Harvesting management: Genrating wood-prices scenarios *

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Abstract. Lately, uncertainty has been introduced explicitly into forest planning models. A main issue in uncertanity is in terms of future wood and wood product prices. A well known and used form is to express this uncertainty is in terms of scenarios. Each scenario defines a price for each period through the horizon. These problems need to introduce the well known non-anticipativity constraints, which make the problems much harder to solve if many scenarios are defined. Algorithms have been developed which are quite successful in solving these problems, mainly by decomposing by scenarios. What has not been tackled in a rigorous form is the definition of the scenarios. Previous papers develop the scenarios in a manual form, using expert knowledge of the problems. This leads to the basic problema being formulated with errors, as the scenarios do not represent well the real underlying uncertainties. This paper introduces a methodology for the construction of stochastic processes that model the price dynamics of wood that will converted into saw-timber (lumber), and pulp wood. The system will be used used to generate (individual) scenarios, Monte Carlo simulation and scenario trees. The scenario trees developed in this form are inputed into decisin making models and represent in a much more rigorous way the basic uncertainty reality.
Keywords: forest planning, price uncertainty, scenario generation, scenario tree

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1 Introduction

Operations Research models have been used successfully for over four decades, starting with LPs for long range harvest planning (Navon 1971 [8]). Later, mixed integer programs were introduced to consider the need to incorporate discrete decisions , such as road building or the need to harvest a unit completely or not at all. In the 80s and 90s operational models for short term harvesting, machine location, and transportation were developed and implemented. All the models used were deterministic , see Martell et al , 1998 [7]. In the 80s some theoretical efforts where developed to introduce explicitly uncertainty, mostly on the rate of tree growth. Chance constrained programming was an approach typically proposed, where demand constraints need to be satisfied with a given confidence level. Under not too restrictive conditions , these problems could be transformed into non-linear deterministic problems (Hof, 1993 [4], Weintraub and Vera,1991 [12]). At stand level, there were proposals based on Stochastic Dynamic Programming (Lohmander 2007 [6]). In the last decade, a new approach for handling uncertainty in forest planning has been proposed. It is based on the notion that uncertainty is best expressed through scenarios. A scenario is the realization of an uncertain parameter or of a collection of parameters, for each period through the horizon. Given a set of scenarios, an optimization problem can be solved that optimizes the expected objective function, or a variant thereof involving a risk measure, subject to the solution satisfying all the constraints under all scenarios taking into account recourse decisions. In order to satisfy these conditions, the nonanticipativity constraints [13] which state that if two scenarios are identical up to a period t, then all decisions under these two scenarios must also be identical, as they share the same information. These constraints, that are inherent to all (dynamic) stochastic optimization problem, when introduced explicitly in the formulation of lead to very large problems, hard to solve, in particular if the number of scenarios is large. This approach to handle uncertainty has been applied in several fields : production (Alonso-Ayuso et al 2003 [1]), finance(King & Wallace 2012 [5]), energy (Ryan et al 2013 [10]) natural resources: mining and, now, in particular forestry. In our work, in a first paper (Alonso-Ayuso et al [2]) we considered a tactical model, with decisions related to harvesting units and road building to access the units for a medium sized forest. The uncertainty was reflected through 16 scenarios, where the variability was in market conditions, expressed as uncertainty in future prices, as well as bounds on demand. When larger number of scenarios were developed, the problem became too large to be solved directly in its extended form, and a decomposition method was used. In this case, it was coordinated branching, where a tree is defined for each scenario and the selection of values 0 or 1 for each variable is the same in all trees, thus implicitly preserving the non-anticipativity constraints. In Badilla et al 2013 [3], a similar forestry problem was used, where uncertainty in tree growth was also incorporated, the problem had up to 324 scenarios. In this case, Progressive Hedging (PH) [Wets89:aggreg, RckW91:scn, WtWW10, WtsW11] was used. In this approach the nonanticipativy constraints are relaxed, thus leaving easy to solve individual problems.

Naturally in this solution, the nonanticipativity constraints are normally not satisfied, and variables that should be the same to satisfy the nonanticipativity are different. A penalization is applied to each individual scenario problem to try to drive these values to the be equal in the next iteration. In can be proved that in the case of continuous variables, this approach will converge, [9]. In the case of integer variables, the method incorporates heuristics [11]. Results using PH in several fields have been successful. In our forestry problems , use of PH allowed to solve problems with up to 324 scenarios, though a much larger number of scenarios could be modeled.

But here is an important modeling challenge that in general has not been well analyzed, if not systematically ignored. How should the scenarios be developed? It is evident that the quality of the solutions obtained depends heavily on how well the scenarios represent the given problem. Note that in our case, considering just uncertainties in prices, the alternative values are in a continuum. In order to (partially) solve the problem we need to discretize, which allows to generate a manageable number of scenarios. Consider a problem with Tperiods. For each period, depending on the starting value, there may be several options, both of increased or decreased values. Considering all combinations, we could reach over many millions of scenarios. In our experimental studies we were much more conservative, mostly motivated by the fact that at this stage they would have to be carried out on laptops or relatively small PCs, creating only 16 scenarios in one case and up to 324 in the other, sizes for which we could solve the problems repeatedly. To generate these scenarios we used ad-hoc methods, relying on 'our' expert knowledge of the problems, and trying to generate scenarios which could reasonably cover a wide spectrum of combinations. But there was no rigor in the selection of these scenarios. In order to represent well the scenario possibilities, many thousands of scenarios would actually be needed. Since the state of the art does not allow to solve problems with such large number of scenarios, aggregation and bundling techniques, where a set of scenarios well chosen are replaced by one, allow to reduce the number to a much smaller number of 'representative' scenarios based on a good approximation of the price process. Given the advantages of solving stochastic problems through expressing the uncertainty through scenarios, and the importance of generating the 'right' scenarios, the main contribution of this paper is to show how such scenarios could be generated via a systematic analysis of the information available. Based on historical information we build a stochastic process that would describe the potential evolution of wood prices for both sawtimber and pulpwood. From the data-analysis, its clear that there are no substitute effects that might be advantageous and so we can deal with saw-timber prices and pulp prices as separate entities. The construction of the two stochastic processes allows us to get a hold of price-distributions at any foreseen future time and from there proceed to build a scenario tree that would appropriately approximate the (estimated) stochastic process. The word 'appropriately' is important in this instance because it allows us to build this scenario tree in such a way to guarantee that the decision problem associated with the harvesting management would actually be 'robust', i.e., would not have ignored critical scenarios.

The next section develops models and derives best estimates for both the sawtimber and pulp prices processes. The following section derives the distribution of the wood prices for any time t that might be of interest which, in turn, is used, in the last section, to illustrate how one generates robust scenarios trees. An appendix reviews when a gaussian distribution might be used as a substitute for a log-gaussian one.

2 Guiding models

This paper introduces a methodology for the construction of stochastic processes that model the price dynamics of wood that will converted to saw-timber (aserrables) and pulp-wood (pulpables). The system will be used used to generate (individual) scenarios, Monte Carlo simulation and scenario trees.

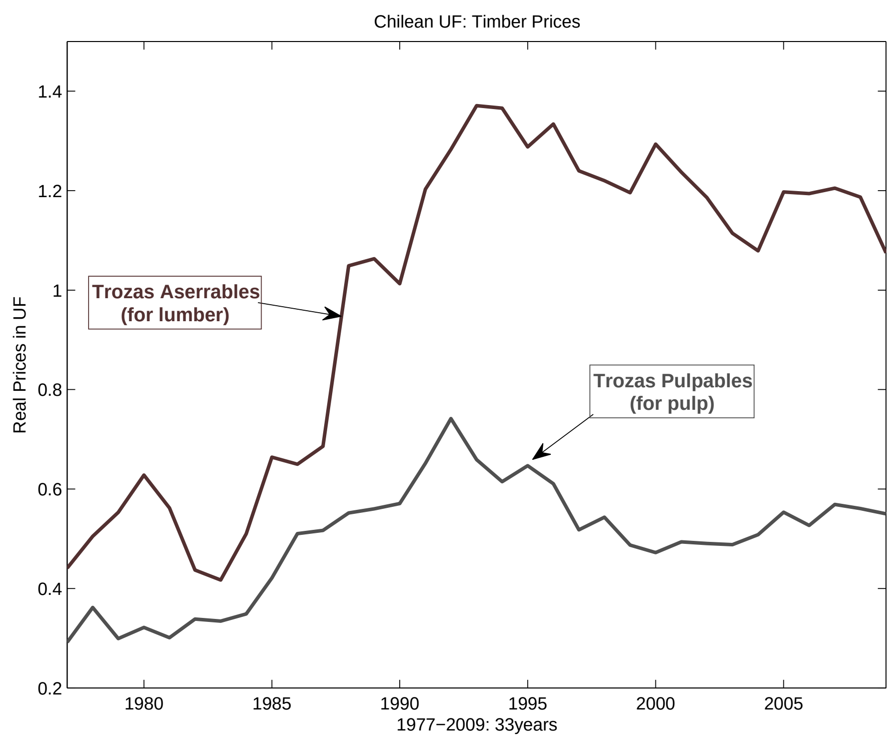

Figure 1: Real Prizes in Chilean UF: 1977-2009

Referring now to the Chilean market, cf. Figure 1, there seems to have been a radical change in the overall structure and trends of the timber industry starting about in 1988. Before 1988, high diameter wood was mostly exported to be processed whereas now, except for an insignificant percentage, the wood is processed locally. So, only the saw-timber and pulp-wood prices starting in 1988 can should reasonably be taken into account. Moreover, saw-timber and pulp wood prices don't seems to be interconnected; we are essentially dealing with two different product, their end use is completely dissociated.

Each price process is modeled as a stochastic differential equation of the type:

$$dp(t) = \mu(\upsilon - p(t))\,dt + \sigma\,dw(t)p(t), \quad p(0) = p_0, \quad t \geq 0,$$

where $p(t)$ is the price at time t, p_0 is the present price (or an estimate of the present price, see below) at time 0, μ, σ are constants that need to be estimated (they do no depend on t), υ is a price to which p reverts in the relatively distant future, and the process 'drifts' at a rate μ to υ; this is essentially a *geometric Brownian motion process* adjusted by a *mean reversion* term. The volatility in the process is modeled through the diffusion term involving w, a (standard) Wiener processes [14]. Note that μ, the drift coefficient only stands for the rate at which the process drifts to a return to 'normalcy.'

The solution is given by

$$p(t) = p_0 \exp\Big[-\Big(\mu + \tfrac{1}{2}\sigma^2\Big)t + \sigma w(t)\Big] + \mu\upsilon \int_0^t e^{r(t,s)}\, ds$$

where

$$r(t,s) = -\big[\mu + \tfrac{1}{2}\sigma^2\big](t-s) + \sigma\big(w(t) - w(s)\big);$$

recall that $w(t)$ is normally distributed (mean = 0, variance = t).

We replace this solution by an approximating one obtained by substituting for the term $\mu\upsilon \int_0^t e^{r(t,s)}\, ds$ its expectation. For all practical purposes the error introduced is negligible and the estimation of the coefficients μ, υ and σ would become very onerous, if not practically impossible, otherwise. So, we accept as 'solution' to the stochastic differential equation:

$$p(t) = \upsilon\big(1 - e^{-\mu t}\big) + p_0 \exp\big[(-\mu - \tfrac{1}{2}\sigma^2)t + \sigma w(t)\big].$$

Since, the increments of a Wiener process are independent gaussian random variables, the exponent of e, the term between $[\,\cdot\,]$, is also gaussian, and thus for each t, $p(t)$ has a 'displaced' log-gaussian distribution (on $\mathbb{R}_+$); the appendix reviews when a log-gaussian distribution can be approximated by a gaussian distribution. Hence, the price process is a *log-gaussian process*; it's not quite geometric brownian motion because of the mean reversion term as well as the initial condition.

To calculate the mean and variance terms of the price $p(\cdot)$ process, we rely again on the properties of Wiener processes [14], see the appendix (later). One obtains,

$$E\{p(t)\} = \upsilon + (p_0 - \upsilon)e^{-\mu t}, \qquad \mathrm{var}\big(p(t)\big) = \big(p_0 e^{-\mu t}\big)^2 \big(e^{\sigma^2 t} - 1\big).$$

In our model, the mean reversion term is taken to be the average of the yearly prices with the drift term based on the fact that 'reversion' will take place at a very slow rate, namely, over a 45 years time span[2].

The figures below illustrate the scenarios generated by these programs.

[2] including such a term doesn't mean that all paths = scenarios will end up at this reversion-mean but that there is a slow drift that influences the overall tendency of the price-paths

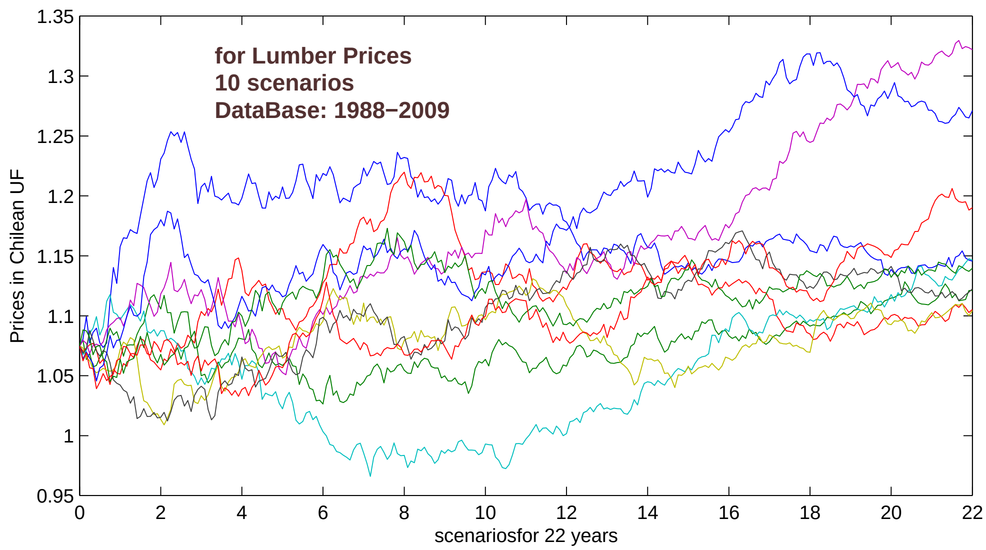

Figure 2: Prices scenarios (in Chilean UF) for saw-timber prices

3 The distribution of $p(\cdot)$

We begin with some formulas for the expectation and the variance of a random variable of the type $Y = \exp(\alpha + \beta X)$ with $X = \mathcal{N}(0, \eta^2)$, i.e., normal with expectation 0 and variance η^2

$$\begin{aligned}
E\{e^{\alpha+\beta X}\} &= e^{\alpha} E\{e^{\beta X}\} = \frac{e^{\alpha}}{\sqrt{2\pi}\eta} \int_{-\infty}^{\infty} e^{\beta x} e^{-x^2/2\eta^2}\, dx \\
&= \frac{e^{\alpha}}{\sqrt{2\pi}\eta} \int_{-\infty}^{\infty} e^{\beta^2\eta^2/2-(x-\beta\eta^2)^2/2\eta^2}\, dx \\
&= \frac{e^{\alpha+\beta^2\eta^2/2}}{\sqrt{2\pi}\eta} \int_{-\infty}^{\infty} e^{-(x-\beta\eta^2)^2/2\eta^2}\, dx \\
&= e^{\alpha+\beta^2\eta^2/2}.
\end{aligned}$$

For example, if X is $\mathcal{N}(0, t)$, $\alpha = (\mu - \frac{1}{2}\sigma^2)t$ and $\beta = \sigma$, then

$$E\big\{ \exp\left[\, (\mu - \tfrac{1}{2}\sigma^2)t + \sigma X \,\right] \big\} = e^{\mu t}.$$

$\mathrm{var}(Y) = E\{Y^2\} - (E\{Y\})^2 = E\{e^{2(\alpha+\beta X)}\} - e^{2\alpha+\beta^2\eta^2}$ from which one concludes

$$\mathrm{var}(Y) = e^{2\alpha+\beta^2\eta^2}\big(e^{\beta^2\eta^2} - 1\big).$$

Again, when X is $\mathcal{N}(0, t)$, $\alpha = (\mu - \frac{1}{2}\sigma^2)t$ and $\beta = \sigma$,then

$$\mathrm{var}(Y) = e^{2\mu t}\big(e^{\sigma^2 t} - 1\big).$$

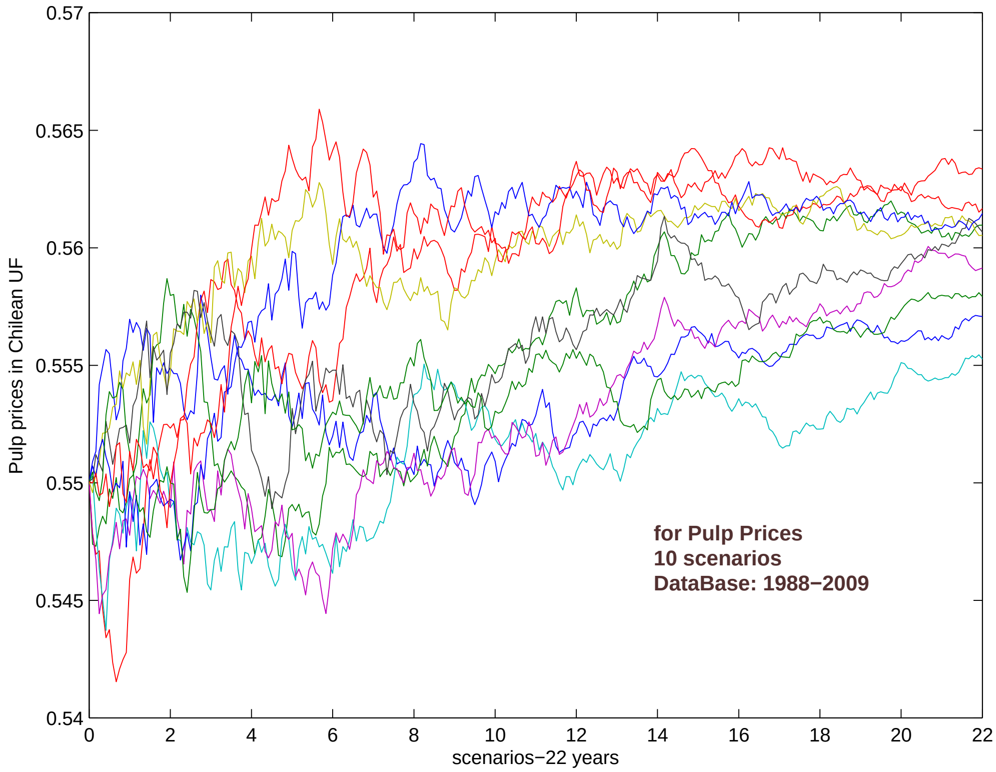

Figure 3: Prices scenarios (in Chilean UF) for wood-pulp prices

Returning to the equation

$$p(t) = \upsilon\big(1 - e^{-\mu t}\big) + p_0 \exp\big[(-\mu - \tfrac{1}{2}\sigma^2)t + \sigma w(t)\big].$$

that defines $p(t)$, one obtains

$$E\{p_0 \exp\big[(-\mu - \tfrac{1}{2}\sigma^2)t + \sigma w(t)\big]\} = p_0 e^{-\mu t}$$

and

$$\mathrm{var}(p_0 \exp\big[(-\mu - \tfrac{1}{2}\sigma^2)t + \sigma w(t)\big]) = p_0^2 e^{-2\mu t}\big(e^{\sigma^2 t} - 1\big),$$

so that

$$E\{p(t)\} = \upsilon\big(1 - e^{-\mu t}\big) + p_0 e^{-\mu t}$$

and

$$\mathrm{var}\big(p(t)\big) = p_0^2 e^{-2\mu t}\big(e^{\sigma^2 t} - 1\big).$$

Thus $p(t)$ has a 'displaced' log-normal distribution with known expectation and variance; the 'displacement' is the constant term $\upsilon\big(1 - e^{-\mu t}\big)$. This allows us to derive the density function of $p(t)$ by relying on the formulas in §2.

The density of the log-normal random variable $Z_t = p_0 \exp\left[(-\mu - \frac{1}{2}\sigma^2)t + \sigma w(t)\right]$ is

$$d_{Z_t}(s) = \left(s\tau\sqrt{2\pi}\right)^{-1} e^{-(\ln s - \theta)^2/2\tau^2}, \quad s \in (0, \infty),$$

where

$$E\{Z_t\} = e^{\theta + \tau^2/2} = p_0 e^{-\mu t + \sigma^2 t/2}$$
$$\operatorname{var}(Z_t) = \left(E\{Z_t\}\right)^2 \left(e^{\tau^2} - 1\right) = p_0^2 e^{-2\mu t}\left(e^{\sigma^2 t} - 1\right).$$

This allows us to compute the coefficients of the density function:

$$\theta = \ln p_0 - \mu t, \qquad \tau = \sigma\sqrt{t}.$$

This yields a complete description of the distribution of $p(t)$ at all t since

$$p(t) = Z_t + \upsilon\left(1 - e^{-\mu t}\right).$$

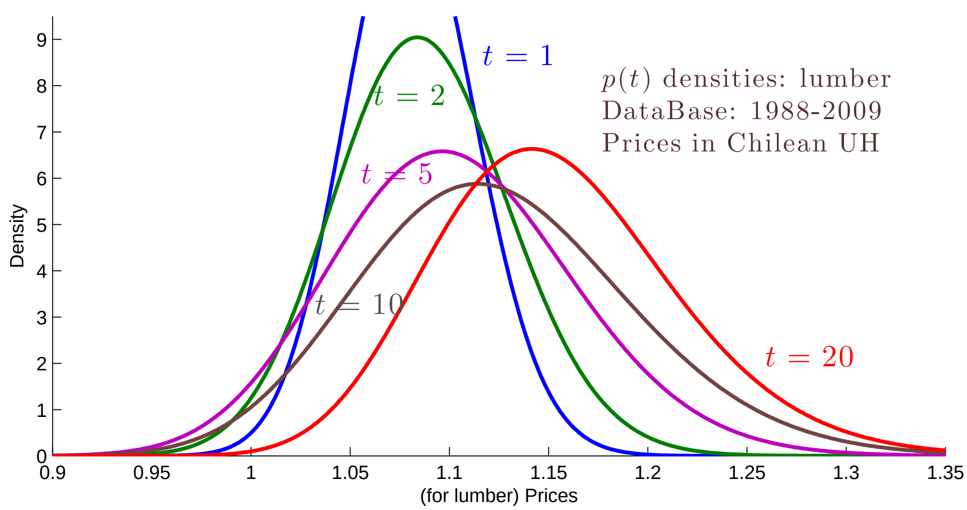

Figure 4: Density functions for $p(\cdot)$ 1988-2009 data

With the 1988-2009 data, for sawtimber, the coefficients are

$$p_0 = 1.0749, \quad \upsilon = 1.1998, \quad \mu = 0.0462, \quad \sigma = 0.0319,$$

hence, when $t = 20$, $\theta = -0.8524$, $\tau = 0.1425$. Plugging in these values in density function allows us to derive the distribution of the prices at any time t. The first one of these figures graphs these densities for $t = 1, 2, 5, 10$ and $t = 20$. The second one suggests the full 'evolution' of these densities for $t \in (0, 10]$. It should be noted that these are definitely not the densities of gaussian distributed random variables.

To 'check' the quality of the forecast process, we mapped both the drift of the this price process as well as the drift $\pm$ the standard deviations, the actual price process stay remarkbly closed to the predicted drift, even if we consider an eleven years horizon; we show just one such graph from 2000 to 2011 (although our data didn't include the 2010-observed prices), a similar figure could be generated for any time span with almost identical characteristics.

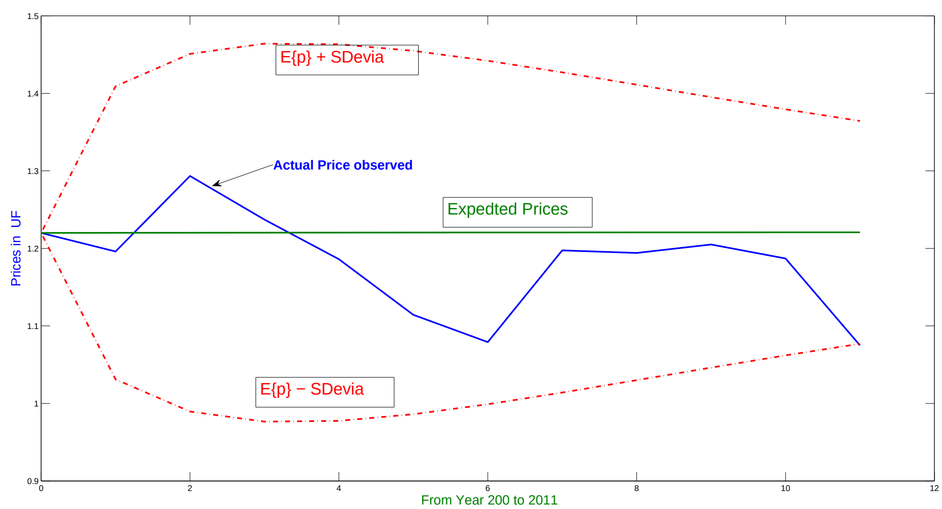

Figure 5: $p(\cdot)$: Forecast drift and associated standard deviations 2000-2011

For pulp-wood, again on the 1988-2009 database, the coefficients are

$$p_0 = 0.5501, \quad \upsilon = 0.5623, \quad \mu = 0.0979, \quad \sigma = 0.0086.$$

Plugging in these values in the density functions yields the distribution of the pulp-prices at any time t. The next figure graphs these densities for $t = 1, 2, 5, 10, 20$.

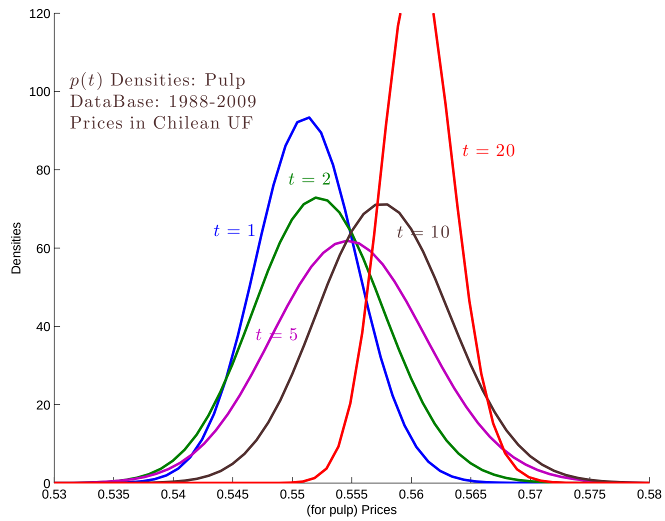

Figure 6: Density functions for prices of timber for pulp

4 Scenario tree construction

Given all the information about the price processes, one can build 'robust' scenario trees. To do this, we begin with an analysis of the cumulative distributions associated to the densities. This will allows us to choose scenarios that represent significant components of the optimization problem we shall be confronted with.

Unfortunately, there is no close-form version of these cumulative distribution, but it's relatively easy to design a numerical procedure that allows you to graph them. Of course, just like the densities these cumulative distributions will depend on the time at which they are calculated, cf. the next figure.

Just looking at the cumulative distribution functions. It's clear how a scenario tree should be built. We should be mostly concerned with the 'tail' events. Assuming we build a scenario tree for a four-stage model with nodes at times 0, 1, 2, 5 and 10, we would begin with a discretization of $p(1)$ (for aserrables and pulpables independently). We continue with just the saw-timber (aserrables) case. Here is a proposal based on choosing $p_{1,1}, \dots, p_{1,5}$ (six discretization points); of course this is just an example and in practice one could consider a much finer discretization.

$$p_{1,k} = \quad \text{conditional expectation of} \quad p(1) \quad \text{given the interval} \quad [UF_k, UF_{k+1}]$$

where

$$UF_k = \text{cum}^{-1}(\alpha_k) \quad \text{for} \quad \alpha = (0, 0.0.5, 0.15, 0.3, 0.6, 0.85, 1).$$

This would give sufficient weight to the tail events. The probability vector associated with

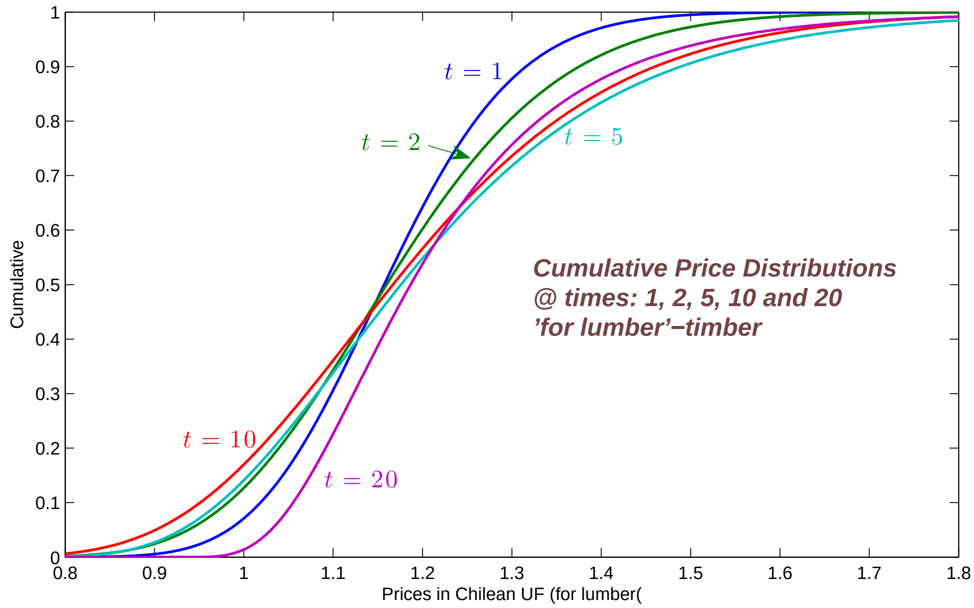

Figure 7: Cumulative distributions for saw-timber prices, 1988-2007

this choice of discretization would turn out to be prob $= (0.05, 0.1, 0.15, 0.3, 0.25, 0.15)$. But, of course, one could well proceed with a different discretization. These percentiles generates a partition of the potential prices 'next' year. For each generated interval one can easily compute the conditional expectation as included in the scenario-tree picture below

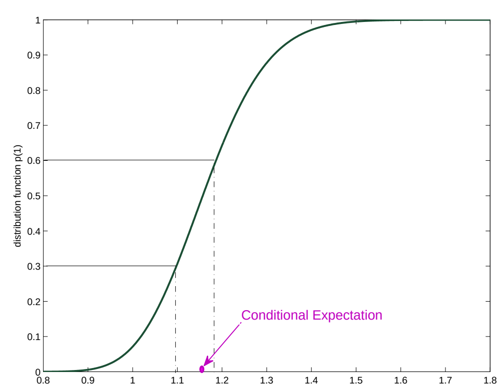

Figure 8: Cumulative distribution saw-timber prices for $t = 1$

To proceed to the next stage, one relies on the guiding stochastic differential equation consid-

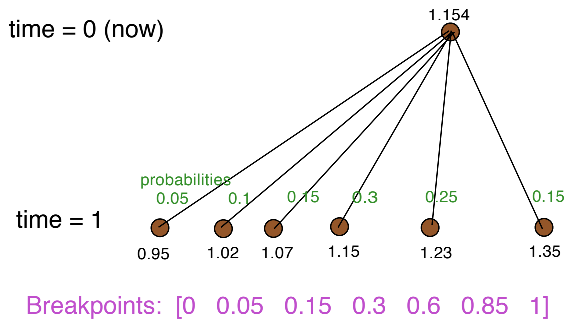

Figure 9: Scenariot Tree saw-timber prices for $t = 1$

ering each $p_{1,k}$ as the initial condition ('current price') to build the distribution of the resulting cumulative distribution at time $= 2$. Each one of the resulting distribution would then be discretized in a similar way as suggested earlier. This would be then be continued for times 5 and 10 but the discretization could get coarser (fewer points). This can all be programmed to be done directly from the available data. The only required input would have to be the discretization scheme to be used, i.e. the α-vector to be used in the discretization at each stage.

Here we proceed with extending the tree to $t = 2$ (3rd stage). We explicitly calculate how this would be done when at time $t = 1$, the price would have been $p(1) = 0.95$UF and $p(1) = 1.35$UF to stress the fact that one would end with a very dissimilar, but natural extension; UF is a common way to express prices in Chile to incorporate inflation. The first one of these figure results from chosing as percentiles $\alpha = 0.1, 0.25 and 0.8$ generating four potentials branches, the second one, branching from $p(1) = 1.35$ picks $\alpha = 0.1, 0.3, 0.6, 0.8$ as the precentiles that will determine the partition.
One could also proceed via sampling but to reach the same quality in the approximation of the stochastic process an extremely large number of scenarios would be required, maybe in the millions! But sampling, more precisely out-of-sample sampling, is important once we have a proposed solution. By generating a large number scenarios as in §1, we would then derive the probability distribution of the return associated with the proposed solution (via nonparametric estimation). The same could be done for any other suggested solution and this would provide the ultimate means to compare solutions.

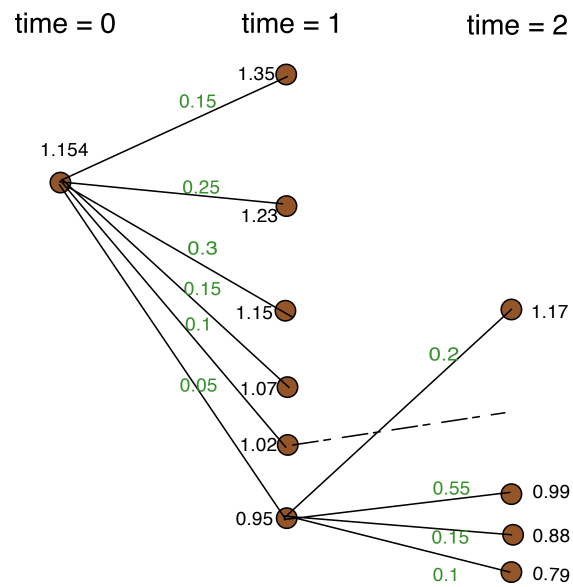

Figure 10: Extending the scenario tree for saw-timber prices @ $t = 2$

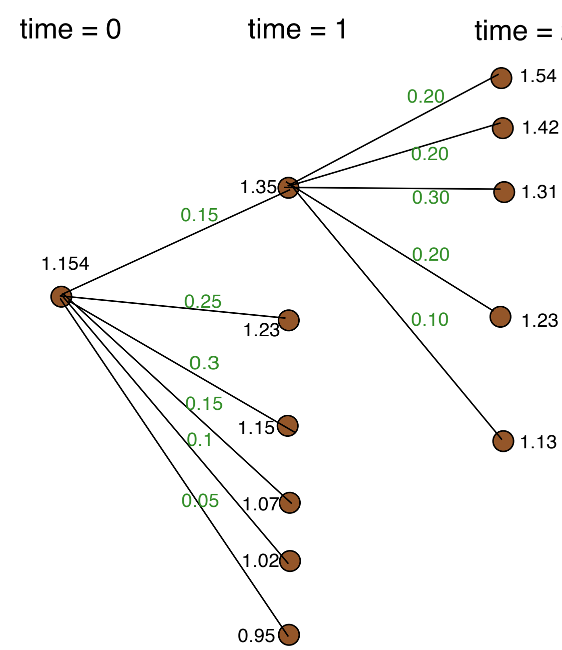

Figure 11: Extending the scenario tree for saw-timber prices @ $t = 2$

A Appendix: Approximating log-gaussian distributions

Suppose Y is real-valued log-gaussian random variable with density:

$$d_Y(z) = \big(z\tau\sqrt{2\pi}\big)^{-1} e^{-(\ln z-\theta)^2/2\tau^2}, \quad z \in (0,\infty).$$

Then

$$E\{Y\} = e^{\theta+\tau^2/2} =: m, \qquad \mathrm{var}(Y) = \big(E\{Y\}\big)^2\big(e^{\tau^2}-1\big) =: s^2.$$

Assuming that s/m is sufficiently small:

$$\begin{aligned} \tau^2 &= \ln(1+s^2/m^2) \approx s^2/m^2, \\ \theta &= \ln m - \tau^2/2 \approx \ln m - \tfrac{1}{2}s^2/m^2. \end{aligned}$$

In practical terms: s/m large means the variance of Y quite large with respect to the mean. For interest rates that is not the case, when interest rates are small the volatility is quite small, and it's only when interest rates are large that volatility gets somewhat larger. So, in the case of interest rate models, we don't have to be concerned with s/m large.

So, with s/m small, Y can be closely approximated by a gaussian random variable X with mean m and variance s^2. One has

$$\int_{-\infty}^{a} \frac{1}{s\sqrt{2\pi}} e^{-(x-m)^2/2s^2}\, dx \approx \int_{-\infty}^{a} \frac{1}{y\tau\sqrt{2\pi}} e^{-(\ln y-\theta)^2/2\tau^2}\, dy;$$

make the change of variable:

$$\frac{x-m}{s} = \frac{\ln y - \theta}{\tau}$$

accepting the values $\tau = s/m$ and $\theta = \ln m - (1/2)\tau^2$ and for values of a near m, one has $m(1+\ln(a/m)+s^2/2m^2) \approx a$ and this for $a \in (m-3.5s, m+3.5s)$ which corresponds to the 'essential' support of the densities of Y and X.

The examples in Figure 2 confirm what we indicated earlier. When s/m is sufficiently small, say $s/m < .25$, the log-normal density is essentially identical to the normal density with the same mean and variance. But when the ratio s/m gets large, like in the example where $s/m = 1.8/2 = 0.9$, then the normal density with the same mean and variance doesn't provide an acceptable approximation.

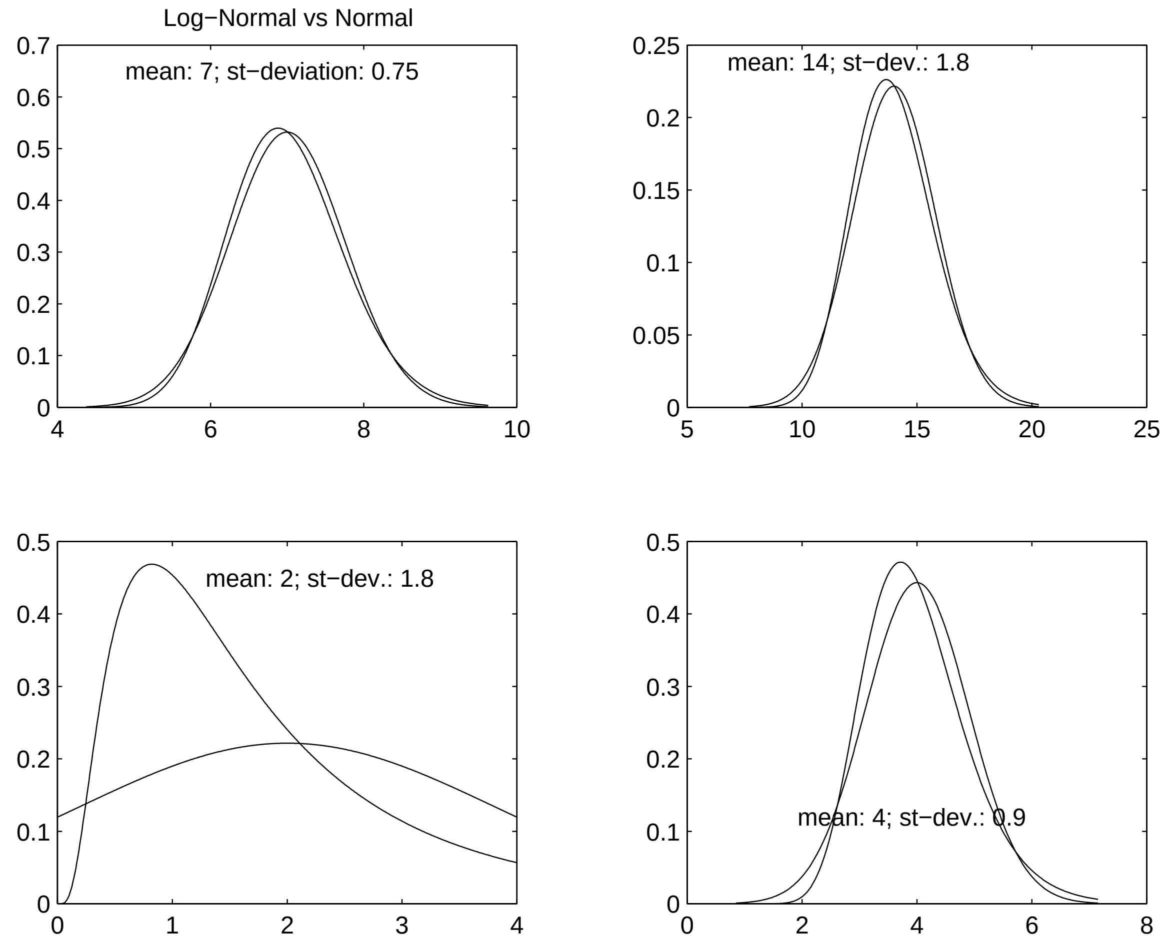

Figure 12: Log-Normal versus Normal densities

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