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STOCHASTIC PROGRAMS WITH RECOURSE*

DAVID W. WALKUP AND ROGER J.-B. WETS†

1. Introduction. A number of authors [1], [3], [4], [9], [12] have considered a particular form of stochastic linear programming called by Dantzig [3] *programming under uncertainty*. Essentially, the problem considered is that of finding the optimum value of the vector x in the program

$$\begin{aligned} z &= \min_x E[cx + \min_y (qy)], \\ (1.1) \quad Ax &= b, \\ Tx + Wy &= p, \\ x &\geq 0, \quad y \geq 0, \end{aligned}$$

where E is expectation with respect to the random vector of resources p . (For a description of problem (1.1) and some of its practical interpretations, see [6], or the various papers quoted above.) In this paper we study the natural extension of the restricted right-hand side problem (1.1) to the general case in which c , q , T and W as well as p are random variables. We call this general problem a *stochastic program with recourse*. The rigorous definition will be given in §2.

With the generalization of the stochastic program to include random W , the problem of attributing a precise meaning to the stochastic constraints $Tx + Wy = p$ becomes significant. One interpretation is to require that x be selected so that the equations $Tx + Wy = p$ are solvable in non-negative y for all values of the random parameters in the support of their joint distribution. This interpretation is computationally convenient and reasonable (as we shall see) when W is fixed. A second interpretation is to require that x be selected so that the equations $Tx + Wy = p$ are solvable almost surely. When W is random these two interpretations can lead to materially different sets of feasible values for x . In §3 we show that these two interpretations are equivalent under a rather weak continuity-type condition—the W -condition—which includes fixed W as a special case. Even when the W -condition is not satisfied, we show that there always exists some subset Σ of this support of the random variables such that for any x , $Tx + Wy = p$ is solvable almost surely if and only if $Tx + Wy = p$ is solvable for all values of the random variables in Σ . In the second part of §3 we show that, if W is fixed and the support of the random variables

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is a polyhedron, then the set of feasible values of x is also a polyhedron. This follows from a more general result, Theorem 3.14, which shows that the set of feasible x is unaffected by a broad class of manipulations on the support of the random variables.

In §4 we show without restrictions that the natural equivalent deterministic form of a stochastic program with recourse is a convex programming problem. Moreover, for the case of fixed W we obtain results on the continuity of, and the existence of supports to, the functional of the equivalent deterministic problem, which generalize and strengthen some of the results previously obtained for the case of p random only.

The fact that the equivalent deterministic problem is convex is encouraging from a computational point of view, but it constitutes only an initial step towards solution methods for stochastic programs with recourse. In another paper [8] we are able to obtain results for some special forms of recourse of more immediate computational significance.

2. Statement of the problem. We suppose a probability space (Ξ, \mathcal{F}, μ) is given, in which Ξ is a Borel subset of R^N , $N = (m+1)(n+\bar{n}+1) - 1$, \mathcal{F} is a σ -field on Ξ which includes the Borel sets, μ is a probability measure defined on \mathcal{F} , and \mathcal{F} is completed with respect to μ . Admittedly the arbitrary Borel sets, singular measures, etc., which the generality of this assumption allows are not to be expected in practical problems. However, a certain amount of nicety must be observed anyway, especially with random W , and the additional abstraction is essentially free. We think of the coordinates of a point ξ of Ξ as the components of a collection of five matrices, c , q , p , T , and W of dimensions $1 \times n$, $1 \times \bar{n}$, $m \times 1$, $m \times n$, and $m \times \bar{n}$ respectively. Thus c , q , p , T , and W are functions of the random variable ξ (specifically projections) which we may write $c(\xi)$, $q(\xi)$, etc.

A *stochastic program with recourse* can be formulated as:

$$(2.1) \quad z = \inf_{x \geq 0} E_{\xi} \{ c(\xi)x + [\min_{y \geq 0} q(\xi)y \mid T(\xi)x + W(\xi)y = p(\xi)] \},$$

where x and y are variable matrices of dimension $n \times 1$ and $\bar{n} \times 1$ respectively, \geq indicates componentwise inequality, and 0 is the appropriate zero matrix. The symbol E_{ξ} (expectation with respect to ξ) remains to be defined precisely.

For fixed x and ξ the expression in square brackets in (2.1) is just the value of the linear program

$$(2.2) \quad \begin{aligned} Q(x, \xi) &= \min_y q(\xi)y, \\ W(\xi)y &= p(\xi) - T(\xi)x, \\ y &\geq 0. \end{aligned}$$

We shall refer to (2.2) as the *second-stage program*. Except in trivial cases, a value y^0 of y which achieves the minimum in (2.2) is clearly dependent on ξ . But because (2.2) may be unbounded, infeasible, or have more than one optimal solution, it is not correct to speak of y^0 as a function of ξ except possibly as a set-valued function. On the other hand, $Q(x, \xi)$ is a legitimate function from Ξ into the extended reals if we let it take the values $+\infty$ or $-\infty$ when (2.2) is infeasible or unbounded below, respectively.

In order to give meaning to E_ξ in (2.1), it is certainly helpful to note the following lemma.

LEMMA 2.3. *For fixed x , $Q(x, \xi)$ is a measurable function from (Ξ, \mathfrak{F}, μ) into $(\bar{R}, \mathfrak{B}, \nu)$, where \mathfrak{B} is the Borel algebra on the extended reals \bar{R} and ν is the Borel measure extended to \bar{R} .*

Proof. The value of the linear program (2.2) is given by one of a finite number of algebraic expressions (or $\pm\infty$) subject to a finite number of optimality, infeasibility, or unboundedness conditions, each of which is an algebraic expression. Thus $Q(x, \xi)$ is piecewise continuous on each of a finite number of subsets D of R^N , which are finitely generated by open and closed subsets of R^N . It follows that each set D is a member of \mathfrak{F} since, in fact, it is a member of the Borel algebra on R^N . Since $Q(x, \xi)$ is continuous on each D , the inverse image of an open ray in \bar{R} is also a member of \mathfrak{F} , and since the open rays of \bar{R} generate \mathfrak{B} , the inverse images of members of \mathfrak{B} are members of \mathfrak{F} .

The foregoing lemma is not quite trivial. There exist functions (see the example used by Carathéodory [2, p. 379]) for which the inverse image of Lebesgue measurable sets need not be Lebesgue measurable but which can plausibly be interpreted as recursively computable functions on the reals.

There are several possibilities for extending the definition of integration to cover the function $z(x, \xi) = cx + Q(x, \xi)$, which for each x maps Ξ into the extended reals. We define

$$(2.4) \quad z(x) = E_\xi\{z(x, \xi)\} = \int z(x, \xi) d\mu$$

to be the sum of the four quantities:

$$(2.5) \quad \begin{aligned} A[z(x, \xi)] &= \int_{0 \leq z(x, \xi) < +\infty} z(x, \xi) d\mu, \\ B[z(x, \xi)] &= \int_{-\infty < z(x, \xi) < 0} z(x, \xi) d\mu, \\ C[z(x, \xi)] &= \begin{cases} +\infty & \text{if } z(x, \xi) = +\infty \text{ on a set of positive measure,} \\ 0 & \text{otherwise,} \end{cases} \\ D[z(x, \xi)] &= \begin{cases} -\infty & \text{if } z(x, \xi) = -\infty \text{ on a set of positive measure,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where we adopt the convention $(+\infty) + (-\infty) = +\infty$. The formal definition of a stochastic program with recourse (2.1) is now complete.

The practical import of the definition of C and D above is that we are willing to ignore an irregular outcome of the second-stage program (namely, infeasibility or unboundedness) if it occurs with zero probability. Since (2.1) is a minimization problem, the convention $(+\infty) + (-\infty) = +\infty$ can be interpreted as taking the conservative view towards "neurotic" values of x which lead to both infinitely good and infinitely bad contributions to $z(x)$. From the definitions (2.5) it follows that we may replace the set Ξ of the probability space (Ξ, \mathcal{F}, μ) in the obvious way by R^N , or any other Borel subset of R^N whose intersection with Ξ has measure 1, without altering the objective $z(x)$. Thus $z(x)$ and consequently the solution to the stochastic program (2.1) depend on the probability distribution given by μ and not at all upon an a priori choice of a set Ξ .

An altogether different situation arises if we replace the definition of C and D in (2.5) by

$$(2.6) \quad \begin{aligned} C'[z(x, \xi)] &= \begin{cases} +\infty & \text{if } z(x, \xi) = +\infty \text{ for some } \xi \in \Xi, \\ 0 & \text{otherwise,} \end{cases} \\ D'[z(x, \xi)] &= \begin{cases} -\infty & \text{if } z(x, \xi) = -\infty \text{ for some } \xi \in \Xi, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where Ξ is now interpreted as the set of possible values of the random variable ξ . In this case, $z(x)$ depends crucially on Ξ . In the next section we prove that there always exists some set Ξ^0 of measure 1 so that, on replacing Ξ by Ξ^0 in (Ξ, \mathcal{F}, μ) , the resulting $z(x)$ as defined using (2.6) is identical to the original $z(x)$ as defined using (2.5). A plausible candidate for the set Ξ^0 is the smallest relatively closed subset $\tilde{\Xi}$ of Ξ having measure 1 (i.e., the support set of the measure μ). We shall show that this intuitively tempting substitution (certainly when $\Xi = R^N$) is correct for a rather broad class of stochastic programs with recourse but fails in general.

Since our principal concern is illuminating difficulties that might arise from random coefficients in (2.2) rather than discussing obvious pathologies of $c(\xi)$, we shall assume throughout the rest of this paper that $\bar{c} = E_{\xi}\{c(\xi)\}$ is finite so that (2.1) is equivalent to

$$(2.7) \quad z = \inf_{x \geq 0} [\bar{c}x + Q(x)],$$

where

$$Q(x) = E_{\xi}\{Q(x, \xi)\}.$$

We call (2.7) the equivalent deterministic form of the stochastic program (2.1).

3. Feasibility sets. In the preceding section we defined the objective function $z(x)$ for all values of x in R^n . Nonetheless, it seems desirable to have specific knowledge about where $z(x)$ or, equivalently, $Q(x)$ is finite, and this requires in part a knowledge of where $Q(x, \xi) < +\infty$, i.e., where the second-stage program (2.2) is feasible. Accordingly we define the *weak feasibility set*:

$$(3.1) \quad K_2 = \{x \mid Q(x, \xi) = +\infty \text{ with zero probability}\},$$

the *strong feasibility set*:

$$K_2^s = \{x \mid Q(x) < +\infty\},$$

and for each $\xi \in R^N$ the *elementary feasibility set*:

$$K_2(\xi) = \{x \mid Q(x, \xi) < +\infty\}.$$

The set K_2 consists of exactly those x for which the term C in (2.5) is finite. Thus, clearly, $K_2 \supset K_2^s$. Like $z(x)$, K_2 and K_2^s are unaffected by replacing Ξ by any set of measure 1, in particular R^N . Provided the solution is not $-\infty$, the equivalent deterministic problem (2.7) is one of finding the infimum of a finite function over the intersection of K_2^s with the set $K_1 = R_{\oplus}^n = \{x \mid x \geq 0\}$.

If some of the rows of p , T , and W are nonstochastic with in fact zero entries in W , it is natural to write the equations $Ax = b$ corresponding to these rows separately as in (1.1) and define K_1 (as in [9]) to be the set of solutions to $Ax = b$, $x \geq 0$. The remaining equations give rise to new functions and sets, $Q(x, \xi)$, $Q(x)$, K_2^s , and K_2 . The results which we shall derive for the unseparated form (2.1) apply with obvious adaptations to the separated form (1.1).

In order to facilitate our discussion of the relationships between the weak feasibility set K_2 and the elementary feasibility sets $K_2(\xi)$, we introduce a more general setting and attendant notation. Let \mathcal{R} be a relation between points x of a set X and points ξ of a set Ξ . For each subset S of X and each subset Σ of Ξ , we define

$$\begin{aligned} \kappa(\Sigma) &= \{x \in X \mid x\mathcal{R}\xi \text{ for all } \xi \in \Sigma\} = \bigcap_{\xi \in \Sigma} \kappa(\xi), \\ \kappa^{-1}(S) &= \{\xi \in \Xi \mid x\mathcal{R}\xi \text{ for all } x \in S\} = \bigcap_{x \in S} \kappa^{-1}(x). \end{aligned}$$

It is easily verified that

$$(3.2) \quad \begin{aligned} \kappa\kappa^{-1}(S) &\supset S, & \kappa^{-1}\kappa(\Sigma) &\supset \Sigma, \\ \kappa^{-1}\kappa\kappa^{-1}(S) &= \kappa^{-1}(S) & \text{and} & \quad \kappa\kappa^{-1}\kappa(\Sigma) = \kappa(\Sigma). \end{aligned}$$

For the problem of this section we set $X = R^n$ and define the relation \mathcal{R}

by setting $x \mathcal{R} \xi$ if and only if $Q(x, \xi) < +\infty$, so that $K_2(\xi) = \kappa(\xi)$. We note the following simple consequence of the definition of K_2 and κ .

PROPOSITION 3.3. *Suppose Σ is any subset of Ξ such that $\mu[\Sigma] = 1$ and $K_2 \subset \kappa(\Sigma)$. Then, in fact, $K_2 = \kappa(\Sigma)$.*

Under the possibility formulation (2.6) of a stochastic program with recourse the set K_2' , consisting of all $x \in R^n$ for which $C' = 0$, would take the form

$$K_2' = \kappa(\Xi) = \bigcap_{\xi \in \Xi} K_2(\xi)$$

rather than (3.1). One of the principal objectives of this section is to determine conditions under which the analogous expression

$$(3.4) \quad K_2 = \kappa(\tilde{\Xi})$$

holds in the probability formulation (2.5). We begin with the following theorem, the proof of which is given in Appendix A.

THEOREM 3.5. *The weak feasibility set K_2 is closed, convex, and can be written in the form $K_2 = \kappa(\Sigma)$, where $\Sigma = \kappa^{-1}(K_2)$. Moreover, $\mu[\Sigma] = 1$.*

Since $K_2(\xi)$ is always a closed convex polyhedron, one obvious consequence of the above theorem is that K_2 is polyhedral if $\tilde{\Xi}$ is a finite set. Moreover, since the intersection of any collection of closed sets in a Lindelöf space is the intersection of a countable subcollection, it follows from Theorem 3.5 that K_2 can always be represented as the intersection of at most countably many sets $K_2(\xi)$.

In order to state the conditions under which (3.4) will hold we must consider some further properties of the second-stage program (2.2). The columns of the $m \times \bar{n}$ matrix W span positively [5] a closed convex cone:

$$\text{pos } W = \{t \mid Wy = t, y \geq 0\}$$

in R^m . As in [7], let us consider the space \mathcal{C} , whose points are the closed convex cones in R^m with apex at the origin, and define a metric on \mathcal{C} by taking as distance $d(C_1, C_2)$ between two members of \mathcal{C} the Hausdorff distance between their intersections with the unit ball of R^m . In [7] it is shown that pos is a continuous map of a subset Z of $R^{m \times n}$ (considered as a set of $m \times n$ matrices) into the metric space \mathcal{C} if and only if it is a closed, lower semicontinuous set-valued mapping of Z into R^m . The following lemma and theorem show that the appropriate continuity conditions on $\text{pos } W$ will ensure (3.4).

LEMMA 3.6. *Suppose the restriction of $\text{pos } W(\xi)$ to a subset Σ of Ξ is continuous. Then for any x , $\kappa^{-1}(x) \cap \Sigma$ is a relatively closed subset of Σ .*

Proof. For any x and ξ , the second-stage program (2.2) is feasible, i.e., $\xi \in \kappa^{-1}(x)$, if and only if $\text{pos } W(\xi)$ includes the point $z(\xi) = p - Tx$.

Thus, $\xi \in \kappa^{-1}(x)$ if and only if the pair $(z(\xi), \xi)$ is in the graph of the set-valued function $\text{pos } W$. By hypothesis, the graph of the restriction of $\text{pos } W(\xi)$ to Σ is a closed subset of $R^m \times \Sigma$. Since $z(\xi)$, and hence $(z(\xi), \xi)$, is a continuous function of ξ , it follows that $\kappa^{-1}(x) \cap \Sigma$ is closed in Σ .

THEOREM 3.7. *If the restriction of $\text{pos } W(\xi)$ to $\tilde{\Sigma}$ is continuous, then*

$$K_2 = \kappa(\tilde{\Sigma}).$$

Proof. Let $x \in K_2$. By definition of K_2 we have $\mu[\kappa^{-1}(x)] = 1$. Applying Lemma 3.6 with $\Sigma = \tilde{\Sigma}$, we have that $\kappa^{-1}(x) \cap \tilde{\Sigma}$ is closed. But a closed subset of $\tilde{\Sigma}$ of measure 1 is just $\tilde{\Sigma}$; hence $\kappa^{-1}(x) \supset \tilde{\Sigma}$. It follows that $K_2 \subset \kappa(\tilde{\Sigma})$, and by Proposition 3.3, $K_2 = \kappa(\tilde{\Sigma})$.

To see the significance of continuity of $\text{pos } W$ in the foregoing theorem, consider the simple stochastic program:

$$\begin{aligned} &\text{minimize } 0 \cdot x + E_w\{y\}, \\ &-x + wy = 0, \\ &x \geq 0, \quad y \geq 0, \end{aligned}$$

where w has a distribution with support $\tilde{\Sigma}_w = [0, 1]$ and $w = 0$ with probability zero. It may be seen that $K_2 = \{x \mid x \geq 0\}$ but $\kappa(\tilde{\Sigma}) = \{x \mid x = 0\}$. Note that $\text{pos } w$ is the half-line $[0, \infty)$ so long as $w > 0$, but at $w = 0$, $\text{pos } w$ abruptly collapses to the origin.

So far the present section has been concerned exclusively with the value $+\infty$ for $Q(x, \xi)$, which corresponds to infeasibility of the second-stage program. We turn now to an examination of the dual of that program, since the dual is infeasible when $Q(x, \xi) = -\infty$. The dual program may be written in the form of equality constraints in nonnegative variables:

$$\begin{aligned} Q^*(x, \xi) &= \max_{\mu} [(p - Tx)^T, -(p - Tx)^T, 0]u, \\ (3.8) \quad &[W^T, -W^T, I]u = q, \\ &u \geq 0, \end{aligned}$$

where u is a column vector of length $2m + \bar{n}$, $[W^T, -W^T, I]$ is the matrix formed by juxtaposing the transpose of W , its negative, and an $\bar{n} \times \bar{n}$ identity, etc. By direct analogy with the treatment of $Q(x, \xi)$, we may define a dual relation \mathcal{Q}^* given by

$$x \mathcal{Q}^* \xi \quad \text{if and only if} \quad Q^*(x, \xi) > -\infty,$$

dual operators κ^* and κ^{*-1} , dual feasibility sets $K_2^*(\xi)$ and K^* and obtain dual results leading up to the following corollary.

COROLLARY 3.9. *If the restriction of $\text{pos } [W^T(\xi), -W^T(\xi), I]$ to $\tilde{\Sigma}$ is con-*

tinuous, then

$$K_2^* = \kappa^*(\tilde{\Xi}).$$

However, since the feasibility of (3.8) is independent of the value of x , for each ξ the set $K_2^*(\xi)$ is either empty or R^n . Thus Corollary 3.9 is actually less substantial than comparison with Theorem 3.7 might suggest.

The hypotheses of Theorem 3.7 and Corollary 3.9 taken together constitute an important regularity condition for stochastic programs which we formalize as follows.

DEFINITION 3.10. A stochastic program with recourse (2.1) is said to *satisfy the W -condition* if the restriction to $\tilde{\Xi}$ of $\text{pos } W(\xi)$ and $\text{pos } [W^T(\xi), -W^T(\xi), I]$ are continuous in the sense of [7], where $\tilde{\Xi}$ is the smallest relatively closed subset of Ξ of measure 1.

Thus Theorem 3.7 and Corollary 3.9 may be restated as the following theorem, which shows that for stochastic programs with recourse satisfying the W -condition, the probability formulation is equivalent to the possibility formulation using the intuitively appealing and convenient set $\tilde{\Xi}$ as the set of possible values of the random variable ξ .

THEOREM 3.11. *If a stochastic program with recourse (2.1) satisfies the W -condition, then $K_2 = \kappa(\tilde{\Xi})$ and either $Q(x) = -\infty$ for all $x \in K_2^\circ$ or $Q(x, \xi)$ is finite for all $x \in K_2$ and all $\xi \in \tilde{\Xi}$.*

It is not very clear from Definition 3.10 just how restrictive the W -condition may be. In Appendix B we give a proof of the following theorem, which in conjunction with certain remarks below suggests that many practical stochastic programs satisfy the W -condition. (We remark that algebraic characterizations of hypotheses (i) and (ii) can be obtained from the results in [11].)

THEOREM 3.12. *Any one of the following constitutes a sufficient condition for a stochastic program with recourse to satisfy the W -condition:*

- (i) *For each $\xi \in \tilde{\Xi}$, $\text{pos } W(\xi)$ is a pointed cone and no column of $W(\xi)$ has zero norm.*
- (ii) *There exists an integer k such that for each $\xi \in \tilde{\Xi}$, $\text{pos } W(\xi)$ is a (not necessarily fixed) subspace of R^m of dimension k .*
- (iii) *$W(\xi)$ is constant throughout $\tilde{\Xi}$.*

COROLLARY 3.13. *A stochastic program satisfies the W -condition if there exists a linear combination of the rows of W which is strictly positive on $\tilde{\Xi}$, in particular, if every component of some row of W is strictly positive for all ξ in $\tilde{\Xi}$.*

Proof. If r is a linear combination of the rows of W , then $r = \pi W$, where π is a row vector of length m . But if r is strictly positive, then every column of W has a positive inner product with π and hypothesis (i) of Theorem 3.12 holds.

An important special case of hypothesis (ii) is the case $\text{pos } W(\xi) = R^m$ (or its variant $\text{pos } W(\xi) = R^{\bar{m}}$, where \bar{m} is the number of rows of W in the separated form (1.1)). We shall say in this case that the stochastic program has *complete* recourse. In case hypothesis (iii) holds we shall say the stochastic program has *fixed* recourse.

In the remainder of this section we assume that W is fixed and develop some of the resulting properties of the feasibility set K_2 . We also assume, as we may, that $\Xi = R^N$. Since feasibility of (2.2) depends only on the (p, T, W) components of ξ , we can define \mathcal{R} as a relation between elements of R^n and Ξ_{pTW} . All the results from Proposition 3.3 through Corollary 3.13 remain valid. Note that by replacing $\tilde{\Xi}$ by $\tilde{\Xi}_{pTW}$ (the support of the marginal distribution) in the definition of the W -condition, we impose a stronger condition since $\tilde{\Xi}_{pTW}$ always contains the projection of $\tilde{\Xi}$ into Ξ_{pTW} . In fact, when W is fixed we can replace $\tilde{\Xi}$ by $\tilde{\Xi}_{pT}$. In this case it follows from the modified version of Lemma 3.6 that any set $\kappa(\Sigma)$, $\Sigma \subset \Xi_{pT}$, is unaltered by the operation of replacing Σ by its closure or by any dense subset of Σ . Also, it is easy to see that if $Tx + Wy = p$, $y \geq 0$, is feasible for two values of (p, T) , then it is also feasible for any convex combination of them or any positive multiple of one of them. Thus we have shown that the following theorem holds.

THEOREM 3.14. *If W is fixed, then $K_2 = \kappa(\Sigma)$, where Σ is any set obtained from $\tilde{\Xi}_{pT}$ by applying the operations: topological closure, convex closure, positive hull closure, positive scalar multiplication, or any of the (not necessarily unique) inverses of these operations.*

It should be borne in mind that there is a difference between the action of the positive hull or positive multiplication operations on the set $\tilde{\Xi}_{pT}$ and the action of the same operations on $\tilde{\Xi}_{pT} \times W$ considered as a set of values of (p, T, W) . From Theorem 3.14 now follows the next proposition.

PROPOSITION 3.15. *If W is fixed and Σ , the closure of the positive hull of $\tilde{\Xi}_{pT}$, is polyhedral, then K_2 is polyhedral.*

Proof. Note that K_2 is polyhedral when Σ is finite and apply Theorem 3.14.

PROPOSITION 3.16. *If W is fixed, p and T are independent, and $\tilde{\Xi}_T$ (or the closure of its positive hull) is polyhedral, then K_2 is polyhedral.*

Proof. Since p and T are independent, $\tilde{\Xi}_{pT} = \tilde{\Xi}_p \times \tilde{\Xi}_T$. For each T in $\tilde{\Xi}_T$ let $K_2(T)$ denote the set of x such that

$$Tx + Wy = p,$$

$$y \geq 0,$$

is feasible for all p in $\tilde{\Xi}_p$. By Theorem 13 of [10], $K_2(T)$ is the set of feasible

x for the equations

$$(3.17) \quad \begin{aligned} W^*Tx + Iz &= \alpha^*, \\ z &\geq 0, \end{aligned}$$

where W^* is the so-called polar matrix of W and α^* is a vector depending on W and $\tilde{\Xi}_p$ only. By Theorem 3.7,

$$\begin{aligned} K_2 &= \bigcap_{\xi \in \tilde{\Xi}_p T} K_2(\xi) \\ &= \bigcap_{T \in \tilde{\Xi}_T} K_2(T) \\ &= \{x \mid \text{equations (3.17) are feasible for all } T \in \tilde{\Xi}_T\}. \end{aligned}$$

Since $\tilde{\Xi}_T$ is polyhedral, so is $W^*(\tilde{\Xi}_T) = \{W^*T \mid T \in \tilde{\Xi}_T\}$. Thus, by Proposition 3.15, K_2 is polyhedral.

In general, K_2 need not be polyhedral even if

- (i) W is fixed and $\tilde{\Xi}_p$ and $\tilde{\Xi}_T$ are polyhedral, or
- (ii) T and p are fixed, $\tilde{\Xi}_W$ is polyhedral, and the W -condition is satisfied.

4. Properties of $Q(x)$. Since the functions $Q(x, \xi)$ and $Q(x)$ have been defined with the extended reals \bar{R} for range, we will need to adapt the usual definition for convex functions.

DEFINITION 4.1. A function f with convex domain \mathfrak{X} and range \bar{R} is *convex* if its epigraph $\{(x, z) \mid x \in \mathfrak{X}, z \in R, z \geq f(x)\}$ is a convex set, or equivalently, if

$$f(x_\lambda) = f[(1 - \lambda)x_0 + \lambda x_1] \leq (1 - \lambda)f(x_0) + \lambda f(x_1), \quad \lambda \in [0, 1],$$

where we adopt the conventions $0 \cdot \infty = 0$ and $(+\infty) + (-\infty) = +\infty$. If the epigraph of f is a convex polyhedron we say that f is convex polyhedral.

In Appendix C we develop some of the consequences of this definition of convexity and the definition (2.4) of integration for functions into the extended reals, including the following proposition.

PROPOSITION 4.2. Suppose $f(x, \xi)$ is a function from $R^n \times R^N$ into \bar{R} , convex in x and measurable in ξ with respect to μ on R^N . Then $F(x) = \int f(x, \xi) d\mu$ is also convex in x .

We now turn to the examination of the specific properties of $Q(x, \xi)$ and $Q(x)$.

PROPOSITION 4.3. $Q(x, \xi)$ is convex polyhedral in x , convex polyhedral in (p, T) , and concave polyhedral in q .

Proof. Since the right-hand sides of the program (2.2) are linear in x and

linear in (p, T) , the proposition follows immediately from the properties of the optimal value of a linear program as a function of the right-hand sides and the coefficients of the objective function summarized in the Appendix to [9].

Thus from Lemma 2.3 and Propositions 4.2 and 4.3 we have immediately the following theorem.

THEOREM 4.4. $Q(x) = E_{\xi}\{Q(x, \xi)\}$ is convex.

From this theorem and the properties of convexity it follows that, like K_2 , the set K_2^s and the set $\{x \mid Q(x) = -\infty\}$ are convex and either $Q(x)$ is finite on K_2^s or $Q(x) = -\infty$ on the relative interior of K_2^s . In general, K_2^s is not closed and $Q(x)$ may be discontinuous on K_2^s , even when $Q(x)$ is finite on K_2^s .

We obtain additional regularity conditions for $Q(x)$ if we impose the plausible assumption that each component of ξ is square integrable ($\xi \in L_2$) and the more stringent requirement of fixed recourse, i.e., W fixed.¹

THEOREM 4.5. If W is fixed and $\xi \in L_2$, then $K_2^s = K_2$ and either $Q(x) = -\infty$ on K_2 or $Q(x)$ is finite and Lipschitz on K_2 .

Proof. It suffices to consider x in K_2 and ξ in $\tilde{\Xi}$, and by Theorem 3.7, $Q(x, \xi) < +\infty$ for these values. Let $\Sigma = \tilde{\Xi} \cap \{\xi \mid -\infty < Q(x, \xi) < +\infty \text{ for all } x \in K_2\}$. For any given x in K_2 and ξ in Σ , $Q(x, \xi)$ can be expressed in terms of a basic solution of a linear program by writing

$$Q(x, \xi) = q^{(i)} W_{(i)}^{-1} (p - Tx),$$

where $W_{(i)}$ is a nonsingular $m \times m$ square submatrix of W and $q^{(i)}$ is the corresponding subvector of q . (Here we have assumed that W is of full rank; if it is not, we make it so by eliminating the dependent rows. Since we have restricted our attention to feasible x and ξ , dependence in W implies dependence in the augmented matrix $[W, p - Tx]$.) Thus for any fixed x , Σ may be partitioned into a finite number of (Borel) subsets on each of which $Q(x, \xi)$ is quadratic in ξ . Since ξ is square integrable it follows that $Q_2(x) = \int_{\Sigma} Q(x, \xi) d\mu$ is finite for all x in K_2 . If x, x^0 are distinct points of K_2 , then since $Q(x, \xi)$ is polyhedral in x ,

$$(4.6) \quad N(x, x^0; \xi) = \frac{|Q(x, \xi) - Q(x^0, \xi)|}{\|x - x^0\|}$$

achieves a maximum for each ξ when x, x^0 belong to the same region of linearity of $Q(x, \xi)$. For x, x^0 in the same linearity region we have $Q(x, \xi)$

¹ Added in proof: In "Qualitative Aussagen zu einigen Problemen der stochastischen Programmierung", Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 6 (1966), pp. 246-272, Peter Kall has obtained related results for stochastic programs with complete recourse with q and W fixed.

$-Q(x^0, \xi) = q^{(i)} W_{(i)}^{-1} T(x^0 - x)$ for some i . Hence for any x^0, x in K_2 ,

$$(4.6) \quad |Q(x, \xi) - Q(x^0, \xi)| \leq |q^{(i)} W_{(i)}^{-1} T(x^0 - x)| \leq M \cdot \|\xi\|^2 \cdot \|x^0 - x\|,$$

where M is a bound that is independent of i . It follows that $Q_\Sigma(x)$ is Lipschitz with constant $M \int \|\xi\|^2 d\mu$. But by Proposition 4.3, $Q(x, \xi) = -\infty$ for some x in K_2 if and only if $Q(x, \xi) = -\infty$ for all x in K_2 . Thus $Q_{\tilde{\Sigma}-\Sigma} = \int_{\tilde{\Sigma}-\Sigma} Q(x, \xi) d\mu$ is either identically $-\infty$ or zero on K_2 . The theorem now follows from a simple application of Lemma C1 in Appendix C.

COROLLARY 4.7. *Theorem 4.5 remains true if the hypothesis $\xi \in L_2$ is replaced by any of the following:*

- (i) q is fixed and $\xi \in L_1$,
- (ii) p and T are fixed and $\xi \in L_1$,
- (iii) $\tilde{\Sigma}$ is bounded.

We remark that (ii) of Corollary 4.7 constitutes a generalization of certain results of a previous paper [9]. Proposition 24 of [9] asserts the existence and continuity of $Q(x)$ under the assumption that $p \in L_1$ and all other parameters are fixed. Proposition 27 of [9] shows the existence of (nonvertical) supporting hyperplanes to the epigraph of $Q(x)$ by giving an explicit formula for them. But Corollary 4.7 shows that $Q(x)$ is Lipschitz and, therefore, there exists a bound N such that for each $x \in K_2$ there exists a hyperplane supporting the epigraph of $Q(x)$ at $(x, Q(x))$ with slope less than N .

COROLLARY 4.8. *If p only is random, then either K_2^* is empty, $Q(x) \equiv -\infty$ on $K_2 = K_2^*$, or $Q(x)$ is finite and Lipschitz on $K_2 = K_2^*$.*

Proof. The right side of (4.6) can be replaced by a bound independent of ξ . Thus $Q(x)$ is Lipschitz where it is finite. But by Proposition C4 of Appendix C, if $x^0, x \in K_2$ and $Q(x^0, \xi) \in L_1$, then $Q(x, \xi) \in L_1$.

Finally, we note the following supplement to Proposition 4.3.

PROPOSITION 4.9. *For fixed W , $Q(x, \xi)$ is Lipschitz in $(x, (q, p, T))$ on every bounded set on which $Q(x, \xi)$ is finite.*

Proof. By (4.6), $Q(x, \xi)$ satisfies a Lipschitz condition in x on each bounded subset of $R^n \times R^N$ on which $Q(x, \xi)$ is finite. Similarly, since $Q(x, \xi)$ is polyhedral in (p, T) and q , it follows that $Q(x, \xi)$ is Lipschitz in (p, T) and q separately on bounded sets where $Q(x, \xi)$ is finite. The conclusion is immediate.

Appendix A. Proof of Theorem 3.5. The theorem follows from the fact that $K_2(\xi)$ is closed and convex, and the slightly more general result,

Proposition A1. A topological space is hereditarily separable if every subset with the induced topology is separable.

PROPOSITION A1. Suppose X is a hereditarily separable topological space, (Ξ, \mathfrak{F}, μ) is an abstract probability space and \mathcal{R} is a relation between X and Ξ such that

$$\kappa(\xi) = \{x \mid x\mathcal{R}\xi\} \text{ is closed for all } \xi \in \Xi,$$

$$\kappa^{-1}(x) = \{\xi \mid x\mathcal{R}\xi\} \text{ is a member of } \mathfrak{F} \text{ for all } x \in X.$$

Suppose further that

$$K_2 = \{x \mid \mu[\kappa^{-1}(x)] = 1\}.$$

Then

$$K_2 = \kappa(\Sigma) = \bigcap_{\xi \in \Sigma} \kappa(\xi),$$

where

$$\Sigma = \kappa^{-1}(K_2) = \bigcap_{x \in K_2} \kappa^{-1}(x).$$

Moreover, $\mu(\Sigma) = 1$.

Proof. If K_2 is empty, then $\Sigma \equiv \kappa^{-1}(\emptyset) \equiv \Xi$, whence $\mu(\Sigma) \equiv 1$. Comparison of the definitions of K_2 and $\kappa(\Xi)$ shows that $\kappa(\Xi) = \kappa(\Sigma) = \emptyset$. Now suppose that $K_2 \neq \emptyset$ and let x_1, x_2, \dots be a sequence of points dense in K_2 , and let $\Sigma' = \bigcap_{i=1}^{\infty} \kappa^{-1}(x_i)$. From the properties of κ and κ^{-1} it follows that $\kappa(\xi) \supset \bigcup x_i$ for each $\xi \in \Sigma'$, and since $\kappa(\xi)$ is closed, $\kappa(\xi) \supset K_2$ for each $\xi \in \Sigma'$, whence it follows that $\Sigma' = \Sigma$ and $K_2 \subset \kappa(\Sigma)$. But since Σ' is the intersection of countably many sets of measure 1, $\mu(\Sigma) = 1$, and hence $K_2 \supset \kappa(\Sigma)$. The proof is complete.

The fact that we can use this abstract method of proof for Theorem 3.5 (without introducing any but the simplest of the properties of the second-stage program) suggests that Theorem 3.5 may not be as strong as possible.

Appendix B. Proof of Theorem 3.12. The result is clearly trivial when hypothesis (iii) holds. Moreover, by Corollaries 1 and 2 of [7], hypotheses (i) and (ii) imply that the restriction of $\text{pos } W(\xi)$ to $\tilde{\Xi}$ is continuous. The following two lemmas will be required to complete the proof.

LEMMA B1. $\text{pos } [W^T, -W^T, I] = R^{\bar{n}}$ if and only if $\text{pos } W$ is a pointed cone and none of the columns of W is zero.

Proof. Suppose $\text{pos } [W^T, -W^T, I] = R^{\bar{n}}$. Then some positive combination of the columns of $[W^T, -W^T, I]$ is a strictly negative vector; in fact, some positive combination of the columns of $[W^T, -W^T]$, or equivalently, some

linear combination of the columns of W^T is a strictly negative vector. But this last condition is equivalent to saying that some vector has a strictly negative inner product with every column of W , which implies that $\text{pos } W$ is pointed and no column of W is zero. A converse argument shows that if $\text{pos } W$ is pointed and no column of W is zero, then $\text{pos } [W^T, -W^T]$ contains a strictly negative vector, from which it follows immediately that $\text{pos } [W^T, -W^T, I] = R^{\bar{n}}$.

LEMMA B2. *$\text{pos } W$ is a subspace of dimension k if and only if $\text{pos } [W^T, -W^T]$ is a subspace of dimension k supporting $\text{pos } I$ at the origin only.*

Proof. Clearly, $\text{pos } [W^T, -W^T]$ is always a subspace. Now subspaces are characterized among the convex cones in that every linear form on the containing space is either zero on the cone or takes on both positive and negative values there. In other words, $\text{pos } W$ is a subspace if and only if $\pi W \geq 0$ implies $\pi W = 0$, where π is a $1 \times m$ matrix. Thus $\text{pos } W$ is a subspace if and only if every linear combination of the columns of W^T which lies in $\text{pos } I$ is actually the zero column. Finally, we observe that the dimension of $\text{pos } W$, the rank of W , the rank of W^T , and the dimension of $\text{pos } [W^T, -W^T]$ are all equal.

Lemma B1 shows that if Theorem 3.12(i) holds, then $\text{pos } [W(\xi)^T, -W(\xi)^T, I]$ is continuous in ξ since it is constant. If (ii) holds, then it follows from Lemma B2 that

- (a) $\dim \mathcal{L} \text{ pos } [W^T, -W^T, I] = \dim \text{pos } [W^T, -W^T] = k$,
- (b) a column of $[W^T, -W^T, I]$ lies in $\mathcal{L} \text{ pos } [W^T, -W^T, I]$ if and only if it is a column of $[W^T, -W^T]$,

for all $W = W(\xi)$, $\xi \in \tilde{\Xi}$, where $\mathcal{L}C$ denotes the maximal linear subspace contained in the cone C . From [7, Theorem 2], we have that $\text{pos } [W^T, -W^T, I]$ is a continuous function on the set $\{[W(\xi)^T, -W(\xi)^T, I] \mid \xi \in \tilde{\Xi}\}$. It follows then that $\text{pos } [W(\xi)^T, -W(\xi)^T, I]$ is a continuous function on $\tilde{\Xi}$. This completes the proof of Theorem 3.12.

We do not know whether the hypotheses of Theorem 3.12 can be replaced by the more general hypotheses of [7, Theorem 2].

Appendix C. Properties of the integral (2.4).

LEMMA C1. *If $f(\xi)$ is any measurable function from R^N into the extended reals \bar{R} , then*

$$(C2) \quad \int_{s_1 \cup s_2} f(\xi) \, d\mu = \int_{s_1} f(\xi) \, d\mu + \int_{s_2} f(\xi) \, d\mu,$$

where S_1 and S_2 are disjoint measurable subsets of R^N and $\int \cdot \, d\mu$ is defined as in (2.4).

Proof. The definition of addition on the extended reals is commutative and associative; thus it suffices to show that an equality of the same form as (C2) holds for each of the four terms (2.5) defining the integral (2.4). From the standard properties of the Lebesgue-Stieltjes integral it follows that $A_{s_1 \cup s_2}[f] = A_{s_1}[f] + A_{s_2}[f]$ and $B_{s_1 \cup s_2}[f] = B_{s_1}[f] + B_{s_2}[f]$, where

$$A_s[f] = \int_{\{0 \leq f(\xi) < +\infty\} \cap s} f(\xi) \, d\mu, \quad \text{etc.}$$

The equality for the terms C and D follows from the fact that the union of two disjoint measurable sets has positive measure if and only if one of them does.

PROPOSITION C3. *The integral (2.4) is order preserving, i.e., if f and g are measurable functions from R^N into \bar{R} , and $f(\xi) \leq g(\xi)$ for all ξ in R^N , then $\int f(\xi) \, d\mu \leq \int g(\xi) \, d\mu$.*

Proof. By Lemma C1 it suffices to show $F_s = \int_s f(\xi) \, d\mu \leq G_s = \int_s g(\xi) \, d\mu$ for each set S in a finite measurable partition of R^N . Note that for any set S of measure zero, $F_s = G_s$ trivially. If $\mu[S_1 = \{f(\xi) = -\infty\}] > 0$, then $-\infty = F_{s_1} \leq G_{s_1}$. If $\mu[S_2 = \{f(\xi) = +\infty\}] > 0$, then $F_{s_2} = G_{s_2} = +\infty$. If $\mu[S_3 = \{-\infty < f(\xi) < +\infty, g(\xi) = +\infty\}] > 0$, then $F_{s_3} \leq G_{s_3} = +\infty$. For the remaining set S_4 , on which f and g are finite, $F_{s_4} \leq G_{s_4}$ by the order-preserving properties of the Lebesgue-Stieltjes integral for finite-valued functions.

PROPOSITION C4. *The integral (2.4) is subadditive, i.e., if f and g are measurable functions from R^N into \bar{R} , then*

$$\int [f(\xi) + g(\xi)] \, d\mu \leq \int f(\xi) \, d\mu + \int g(\xi) \, d\mu$$

with equality if either of the integrals on the right is finite.

Proof. The proof is similar to that for Proposition C3.

Proof of Proposition 4.2. If $f(x, \xi)$ is convex in x , then by definition,

$$f(x_\lambda, \xi) = f[(1 - \lambda)x_0 + \lambda x_1, \xi] \leq (1 - \lambda)f(x_0, \xi) + \lambda f(x_1, \xi),$$

$$\lambda \in [0, 1], \quad \xi \in R^N.$$

From this it follows by Propositions C3 and C4 that

$$F(x_\lambda) \leq (1 - \lambda)F(x_0) + \lambda F(x_1),$$

where $F(x) = \int f(x, \xi) \, d\mu$, which is exactly the condition of convexity for $F(x)$.

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