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STOCHASTIC PROGRAMS WITH RECOURSE: SPECIAL FORMS

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I. Introduction

This paper is a sequel to Stochastic Programs with Recourse [20] in which we defined stochastic programs with recourse and developed some of their theoretical properties. In this paper we consider some special forms of stochastic programs with recourse, which, because they are less general, may prove to be more amenable to computational solution. We also show how certain problems studied by others, including the active approach of G. Tintner [15, 17] and the conditional probability model of chance constrained programming treated by A. Charnes and M. Kirby in [6, 7], can be represented as stochastic programs with recourse.

In [20] we have defined a stochastic program with recourse to be essentially

$$\operatorname*{Inf}_{x} \quad z(x) = E\{ cx + \operatorname*{Min}_{y} qy\} \tag{1.1a}$$

$$\text{subject to} \quad Ax \qquad = b \tag{1.1b}$$

$$Tx + Wy = p \tag{1.1c}$$

$$x \geq 0 \qquad y \geq 0 , \tag{1.1d}$$

where A and b are fixed matrices of size m x n and m x 1 respectively and c, q, p, T, and W are matrices of size $1 \times n$, $1 \times \bar{n}$, $\bar{m} \times 1$, $\bar{m} \times n$, and $\bar{m} \times \bar{n}$ respectively whose elements are components of a random variable ξ defined on R^N, $N = (\bar{m} + 1)(n + \bar{n} + 1) - 1$, with an associated distribution function F.

Strictly speaking, in [20] we considered the possibility that R^N could be replaced by some Borel subset Ξ of R^N with probability measure 1, thus generating an abstract probability space $(\Xi, \mathcal{F}, \mu)$, where $\mathcal{F}$ is a σ-field on Ξ including the Borel sets, μ is the probability measure defined on $\mathcal{F}$ derived from F, and $\mathcal{F}$ is completed with respect to μ. As observed in [20], such a replacement has no effect on the objective function z(x) in (1.1a). When we wish to consider the case $\Xi \neq R^N$ in this paper we shall use the phrase "relative to Ξ" where Ξ is some particular set under consideration.

By $\tilde{\Xi}$ we denote the support set of the random variable ξ, i.e., the smallest closed subset of R^N (smallest relatively closed subset Ξ when speaking relative to Ξ) of measure 1. When $\Xi = R^N$ any component or group of components of ξ, such as the elements of W or the elements of p and T, may be considered a marginal random variable, denoted simply by W or (p, T), with an associated marginal distribution function and an associated support set denoted by $\tilde{\Xi}_W$ or $\tilde{\Xi}_{pT}$.[1,2]

For each value of the decision variable x, and for each value of the random variable ξ, the second term on the right in (1.1a) is the optimal value $Q(x,\xi)$ of the second-stage linear program

$$\begin{array}{lll} \text{Minimize} & qy & \\ \text{subject to} & Wy = (p - Tx) & (1.2) \\ & y \geq 0. & \end{array}$$

If (1.2) is infeasible or unbounded below, we set $Q(x,\xi)$ equal to $+\infty$ or $-\infty$ respectively. In Section 2 of [20] we give a precise definition of the expectation operator E which accommodates values $+\infty$ and $-\infty$ for $Q(x,\xi)$. In particular, $E\{cx + Q(x,\xi)\} = +\infty$ if $Q(x,\xi) = \infty$ with positive probability or if the positive contribution to the integral $E\{cx + Q(x,\xi)\}$ diverges. A vector x is a feasible solution to the stochastic program (1.1) if $Ax = b$ and $E\{cx + Q(x,\xi)\} < +\infty$. As in [20] we shall assume that $E\{c\} = \bar{c}$ exists and is finite. This allows us to rewrite the objective (1.1a) as $z(x) = \bar{c}x + Q(x)$, where $Q(x) = E\{Q(x,\xi)\}$. The effect of this result is that it is unnecessary to consider c as part of the components of ξ. Essentially this same observation may be found in [9].

One of the principal results of [20] is that the equivalent deterministic program

$$\begin{array}{lll} \text{Inf} & z(x) = \bar{c}x + Q(x) & \\ \text{subject to} & Ax = b & (1.3) \\ & x \geq 0 & \end{array}$$

associated with (1.1) is a convex programming problem. In fact, it can be shown [21] that $Q(x)$ is a "closed" convex function, i.e. $Q(x)$ is convex and lower semicontinuous. Thus the set $\{x \mid Q(x) < +\infty\}$ is convex, though it need not be closed. Since the first-stage feasibility set $K_1 = \{x \mid Ax = b,\ x \geq 0\}$ is a closed convex polyhedron, it follows that $\{x \mid Q(x) < \infty\} \cap K_1$, i.e., the set of feasible solutions, is convex.

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[1] When certain of the matrices c, q, p, T, or W are not random we may continue to define ξ as a random variable on R^N with certain components degenerate, or we may define ξ as a random variable on the coordinate subspace of R^N associated with the random matrices. The difference between these two approaches is that in the former $\tilde{\Xi}$ is a subset of a flat parallel to a subspace of R^N whereas in the latter $\tilde{\Xi}$ is the projection of this set into the subspace. In any case we shall speak of the nonrandom parameters of (1.1) or degenerate components of ξ as fixed. The distinction will not be important in this paper although it can be in some contexts, as in Theorem (3.14) of [20].

[2] Recall here that $\tilde{\Xi}_W$ (etc.) is not just the projection of $\tilde{\Xi}$ into the subspace of R^N spanned by the coordinates corresponding to the components of W, but the closure of this projection.

In view of the two ways in which $Q(x)$ may be $+\infty$, $\{x \mid Q(x) < +\infty\}$ is contained in, but not necessarily equal to, the second-stage feasibility set

$$K_2 = \{x \mid \quad (1.2) \text{ is feasible with probability } 1\} . \qquad (1.4)$$

However, it can be shown that, unlike $\{x \mid Q(x) < \infty\}$, K_2 is a closed convex set: see Theorem (3.5) of [20]. In Section II we shall quote some conditions sufficient to insure that K_2 is actually equal to the set of all x for which $Q(x)$ is finite. Under these conditions x is feasible if and only if it belongs to the closed convex set $K_1 \cap K_2$.

A notion convenient in discussing the feasibility of a linear program such as (1.2) is the positive hull operator, "pos" [19]. If W is any $\bar{m} \times \bar{n}$ matrix, pos W is the closed convex cone consisting of all points in $R^{\bar{m}}$ which can be represented as nonnegative weighted sums of columns of W, i.e., pos $W = \{t \mid \exists y \geq 0, \; t = Wy\}$. Thus (1.4) may be rewritten

$$K_2 = \{x \mid (p - Tx) \in \text{ pos } W \text{ with probability } 1\} . \qquad (1.5)$$

* * * * *

Some special forms of stochastic programs with recourse to be discussed in this paper are stochastic programs with --

(i) relatively complete recourse: $K_2 \supset K_1$. This holds if and only if for all values of x in K_1, $p - Tx$ belongs to pos W with probability 1.

(ii) complete recourse: pos $W = R^{\bar{m}}$ with probability 1. This is a special case of relatively complete recourse.

(iii) fixed recourse: W is fixed, i.e., for all values of ξ in $\tilde{\Xi}$, W is constant.

(iv) simple recourse: W is fixed and equal to $[I, -I]$, the $\bar{m} \times (2\bar{m})$ matrix formed by juxtaposing an $\bar{m} \times \bar{m}$ identity matrix and its negative. This is a special case of both fixed recourse and complete recourse. (In [23] the term "complete" was used in connection with this special form of recourse in recognition of the fact that it is a special case of (ii). Subsequent development of the subject has suggested the distinctions given here.)

(v) stable recourse: W is square and nonsingular with probability 1.

Sections II and III of this paper are devoted primarily to simple recourse and stable recourse respectively. Sections IV and V are devoted to showing how the active approach of G. Tintner and the conditional probability model of chance constrained programming due to A. Charnes and M. Kirby can be formulated as stochastic programs with special forms of recourse. A few properties of fixed recourse necessary for the discussion of simple recourse appear as introductory remarks in Section II. The remaining terms in the above list, namely complete recourse and relatively complete recourse, are used in the discussion of simple recourse in Section II and the active approach in Section IV.

For some results on the existence of continuous piecewise linear decision rules for programs with only right-hand sides random (a special case of fixed recourse) see [22].

II. Stochastic Programs with Simple Recourse

By definition a stochastic program with simple recourse reads

$$\operatorname*{Inf}_{x} \qquad z(x) = \bar{c}x + E\,\mathrm{Min}(q^{+}y^{+} + q^{-}y^{-}) \qquad (2.1a)$$

$$\begin{array}{lllll} \text{subject to} & Ax & & = b & (2.1b) \\ & Tx \quad + & Iy^{+} - Iy^{-} & = p & (2.1c) \\ & x \geq 0 & y^{+} \geq y^{-} & \geq 0, & (2.1d) \end{array}$$

where I is an $\bar{m} \times \bar{m}$ identity matrix; $\bar{c}$, A, and b are fixed matrices of dimension $1 \times n$, $m \times n$, and $m \times 1$ respectively; q^{+}, q^{-}, p, and T are matrices of dimensions $1 \times \bar{m}$, $1 \times \bar{m}$, $\bar{m} \times 1$, and $\bar{m} \times n$ respectively whose elements are components of a random variable ξ defined on R^{N}, $N = \bar{m}(n + 3)$; and E denotes expected value with respect to ξ.

In [23] the special case of (2.1) in which p is the only random variable is examined in detail. In Section 2 of [23] theoretical results are developed for this special case. Other sections of that paper discuss practical solution methods when the random variables are discrete or continuous. One of our principal objectives here will be to show that essentially all the theoretical results obtained in Section 2 of [23] hold for the general stochastic program with simple recourse, (2.1). This success in generalizing the theoretical results suggest that it may be possible to adapt some of the computational methods to the general case also, but we shall touch on this possibility only briefly.

Before beginning our discussion of programs with simple recourse we list in the omnibus theorem below some of the properties which (2.1) has in common with all stochastic programs with fixed recourse.

Theorem (2.2). Suppose the stochastic program (1.1) has fixed recourse (i.e., W is fixed) and each component ξ_i of ξ has finite variance (i.e., $E\{\xi_i^2\}$ finite). Then:

(a) The second-stage feasiblity set K_2 given by (1.4) is a closed convex set which may be written

$$\begin{aligned} K_2 &= \{x \mid p - Tx \in \text{ pos } W \text{ for all } (p, T) \in \tilde{\Xi}_{pT}\} \\ &= \bigcap_{[(p,T)\in\tilde{\Xi}_{pT}]} \{x \mid \exists y \geq 0,\ Wy = p - Tx\}. \end{aligned} \qquad (2.3)$$

Moreover, if we have complete recourse, i.e., if pos $W = R^{\bar{m}}$, then $K_2 = R^n$.

(b) The equivalent deterministic form of (1.1) is

$$\begin{aligned} &\inf \qquad z(x) = \bar{c}x + Q(x) \\ &\text{subject to} \qquad x \in K_1 \cap K_2 , \end{aligned} \tag{2.4}$$

where K_1 is the closed convex polyhedron $\{x \mid Ax = b,\ x \geq 0\}$. Moreover, $-\infty \leq Q(x) < +\infty$ for all x in K_2. Thus x is a feasible solution to (2.4) or the original stochastic program (1.1) if and only if $x \in K_1 \cap K_2$.

(c) Let $x \in R^n$ and $\xi \in R^N$ and suppose either $x \in K_2$ and $\xi \in \tilde{\Xi}$ or we have complete recourse. Then $Q(x, \xi) < +\infty$. Moreover, $Q(x, \xi) > -\infty$ if and only if the vector q^T associated with ξ lies in the convex polyhedral cone $\psi = \text{pos}[W^T, -W^T, I]$, where $[W^T, -W^T, I]$ is the matrix formed by juxtaposing the transpose of W, its negative, and an $\bar{n} \times \bar{n}$ identity matrix.

(d) If ψ does not contain $\tilde{\Xi}_q$, then $Q(x, \xi) = -\infty$ with positive probability and $Q(x) = -\infty$ for all x in K_2. If ψ does contain $\tilde{\Xi}_q$, then $Q(x, \xi)$ is finite for all x in K_2 and all ξ in $\tilde{\Xi}$ and the restriction of $Q(x)$ to K_2 is a finite, Lipschitz, convex function.

Proof. In Theorem (3.14) of [20] it is shown that the expression "with probability 1" in (1.5) can be replaced by "for all $(p, T) \in \tilde{\Xi}_{pT}$" if W is fixed. We have already remarked in Section I that K_2 is always closed and convex, but this follows easily in this case from the second line of (2.3) and the fact that $\{x \mid \exists\, y \geq 0,\ Wy = p - Tx\}$ is a closed convex polyhedron. The rest of the proof of (a) is easy. Theorem (4.5) of [20] establishes (b). Now consider (c). If we have complete recourse the second-stage program (1.2) is always feasible. On the other hand, if $x \in K_2$ and $\xi \in \tilde{\Xi}$, then $(p, T) \in \tilde{\Xi}_{pT}$, and in view of (2.3) the second-stage program is again feasible. Thus in either case $Q(x, \xi) < +\infty$. Since the second-stage program (1.2) is feasible, it is not possible that (1.2) and its dual are both infeasible. Thus $Q(x, \xi)$ is the value of the dual. The natural dual of (1.2) is a maximization subject to inequality constraints on unconstrained variables; it may be rewritten as a maximization subject to equality constraints on nonnegative variables, namely

$$\begin{aligned} &\text{Maximize} \qquad [(p - Tx)^T,\ -(p - Tx)^T,\ 0]u \\ &\text{subject to} \qquad [W^T,\ -W^T,\ I]u = q^T \\ &\qquad\qquad\qquad u \geq 0 . \end{aligned}$$

The remainder of (c) is now obvious. Now consider (d). Since ψ is a closed and $\tilde{\Xi}_q$ is the smallest closed set of measure 1, the statements concerning $Q(x, \xi)$ alone are immediate. The remainder of (d) follows from Theorem (4. 5) of [20]. ||

The hypothesis of this theorem that each component of ξ has finite variance is a very weak one from a practical viewpoint. However, it is still stronger than absolutely necessary. What is needed, as an examination of the proof of Theorem (4. 5) of [20] will show, is that certain bilinear forms $f(q;(p, T))$ must have expected values. Thus an alternate hypothesis is that the components of q, p, and T have expected values and q is independent of (p, T). In particular, existence of the expectation of (p, T) will suffice if q is fixed, see Corollary (4. 7) of [20].

We return now to a discussion of programs with simple recourse. As in Theorem (2. 2) we shall suppose where necessary that ξ satisfies suitable moment conditions. Several properties of the program (2. 1) can be deduced directly from Theorem (2. 2). Since pos $W = \text{pos}[I, -I] = R^{\bar{m}}$, we have complete recourse, and $K_2 = R^n$. Thus by (b) of Theorem (2. 2), the equivalent deterministic program of (2. 1) reads

$$\begin{array}{lll} \text{Inf} & z(x) = \bar{c}x + Q(x) & \\ \text{subject to} & Ax = b & \qquad (2.5) \\ & x \geq 0\,, & \end{array}$$

with $Q(x) < +\infty$ for all x. As in [23] and [25] where the case p only random is considered, we see that it is possible to attribute portions of the quantity $Q(x, \xi)$ to different rows of (2. 1c). Specifically,

$$Q(x, \xi) = \sum_{i=1}^{\bar{m}} Q_i(x, \xi_{(i)}), \text{ where}$$

$$\begin{array}{llll} Q_i(x, \xi_{(i)}) = & \text{Min} & q_i^+ y_i^+ + q_i^- y_i^- & \\ & \text{subject to} & y_i^+ - y_i^- = p_i - T_i x & (2.6) \\ & & y_i^+ \geq 0 \quad y_i^- \geq 0 & \end{array}$$

and $\xi_{(i)}$ is the random $(n + 3)$-vector whose components consist of the i-th components of q^+, q^-, p and the components of the i-th row, T_i, of T. Note that (2. 6) may itself be considered the second-stage program of a stochastic program with recourse, to which Theorem (2. 2) may be applied. The cone ψ_i obtained in applying part (c) of Theorem (2. 2) to this program is just the set of values of q_i^+ and q_i^- such that $q_i^+ + q_i^- \geq 0$. Thus, if $q_i^+ + q_i^- \geq 0$ almost surely, then

$$Q_i(x) = E\{Q_i(x, \xi_{(i)})\}$$

is a finite, Lipschitz, convex function defined for all x. Now the cone obtained in applying part (c) of Theorem (2.2) to the whole program (2.1) is the product of the cones ψ_i. Also, since each $\mathcal{Q}_i(x)$ is finite, expectation and summation on i may be interchanged to yield

$$\mathcal{Q}(x) = E_\xi \Sigma_i Q_i(x, \xi) = \Sigma_i E_\xi Q_i(x, \xi) = \Sigma_i E_{\xi(i)} Q_i(x, \xi_{(i)})$$
$$= \Sigma_i \mathcal{Q}_i(x) .$$

(This may fail if the $\mathcal{Q}_i(x)$ are not finite. It is easy to construct examples of distributions without finite variances such that the integrals $\mathcal{Q}_i(x) = E_\xi Q_i(x, \xi) = E_{\xi(i)} Q_i(x, \xi_{(i)})$ diverge variously to $+\infty$ and $-\infty$ but such that $Q(x, \xi) = \Sigma_i Q_i(x, \xi) = 0$ for all x and all $\xi \in \tilde{\Xi}$. In this case $\mathcal{Q}(x) \equiv 0$, but according to the definition of the expectation operator and the conventions introduced in [20], $\Sigma_i \mathcal{Q}_i(x) = +\infty$.) Thus we have proven:

Proposition (2.7). Suppose (2.1) is a stochastic program with simple recourse such that $q_i^+ + q_i^- \geq 0$ almost surely, $1 \leq i \leq \bar{m}$, and each component of ξ has a finite variance. Then (2.5) is the equivalent deterministic program for (2.1), where

$$\mathcal{Q}(x) = \Sigma_i \mathcal{Q}_i(x) = \Sigma_i E\{Q_i(x, \xi_{(i)})\}$$

is a finite, Lipschitz, convex function defined for all x.

The importance of the representation of $\mathcal{Q}(x)$ as a sum of functions $\mathcal{Q}_i(x) = E\{Q_i(x,\xi_{(i)})\}$ lies in the fact that the $Q_i(x, \xi_{(i)})$ and hence $\mathcal{Q}(x)$ depend only on the marginal distributions of the random variables $\xi_{(i)}$. By no means a trivial part of the problem of obtaining a practical solution of a general stochastic program would be the experimental determination of the joint distribution of the random variables. Even the specification and manipulation of the resulting joint distribution function would be extremely difficult for a very few variables. But it might not be too much to hope for that a practical simple recourse problem would involve only one or two random components in each $\xi_{(i)}$. The case studied in [23] in which p only is random is just one example of a simple recourse problem in which each $\xi_{(i)}$ has only one random component.

In Section 4 of [23] it is shown how a descent method can be used to solve a stochastic program with simple recourse when p is the only random variable and each of its components has a continuous distribution. The algorithm makes use of the fact that in this case $\mathcal{Q}(x)$ has a continuous gradient. When the distribution contains points of positive mass, so that at certain values of x the graph of $\mathcal{Q}(x)$ has more than one supporting hyperplane, a descent algorithm which uses the gradient of an arbitrary supporting hyperplane can run into serious difficulties. The following proposition shows that the result on the existence of a continuous gradient has a natural extension for the general stochastic program with simple recourse. In view of this result, there would appear to be no serious obstacle to adapting the algorithm of [23] to the general case provided each random vector $\xi_{(i)}$ has one (or very few) random components.

Proposition (2.8). Suppose in addition to the hypotheses of Proposition (2.7) that each of the marginal variables $\xi_{(i)}$ has an absolutely continuous distribution. Then $Q(x)$ has bounded continuous first partial derivatives and

$$\nabla Q(x) = \frac{\partial Q(x)}{\partial x_1}, \ldots, \frac{\partial Q(x)}{\partial x_n} = -E\{\pi(x,\xi)T\}, \tag{2.9}$$

where $\pi(x,\xi)$ is the row $\bar{m}$-vector whose i-th component is

$$\pi_i(x,\xi) = \pi_i(x,\xi_{(i)}) = \begin{cases} q_i^+ \text{ if } (p_i - T_i x) \geq 0 \\ \\ -q_i^- \text{ if } (p_i - T_i x) < 0. \end{cases} \tag{2.10}$$

Proof. The second-stage program is

$$\begin{aligned} Q(x,\xi) = \operatorname{Min} q^+ y^+ &+ q^- y^- \\ Iy^+ - Iy^- &= p - Tx \\ y^+ \geq 0 \quad y^- &\geq 0 \end{aligned}$$

and its dual is

$$\begin{aligned} Q^*(x,\xi) &= \max \pi \cdot (p - Tx) \\ -q^- &\leq \pi \leq q^+ . \end{aligned} \tag{2.11}$$

The primal problem is always feasible, hence $Q^*(x,\xi) = Q(x,\xi)$. By assumption, $-q^- \leq q^+$ almost surely; and hence $-q^- \leq q^+$ for all $\xi \in \tilde{\Xi}$. Thus, provided $\xi \in \tilde{\Xi}$ so that (2.11) is feasible, the optimal value of in (2.11) is given by (2.10), with ambiguity only when some $p_i - T_i x$ is zero. On differentiating the objective in (2.11) we find $\nabla_x Q(x,\xi) = -\pi(x,\xi) \cdot T$ for all x and all $\xi \in \tilde{\Xi}$ such that no component of $p - Tx$ is zero. The same argument applies to each of the functions $Q_i(x,\xi)$, so that $\nabla_x Q_i(x,\xi) = -\pi_i(x,\xi)T_i = -\pi_i(x,\xi_{(i)}) \cdot T_i$ provided $\xi_{(i)}$ is a member of the set $\Omega_i(x) = \tilde{\Xi}_{(i)} \cap \{\xi_{(i)} | (p_i - T_i x) \neq 0\}$, where $\tilde{\Xi}_{(i)}$ is the support of $\xi_{(i)}$. Now since $\xi_{(i)}$ has an absolutely continuous distribution, $\Omega_i(x)$ has measure 1. Thus

$$\nabla_x Q_i(x,\xi_{(i)}) = -\pi_i(x,\xi_{(i)}) \cdot T_i \text{ almost surely.} \tag{2.12}$$

Also, for each value of $\xi_{(i)}$, $Q_i(x, \xi_{(i)})$ is a continuous piecewise linear function satisfying the Lipschitz condition in the direction of x_j, with Lipschitz constant $B_{ij}(\xi_{(i)}) = \max \{ | q_i^+ T_{ij} |, | q_i^- T_{ij} | \}$. Since each component of $\xi_{(i)}$ has finite variance,

$$E\{ B_{ij}(\xi_{(i)}) \} \text{ is finite} \tag{2.13}$$

It follows from (2.12) and (2.13) that the interchange of expectation and partial differentiation in

$$\nabla Q_i(x) = \nabla E\{Q_i(x, \xi_{(i)})\} = E\{ \nabla Q_i(x, \xi_{(i)})\} = - E\{ \pi_i(x, \xi_{(i)}) \cdot T_i \} \tag{2.14}$$

is valid and all quantities are finite; see for example Cramer [8], who treats the case in which the parameter x and the random variable $\xi_{(i)}$ are both one-dimensional. Summing (2.14) on i and interchanging the summation with ∇_x and E we get (2.9). From (2.13) it also follows that Q(x) has bounded partial derivatives. Continuity of the partial derivatives can be proven by a similar method (see Cramer [8] again) or deduced from the fact that a convex function with partial derivatives has continuous partial derivatives [12].

It is easily verified that Propositions (2.7) and (2.8) remain true when the alternate moment conditions discussed in connection with Theorem (2.2) are used. An alternate version of Proposition (2.8) in which it is assumed q is fixed and (p, T) has an expected value has been obtained by Kall, who gives a detailed proof in [13].

Rather than seeking to define stochastic programs with simple recourse as the most general programs having the properties exhibited in Propositions (2.7) and (2.8), we have tried to give the simplest definition consistent with the spirit of these results. Naturally there are a number of minor generalizations possible. For example, we could take any recourse matrix W obtained from [I, -I] by permutation, positive scalar multiplication, or duplication of columns. One such generalization of simple recourse has been mentioned in the literature; it is discussed in the following paragraph.

In Case (B) on page 99 of [5] it is shown (using different terminology) that under certain conditions the optimum value of the second-stage linear program can be expressed in closed form (using the absolute-value function) as a function of the random variable ξ and the decision variable x. (It is supposed that p is the only random variable, but the same argument applies obviously so long as **W** is fixed.) The condition imposed is that there exist square nonsingular matrices D, F and a nonnegative real number h such that W can be partitioned in the form $W = [D, -F]$, $D^{-1}F$ is a nonnegative matrix, and

$$hD^{-1}F = (F^{-1}D)^T . \tag{2.15}$$

We shall show here that these conditions are equivalent to a more explicit condition which is only slightly less stringent than simple recourse. Let $G = D^{-1}F$. Then G is a square, nonnegative, nonsingular matrix. Moreover, since $G^{-1} = F^{-1}D$, we have from (2.15) that G^{-1} is nonnegative. But it is easily proven that if a matrix and its inverse are nonnegative then $G = \Delta P$, where P is a permutation matrix and Δ is a diagonal matrix with positive elements on the diagonal. In fact, substituting ΔP for $D^{-1}F$ in (2.15) we readily conclude that $\Delta = h^{-\frac{1}{2}}I$. Thus we have that $F = DG = h^{-\frac{1}{2}}DP$, or $W = [D, -h^{-\frac{1}{2}}DP]$. Now recall that W is the recourse matrix of a stochastic program. By reordering the components of q and y associated with the second half of the columns of W and multiplying these components by $h^{\frac{1}{2}}$, we obtain an equivalent stochastic program

$$\text{Minimize } \bar{c}x + E\{q'y' + q''y''\} \tag{2.16a}$$

$$Ax = b \tag{2.16b}$$

$$Tx + Dy' - Dy'' = p \tag{2.16c}$$

$$x \geq 0 \quad y' \geq 0 \quad y'' \geq 0\,. \tag{2.16d}$$

Note that this permutation and multiplication in no way increases the difficulties of solving a practical problem. We can now multiply through the equations (2.16c) by D^{-1} to obtain an equivalent stochastic program with simple recourse in which the second-stage constraints read

$$(D^{-1}T)x + Iy' - Iy'' = D^{-1}p\,. \tag{2.16c'}$$

We remark that the reduction from (2.16c) to (2.16c') is not as trivial as the elimination of P and $h^{-\frac{1}{2}}$. For simple recourse, a knowledge of the distribution of each of the marginal variables $\xi_{(i)}$ is sufficient in theory to obtain a solution. But, a knowledge of the marginal distribution of the variables associated with each row of (2.16c) is not sufficient to solve (2.16) even if q and T are fixed; the marginal distributions of the components of $D^{-1}p$ cannot be deduced from the marginal distributions of the components of p. Of course if p is the only random variable and its components are independent, the marginal distributions of the components of $D^{-1}p$ can be obtained with a moderate amount of effort from the marginal distributions of the components of p.

A model of stochastic programming related to the simple recourse model considered in this section has been studied by Evers [11]. In the terminology of this paper, his model assumes $W = [I, -I]$, q fixed, and (p, T) random, but includes as well interrelated chance constraints and set-up costs.

III. Stochastic Programs with Stable Recourse

A stochastic program is said to have stable recourse if the matrix W is square and W is nonsingular with probability one. An assumption frequently encountered in papers devoted to the so-called distribution problem is that the optimal basis is stable, i.e., the same set of columns is an optimal basis with probability one, see e.g., [1], [3], [4], [15]. Obviously there is no material difference between stochastic programs satisfying such a condition and stochastic programs with stable recourse. This observation is behind our choice of the term "stable".

Even the simplest stochastic programs with stable recourse may have some unpleasant properties. Consider for example the program

$$\begin{array}{lllll} \text{Minimize} & x_1 & & + E\{y\} & \\ & x_1 + x_2 & & & = 1 \\ & x_1 & & + wy & = 1 \\ & x_1 \geq 0 \quad x_2 \geq 0 \quad y \geq 0, & & & \end{array} \tag{3.1}$$

where w is the only element of a 1 x 1 matrix and has a distribution with support $\Xi_w = [0, 1]$. Provided w = 0 with probability zero, this is a stochastic program with stable recourse. The second-stage program for (3.1) is just

$$\begin{array}{ll} \text{Minimize} & y \\ & wy = 1 - x_1 \\ & y \geq 0 . \end{array}$$

If $x_1 < 1$ this program is feasible for all $w > 0$, if $x_1 = 1$ it is feasible for all w, and if $x_1 > 1$ it is feasible for $w < 0$ only. Thus from (1.4) we have

$$K_2 = \{x \mid x_1 \leq 1\} . \tag{3.2}$$

Note that (3.1) is an example of relatively complete recourse, since $K_1 = \{x \mid 0 \leq x_1 \leq 1,\ x_2 = 1 - x_1\} \subset K_2$.

A first unpleasant property of (3.1) is that the expected value may all too easily fail to exist. The uniform distribution on [0, 1] is absolutely continuous and possesses all moments, yet if w has this distribution and $x_1 \neq 1$, then $E\{y\}$ is $+\infty$. On the other hand, if the distribution of w on [0, 1] is triangular, i.e., given by $P\{w \leq \lambda\} = \lambda^2$, then $E\{y\}$ is finite for all values of x in K_2.

A second unpleasant property of (3.1) is that it does not satisfy the W-condition. A stochastic program with recourse is said to satisfy the W-condition if pos W and pos $[W^T, -W^T, I]$, considered as set-valued functions of ξ, are continuous on $\tilde{\Xi}$; see definition (3.10) in [20]. Here $[W^T, -W^T, I]$ is the matrix formed by juxtaposing the transpose of W, its negative, and an $\bar{n} \times \bar{n}$ identity: it is associated with the dual of the second-stage program (1.2) in the same way W is associated with (1.2). A discussion of pos W as a set-valued function and the meaning of continuity for such functions can be found in [19]. It is readily seen in example (3.1) that the restriction of pos w to the set [0, 1] is not continuous: for all w in the half-open interval (0, 1], pos w is the non-negative reals, but pos w abruptly collapses to the origin when w = 0. One of the values of the W-condition, as shown in Theorem (3.7) of [20], is that when it is satisfied we may write

$$
\begin{aligned}
K_2 &= \{x \mid \quad (1.2) \text{ feasible for all } \xi \in \tilde{\Xi}\} \\
&= \bigcap_{\xi \in \tilde{\Xi}} \{x \mid \quad (1.2) \text{ feasible for } \xi\}
\end{aligned}
\tag{3.3}
$$

in place of (1.4). This identity is not valid for (3.1), since the right-hand side is $\{x \mid x_1 = 1\}$, contradicting (3.2).

The following two theorems show how (3.3) can be reclaimed under certain conditions even if the W-condition is not satisfied. The proof are given at the end of this section.

Definition (3.4). A collection $W^1, \ldots, W^n$ of n points (or column vectors) in R^m are said to be in linear general position if none can be expressed as a linear combination of fewer than m of the others.

Obviously, if the columns of a matrix W are in linear general position, then W has the maximum rank consistent with its size, but the converse need not hold. However, if W is square then the columns of W are in linear general position if and only if W is nonsingular.

Theorem (3.5). For any stochastic program with recourse, let Σ denote the subset of R^N consisting of all points ξ such that the columns of the recourse matrix W are in linear general position. Then the set Σ is open, and if the marginal distribution of W is absolutely continuous, Σ has probability measure 1.

Theorem (3.6). If for some stochastic program with recourse the set Σ defined above has probability measure 1, then the stochastic program satisfies the W-condition relative to Σ and (3.4) hold with $\tilde{\Xi}$ replaced by $\tilde{\Sigma} = \Sigma \cap \tilde{\Xi}$.

As a corollary to Theorem (3.6) we have that a stochastic program with stable recourse satisfies (3.3) with $\tilde{\Xi}$ replaced by $\tilde{\Sigma}$. For the example (3.1) $\Sigma = \{w \mid w \neq 0\}$ and $\tilde{\Sigma} = \{w \mid w \in (0, 1]\}$. Substituting

$\tilde{\Sigma}$ into (3.3) correctly yields (3.2).

We now observe that any stochastic program with stable recourse can be rewritten in the equivalent form

$$\begin{array}{llll} \text{Minimize} & \bar{c}x + E \min_y qy & & (3.7a) \\ & Ax & = b & (3.7b) \\ & W^{-1}Tx + \quad Iy & = W^{-1}p & (3.7c) \\ & x \geq 0 \qquad\quad y \geq 0\,, & & (3.7d) \end{array}$$

where ξ is considered to be defined on Σ only. An expression for K_2 is readily obtained from (3.7c) and (3.3). Specifically,

$$K_2 = \bigcap_{\xi \varepsilon \tilde{}} \{x \mid (W^{-1}T)\, x \geq W^{-1}p\}\ .$$

From (3.7c) the optimal y is uniquely determined for all ξ in Σ as $y = W^{-1}(p - Tx)$. Thus the original stochastic program with stable recourse is equivalent to

$$\begin{array}{ll} \text{Minimize} & z(x) = E\{qW^{-1}p\} + (\bar{c} - E\{qW^{-1}T\})x \\ \text{subject to} & Ax = b \\ & x \,\varepsilon\, K_2 \\ & x \geq 0\,, \end{array} \qquad (3.8)$$

(provided of course that the necessary interchanges of expectation with finite sum and multiplication by components of x are valid--a provision clearly satisfied if $E\{qW^{-1}p\}$ and $E\{qW^{-1}T\}$ are finite.) Note that (3.8) is a minimization of a linear function over a closed convex set.

A particularly simple special form of recourse occurs when W is fixed, square, and nonsingular. This form of recourse has been discussed in detail in Section 3, Case 2 of [24] for the case p only random. This is a special case of both fixed recourse and stable recourse, and we have

$$K_2 = \bigcap_{(p,T)\,\varepsilon\, \tilde{\Xi}_{pT}} \{x \mid W^{-1}(p - Tx) \geq 0\}\ . \qquad (3.9)$$

It can be shown that an analogue of (3.9) holds for any fixed recourse matrix whether stable or not, but with W^{-1} replaced by a typically very large matrix, $-W^*$, the (positive) polar matrix of W. This and related results will be discussed in [~~28~~]. R. Wets: S.P. w Fixed Recourse: The Equivalent Program.

We conclude this section with the proofs of Theorems (3.5) and (3.6).

Proof of Theorem (3.5). The theorem is an immediate consequence of the intuitively obvious fact that the set Σ of all m by n matrices whose columns are in linear general position has a closed complement D in R^{mn} of Lesbegue measure zero. The set D is a (not necessarily disjoint) union of a finite number of sets D_i, a typical one of which consists of those matrices such that a particular column is a linear combination of a particular set of m-1 or fewer other columns. Each set D_i may be parametrized by n-1 arbitrary column vectors of length m and m-1 or fewer scalars acting as coefficients in the expression for the selected dependent column. The dimension of the parametrization space is at most mn-1, and the mapping of this space onto D_i is of class C^1 (in fact bilinear). Hence D_i is a closed set of Lesbegue measure zero. ||

Lemma (3.10). The set Σ of all m x n matrices W whose columns are in linear general position is the disjoint union of two open subsets $\Sigma \cap \Sigma_1$ and $\Sigma \cap \Sigma_2$ of R^{mn}, where

$$\Sigma_1 = \{W \mid \text{pos } W = R^m\}$$

and

$$\Sigma_2 = \{W \mid \text{pos } W \text{ is a pointed cone and no column of } W \text{ is zero}\}$$

are also disjoint open subsets of R^{mn}.

Proof. Clearly Σ_1 and Σ_2 are disjoint. Consider the proposition that Σ_1 is open. If $n < m + 1$, then Σ_1 is empty. Otherwise, let $\hat{W}$ be a particular point of Σ_1. Then some set of m columns of $\hat{W}$ constitutes a nonsingular submatrix $\hat{B}$ of $\hat{W}$. For the convenience of notation assume $\hat{B}$ consists of the first m columns so that $\hat{W} = [\hat{B}, \hat{C}]$. For W within a sufficiently small neighborhood $\mathcal{N}_1$ about W, the matrix B in the partition $W = [B, C]$ is nonsingular and B^{-1} is a continuous function of W. Thus $B^{-1} W = [I, D]$ is continuous in W on $\mathcal{N}_1$. Since pos $\hat{W} = R^m$, there exists some vector d such that the vector $\hat{D}d$ has strictly negative entries. For a sufficiently small neighborhood $\mathcal{N}_2 \subset \mathcal{N}_1$ about $\hat{W}$, $Dd < 0$. If follows that post $B^{-1} W = \text{pos } W = R^m$ for all W in $\mathcal{N}_2$, i.e., Σ_1 is open. Next consider the proposition that Σ_2 is open. If $\hat{W}$ is any point of Σ_2 then post $\hat{W}$ is contained in some halfspace $\{z \mid hz \leq 0\}$ whose bounding hyperplane supports pos $\hat{W}$ at the origin only, i.e., $h\hat{W}$ is a strictly negative vector. (Recall that no column of $\hat{W}$ is zero.) It follows that there is a neighborhood $\mathcal{N}$ about $\hat{W}$ such that hW is strictly negative for all W in $\mathcal{N}$. Thus $\mathcal{N} \subset \Sigma_2$, and hence Σ_2 is open. Since Σ is open by Proposition (3.5), all that remains to be shown is that Σ_1

and Σ_2 together cover Σ. Suppose, therefore, that $W \in \Sigma$ but pos W is neither pointed nor all of R^m. Then the lineality space $\mathcal{L}$ pos W of pos W has dimension k, $0 < k < m$. It is readily shown that the columns of W contained in pos W are a positive basis for $\mathcal{L}$ pos W, and hence some nonempty set of $k + 1$ or fewer of them are linearly dependent. Since $k < m$, this contradicts the assumption that the columns of W are in linear general position. ||

Proof of Theorem (3.6). We must show that the restrictions of pos W and pos$[W^T, -W^T, I]$ to the set Σ of Lemma (3.10) are continuous set-valued functions in the sense of [19]. From the proof of Theorem (3.12) of [20] it is apparent that the restrictions of these functions to $\Sigma \cap \Sigma_1$ and $\Sigma \cap \Sigma_2$ are separately continuous. But since $\Sigma \cap \Sigma_1$ and $\Sigma \cap \Sigma_2$ are open, the restrictions of pos W and pos$[W^T, -W^T, I]$ to their union are continuous. ||

IV. Tintner's Active Approach

In his paper "A Note on Stochastic Linear Programming" [16], Tintner formulated a model for stochastic programming which was later [17] given the name of active approach. The typical problem can be summarized as follows: A decision-maker has n activities in which he may engage and has m resources to allocate to them. The amount d_i of resource i may be known or known only in probability. He must decide upon a fixed fraction x_{ij} of each resource i to allocate to each activity j. The level y_j of activity j actually achieved when the values d_i are known must satisfy

$$0 \leq y_j \leq \operatorname*{Min}_i (d_i \, x_{ij} \, r_{ij}^{-1}), \qquad (4.1)$$

where r_{ij} is a technology coefficient relating the activity j to the resource i. Again r_{ij} may be fixed or known in distribution. The objective of the decision-maker is to select the allocation coefficients x_{ij} so as to optimize a given criterion on the distribution of the return $z = \Sigma_j \, q_j \, y_j$. Once more the q_j's can be random or fixed. One of the possible criteria considered by Tintner is the expected value of z. We shall consider this criterion only, although there are ways of handling certain other criteria within the framework of stochastic programs with recourse.

From the statement of the problem given above we see immediately that we have a stochastic program with recourse. The decision process occurs in two steps: First the x_{ij}'s are selected, then the values of the random variables d_i, r_{ij}, and q_j are observed and values of y_j minimizing $\sum_j q_j \, y_j$ subject to (4.1) are selected. Moreover, the constraints are linear. Thus the problem can be given in the form

$$\text{Minimize} \quad Q(x) = \quad E\left\{\text{Min} \sum_{j=1}^{n} -q_j\, y_j\right\} \tag{4. 2a}$$

$$\text{subject to} \quad \sum_{j=1}^{n} x_{ij} = 1, \quad 1 \leq i \leq m \tag{4. 2b}$$

$$-d_i x_{ij} + r_{ij} y_j \leq 0, \quad 1 \leq i \leq m, \ 1 \leq j \leq n, \tag{4. 2c}$$

$$x_{ij} \geq 0 \qquad y_j \geq 0, \quad 1 \leq i \leq m, \ 1 \leq j \leq n. \tag{4. 2d}$$

Not the least significant fact to be derived from this representation of an active-approach problem as a stochastic program with recourse is that the active-approach problem has an equivalent deterministic program which is a maximization of a concave objective.

When (4. 2) is written out completely in the form (1. 1), the matrix A is a sparse matrix of zeros and ones with m rows and mn columns, T is a diagonal matrix of size mn by mn with d's as diagonal elements, and W can be arranged as a block-diagonal matrix, whose j-th block has the form

$$\begin{bmatrix} r_{1j} & 1 & & & 0 \\ r_{2j} & & 1 & & \\ \cdot & & & \cdot & \\ \cdot & 0 & & & \cdot \\ \cdot & & & & \cdot \\ r_{mj} & & & & -1 \end{bmatrix}.$$

In an example considered by Tintner, the y_j's are the amounts of various crops raised on a farm, the q_j's are their market prices, and the d_j's are the amounts of labor, land, etc. available. In this case it is reasonable to assume that the q_j's, d_i's, and r_{ij}'s are nonnegative, and we shall make this assumption in the rest of this section. Under this assumption, (4. 2) is an example of relatively complete recourse as defined in the introduction. That is, if the x_{ij} satisfy (4. 2b) and (4. 2d) then (4. 2c) is solvable in nonnegative y_j's almost surely. It is apparent from our initial description of the problem that the choice of y_j depends only on $x_{1j}, \ldots, x_{nj}$ and the value of a random variable $\xi_{(j)}$ whose components are the components q_j; $d_1, \ldots, d_m$; $r_{1j}, \ldots, r_{mj}$ of ξ. This is reflected in the block structure of W and the consequent fact that the second-stage program for (4.2) is essentially a collection of independent subproblems. The optimum value of the j-th subproblem is

$$Q_j(x_{1j}, \ldots, x_{mj}; \xi_{(j)}) = \text{Min}\{-q_j y_j \mid y_j \text{ satisfying (4.1)}\} \quad (4.3)$$

and if we write

$$Q_j(x_{1j}, \ldots, x_{mj}) = E\{Q_j(x_{1j}, \ldots, x_{mj}; \xi_{(j)})\} \quad (4.4)$$

then (provided the expectations in (4.4) are finite)

$$Q(x) = \sum_{j=1}^{n} Q_j(x_{1j}, \ldots, x_{mj})$$

and the equivalent convex program for (4.2) may be written

$$\begin{aligned} &\text{Minimize} && \sum_{j=1}^{n} Q_j(x_{1j}, \ldots, x_{mj}) \\ &\text{subject to} && \sum_{j=1}^{n} x_{ij} = 1, \quad 1 \leq i \leq m, \\ & && x_{ij} \geq 0 . \end{aligned} \quad (4.5)$$

Obviously only the marginal distributions associated with the variables $\xi_{(1)}, \ldots, \xi_{(n)}$ are needed in solving the problem.

We conclude this section by showing that under certain moderate assumptions a numerical solution of an active-approach problem is possible. We suppose henceforth for each j, $1 \leq j \leq n$, that (i) d_j is fixed and positive, (ii) $r_{ij}, \ldots, r_{mj}$ are positive random variables so that $r_{1j}^{-1}, \ldots, r_{mj}^{-1}$ are also positive random variables, (iii) the distribution function F_{ij} of r_{ij}^{-1} is absolutely continuous, and (iv) q_j; $r_{ij}^{-1}, \ldots, r_{mj}^{-1}$ are independent and have finite expected values. Then from (4.3) it follows

$$Q_j(x_{1j}, \ldots, x_{mj}) = -P\{q_j \geq 0\} \cdot E\{\min_i(x_{ij}\, d_i\, r_{ij}^{-1})\}. \quad (4.6)$$

If F_j denotes the distribution function for $\min_i(x_{ij}\, d_i\, r_{ij}^{-1})$, then

$$\begin{aligned} 1 - F_j(t) &= P\{\min_i(x_{ij}\, d_i\, r_{ij}^{-1}) > t\} \\ &= \prod_i P\{x_{ij}\, d_i\, r_{ij}^{-1} > t\} \\ &= \prod_i [1 - F_{ij}(t/x_{ij}\, d_i)] \end{aligned}$$

and

$$E\{\min_i(x_{ij}\, d_i\, r_{ij}^{-1})\} = \int_0^\infty [1 - F_j(t)]dt$$
$$= \int_0^\infty \prod_i [1 - F_{ij}(t/x_{ij}\, d_i)]dt \tag{4.7}$$

Of course (4.7) is finite since each r_{ij}^{-1} has finite expected value. Equations (4.7) and (4.6) together give an expression for $Q_j(x_{1j}, \ldots, x_{mj})$ which involves only a one-parameter integral. Thus the numerical evaluation of the objective of (4.5) is within practical limits of difficulty. Moreover, it is not difficult to see that the partial derivatives of $Q_j(x_{1j}, \ldots, x_{mj})$ exist and are given by

$$\frac{\partial}{\partial x_{kj}} Q_j(x_{1j}, \ldots, x_{mj}) = \frac{-P\{q_j \geq 0\}}{x_{kj}^2\, d_k} \cdot \int_0^\infty t\, F'_{kj}\left(\frac{t}{x_{kj} d_k}\right) \prod_{i \neq k} \left[1 - F_{ij}\left(\frac{t}{x_{ij}\, d_i}\right)\right] dt.$$

Thus an active-approach problem satisfying the assumptions given above when written as a minimization has a deterministic equivalent (4.5) which is a convex programming problem with linear constraints and a convex objective whose value and gradient are computable at any point.

V. The Conditional Probability (E-) Model for Stochastic Programs with Chance Constraints

In this brief section we show how a problem formulated using the conditional probability (E-) model for stochastic programs with chance constraints can be reduced to a stochastic program with recourse.[1] For this point it suffices to consider a two-period model. The generalization to an n-period model does not present any theoretical difficulty but it is notationally cumbersome and does not provide any additional insights.

The conditional probability (E-) model for stochastic programs with chance constraints defined by Charnes and Kirby in [6, 7] may be written for the special case of two-periods in the form

[1]This result should not be taken as implying that the two models have the same economic interpretation or that they arise from identical practical situations. In this connection see [2].

$$\text{Minimize} \quad z = E\ \{ c^1 x + \text{Min}\, E\{ c^2 y \quad | \xi^1 \} \} \tag{5.1a}$$

$$\text{subject to} \quad \Pr\{ A\ x \qquad \geq b^1 \quad \} \geq \alpha^0 \tag{5.1b}$$

$$\Pr\{ T\ x + \qquad Wy \geq b^2 | \xi^1 \} \geq \alpha^1 \tag{5.1c}$$

$$x \geq 0 \qquad y \geq 0\ ,$$

where A, T, and W are fixed matrices of size m x n, $\bar{m}$ x n, and $\bar{m}$ x $\bar{n}$ respectively; c^1 and c^2 are random row vectors of length n and $\bar{n}$ respectively; b^1 and b^2 are random column vectors of length m and $\bar{m}$ respectively; α^0 is a fixed column vector of length m; and α^1 is a random column vector of length $\bar{m}$. The expressions of the form $\Pr\{ b \leq t\} \geq \alpha$ in (5.1b) and (5.1c) are intended as abbreviations for the componentwise expressions $\Pr\{ b_i \leq t_i\} \geq \alpha_i$. For brevity we write ξ^1 for the random variables (c^1, b^1, α^1) and ξ^2 for the random variables (c^2, b^2). The model may be interpreted briefly as follows: First, the decision maker selects his first-period decision x subject to (5.1b). Second, he observes the values of the random variables c^1, b^1, and α^1 making up ξ^1. Third, with ξ^1 known and x already chosen, he selects the second-period decision y subject to (5.1c) so as to minimize $E\{ c^2 y | \xi^1\}$. We assume that the joint distribution of (ξ^1, ξ^2) is available, the conditional distribution of ξ^2 given ξ^1 is available or computable, and all random components have finite expectation. A description of the n-period model can be found in [14] as well as in [6, 7].

Much the same type of reasoning used in [20] on the recourse model applies to the conditional probability model described above. Thus we may define a second-period problem:

$$Q(x,\ \xi^1) = \quad \min_y E\{ c^2 y | \xi^1\} \tag{5.2a}$$

$$\Pr\{ Tx + Wy \geq b^2 | \xi^1\} \geq \alpha^1 \tag{5.2b}$$

$$y \geq 0\ ,$$

and as in [20] we define $Q(x,\ \xi^1)$ to be $+\infty$ or $-\infty$ if (5.2) is infeasible or unbounded below. We also adopt the definition of the integral (expectation) E given in [20] which accommodates infinite integrands. Since we assume that c^2 has finite expectation, $q(\xi^1) = E\{ c^2 | \xi^1\}$ is finite for almost all ξ^1 and hence $E\{ c^2 y | \xi^1\} = q(\xi^1)y$ for almost all ξ^1. Also, (5.2b) is equivalent to

$$Wy \geq p(\xi^1) - Tx \tag{5.3}$$

where

$$p_i(\xi^1) = \min \{t | F_i^2(t | \xi^1) \geq \alpha_i^1 \}$$

and $F_i^2(t \mid \xi^1)$ is the (right-continuous) conditional distribution function given by

$$\Pr\{b_i^2 \le t \mid \xi^1\} = F_i^2(t \mid \xi^1).$$

In the event $\alpha_i^1 = 0$, this gives $p_i(\xi^1) = -\infty$ and the i-th constraint of (5.3) is satisfied for all y. In the event $\alpha_i^1 = 1$ and the support of $(b_i^2 \mid \xi^1)$ is not bounded above, $p_i(\xi^1) = +\infty$ and (5.3) is infeasible. In what follows we make the assumption that neither of these events occur with positive probability. Under this assumption we may define functions q* and p* which are finite and differ from q and p at most on a set of measure zero. Then $Q(x, \xi^1)$ is equal almost surely to $Q^*(x, \xi^1)$ where

$$Q^*(x, \xi^1) = \text{Min } q^*(\xi^1)y$$
$$Wy \ge p^*(\xi^1) - Tx$$
$$y \ge 0 .$$

The same procedure used in converting (5.1c) to (5.3) can be applied to (5.1b) to obtain the equivalent constraint

$$Ax \ge b^*$$

where b* is the fixed vector given by

$$b_i^* = \min\{t \mid F_i^1(t) \ge \alpha_i^0\}$$

and $F_i^1(t)$ is the marginal distribution function for b_i^1. Again we assume that $\alpha_i^o \ne 0$ and if $\alpha_i^o = 1$ then the support of b_i^1 is bounded above. It is now apparent that (5.1), subject to the given assumptions, is equivalent to a stochastic program with fixed recourse:

$$\begin{array}{llllll} \text{Minimize} & E\{c^1x + \text{Min } q^*y\} & & & & (5.4a) \\ & Ax & & \ge b^* & & (5.4b) \\ & Tx + & Wy & \ge p^* & & (5.4c) \\ & x \ge 0 & y & \ge 0 . & & (5.4d) \end{array}$$

Note that the same matrices A, T, and W given in the stochastic program with chance constraints (5.1) are involved in the equivalent stochastic program with recourse (5.4). Note also that it is not really necessary to assume $\alpha_i^o \ne 0$. The stochastic program with recourse (5.4) will still be equivalent to (5.1) if the i-th row of (5.4b) is deleted when $\alpha_i^o = 0$. Of course the matrix A of (5.4) will be different from the matrix A of (5.1), but this is no handicap. However, the requirement that $p_i(\xi^1) = -\infty$ with probability zero is not so easily overcome. In this case we could replace the i-th row of (5.4c) by a trivial equation satisfied for all $x \geq 0$ and all $y \geq 0$, but the resulting equivalent stochastic program with recourse would have a nonconstant recourse matrix W. The conditions $b_i^* = +\infty$ and $p_i(\xi^1) = \infty$ are of less concern since they cannot occur without leading to infeasible programs.

In deriving the stochastic program with recourse (5.4) from the stochastic program with chance constraints (5.1) we have spoken of $p(\xi^1)$ as a function of ξ^1 and defined $p^*(\xi^1)$ by altering $p(\xi^1)$ on a set of measure zero. But in order to solve the stochastic program with recourse (5.4) it suffices to know the expected value of c^1 and the joint distribution of q^* and p^* which in theory are computable from the joint distribution of c^2, α^1, and b^2. Where it is possible to perform these computations, the solution method developed to solve stochastic programs with recourse will apply as well to the solution of programs of the form (5.1); see e.g. [10, 18]. In fact, the inequality form of the second-stage constraints of (5.4) may allow for substantial simplifications; see section 2B of [24].

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