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Vol. 24, No. 2, February 1988
Printed in Belgium

Approximating the Integral of a Multifunction

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Communicated by M. M. Rao

Given a weakly converging sequence of measures, we study the convergence of the corresponding integrals of a continuous unbounded multifunction. We also study the implication of these results to variational problems, and provide further approximating results for the integral of a multifunction, involving both truncation of the multifunction and measure approximation. © 1988 Academic Press, Inc.

1. INTRODUCTION

The problems addressed in this work stem from approximation considerations in some variational problems, stochastic optimization in particular. When dealing with variational problems in practice, one is often confronted with the fact that the data available provide us with only a sample of the possible values that could be assumed by the parameters of the problem. For example, suppose that we are dealing with a stochastic optimization problem, but a limited number of observations is the only information we have about the distribution of the random coefficients of the model. This means that instead of the actual probability distribution of the random parameters, we can only use an approximate distribution in the formulation of the stochastic optimization problem. As more information is collected about the random coefficients, we can use a more

Received September 2, 1986; revised June 15, 1987.

AMS subject classifications: 28B20, 44D99.

Key words and phrases: integrating multifunctions, approximation of variational problems.

* The major part of this work was done while visiting the Department of Mathematics, University of California at Davis.

† Partially supported by a grant from the National Science Foundation.

0047-259X/88 $3.00

accurate distribution but only in a probabilistic sense. We are thus dealing with a collection of approximate problems, each one generated by sampling the space of possible optimization problems of a given type. In an abstract setting, solving the variational problem often amounts to computing the integral of a multifunction,

$$\int_{\Omega} \Gamma(\omega)\, dP(\omega), \tag{1.1}$$

with $\Gamma(\omega) \subset R^n$ and P a probability measure. (The technical framework and terminology are introduced in the next section, along with some preliminary results.) See Aumann and Perles [3], Rockafellar [9], or Rockafellar and Wets [10] and references therein for examples. In the aforementioned case, where only a sample is given, or when there are limits on the computational power, the distribution P in (1.1) ought to be replaced by an approximation or by a sequence of approximations. Then, instead of computing (1.1) we face a sequence

$$\int_{\Omega} \Gamma(\omega)\, dP^{\nu}(\omega). \tag{1.2}$$

The measures P^{ν} are determined typically by samplings, or via discretization, and hence converge weakly to P. The quality of the approximation is then reflected in the convergence properties of the sets determined by (1.2) to the set defined by (1.1). In the case of bounded continuous and point-valued Γ, the convergence is guaranteed by the weak convergence of the P^{ν} to P. It is the set-valued characteristics of Γ that create the difficulties and the interest. Indeed, in the applications of interest, the multifunctions have unbounded values. In this work we determine conditions that yield the convergence, and we verify semiconvergence properties under more relaxed conditions. This analysis is done in Sections 3 through 7, starting with the continuous and bounded case, through counterexamples, lower semicontinuous convergence, and ending with a general convergence result.

A particular case, and a prime application in optimization problems, is when $\Gamma(\omega)$ is the epigraph of a normal integrand $f(x, \omega)$ (the notions concerning these problems are reviewed at the beginning of Section 8). The aforementioned integral amounts then to the inf-convolution as

$$\mathop{\int\!\!\!\!\!\!E} f(x, \omega)\, dP(\omega) \tag{1.3}$$

(the letter E on the integral sign stands for “in the sense of epigraphs”; see

Section 8, where we elaborate on the variational problem involved.) The desired convergence here is the epi-convergence of the functions

$$\oint f(x, \omega)\, dP^{\nu}(\omega) \tag{1.4}$$

to the function determined by (1.3). The particular structure of epigraphs helps in expressing conditions for this convergence; we do this in Section 8.

Another type of approximation is encountered in optimization procedures, especially in numerical computation techniques. It involves the truncation of the values of Γ, namely, replacing $\Gamma(\omega)$ by $\Gamma(\omega) \cap \lambda B$ with B the unit ball in R^n and λ a finite, large enough, number. The analogous operation for the normal integrand is to perform the inf-convolution only for x with $|x| \leqslant \lambda$. For a fixed probability measure P, the integral

$$\int_{\Omega} (\Gamma(\omega) \cap \lambda B)\, dP(\omega) \tag{1.5}$$

is a good approximation of (1.1) if λ is large. If, however, P is replaced by a sequence P^{ν}, even converging weakly to P, it may not be true that the approximation provided by λ is good uniformly for all P^{ν}. In Section 9 we give conditions that guarantee this uniformity. Even more interesting is the observation, also proved in Section 9, that the sets

$$\int_{\Omega} (\Gamma(\omega) \cap \lambda B)\, dP^{\nu}(\omega) \tag{1.6}$$

give a good approximation to (1.1) under very mild conditions, provided $\nu \to \infty$ and $\lambda \to \infty$ in a suitable order; this occurs even in cases when (1.2) fails to produce good approximations. The results of Section 9, in particular Theorem 9.4, suggest practical guidelines for the design of numerical procedures to calculate the integral of a multifunction and how to use the approximations in a variational setting.

2. Framework, Terminology, and Preliminaries

In this section we introduce our terminology and technical assumptions concerning multifunctions; normal integrands are introduced in Section 8. In both cases we follow the standard literature, where the basic facts we quote can be found, e.g., in Rockafellar [9], Castaing and Valadier [5], and Rockafellar and Wets [10].

The norm of a vector x in the n-dimensional euclidean space is $|x|$, and $x \cdot v$ denotes the scalar product of x and v. We denote by B the closed ball in R^n of radius one. For a sequence C_i of nonempty subsets of R^n, the *topological limit inferior* is given by

$$\liminf C_i = \{x: x = \lim x_i, \text{ with } x_i \text{ in } C_i \text{ for all } i\},$$

and the *topological limit superior* is given by

$$\limsup C_i = \{x: x = \lim\ x_{i_j}, \text{ with } x_{i_j} \text{ in } C_{i_j}, \text{ a subsequence of } C_i\}.$$

We say that C_i *lower semiconverges* to C if lim inf C_i includes C; we say that C_i *upper semiconverges* to C if lim sup C_i is included in the closure of C. (These semiconvergences are sometimes called lower and upper convergences.) We say that C_i *converges* to C if it is both lower and upper semiconvergent to C; then we write $C = \lim C_i$. Note that the limit does not distinguish between a set and its closure.

The convergence of sets is metrizable, or rather semi–metrizable since we allow non–closed sets. The induced topology is the convergence in the Hausdorff distance for a one-point compactification of R^n. We seldom need an explicit metric; when needed, we adopt the stereographic Hausdorff distance (see [10, p. 25]), denoted by $\text{haus}^s(\cdot, \cdot)$. The Hausdorff distance between two bounded sets is denoted by $\text{haus}(\cdot, \cdot)$.

Let Ω be a complete separable metric space, with metric $d(\cdot, \cdot)$. A multifunction Γ is a mapping that assigns to each ω in Ω a subset $\Gamma(\omega)$ of R^n. The multifunction is upper semicontinuous (respectively lower semicontinuous, or continuous) at ω_0 if $\omega_i \to \omega_0$ in Ω implies that $\Gamma(\omega_i)$ upper semiconverge (respectively lower semiconverge, or converge) to $\Gamma(\omega_0)$.

Consider now the space Ω with its Borel σ-field Σ. A multifunction Γ is measurable if for every open set G in R^n the set $\Gamma^-(G) = \{\omega: \Gamma(\omega) \cap G \neq \varnothing\}$ is in Σ.

For a set D in R^n we denote by int D its interior, by cl D its closure, and by co D its convex hull (namely the set of convex combinations of elements in D, which may not be closed even if D is closed). We set $\|D\| = \sup\{|x|: x \text{ in } D\}$ and leas $D = \inf\{|x|: x \text{ in } D\}$. If Γ is a multifunction we write co Γ for the multifunction $(\text{co}\,\Gamma)(\omega) = \text{co}(\Gamma(\omega))$; the multifunctions cl Γ and leas Γ are defined similarly. We say that Γ is bounded if $\|\Gamma(\omega)\|$ is a bounded real function.

The support function of a set D, denoted by $s(v, D)$ and defined for v in R^n, is given by

$$s(v, D) = \sup\{v \cdot x: x \text{ in } D\}.$$

The Minkowski sum $C + D$ of two subsets in R^n is $\{x + y: x \text{ in } C, y \text{ in } D\}$. A generalization of that is the integral of a multifunction Γ with respect to a probability measure P, introduced by Aumann [2]. We denote it by $\int_\Omega \Gamma(\omega)\, dP(\omega)$, or by $\int \Gamma dP$ when no confusion can arise; it is defined by

$$\int \Gamma dP = \left\{ \int \gamma dP \colon \gamma \text{ is a } P\text{-integrable selection of } \Gamma \right\}$$

(P-integrable selection means that γ is P-integrable and that $\gamma(\omega)$ is in $\Gamma(\omega)$ for P-almost every ω).

The multifunction Γ is *P-atom convex* if $\Gamma(a)$ is a convex set whenever $\{a\}$ is an atom of P. A basic property is the following:

If Γ is P-atom convex, then $\int \Gamma dP$ is a convex set. (2.1)

The atomless case is covered in [2]; the integration on the purely atomic part of P amounts to a summation of convex sets, which obviously preserves convexity. We need also the following property:

If Γ has closed values and is measurable, and if $\int \Gamma dP$ is not empty and co $\int \Gamma dP$ *does not contain a line, then* co $\int \Gamma dP = \int$ co ΓdP. (2.2)

The result is a variation of Aumann [2, Theorem 3]. The latter assumes that $\Gamma(\omega)$ is contained in the positive orthant of R^n, which implies that co $\int \Gamma dP$ does not contain a line, and that the nonemptiness assumption can be removed. (In our case we cannot remove the nonemptiness assumption, as the same example that Aumann uses [2, p. 7] shows.) In fact, (2.2) holds under Aumann's measurability condition, which does not require the function to be closed-valued. One way to prove (2.2) is to mimic Aumann's proof of [2, Theorem 3], and note that the induction works. Another possibility is to check that the necessary and sufficient conditions of Wagner [11], for co $\int \Gamma = \int$ co Γ, hold in our case.

We use, throughout, the weak convergence of probability measures; see, e.g., Billingsley [4]. We also need the following modification. Let $h(\omega)$ be a measurable real-valued function. We say that the sequence P^ν of probability measures is *h-tight* if, for every $\varepsilon > 0$, there exists a compact subset K_ε of Ω such that h is bounded on K_ε and

$$\int_{\Omega \setminus K_\varepsilon} |h(\omega)|\, dP^\nu(\omega) < \varepsilon$$

for all ν. (Note that for h continuous, the boundedness of h on K_ε is automatically satisfied.)

3. Continuous and Bounded Integrands

Our first result could be derived as a corollary from later results; a direct proof is easier.

THEOREM 3.1. *Suppose that Γ has convex compact values and that it is bounded and continuous. If $P^\nu \to P$ weakly, then $\int \Gamma dP^\nu$ converge to $\int \Gamma dP$ in the Hausdorff metric.*

Proof. The convex compact sets in R^n with the Hausdorff distance can be linearly and isometrically embedded as a convex cone in a Banach space; see, e.g., [7, Theorem 17.2.1]. Furthermore, a uniformly bounded collection of sets is precompact in this Banach space. Therefore all the values $\Gamma(\omega)$ and all the integrals $\int \Gamma dP^\nu$ and $\int \Gamma dP$ belong to a compact set. Hence convergence in norm, namely in the Hausdorff distance, is implied by convergence in the weak topology in the Banach space. Let L be a continuous linear functional. Then $L(\Gamma(\omega))$ is a continuous bounded real function, and hence $P^\nu \to P$ weakly implies that $\int L(\Gamma(\omega))\, dP^\nu$ converge to $\int L(\Gamma(\omega))\, dP$. We claim that the integration commutes with taking continuous linear functionals. The reason is that the integration in our case coincides with the Bochner integral into the Banach space (see, e.g., [7, Theorem 17.3.2)]), and all the values are in a compact set. Therefore the previous convergence implies that $L(\int \Gamma dP^\nu)$ converge to $L(\int \Gamma dP)$, which is the desired weak convergence. Hence the strong convergence holds and the proof is complete.

The conclusion of the previous result may fail if Γ is not convex-valued. Here is a simple counterexample.

EXAMPLE 3.2. Let $\Omega = [0, 1]$ and let Γ be a constant multifunction, in particular continuous, say $\Gamma(\omega) = \{0, 1\}$ for all ω. Let P be a probability measure concentrated at one atom, say $P\{0\} = 1$. Let P^ν be a sequence of atomless probability measures converging weakly to P, say $dP^\nu(\omega) = \nu$ if $0 \leqslant \omega \leqslant \nu^{-1}$ and $dP^\nu(\omega) = 0$ otherwise. Then $\int \Gamma dP^\nu = [0, 1]$ for all ν, by (2.2). Clearly, $\int \Gamma dP^\nu$ do not converge to $\int \Gamma dP = \{0, 1\}$.

With further conditions the convergence holds as follows.

THEOREM 3.3. *Let Γ be a bounded continuous multifunction with compact values, and suppose it is P-atom convex. Then $P^\nu \to P$ weakly implies that $\int \Gamma dP^\nu$ converge to $\int \Gamma dP$ in the Hausdorff metric.*

Proof. Taking convex hulls is nonexpansive, namely haus(co C, co D) $\leqslant$ haus(C, D); see, e.g., [7, Theorem 7.2.5]. It then follows from Theorem 3.1 that $\int \text{co}\, \Gamma dP^\nu$ converge to $\int \text{co}\, \Gamma dP$; the latter being equal to

$\int \Gamma dP$ in view of (2.1). The result would then follow once we show that the distance haus($\int$ co ΓdP^ν, $\int \Gamma dP^\nu$) is small when ν is large. (This distance is not zero since we have not assumed that Γ is also P^ν-atom convex.)

To this end let K be a compact subset of Ω such that $P^\nu(\Omega \backslash K) < \delta$ for all ν, where $\delta > 0$ is a small number to be determined later; given δ, existence of such K is guaranteed by the tightness (see [4, p. 37]). For each ω in K let B_ω be a small ball in Ω around ω such that: If ω is an atom of P then haus$(\Gamma(\sigma), \Gamma(\omega)) < \delta$ whenever σ is in B_ω; if ω is not an atom of P then $P(B_\omega) < \delta$. Existence of such B_ω follows in the first case from the continuity of Γ, and in the second case from the continuity of P on a decreasing sequence of sets. A finite number of such balls, say $B_1, ..., B_k$, corresponding to $\omega_1, ..., \omega_k$, cover K. Define $C_1 = B_1$ and successively $C_i = B_i \backslash (B_1 \cup \cdots \cup B_{i-1})$, this for $i = 2, ..., k$. Then let $C_0 = \Omega \backslash (C_1 \cup \cdots \cup C_k)$. Then each C_i is measurable, and

$$\int_\Omega \Gamma dP^\nu = \sum_{i=0}^{k} \int_{C_i} \Gamma dP^\nu.$$

We claim first that for all $i = 0, ..., k$ and for ν large enough, haus$(\int_{C_i} \Gamma dP^\nu$, $\int_{C_i}$ co $\Gamma dP^\nu) \leqslant 2b\delta$, with $b = 1 + \sup(\|\Gamma(\omega)\|)$. To verify this consider first the case $i = 0$. The choice of K implies that all the sets $\int_{C_0} \Gamma dP^\nu$ and $\int_{C_0}$ co ΓdP^ν are bounded by $b\delta$, hence the inequality. The same estimate, namely $\| \int_{C_i}$ co $\Gamma dP^\nu \| \leqslant b\delta$, applies also for ν large enough when ω_i is not an atom of P. This follows from the weak convergence of P^ν to P. To complete verifying the claim it is enough to check the case where ω_i is an atom of P. But then the choice of B_i implies that haus$(\Gamma(\sigma)$, co $\Gamma(\sigma)) \leqslant 2\delta$ for σ in C_i. A standard argument implies then that haus$(\int_{C_i} \Gamma dP^\nu$, $\int_{C_i}$ co $\Gamma dP^\nu) \leqslant 2\delta \leqslant 2\delta b$, and the proof of the claim is complete.

We employ now the Shapley–Folkmann lemma (see, e.g., [1, p. 396]), which implies in our case that

$$\text{haus}\left(\int_\Omega \text{co } \Gamma dP^\nu, \int_\Omega \Gamma dP^\nu\right) \leqslant n^{1/2} \max \text{haus}\left(\int_{C_i} \text{co } \Gamma dP^\nu, \int_{C_i} \Gamma dP^\nu\right).$$

In view of the previous claim, the right-hand side is less than or equal to $2n^{1/2}b\delta$. Since n, b, and 2 are fixed, and since δ can be chosen arbitrarily small, the proof of the theorem is complete.

COROLLARY 3.4. *Suppose Γ is a bounded continuous multifunction with compact values, and P is atomless. Then $P^\nu \to P$ weakly implies that* haus$(\int \Gamma dP^\nu, \int \Gamma dP) \to 0$.

Proof. This is a particular case of the previous result.

Another condition that implies the convergence, and prevents counerexamples of the type in Example 3.2, is a correlation between atoms of P and atoms of P^v, as follows.

THEOREM 3.5. *Suppose that Γ is a bounded continuous multifunction with compact values, and suppose that $P^v \to P$ weakly. Let $\Psi = \{a_1, a_2, ...\}$ be the set of atoms of P for which $\Gamma(a_i)$ is not convex. Suppose that for each v a set $\Psi_v = \{a_{1,v}, a_{2,v}, ...\}$ exists, which consists of atoms of P^v, and such that for each i we have both that $d(a_i, a_{i,v}) \to 0$ and $P^v(a_{i,v}) \to P(a_i)$ as $v \to \infty$. Then $\int \Gamma dP^v$ converge to $\int \Gamma dP$ in the Hausdorff metric.*

Proof. We write

$$\int_\Omega \Gamma dP = \sum \Gamma(a_i)\, P(a_i) + \int_{\Omega \backslash \Psi} \Gamma dP \tag{3.1}$$

and likewise

$$\int_\Omega \Gamma dP^v = \sum \Gamma(a_{i,v})\, P^v(a_{i,v}) + \int_{\Omega \backslash \Psi_v} \Gamma dP^v. \tag{3.2}$$

The integral part of (3.2) converges to the integral part of (3.1), this in view of Corollary 3.4. Indeed, on $\Omega \backslash \Psi$ the multifunction Γ is P-atom convex, and the conditions on Ψ_v and P^v imply that P^v restricted to $\Omega \backslash \Psi_v$ converge weakly to the restriction of P to $\Omega \backslash \Psi$. Each of the summands in (3.2) converges as $v \to \infty$ to the corresponding summands in (3.1), this in view of the continuity of Γ and the convergence of $d(a_i, a_{i,v})$ and $P^v(a_{i,v}) - P(a_i)$ to zero. It is now a simple exercise in converging series to show that the sum in (3.2) converges to the sum in (3.1), and then the result follows.

4. BOUNDED AND SEMICONTINUOUS INTEGRANDS

If Γ is a bounded semicontinuous multifunction, and not continuous, then the results of the previous section fail, as the following simple examples show.

EXAMPLE 4.1. *Let* $\Omega = [0, 1]$, and let $P(\{0\}) = 1$ and $P^v(\{v^{-1}\}) = 1$ for $v = 1, 2, ...$ Then $P^v \to P$ weakly. Let $\Gamma(0) = [0, 1]$ and $\Gamma(\omega) = \{0\}$ for $\omega > 0$. Then Γ is upper semicontinuous. Clearly $\int \Gamma dP^v$, which is equal to $\{0\}$, does not converge to $\int \Gamma dP = [0, 1]$. If we change $\Gamma(\omega)$ for $\omega > 0$ and make it equal to $[0, 2]$ we get a lower semicontinuous multifunction. Again, $\int \Gamma dP^v = [0, 2]$ does not converge to $[0, 1]$.

Note that in both cases of the preceding example the corresponding semiconvergences hold. This reflects general properties, which we state and prove next.

THEOREM 4.2. *Suppose that Γ is a bounded and upper semicontinuous multifunction with compact values. Suppose also that Γ is P-atom convex. If $P^\nu \to P$ weakly, then* $\limsup \int \Gamma dP^\nu$ *is contained in* $\int \Gamma dP$.

Proof. We start with the case where the values $\Gamma(\omega)$ are convex. Consider the support function $s(v, \Gamma(\omega))$, which we denote by $s(v, \omega)$. For each v the function $s(v, \cdot)$ is an upper semicontinuous real function, namely $s(v, \lim \omega_i) \geqslant \limsup s(v, \omega_i)$ whenever ω_i is convergent. This is implied by the upper semicontinuity and the boundedness of Γ. The weak convergence of P^ν to P implies then that

$$\limsup_{\nu \to \infty} \int s(v, \omega)\, dP^\nu(\omega) \leqslant \int s(v, \omega)\, dP(\omega). \tag{4.1}$$

To verify (4.1) recall that an upper semicontinuous function, here $s(v, \cdot)$, is the pointwise limit of a decreasing sequence of continuous functions, say $s_i(v, \cdot)$, which can be chosen bounded since $s(v, \cdot)$ is bounded. Let $\varepsilon > 0$ be given. The Lebesgue dominated convergence theorem implies that an index j can be chosen with

$$\int s_j(v, \omega)\, dP(\omega) - \int s(v, \omega)\, dP(\omega) < \varepsilon.$$

The weak convergence of P^ν to P implies that for ν large enough

$$\left| \int s_j(v, \omega)\, dP^\nu(\omega) - \int s_j(v, \omega)\, dP(\omega) \right| < \varepsilon.$$

The obvious inequality $\int s(v, \omega)\, dP^\nu(\omega) \leqslant \int s_j(v, \omega)\, dP^\nu(\omega)$ with the two displayed inequalities implies (4.1), since ε is arbitrarily small. An upper semicontinuous multifunction with closed values is measurable (see, e.g., [10]). Therefore, for any measure Q, the equality $s(v, \int \Gamma dQ) = \int s(v, \Gamma(\omega))\, dQ$ holds (see, e.g., [9]). Applying this with Q being P, P^1, P^2, etc., together with (4.1), implies

$$\limsup_{\nu \to \infty} s\left(v, \int \Gamma dP^\nu\right) \leqslant s\left(v, \int \Gamma dP\right). \tag{4.2}$$

In the convex case, since $\int \Gamma dP$ is closed (see [2]), the inequality (4.2) implies the desired conclusion of the theorem.

We turn now to the general case, where Γ may not have convex values, but it is still assumed to be P-atom convex. First note that $\int \Gamma dP$ is a convex set which coincides with $\int \text{co}\, \Gamma dP$ (see (2.1) and (2.2)). Since Γ is bounded, the upper semicontinuity of Γ implies that co Γ is upper semicontinuous (see Remark 4.3). The first part of the present proof then yields

$$\limsup \int \text{co}\, \Gamma dP^{\nu} \subset \int \text{co}\, \Gamma dP. \tag{4.3}$$

Since $\int \text{co}\, \Gamma dP = \int \Gamma dP$ and since $\int \Gamma dP^{\nu} \subset \int \text{co}\, \Gamma dP^{\nu}$, the conclusion of the theorem follows from (4.3). This completes the proof.

Remark 4.3. We have used the fact that co Γ is upper semicontinuous if Γ is upper semicontinuous and bounded. The proof is simple: If x_i are in co $\Gamma(\omega_i)$ then each x_i can be written as a convex combination $\sum \alpha_{i,j} y_{i,j}$ with $j = 1, ..., n+1$ (n being the dimensionality of the space) and $y_{i,j}$ in $\Gamma(\omega_i)$. A subsequence of $\{i\}$, which we denote by $\{k\}$, exists such that $y_{k,j}$ and $\alpha_{k,j}$ converge as $k \to \infty$, say to y_j and α_j. This follows from the boundedness. If ω_i converges, say to ω_0, then by the upper semicontinuity each y_j is in $\Gamma(\omega_0)$, and the corresponding subsequence of x_i converges, say to x_0 with $x_0 = \sum \alpha_j y_j$. The latter belongs therefore to co $\Gamma(\omega_0)$. This verifies the upper semicontinuity of co Γ. The argument fails if Γ is unbounded; then co Γ may not be upper semicontinuous. Here is a counterexample: $\Gamma(\omega) = \{0, \omega^{-1}\}$ for $\omega > 0$, and $\Gamma(0) = \{0\}$. For lower semicontinuity the situation is different, and Γ lower semicontinuous implies co Γ lower semicontinuous regardless of whether Γ is bounded. Indeed, if $x = \sum \alpha_j y_j$ with y_j in $\Gamma(\omega_0)$, then $|x - \sum \alpha_i y_{j,\omega}|$ is small when $|y_j - y_{j,\omega}|$ are small, and such $y_{j,\omega}$ exist if Γ is lower semicontinuous.

Remark 4.4. The assumption that Γ is P-atom convex cannot be dropped from the statement of Theorem 4.2, as Example 3.2 shows. The assumption can, however, be replaced by a correlation between atoms of P with nonconvex values of Γ and atoms of P^{ν}, exactly as is stated in Theorem 3.5. The proof is also the same (only using Theorem 4.1 instead of Theorem 3.3). We leave out the details.

The assumption of P-atom convexity is not needed for the lower semiconvergence part, as follows.

THEOREM 4.5. *Suppose that Γ is a bounded and lower semicontinuous multifunction with compact values. Then $P^{\nu} \to P$ weakly implies that* $\liminf \int \Gamma dP^{\nu}$ *includes* $\int \Gamma dP$.

Proof. Consider first the case where for each ω the set $\Gamma(\omega)$ is convex. For each vector v the support function $s(v, \omega) = s(v, \Gamma(\omega))$ is a lower semicontinuous function of ω, namely $\liminf s(v, \omega_i) \geqslant s(v, \lim \omega_i)$; this follows immediately from the lower semicontinuity of Γ. Since $P^\nu \to P$ weakly it follows that

$$\liminf_{\nu \to \infty} \int s(v, \omega)\, dP^\nu(\omega) \geqslant \int s(v, \omega)\, dP(\omega). \tag{4.4}$$

To verify (4.4) recall that a lower semicontinuous function is the supremum of a sequence of continuous functions, say $s_i(v, \omega)$, and if s_i replaces s in (4.4), then equality holds. In the limit we get the inequality (4.4) for s (compare with the verification of (4.1)). A lower semicontinuous multifunction with closed values is measurable (see [10]). Therefore $\int s(v, \omega)\, dP^\nu = s(v, \int \Gamma dP^\nu)$, and likewise for the measure P. Hence (4.4) implies that for all v,

$$\liminf_{\nu \to \infty} s\left(v, \int \Gamma dP^\nu\right) \geqslant s\left(v, \int \Gamma dP\right). \tag{4.5}$$

The latter inequality is equivalent to the desired conclusion for the convex case.

The next case that we examine is that of P atomless. Then $\int \Gamma dP = \int \operatorname{co} \Gamma dP$ (see (2.2)). Since $\operatorname{co} \Gamma$ is also lower semicontinuous (see Remark 4.3) it follows from the first case that we examined that $\liminf \int \operatorname{co} \Gamma dP^\nu$ contains $\int \Gamma dP$. Therefore, in order to verify the present case, it suffices to show that for ν large the Hausdorff distance between $\int \Gamma dP^\nu$ and $\int \operatorname{co} \Gamma dP^\nu$ is small. To this end we partition Ω into a finite number of disjoint sets, say Ω_i, such that $P(\Omega_i) < \delta$; this can be done since P is atomless. Then $P^\nu(\Omega_i) < 2\delta$ if ν is large enough. In particular $\| \int_{\Omega_i} \Gamma dP^\nu \| < 2\delta b$ with $b = \max \|\Gamma(\omega)\|$. By the Shapley–Folkmann lemma, see [1], the Hausdorff distance between $\sum \int_{\Omega_i} \Gamma dP^\nu$ and $\sum \int_{\Omega_i} \operatorname{co} \Gamma dP^\nu$ is less than $2n^{1/2}\delta b$. If δ is then small, the Hausdorff distance is small, and the case of P atomless is also covered.

To cover the general case let $\Psi = \{a_1, a_2, ...\}$ be the collection of atoms of P. Then for each ν there are a number $N(\nu)$ and disjoint neighborhoods $B_{i,\nu}$ of a_i for $i = 1, ..., N(\nu)$ such that: For each i the restrictions of P^ν to, respectively, $B_{i,\nu}$ converge weakly to the restriction of P to the atom a_i, and the restrictions of P^ν to $\Omega \backslash (B_{1,\nu} \cup \cdots \cup B_{N(\nu),\nu})$ converge weakly, as $\nu \to \infty$, to the restriction of P to $\Omega \backslash \Psi$. The existence of such partitions is implied by the weak convergence of P^ν to P.

When we apply the atomless case, which was verified earlier, to the restriction of P to $\Omega \backslash \Psi$, we get

$$\liminf_{\nu \to \infty} \int_{\Omega \backslash (B_{1,\nu} \cup \cdots \cup B_{N(\nu),\nu})} \Gamma dP^\nu \text{ includes } \int_{\Omega \backslash \Psi} \Gamma dP. \tag{4.6}$$

On the other hand, the lower semicontinuity of Γ implies that for each i,

$$\liminf_{\nu \to \infty} \int_{B_{i,\nu}} \Gamma dP^\nu \text{ includes } P(a)\,\Gamma(a). \tag{4.7}$$

Summing together the expressions in (4.6) and in (4.7) for each i yields the desired conclusion and completes the proof.

5. Two Counterexamples with Unbounded Sets

The convergence results of Section 3 fail if Γ is allowed unbounded values. The upper semiconvergence result of Section 4 fails then as well, while the lower semiconvergence result extends to the unbounded case under an additional mild condition, which we display in the next section. In this section we wish to identify two sources for the failure of the convergence. By excluding these two possibilities we get, in Section 7, our convergence result.

Example 5.1. Let $\Omega = [0, 1]$ and $\Gamma(\omega) = [0, \omega^{-1}]$ for $\omega > 0$, and $\Gamma(0) = [0, \infty)$. Then Γ is continuous. Let $P(\{1\}) = 1$ while $P^\nu(\{1\}) = 1 - \nu^{-1}$ and $P^\nu(\{0\}) = \nu^{-1}$. Then $\int \Gamma dP = [0, 1]$ while $\int \Gamma dP^\nu = [0, \infty)$, this despite the weak convergence of P^ν to P. A variant that does not use unbounded sets explicitly is to let the preceding Γ be defined on $(0, 1]$ and let $P^\nu(\{1\}) = 1 - \nu^{-1}$ and $P^\nu(\{\nu^{-1}\}) = \nu^{-1}$; then $\int \Gamma dP^\nu = [0, 2]$, and does not converge to $\int \Gamma dP$.

Example 5.2. Let $\Omega = [0, 1]$ and let the values $\Gamma(\omega)$ be subsets of R^2 defined by $\Gamma(\omega) = \{(\xi_1, \xi_2)\colon \xi_2 \geqslant \max(-\omega\xi_1, -1)\}$. Let P be determined by $P(\{0\}) = 1$. Let P^ν be defined by $P^\nu(\{0\}) = \frac{1}{2}$ and $P^\nu(\{\nu^{-1}\}) = \frac{1}{2}$. Then $\int \Gamma dP = \Gamma(0)$ and it is the upper half plane. The set $\int \Gamma dP^\nu$ is the Minkowski sum of $\frac{1}{2}\Gamma(0)$ and $\frac{1}{2}\Gamma(\nu^{-1})$. The latter, although converging to $\Gamma(0)$ as $\nu \to \infty$, contains the point $\frac{1}{2}(\nu, -1)$. Since $\Gamma(0)$ contains $(-\nu, 0)$, it follows that for every ν the set contains $(0, -\frac{1}{2})$; the latter point does not belong to $\int \Gamma dP$.

We wish to identify here the causes of the phenomena in the two examples. What makes the first example work is that although $P^\nu \to P$ weakly, each P^ν can draw a sizable contribution to the integral in directions unrelated to the set $\int \Gamma dP$. The reason the second example works is the possibility of a summation of large quantities, i.e., $\frac{1}{2}(\nu, -1)$ and $\frac{1}{2}(-\nu, 0)$, that results in small vectors. This possibility is reflected in the integral $\int \Gamma dP$ which contains a line.

6. Lower Semiconvergence

In the two examples of the previous section, $\liminf \int \Gamma dP^\nu$ includes $\int \Gamma dP$. We show in this section that this inclusion holds in general, provided that leas Γ is bounded, i.e., provided that there is a bounded set D in R^n such that $\Gamma(\omega) \cap D \neq \emptyset$ for all ω. Without a condition of that type, semiconvergence may fail. For example, let $\Gamma(\omega) = \{\omega^{-1}\}$ on $(0, 1]$, $P(\{1\}) = 1$, $P^\nu(\{1\}) = 1 - \nu^{-1}$, and $P^\nu(\{\nu^{-1}\}) = \nu^{-1}$. In this example leas Γ is not bounded. We can also see from this example that the boundedness of leas Γ plays a role similar to the boundedness of continuous functions in the theory of weak convergence of probability measures (but in the present case only lower semiconvergence can be deduced). We need the following two lemmas. Recall that int λB is the interior of the ball of radius λ.

Lemma 6.1. *Let Γ be a lower semicontinuous multifunction and suppose* leas Γ *is a bounded function, bounded say by β. Then for any $\lambda > \beta$, the multifunction*

$$\Gamma_1(\omega) = \mathrm{cl}(\Gamma(\omega) \cap \mathrm{int}\, \lambda B)$$

has nonempty values and is lower semicontinuous.

Proof. $\Gamma_1(\omega)$ is clearly nonempty. To prove lower semicontinuity, suppose x belongs to $\Gamma_1(\omega_0)$ and $|x| < \lambda$. Then for ω near ω_0, the set $\Gamma_1(\omega)$ contains elements near x. This follows from the lower semicontinuity of Γ. If x is in $\Gamma_1(\omega_0)$ and $|x| = \lambda$, then $\Gamma(\omega_0)$ contains elements near x, say y, with $|y| < \lambda$ (from the definition of Γ_1) and we can repeat the previous argument with respect to y. This completes the proof.

Lemma 6.2. *Let Γ be a closed-valued measurable multifunction with* leas Γ *bounded. Let P be a probability measure. Then*

$$\lim_{\lambda \to \infty} \int (\Gamma(\omega) \cap \lambda B)\, dP(\omega) = \mathrm{cl} \int \Gamma(\omega)\, dP(\omega).$$

Proof. This is basically the content of Theorem 5 of Jacobs [6]. Here is an outline of the proof: A measurable multifunction Γ, with leas Γ bounded, has a bounded measurable selection, say γ_0. If γ is an integrable selection, then $\int \gamma dP$ is approximated, for λ large, by $\int \gamma_\lambda dP$ with $\gamma_\lambda(\omega) = \gamma(\omega)$ if $|\gamma(\omega)| \leqslant \lambda$, and $\gamma(\omega) = \gamma_0(\omega)$ otherwise. The sequence $\{\int (\Gamma \cap \lambda B)\, dP\}$ is increasing, and for all λ we have $\int (\Gamma \cap \lambda B)\, dP \subset \mathrm{cl} \int \Gamma dP$. Moreover, by the preceding argument any point in $\mathrm{cl} \int \Gamma dP$ can be approximated arbitrarily close by points in $\int (\Gamma \cap \lambda B)\, dP$ for λ sufficiently large.

THEOREM 6.3. *Let Γ be a closed-valued lower semicontinuous multifunction with* leas Γ *bounded. Then P^ν converging to P weakly implies*

$$\liminf_{\nu \to \infty} \int \Gamma dP^\nu \text{ includes } \int \Gamma dP.$$

Proof. Let $\varepsilon > 0$. By Lemma 6.2 there exists a $\lambda_0 > 0$ with $\text{haus}^s(\int \Gamma dP, \int (\Gamma \cap \lambda B) dP) < \varepsilon$. In particular, if $\Gamma(\omega) \cap \lambda_0 B \subset \Gamma_1(\omega) \subset \Gamma(\omega)$, then $\text{haus}^s(\int \Gamma dP, \int \Gamma_1 dP) < \varepsilon$. A multifunction that satisfies these inclusions is

$$\Gamma_1(\omega) = \text{cl}(\Gamma(\omega) \cap \text{int}(\lambda B)) \qquad \text{for} \quad \lambda > \lambda_0.$$

The latter multifunction is lower semicontinuous and has nonempty values, as Lemma 6.1 implies. Then Theorem 4.4 implies

$$\liminf_{\nu \to \infty} \int \Gamma_1 dP^\nu \text{ includes } \int \Gamma_1 dP.$$

Since $\varepsilon > 0$ is arbitrarily small and $\liminf_{\nu \to \infty} \int \Gamma dP^\nu$ includes $\liminf_{\nu \to \infty} \int \Gamma_1 dP^\nu$, the proof follows.

7. A CONVERGENCE RESULT

We start this section with the statement of the main convergence result. Some of the conditions we use have an implicit form; we therefore accompany the main result with some more explicit alternatives and particular cases. Only then we proceed with a lemma and the proofs. Recall that $s(v, C)$ denotes the support function of the set C, and when $\Gamma(\omega)$ is a multifunction then we write $s(v, \omega)$ for $s(v, \Gamma(\omega))$.

THEOREM 7.1. *Let Γ be a multifunction with closed values and let P^ν converge weakly to P. Suppose that the following conditions hold.*

(i) *Γ is continuous, and* leas $\Gamma(\omega)$ *is bounded.*

(ii) *Γ is P-atom convex.*

(iii) *$\int \Gamma dP$ does not contain a line.*

(iv) *For each v with $s(v, \int \Gamma dP) < \infty$, the sequence P^ν is $s(v, \cdot)$-tight.*

Then $\lim \int \Gamma dP^\nu = \int \Gamma dP$.

We need to comment on the conditions. The boundedness of leas $\Gamma(\omega)$ was already needed in the semiconvergence result of the previous section; without it the real-valued counterexample given at the beginning of Section

6 applies. Condition (ii) was needed already in the bounded case (see Theorem 3.3); without it, Example 3.2 furnishes a counterexample. We comment later, in Remark 7.6, on the impossibility of extending the present result along the lines of Theorem 3.5.

Condition (iii) is needed to prevent the phenomenon demonstrated in Example 5.3. We may wish to know that this condition holds without necessarily computing $\int \Gamma dP$. Here are some sufficient conditions that may be of help.

PROPOSITION 7.2. *Let C be a convex cone which does not contain a line. Let $\Gamma_0(\omega)$ be a multifunction with $\|\Gamma_0(\omega)\|$ being P-integrable (in particular, Γ_0 bounded suffices). If $\Gamma(\omega) \subset \Gamma_0(\omega) + C$ then $\int \Gamma dP$ does not contain a line.*

Proof. $\int \Gamma dP$ is contained in $C + \int \Gamma_0 \, dP$. The latter is a sum of a bounded set with a set that does not contain a line; therefore the sum does not contain a line.

PROPOSITION 7.3. *If the support function $s(v, \omega)$ of the multifunction $\Gamma(\omega)$ is P-integrable for an open set of vectors v, then $\int \Gamma dP$ does not contain a line.*

Proof. This follows from the equality $s(v, \int \Gamma dP) = \int s(v, \omega) \, dP(\omega)$.

Condition (iv) prevents the occurrence of the phenomenon in Example 5.1; indeed, in this example the measures P^{ν} are not $s(1, \cdot)$-tight. It is an implicit condition; here is a geometrical condition, often encountered in applications, that guarantees $s(v, \cdot)$-tightness.

PROPOSITION 7.4. *Suppose there exist a convex cone C and two bounded multifunctions $\Gamma_0(\omega)$ and $\Gamma_1(\omega)$ such that $\Gamma_0(\omega) + C \subset \Gamma(\omega) \subset \Gamma_1(\omega) + C$. Then for each v with $s(v, C) < \infty$, the function $s(v, \Gamma(\omega))$ is bounded; in particular, if P^{ν} is a tight family then it is $s(v, \Gamma(\omega))$-tight.*

Proof. The result follows directly from the obvious inequality

$$|s(v, C) - s(v, \Gamma(\omega))| \leqslant |v| \cdot \max(\|\Gamma_0(\omega)\| + \|\Gamma_1(\omega)\|).$$

LEMMA 7.5. *Let C be a closed convex set in R^n, containing no lines. Then the effective domain of $s(\cdot, C)$, or equivalently the barrier cone $\mathrm{ba}(C) = \{v: s(v, C) < \infty\}$ of C, has a nonempty interior. In addition, if x is not in C then there is a vector v in the interior of $\mathrm{ba}(C)$ with $v \cdot x > s(v, C)$.*

Proof. $\mathrm{ba}(C)$ contains the interior of the polar of the recession cone of C; see Rockafellar [8, p. 123]. Since the recession cone has no lines, its polar has a nonempty interior (see [8, p. 126]) and the first claim follows.

The vector x can be separated from C, namely, there exists v_0 such that $v_0 \cdot x > s(v_0, C)$. The inequality is maintained if v_0 is replaced by $v_0 + \varepsilon v_1$ with ε small and v_1 in the interior of $\mathrm{ba}(C)$. Then $v = v_0 + \varepsilon v_1$ is in this interior, and this verifies the second claim.

Proof of Theorem 7.1. In Theorem 6.3 we have established the inclusion

$$\liminf \int \Gamma dP^\nu \text{ includes } \int \Gamma dP. \tag{7.1}$$

We therefore have to verify only that

$$\limsup \int \Gamma dP^\nu \text{ is included in } \mathrm{cl} \int \Gamma dP. \tag{7.2}$$

We assume that (7.2) is false and reach a contradiction. The set on the right-hand side of (7.2) is convex, as implied by conditions (i) and (ii), see (2.1). Suppose that (7.2) is false, then there exists a vector x_0 which does not belong to $\mathrm{cl} \int \Gamma dP$, but such that $x_0 = \lim x_\nu$ with vectors x_ν belonging to $\int \Gamma dP^\nu$ for $\nu = 1, 2, \ldots$. It follows then from condition (iii) and Lemma 7.5 that for some v_0,

$$s\left(v_0, \int \Gamma dP\right) < v_0 \cdot x_0, \tag{7.3}$$

and that v_0 can be chosen an interior point of $\{v: s(v, \int \Gamma dP) < \infty\} = \mathrm{ba}(\int \Gamma dP)$. We plan to show that (7.3) is impossible; this would mean that such an x_0 cannot exist and hence (7.2) is not false. In order to show that (7.3) is impossible we prove that

$$\int s(v_0, \omega)\, dP^\nu(\omega) \text{ converge to } \int s(v_0, \omega)\, dP(\omega) \quad \text{as} \quad \nu \to \infty. \tag{7.4}$$

This would indeed contradict (7.3) since $\int s(v_0, \omega)\, dQ = s(v_0, \int \Gamma dQ)$ for $Q = P, P^1, P^2, \ldots$, and therefore $\int s(v_0, \omega)\, dP^\nu \geqslant v_0 \cdot x_\nu$; the latter numbers, however, converge to $v_0 \cdot x_0$, hence the reverse of inequality (7.3) holds.

We now start proving (7.4). Let $v_1, \ldots, v_{n+1}$ be vectors in $\mathrm{ba}(\int \Gamma dP)$ that form a simplex containing v_0 in its interior. Such v_i exist since v_0 is in the interior of $\mathrm{ba}(\int \Gamma dP)$. For each one of the v_i we use now the $s(v_1, \cdot)$-tightness guaranteed by condition (iv). Since the support function is convex in the v-variable, it follows that for any given $\varepsilon > 0$ there is a compact subset $K \subset \Omega$ such that (a) $s(v, \omega)$ is uniformly bounded for ω in K and v in $\mathrm{co}\{v_1, \ldots, v_{n+1}\}$, and (b)

$$\int_{\Omega \setminus K} s(v, \omega)\, dP^\nu < \varepsilon \tag{7.5}$$

for all ν and all v in $\text{co}\{v_1, ..., v_{n+1}\}$. We plan to prove that $s(v_0, \cdot)$ is continuous on K. To this end, note first that the continuity of Γ implies that

$$s(v_0, \Gamma(\omega_0)) \leqslant \liminf_{i \to \infty} s(v_0, \Gamma(\omega_i)) \qquad \text{when } \omega_i \to \omega_0. \tag{7.6}$$

Therefore the continuity of $s(v_0, \cdot)$ on K would follow if we show that $\omega_i \to \omega_0$ in K implies that

$$s(v_0, \Gamma(\omega_0)) \geqslant \limsup_{i \to \infty} s(v_0, \Gamma(\omega_i)). \tag{7.7}$$

If (7.7) fails, then for some x_j in $\Gamma(\omega_j)$ we have $\limsup v_0 \cdot x_j > s(v_0, \Gamma(\omega_0))$. Then x_j is an unbounded sequence, since otherwise a subsequence of x_j would have a limit point, say y_0, and y_0 is in $\Gamma(\omega_0)$ (by the continuity of Γ). Then $v_0 \cdot y_0 > s(v_0, \Gamma(\omega))$, a contradiction. Unbounded x_j, however, imply that the numbers $v \cdot x_j$ cannot be bounded for v in $\text{co}\{v_1, ..., v_{n+1}\}$. Indeed $v_0 \cdot x_j$ is bounded from below, therefore a perturbation of the form $v_0 + \varepsilon v$, within the interior of $\text{co}\{v_1, ..., v_{n+1}\}$, can be formed so that $\varepsilon v \cdot x_j$ tends to infinity, at least on a subsequence. But this contradicts the boundedness of $s(v, \omega)$ for ω in K and v in $\text{co}\{c_1, ..., v_{n+1}\}$. Therefore x_j cannot be bounded and cannot be unbounded; hence such a sequence does not exist and (7.7) holds.

Once it is proved that $s(v_0, \omega)$ is continuous for ω in K, the weak convergence of P^ν to P implies that (7.4) holds when the integration is done on K. But on $\Omega \backslash K$ the estimate (7.5) holds, with ε arbitrarily small. Hence (7.4) is valid and the proof is complete.

Remark 7.6. The P-atom convexity in condition (ii) cannot be replaced by the compatibility of atoms of P^ν with those of P, as was sufficient in Theorem 3.5 for the bounded case. Here is a counterexample: Let $\Omega = [0, 1] \cup \{2\}$. Let Γ have values in R^2 and define $\Gamma(2) = \{(\xi_1, \xi_2): \xi_i \geqslant 0,\ \xi_1 \cdot \xi_2 = 0\}$, $\Gamma(0) = \{(0, 0)\}$, and $\Gamma(\omega) = \{(0, j\omega^{-1}),\ j = 0, 1, 2, ...\}$ for $0 < \omega < 1$. Let $P(\{2\}) = 1$ and define $P^\nu(\{2\}) = 1 - \nu^{-1}$ and $P^\nu(\{\nu^{-1}\}) = \nu^{-1}$. It is easy to see that conditions (i), (iii), and (iv) of Theorem 7.1 are fulfilled, as well as the compatibility condition in Theorem 3.5. Yet $\int \Gamma dP = \Gamma(2)$ is not the limit of $\int \Gamma dP^\nu$; the latter sets are identical, and equal to $\Gamma(2) \cup \{(\xi_1, \xi_2): \xi_i \geqslant 0,\ \xi_1 = 1, 2, ...\}$.

8. The Case of Epigraphs

As mentioned in the Introduction, a prime example and an application of the analysis of multifunctions is the case of epigraphs; see Rockafellar

[9] and Rockafellar and Wets [10]. We start this section with recalling the basic notions. Then we interpret some of the general results of the preceding sections within the framework of epigraphs, adding also some new results which exploit the particular structure of epigraphs.

Let $g: R^n \to (-\infty, \infty]$ be a lower semicontinuous real function, namely $g(\lim x_i) \leqslant \liminf g(x_i)$. The epigraph of g is the set in R^{n+1} given by

$$\operatorname{epi} g = \{(x, \alpha): \alpha \geqslant g(x)\}.$$

Then epi g is a closed set and this set is unbounded if g is not identically equal to $+\infty$. We denote by $\operatorname{Dom}(g)$ the effective domain of g, namely $\{x: g(x) < \infty\}$.

We say that the sequence g_k *epi-converges* to g (or converges in the sense of epigraphs) if epi g_k converges to epi g as sets in R^{n+1}. (The convergence of sets was introduced in Section 2.) We write then $g = \operatorname{epi} \lim g_k$. This notion of convergence of functions plays an essential role in variational problems; see [10] and references therein. Note, however, that epi-convergence is not comparable to pointwise convergence. A normal integrand (see [9, 10]) is a mapping $f(x, \omega)$: $R^n \times \Omega \to (-\infty, \infty]$, such that $f(x, \cdot)$ is measurable, $f(\cdot, \omega)$ is lower semicontinuous, and $\omega \to \operatorname{epi} f(\cdot, \omega)$ is a measurable multifunction. We say that a normal integrand is *epi-continuous* if $\omega \to \operatorname{epi} f(\cdot, \omega)$ is a continuous multifunction.

In analogy with the multifunction case, we denote by leas f the function $(\operatorname{leas} f)(\omega) = \inf\{|x| + |f(x, \omega)|: x \text{ in } R^n\}$. Note that $(\operatorname{leas} f)(\omega)$ is bounded when there exists a compact set D such that $\inf\{|f(x, \omega)|: x \text{ in } D\}$ is a bounded function.

Let P be a probability measure on Ω. The inf-convolution of the normal integrand $f(x, \omega)$ with respect to P is denoted by $⨍ f(x, \omega)\, dP$, and it is a function, say $F(x)$, of the variable x given by

$$⨍ f(x, \omega)\, dP = \inf\left\{\int f(y_i(\omega), \omega)\, dP: \int y_i(\omega)\, dP \to x\right\}.$$

The function $F(x) = ⨍ f(x, \omega)\, dP$ is related to the variational problem

$$\text{(VP)} \qquad \text{minimize} \int f(y(\omega), \omega)\, dP \text{ subject to } \int y(\omega)\, dP = x.$$

Indeed, $F(x)$ is the relaxed optimal value of (VP) (relaxed since we allow small perturbations in the constraint x). This variational problem is common in stochastic optimization and in applications; see [3, 9, 10]. There is a very useful relation between the inf-convolution operation and epigraphs:

$$\operatorname{epi}\left(⨍ f(x, \omega)\, dP\right) = \operatorname{cl} \int_\Omega \operatorname{epi} f(x, \omega)\, dP, \tag{8.1}$$

where the integration in the right-hand side is of the multifunction as recalled in Section 2 and used throughout. (Relation (8.1) is the reason why we choose to denote the inf-convolution operation by a letter E on the integral; indeed, this is integration in the epigraph sense.) This relation between the variational problem and the integration of epigraphs has been an important tool; it was introduced by Aumann and Perles [3]; see also Rockafellar [9].

Motivated by the analysis of previous sections we say that the normal integrand f is *P-atom convex* if $f(\cdot, a)$ is a convex function whenever $\{a\}$ is an atom of P. A consequence of (2.1) and (8.1) is then: If f is P-atom convex then $F(x) = \oint f\, dP$ is a convex function.

In the sequel we are interested in convergence properties of $F^\nu(x) = \oint f\, dP^\nu$ to $F(x) = \oint f\, dP$ when P^ν converges weakly to P. As explained in the Introduction, the motivation is that in solving the variational problem (VP), one sometimes has to replace the underlying measure P by an estimate P^ν. It is desirable then to know to what extent this changes the value of the problem. We plan to employ the results of the previous sections. Note, however, that epi f is never a bounded set (unless empty). In particular, the two counterexamples of Section 5 can be modified to counterexamples for epigraphs as follows.

Let $f(x, \omega)$ be defined on $R^2 \times \Omega$ such that $f(x, \omega) = 0$ if x in $\Gamma(\omega)$ and $f(x, \omega) = \infty$ otherwise. Then $\oint f\, dQ$ is equal to 0 if x is in $\operatorname{cl} \int \Gamma\, dQ$, and is equal to ∞ otherwise. In particular, if we apply this to $Q = P, P^1, P^2, \ldots$ in the two examples of Section 5 we get that $\oint f\, dP^\nu$ do not epi-converge to $\int f\, dP$.

We can, however, draw positive conclusions, using the results of Sections 6 and 8.

We say that epi lim sup $g^\nu(x) \leqslant g(x)$ if lim inf epi g^ν contains epi g. This semiconvergence has a clear interpretation in the variational framework (VP). Indeed, if g^ν are values of variational problems then, in the relaxed sense, limits of g^ν are not inferior to g. Note that epi lim sup $g^\nu(x) \leqslant g(x)$ is implied by the pointwise condition $\limsup g^\nu(x) \leqslant g(x)$, but does not imply it.

THEOREM 8.1. *Let f be an epi-continuous normal integrand such that* $(\operatorname{leas} f)(\omega)$ *is a bounded function. If $P^\nu \to P$ weakly then*

$$\text{epi}\,\limsup \oint f(x, \omega)\, dP^\nu(\omega) \leqslant \oint f(x, \omega)\, dP(\omega).$$

Proof. The result is a translation of Theorem 6.3 to the language of epigraphs (and in particular the epi-continuity of f can be eased and replaced by epi-lower semicontinuity; we leave out the details).

Note that f in the previous result can be quite general, and it is not difficult to find examples in which the epi lim sup cannot be replaced by pointwise lim sup. Further conditions yield such a condition for the pointwise lim sup, as follows.

THEOREM 8.2. *Let f be an epi-continuous normal integrand, with* $(\text{leas } f)(\omega)$ *bounded. Suppose that $P^\nu \to P$ weakly and that f is P-atom convex and P^ν-atom convex for every ν. Then*

$$\limsup \oint f(x, \omega)\, dP^\nu(\omega) \leqslant \oint f(x, \omega)\, dP(\omega)$$

for every x in the interior of $\text{Dom}(\oint f\, dP)$.

Proof. With the P^ν-atom convexity, the functions $F^\nu(x) = \oint f(x, \omega)\, dP^\nu$ are all convex functions, therefore the result follows from Theorem 8.1.

For epigraphs of functions it is convenient to use the conjugate function instead of the support function that we have used for multifunctions; see [10]. Recall that if $g(x)$ is a function of x then its conjugate function $g^*(v)$ is a function of v defined by

$$g^*(v) = \sup_x (v \cdot x - g(x)).$$

(Note that $g^*(v) = s((-v, 1), \text{epi } g)$. In particular $\text{Dom}(g^*)$ is a convex set, and a vector v is in its interior if and only if $(-v, 1)$ is in the interior of the barrier cone of epi g.)

THEOREM 8.3. *Let f be a normal integrand and suppose that $P^\nu \to P$ weakly. Suppose also that the following conditions hold (we write $F(x)$ for $\oint f dP$).*

(i) *f is epi-continuous and* $(\text{leas } f)(\omega)$ *is bounded,*

(ii) *f is P-atom convex,*

(iii) $\text{Dom } F$ *does not contain a line and F is proper, namely, $F(x) > -\infty$ and* $\text{Dom } F$ *is not empty, and*

(iv) *for all v with $F^*(v) < \infty$, the sequence P^ν is $f^*(v, \cdot)$-tight.*

Then epi limit $\oint f(x, \omega)\, dP^\nu = \oint f(x, \omega)\, dP$.

Proof. It is straightforward to check that conditions (i)–(iv) of Theorem 7.1 hold when the multifunction is generated by the epigraphs of the normal integrand. Taking into account also (8.1), the proof is complete.

COROLLARY 8.4. *Let f be an epi-continuous normal integrand such that*

$\mathrm{Dom}(f(\cdot, \omega))$ *is a bounded multifunction, and* $\inf_x f(x, \omega)$ *is bounded. Suppose that* $P^\nu \to P$ *weakly and that* f *is P-atom convex. Then*

$$\text{epi lim} \oint f(x, \omega)\, dP^\nu = \oint f(x, \omega)\, dP.$$

Proof. Conditions (i) and (ii) of Theorem 8.3 are assumed. Condition (iii) follows from the boundedness of $\mathrm{Dom}\, f$ and $\inf f(\cdot, \omega)$. The last condition of Theorem 8.3 follows from the boundedness of $\mathrm{Dom}(\cdot, \omega)$ and Proposition 7.4, where in applying the latter we use $C = \{(0, r) : r \geq 0\}$. This shows that all the conditions of Theorem 8.3 hold; hence the corollary follows.

9. APPROXIMATION BY TRUNCATION

As explained in the Introduction, when the integral (1.1) is computed, it is very often approximated by

$$\int_\Omega (\Gamma(\omega) \cap \lambda B)\, dP(\omega) \tag{9.1}$$

with λ a number, usually large. The analogous truncation for the inf-convolution is to replace (1.3) by

$$\oint f_\lambda(x, \omega)\, dP(\omega) \tag{9.2}$$

with $f_\lambda(x, \omega) = f(x, \omega)$ if $|x| \leq \lambda$ and $f_\lambda(x, \omega) = \infty$ if $|x| > \lambda$. Approximations of the type (9.1) were studied by Jacobs [6]; here is one useful result.

LEMMA 9.1. *If* Γ *has a bounded selection then* $\int_\Omega (\Gamma(\omega) \cap \lambda B)\, dP$ *converge to* $\int_\Omega \Gamma(\omega)\, dP$ *as* $\lambda \to \infty$.

Proof. See Jacobs [6, Theorem 5]. A similar result holds for the approximations in (9.2).

The first question that we take on in this section is whether this approximation, determined by λ, holds uniformly for all elements in a sequence P^ν which converges weakly to P; namely, we wish to know whether in a situation when P has to be approximated, say by P^ν, the approximation by truncation, (9.1), holds for the approximation P^ν as well. Without some additional conditions the approximation is not uniform. As a counterexample consider Example 5.1; there, for any given λ, the integrals $\int (\Gamma(\omega) \cap \lambda B)\, dP^\nu$ cease to furnish good approximations for ν

large. In this counterexample, however, the integrals $\int \Gamma dP^{\nu}$ do not converge to $\int \Gamma dP$. With such convergence we do get a positive result, as follows.

THEOREM 9.2. *Suppose that Γ satisfies all the conditions of Theorem 7.1 and that $P^{\nu} \to P$ weakly. Then for every $\varepsilon > 0$ there exists a $\lambda_0 > 0$ such that*

$$\text{haus}^{s}\left(\int (\Gamma(\omega) \cap \lambda B)\, dP^{\nu}(\omega),\ \int \Gamma(\omega)\, dP^{\nu}(\omega)\right) < \varepsilon \tag{9.3}$$

for all ν and all $\lambda \geqslant \lambda_0$.

Proof. Let λ_1 be such that $\text{haus}^{s}(\int \Gamma dP, \int \Gamma_1\, dP) < \varepsilon/4$ whenever $\Gamma(\omega) \cap \lambda_1 B \subset \Gamma_1(\omega) \subset \Gamma(\omega)$; such λ_1 exists by Lemma 9.1 and a simple observation concerning the haus^{s} metric. Define $\Gamma_2(\omega) = \text{cl}(\Gamma(\omega) \cap \text{int}(\lambda_1 + 1)\, B)$ and $\Gamma_3(\omega) = \Gamma(\omega) \cap \lambda_1 B$. By Lemma 6.1, the multifunction Γ_2 is lower semicontinuous, and it is clear that Γ_3 is upper semicontinuous. Using Theorems 4.5 and 4.2, respectively, we get that for ν large enough, say $\nu \geqslant \nu_1$, the set $\int \Gamma_2\, dP^{\nu}$ contains an $\varepsilon/4$-steoreographic neighborhood of $\int \Gamma_2 dP$, and $\int \Gamma_3 dP^{\nu}$ is contained in an $\varepsilon/4$-stereographic neighborhood of $\int \Gamma_2 dP$. Since, however, $\Gamma(\omega) \cap \lambda_1 B \subset \Gamma_2(\omega) \subset \Gamma_3(\omega) \subset \Gamma(\omega)$, it follows from the choice of λ_1 that for $\nu \geqslant \nu_1$ the set $\int \Gamma_3(\omega)\, dP^{\nu}$ is within $\varepsilon/2 - \text{haus}^{s}$ distance of $\int \Gamma dP$. Since the conditions of Theorem 7.1 hold, there is a ν_2 such that $\nu \geqslant \nu_2$ implies that $\int \Gamma dP^{\nu}$ is within $\varepsilon/2 - \text{haus}^{s}$ distance of $\int \Gamma dP$. If we choose $\lambda_0 = \lambda_1 + 1$ and $\nu_0 = \max(\nu_1, \nu_2)$, we get from the inclusions $\Gamma_3(\omega) \subset \Gamma(\omega) \cap \lambda B \subset \Gamma(\omega)$ for $\lambda > \lambda_0$, and from the aforementioned estimates for $\nu \geqslant \nu_0$, that $\text{haus}^{s}(\int (\Gamma \cap \lambda B)\, dP^{\nu}, \int \Gamma dP^{\nu}) < \varepsilon$ for $\lambda \geqslant \lambda_0$ and $\nu \geqslant \nu_0$. This verifies the claim for all but a finite number of P^{ν}, but since for a single ν the result holds, by Lemma 9.1, the finite number of $\nu = 1,..., \nu_0$ can be accounted for by an appropriate increase of λ_0. This completes the proof.

A similar result holds for normal integrands, assuming the conditions of Theorem 8.3. We leave out the details.

Our next step is to verify the observation promised in the closing paragraph of the Introduction. We consider approximations that involve at the same time approximations of the measures and truncations of the multifunction.

THEOREM 9.3. *Let $\Gamma(\omega)$ be a continuous multifunction with closed values and such that* leas *$\Gamma(\omega)$ is bounded. Suppose that Γ is P-atom convex and that $P^{\nu} \to P$ weakly. Then*

$$\lim_{\lambda \to \infty} \lim_{\nu \to \infty} \int (\Gamma(\omega) \cap \lambda B)\, dP^{\nu} = \int \Gamma(\omega)\, dP(\omega). \tag{9.4}$$

Proof. We note from Theorem 4.2 and the observation that $\Gamma(\omega) \cap \lambda B$ is upper semicontinuous that

$$\limsup_{\nu \to \infty} \int (\Gamma(\omega) \cap \lambda B)\, dP^{\nu} \text{ is included in } \int (\Gamma(\omega) \cap \lambda B)\, dP. \tag{9.5}$$

We conclude from Theorem 4.5 and Lemma 6.1 that

$$\liminf_{\nu \to \infty} \int \operatorname{cl}(\Gamma(\omega) \cap \operatorname{int} \lambda B)\, dP^{\nu} \text{ includes } \int \operatorname{cl}(\Gamma(\omega) \cap \operatorname{int} \lambda B)\, dP. \tag{9.6}$$

Using the trivial observation that $\operatorname{cl}(\Gamma(\omega) \cap \operatorname{int} \lambda B)$ includes $\Gamma(\omega) \cap (\lambda - 1) B$, we can conclude from (9.6) that

$$\liminf_{\nu \to \infty} \int (\Gamma(\omega) \cap \lambda B)\, dP^{\nu} \text{ includes } \int (\Gamma(\omega) \cap (\lambda - 1) B)\, dP. \tag{9.7}$$

Relations (9.6) and (9.7) together with

$$\lim_{\lambda \to \infty} \int (\Gamma(\omega) \cap \lambda B)\, dP = \int \Gamma(\omega)\, dP \tag{9.8}$$

(Lemma 9.1) imply (9.4) and complete the proof.

As already mentioned, the previous result has an immediate and significant interpretation in the approximation procedures for variational problems. We therefore state the analogous result for the inf-convolution of normal integrands; the proof is very similar and is omitted.

THEOREM 9.4. *Let $f(x, \omega)$ be an epi-continuous normal integrand, with* $(\operatorname{leas} f)(\omega)$ *bounded. Suppose that f is P-atom convex and that $P^{\nu} \to P$ weakly. Then*

$$\operatorname*{epi\,lim}_{\lambda \to \infty} \operatorname*{epi\,lim}_{\nu \to \infty} ⨏ f_{\lambda}(x, \omega)\, dP^{\nu} = ⨏ f(x, \omega)\, dP.$$

We finally note that the counterexamples given before (Example 3.2 and Section 6) show that neither of the, rather weak, conditions of Theorems 9.2 and 9.3 can be dropped, nor can the order in which the limits are taken be reversed.

ACKNOWLEDGMENT

We thank an anonymous referee for very helpful comments.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium