ is a convex function defined on the convex polyhedral D^{m-1}. Using repeatedly (5) and Corollary 1, we can find the equivalent convex program to (7), which reads:

Minimize $c(x^1) + Q(x^1)$ subject to

$$Ax^1 = b,$$

$$x^1 \in D^1,$$

where D^1 is a convex polyhedron and $Q(x^1)$ is convex. This completes the proof of the following proposition.

PROPOSITION 6. The equivalent convex program of (7) is of the form: Find an x which minimizes a convex function subject to linear constraints.

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