EPI-CONSISTENCY OF CONVEX STOCHASTIC PROGRAMS

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Abstract. This paper presents consistency results for sequences of optimal solutions to convex stochastic optimization problems constructed from empirical data, by applying the strong law of large numbers for random closed sets to the epigraphs of the conjugate functions. Because of the special properties of convexity and empirical measures, epi-consistency is obtained under very simple assumptions; nevertheless the results are broadly applicable to many situations arising in stochastic programming. A new epi-consistency result for stochastic linear programs with recourse is obtained.

Keywords: random sets, epi-convergence, statistical consistency, empirical measures, law of large numbers

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1. Introduction.

To solve the stochastic program

$$\text{minimize} \int f(\xi, x)P(d\xi) := Ef(x), \text{ over all } x \in \mathbb{R}^n, \tag{1.1}$$

it is frequently necessary to solve instead an approximating problem,

$$\text{minimize} \frac{1}{\nu}\sum f(\xi_i, x) := E^\nu f(x), \text{ over all } x \in \mathbb{R}^n, \tag{1.2}$$

where the probability measure P is replaced by an empirical measure derived from an independent series of random observations $\{\xi_1, \ldots, \xi_\nu\}$ each with common distribution P. Generally speaking, this arises for one of two reasons: either the measure P itself is known only through the observations; or the numerical solution of (1.1) requires the discretization of P, and one very simple technique is to generate a set of "pseudo-random observations" from the distribution of P. As the number ν of sample observations grows large, we demand that the approximations (1.2) approach the true problem in the sense that the functions $E^\nu f$ be *epi-consistent* with limit Ef — that is, $E^\nu f$ *epi-converges* to Ef almost surely. This would imply the essential property that cluster points of sequences of minimizers (or ε^ν-minimizers, for $\varepsilon^\nu \to 0$) of the approximate functions $E^\nu f$ are, with probability one, minimizers of the function Ef and the corresponding subsequences of values of $E^\nu f$ converge to the minimum value of Ef. As is usual in optimization, we allow f to take values in the *extended reals* $\overline{\mathbb{R}}$ to include the possibility that f can equal $+\infty$ or $-\infty$.

In this paper, we present an epi-consistency theorem based on the strong law of large numbers for sums of independent and identically distributed random closed sets of Artstein and Hart [**1**]. The idea is simple: we apply the strong law to the sums of the conjugate epigraphs and then use standard epi-convergence and convexity procedures in order to obtain epi-consistency. The requirement of the strong law as applied to our problem is that there exists an integrable selection of the random epigraph $\operatorname{epi} f(\xi_1, \cdot)$—an attractively simple assumption that can be implied by readily verifiable conditions on $f(\xi_1, \cdot)$ and its subgradient, as we shall see. This argument is extended virtually without change to reflexive Banach spaces, based on the strong law of Hess [**5**] and Hiai [**4**], but the conditions placed on f will be less simple.

Readers familiar with the the strong law for sequences of random closed sets, $\{A_i\}$, will recall that the pointwise convergence of the average of the support functions $\frac{1}{\nu}\sum \sigma_{A_i}$ is used to demonstrate the limsup inclusion

$$\limsup \frac{1}{\nu}\sum A_i \subset EA_1,$$

and then the liminf inclusion

$$EA_1 \subset \liminf \frac{1}{\nu} \sum A_i$$

is proved by directly constructing a converging sequence of selections to every point in EA_1. Now, of course, the strong law is *equivalent* to the epiconsistency of the support functions; however, at this writing, there does not seem to be a direct proof of epi-consistency that avoids this excursion into the strong law for random closed sets.

Epi-consistency results have appeared in Dupačová and Wets [**3**], Kall [**6**], and Robinson and Wets [**7**]. These papers present sufficient conditions on P and f such that $P \mapsto Ef$ is continuous as a map from the space of probability measures topologized by convergence in distribution into the space of lower semicontinuous functions topologized by epi-convergence. Since the empirical measures converge in distribution to P, then epi-consistency would follow from such assumptions. However, the continuity requirements of this approach are so strong that most practically formulated stochastic programs—including, especially, the linear recourse problems—cannot meet them. We demonstrate the scope of our theorem by proving a new epi-consistency result for stochastic linear programs with recourse under more realistic assumptions than can be managed using continuity techniques.

The situation considered in this paper has much in common with estimation problems in statistics, where x^ν would be called a *statistic* and one would try establish the *consistency* of x^ν, i.e. the existence of a constant $\bar{x}$ such that $x^\nu \to \bar{x}$ with probability one. This concept does not transfer very well to optimization, where a unique minimizing $\bar{x}$ is unlikely due to the presence of constraints. In such cases, epi-consistency is the only possibility. A detailed discussion of these similarities and contrasts is given in [**3**].

2. Epi-consistency

For background on lower semicontinuity and measurability, we refer the reader to Rockafellar [**10**], and for epi-convergence Attouch [**2**].

A function g is called *lower semicontinuous* (lsc) if its *epigraph*

$$\operatorname{epi} g := \{(x, \alpha) \,\big|\, \alpha \geq g(x)\}$$

is a closed subset of $\mathbb{R}^n \times \mathbb{R}$, *convex* if $\operatorname{epi} g$ is convex, and *proper* if $\operatorname{epi} g$ is neither the empty set nor the whole space. The *domain* of g, denoted $\operatorname{dom} g$, is the set $\{x \,|\, g(x) < \infty\}$. Let $(\Xi, \mathcal{A}, P)$ be a probability space completed with respect to P. A closed-valued multifunction $G : \Xi \rightrightarrows \mathbb{R}^n$ is *measurable* if for all closed subsets $C \subset \mathbb{R}^n$ one has

$$G^{-1}(C) := \{\xi \in \Xi \,\big|\, G(\xi) \cap C \neq \emptyset\} \in \mathcal{A}.$$

Following usual practice, we shall also call G a *random closed set.* A function $f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is a *normal integrand* if the epigraphical multifunction $\xi \mapsto \operatorname{epi} f(\xi, \cdot)$ is closed-valued and measurable. A convex normal integrand is one whose epigraph is almost everywhere convex; a proper normal integrand is similarly defined. If the probability space is implicit, then we shall call f a *random lower semicontinuous function* (random lsc function). Thus, if f is normal and ξ_1 is a random variable, then $f(\xi_1, \cdot)$ is a random lsc function.

Let $\{A_\nu\}$ be a sequence of subsets of $\mathbb{R}^n$. We define the (Painlevé-Kuratowski) set limits:

$$\limsup_\nu A_\nu = \{x = \lim_\nu x_\nu \,\big|\, x_\nu \in A_\nu \text{ for infinitely many } \nu\},$$

$$\liminf_\nu A_\nu = \{x = \lim_\nu x_\nu \,\big|\, x_\nu \in A_\nu \text{ for all but finitely many } \nu\}.$$

This sequence *converges* to $A = \lim_\nu A_\nu$ if both limits are equal to A. A sequence $\{g^\nu\}$ of extended real-valued functions *epi-converges* to g if

$$\operatorname{epi} g = \lim_\nu (\operatorname{epi} g^\nu).$$

This is the property we wish to prove for the sequence $\{E^\nu f\}$. The objective functions $E^\nu f$ are random lsc on the sample probability space. Epi-convergence, therefore, need only take place on a set of probability one. We formalize this in a definition.

Definition. A sequence $\{h^\nu\}$ of random lower semicontinuous functions is *epi-consistent* if there is a (necessarily) lower semicontinuous function h such that $\{h^\nu\}$ epi-converges to h with probability one.

The importance of epi-consistency is summarized in the following proposition.

Proposition 2.1. *([**2**], Theorem 1.10) Suppose the sequence of random lower semicontinuous functions $\{h^\nu\}$ is epiconsistent with limit h. Let $\{\varepsilon^\nu\}$ be a sequence of non-negative numbers converging to* 0*, and $\{x^\nu\}$ a sequence of points such that*

$$h^\nu(x^\nu) \le \inf h^\nu + \varepsilon^\nu.$$

If there is a subsequence $\{x^{\nu_k}\}$ converging to a point $\bar{x}$, then, with probability one, $\bar{x}$ minimizes h and $\lim_k h^{\nu_k}(x^{\nu_k}) = \inf h$.

We will need the following (epigraphical) operation: by $+^{\mathrm{e}}$ we denote the *epi-sum* defined by the identity:

$$(f +^{\mathrm{e}} g)(x) = \inf\{f(u) + g(v) \,\big|\, u + v = x\}$$

with, as usual, $\infty - \infty = \infty$. In the literature one also finds the epi-sum $f +^{\mathrm{e}} g$ denoted by $f \mathbin{\square} g$ (or $f \nabla g$) and called the inf-convolution of f and g. By [**8**], Theorem 16.4, if f and g are proper convex functions such that $\mathrm{ri}(\mathrm{dom}\, f)$ and $\mathrm{ri}(\mathrm{dom}\, g)$ have a point in common (for a convex set C, $\mathrm{ri}\, C$ is the interior relative to the smallest affine subspace containing C; it is always nonempty by [**8**] Theorem 6.2), then we have

$$\mathrm{epi}(f+g)^* = \mathrm{epi}[f^* +^{\mathrm{e}} g^*] = \mathrm{epi}\, f^* + \mathrm{epi}\, g^*, \tag{2.2}$$

where $f^* : \mathbb{R}^n \to \mathbb{R}$ is the conjugate function

$$f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x\rangle - f(x)\}\,.$$

The next lemma shows that the $(E^\nu f)^*$ are random lsc functions such that the epi-sum formula (2.2) holds.

Lemma 2.2. *Let $f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex normal integrand, and suppose that there exists a point $\bar{x}$ in $\mathbb{R}^n$ such that*

$$\bar{x} \in \mathrm{ri}(\mathrm{dom}\, f(\xi, \cdot)) \qquad \text{a.s.} \tag{2.3}$$

Then for all $\nu = 1, 2, \ldots$ the functions $E^\nu f$ and $(E^\nu f)^*$ are random lower semicontinuous proper convex functions and, moreover,

$$\mathrm{epi}(E^\nu f)^* = \frac{1}{\nu} \sum_{i=1}^{\nu} \mathrm{epi}\, f^*(\xi_i, \cdot) \qquad \text{a.s.} \tag{2.4}$$

Proof. The assumption (2.3) implies that with probability one

$$\bar{x} \in \bigcap_{i=1}^{\nu} \mathrm{ri}(\mathrm{dom}\, f(\xi_i, \cdot))$$

for all ν, since the random sets $\mathrm{dom}\, f(\xi_i, \cdot)$ are independent and identically distributed. This allows us to invoke the epi-sum equality (2.2). Applying the conjugacy formula and this equality, we find that

$$(E^\nu f)^*(x^*) = \frac{1}{\nu}[f^*(\xi_1, \cdot) +^{\mathrm{e}} \cdots f^*(\xi_\nu, \cdot)](\nu x^*) \qquad \text{a.s. }\,,$$

and also,

$$\mathrm{epi}[f^*(\xi_1, \cdot) +^{\mathrm{e}} \cdots +^{\mathrm{e}} f^*(\xi_\nu, \cdot)] = \sum_i \mathrm{epi}\, f^*(\xi_i, \cdot) \qquad \text{a.s.}$$

All the conclusions of the lemma follow from these equalities. □

The strong law of large numbers of Artstein and Hart states that if $\{A_\nu\}$ is an independent and identically distributed sequence of random closed sets in $\mathbb{R}^n$ such that the distance function $d(0, A_1)$ is integrable, then one has with probability one

$$EA_1 = \lim_\nu \frac{1}{\nu} \sum_{i=1}^{\nu} A_i$$

where

$$EA_1 = \overline{\mathrm{co}}\{Ex_1 \,\big|\, x_1 \text{ is an integrable selection of } A_1\}.$$

Imposing an appropriate integrability condition on the random epigraph $\operatorname{epi} f^*(\xi_1, \cdot)$ and applying the strong law would yield convergence with probability one of the sets on the right-hand side of (2.4) to some closed convex epigraph. This would be epi-consistency of the random lsc functions $[f^*(\xi_1, \cdot) +^{\mathrm{e}} \cdots +^{\mathrm{e}} f^*(\xi_\nu, \cdot)]$, but not epi-consistency of the $E^\nu f$. Our main result spells out the conditions under which the strong law implies the epi-consistency we are after.

Theorem 2.3. *Let $f : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex normal integrand such that there is a point $\bar{x}$ with*

$$Ef(\bar{x}) \text{ finite,} \tag{2.5}$$

and a measurable selection $\bar{u}(\xi) \in \partial f(\xi, \bar{x})$ with

$$\int \|\bar{u}(\xi)\| P(d\xi) \text{ finite.} \tag{2.6}$$

Then the function Ef is proper, convex, and lower semicontinuous, and $\{E^\nu f\}$ is epi-consistent with limit Ef.

Proof. The existence of a measurable selection $\bar{u}(\xi) \in \partial f(\xi, \bar{x})$ is assured by assumption (2.5), which implies that $f(\xi, \bar{x})$ is finite and $\partial f(\xi, \bar{x})$ is nonempty for P-almost all ξ. Thus condition (2.6) only requires that among the measurable selections there exists one that is integrable. Assumption (2.5) also implies that f is a proper normal integrand, and $\bar{x} \in \operatorname{dom} f(\xi_1, \cdot)$ almost surely. If $\bar{x}$ is the only point in $\operatorname{dom} f(\xi_1, \cdot)$ a.s. , then with probability one $\lim E^\nu f(x) = +\infty$ for every $x \neq \bar{x}$. The ordinary strong law of large numbers implies $\lim E^\nu f(\bar{x}) = Ef(\bar{x})$ and epi-consistency follows from [**2**], Proposition 1.14. On the other hand, if there are points other than $\bar{x}$ in $\operatorname{dom} f(\xi_1, \cdot)$ a.s. , then with probability one $\operatorname{ri}(\operatorname{dom} f(\xi_1, \cdot))$ is nonempty ([**8**], Theorem 6.2) and Lemma 2.2 implies that the epi-sum equality (2.4) holds. We apply the strong law of large numbers to the sets in (2.4). The random closed sets $\operatorname{epi} f^*(\xi_i, \cdot)$ are independent and identically distributed subsets of $\mathbb{R}^n$. We have

$$d(0, \operatorname{epi} f^*(\xi_1, \cdot)) \leq \|\bar{u}(\xi_1)\| + |f^*(\xi_1, \bar{u}(\xi_1))|;$$

the first term is integrable by (2.6), and by convex analysis ([**8**], Theorem 23.5) we have

$$f^*(\xi_1, \bar{u}(\xi_1)) = \langle \bar{u}(\xi_1), \bar{x} \rangle - f(\xi_1, \bar{x})$$

which is integrable by (2.5) and (2.6). Thus the strong law of large numbers of Artstein and Hart implies that the sums $\frac{1}{\nu} \sum \operatorname{epi} f^*(\xi_i, \cdot)$ converge almost surely to the set $\overline{\operatorname{co}}\, F^*$, where

$$F^* = \left\{ E(u(\xi_1),\, \alpha(\xi_1)) \,\middle|\, (u(\xi_1),\, \alpha(\xi_1)) \text{ is an integrable selection of } \operatorname{epi} f^*(\xi_1, \cdot) \right\}.$$

We now show that $\overline{\operatorname{co}}\, F^* = \operatorname{epi}(Ef)^*$ or, equivalently, that

$$Ef(x) = \sup\{\langle x^*, x \rangle - \alpha \,|\, (x^*, \alpha) \in F^*\}.$$

Let $\mathcal{L}^1$ be the space of integrable functions from Ξ into $\mathbb{R}^n$. We may rewrite the right-hand side as

$$\sup \left\{ \int [\langle u(\xi), x \rangle - f^*(\xi, u(\xi))]\, P(d\xi) \,\middle|\, u \in \mathcal{L}^1 \right\}.$$

This is equal to $(Ef^*)^*(x)$, considering (Ef^*) as a function on the decomposable space $\mathcal{L}^1$. By [**10**], Theorem 3C, $(Ef^*)^*$ equals Ef—which must then be a proper convex lower semicontinuous function. (The integrability of $f^*(\xi_1, \bar{u}(\xi_1))$ is used here.) We conclude that the $(E^\nu f)^*$ epi-converge almost surely to $(Ef)^*$. The bi-continuity of the Fenchel transform ([**2**], Theorem 3.18) implies that the $E^\nu f$ epi-converge almost surely to Ef. $\square$

The setting of this theorem can be extended to a reflexive Banach space X with separable dual X^* by applying instead the strong law for random closed sets of Hess [**5**] and Hiai [**4**]. (Examples of estimation problems demanding such generality are provided by non-parametric statistical estimation.) We provide the details in a separate theorem. Everything goes through as above, using Rockafellar [**9**] as a guide for the infinite-dimensional convex analysis, but with one important exception: the argument used to establish the epi-sum equality (2.4) has to be adjusted to reflect the greater complexity of the notion of interior in infinite dimensional spaces. Mere finiteness of $Ef(\bar{x})$ no longer suffices.

The definitions also must be slightly modified to accomodate the greater number of possible topologies for X. A sequence of subsets of a Banach space $\{C_i\}$ converges *in the Mosco sense* to a set C if

$$C = s\text{-}\liminf C_i = w\text{-}\limsup C_i,$$

where the prefix w- or s- causes limits to be taken in the weak or strong topology, respectively. This Mosco-limit is denoted $M\text{-}\lim C_i$. A sequence of extended real-valued functions $\{g^\nu\}$ *Mosco-epi-converges* to g if

$$\operatorname{epi} g = M\text{-}\lim_{\nu}(\operatorname{epi} g^\nu).$$

Mosco-epi-consistency is defined analogously to the finite-dimensional case, and in this case, the conclusions of Proposition 2.1 will hold for *weak* cluster points of ε^ν-minimizers.

The strong law of Hess and Hiai states that if $\{A_\nu\}$ is a sequence of independent and identically distributed random sets in a separable relexive Banach space (which, in our application, will be $X^* \times \mathbb{R}$) with $d(0, A_1)$ integrable, then

$$M\text{-}\lim_{\nu} \frac{1}{\nu} \sum A_i = EA_1,$$

where EA_1 is defined just as for the finite-dimensional situation.

Theorem 2.4. *Let X be a reflexive Banach space with separable dual X^*, and let $f : \Xi \times X \to \overline{\mathbb{R}}$ be a convex normal integrand with a point $\bar{x}$ in X such that $Ef(\bar{x})$ is finite, and a measurable selection $\bar{u}(\xi) \in \partial f(\xi, \bar{x})$ such that $\int \|\bar{u}(\xi)\|_* P(d\xi)$ is finite ($\|\cdot\|_*$ is the dual norm on X^*). Suppose, moreover, that there is a point $\hat{x}$ such that*

$$\hat{x} \in \operatorname{int} \operatorname{dom} f(\xi_1, \cdot) \text{ a.s.} \tag{2.7}$$

Then $\{Ef^\nu\}$ is a sequence of random lower semicontinuous proper convex functions that is Mosco-epi-consistent with limit Ef.

Proof. Condition (2.7) allows us to apply [**9**], Theorem 20, to determine via the argument of Lemma 2.2 that the $E^\nu f$ are random lower semicontinuous proper convex functions such that $\operatorname{epi}(E^\nu f)^* = \frac{1}{\nu} \sum \operatorname{epi} f^*(\xi_i, \cdot)$. Now apply the strong law of Hess and Hiai just as in Theorem 2.3; then apply [**9**], Theorem 21(a), to get $(Ef^*)^* = Ef$, and conclude that $\overline{\operatorname{co}}\, F^* = \operatorname{epi}(Ef^*) = \operatorname{epi}(Ef)^*$. This shows that the $(E^\nu f)^*$ epi-converge to $(Ef)^*$ almost surely. The bi-continuity of the Fenchel transform ([**2**] Theorem 3.18) applies once again to complete the proof. $\square$

Remark. Condition (2.7) is only the most easily stated one of the many that could be invoked in order to obtain the conclusions of Lemma 2.2. See [**9**], Theorem 20, for more ideas.

3. Application to Stochastic Linear Recourse Problems.

In this section, we show how the conditions of the epi-consistency theorem may be satisfied in the important class of *stochastic linear programs with recourse*:

$$
\begin{aligned}
(3.1) \qquad & \text{minimize} \quad c'x + \int Q(\xi, x) P(d\xi) \quad \text{over all } x \in \mathbb{R}^n \\
& \text{subject to} \quad Ax = b, \\
& \qquad\qquad\quad x \geq 0,
\end{aligned}
$$

where the function $Q : \Xi \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is the minimum value in the *second stage linear program*

$$
(3.2) \qquad Q(\xi, x) = \inf\{q'y \,|\, Wy = Tx - h,\ y \in \mathbb{R}^m,\ y \geq 0\},
$$

and ξ is the vector consisting of the vectors and matrices in the second stage program, i.e. $\xi = (q, W, T, h)$. This class of problems models decisions that must take into account future costs $Q(\xi, x)$, represented as linear programs, responding to presently uncertain values of ξ; see, for example, [**11**]. As in the introduction, we suppose that (3.1) cannot be solved as stated, because either P is not known or must be made discrete. Instead, one solves the problems

$$
\begin{aligned}
(3.3) \qquad & \text{minimize} \quad c'x + \frac{1}{\nu} \sum_{i=1}^{\nu} Q(\xi_i, x) \quad \text{over all } x \in \mathbb{R}^n \\
& \text{subject to} \quad Ax = b \\
& \qquad\qquad\quad x \geq 0,
\end{aligned}
$$

where the ξ_i are independent random vectors distributed according to P.

We shall show that the problems (3.3) are epi-consistent with limit equal to (3.1), under fairly weak assumptions that are prevalent in the stochastic programming literature. The perspective afforded by the strong law in Theorem 2.3 leads to a better theorem than can be derived from the epi-consistency results in [**3**], [**6**] and [**7**]—which are intended to apply to more general perturbations of probability measures than considered here. In particular, we do not need restrictive assumptions concerning continuities of Q. All the above papers require the function $\xi \mapsto Q(\xi, x)$ to be continuous for every x, because the common theoretical approach is via the weak-convergence of probability measures. While this is an unnatural assumption, it is not necessarily a burdensome one. More seriously, they also all require continuity in x: [**3**] requires that for all ξ the function $x \mapsto Q(\xi, x)$ satisfies a lower Lipschitz continuity property on the entire feasibility region; [**6**] and [**7**] demand that for all ξ the function $x \mapsto Q(\xi, x)$ is continuous on an open set containing the minimizers of (3.1). In effect these assumptions all insist that (3.1) include constraints that force the feasible region to be *strictly* contained within the domain of finiteness of $Q(\xi, \cdot)$ for all ξ. The only way such a condition could hold in a practical situation is when the second stage linear program (3.2) is known to be feasible for any choice of x—this is called *complete recourse.* If not, one would then have to employ feasibility cuts to detect the domain of finiteness and try to make sure that the explicit constraints of (3.1) define a region in the *interior* of this domain. The theorem we state allows us to remove all such requirements on the second stage linear program.

A comprehensive review of the properties of Q may be found in Wets [**11**]. Denote by K_1 the set of x satisfying the constraints of (3.1), i.e.

$$K_1 = \{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}.$$

When the matrix W is *fixed,* i.e. nonrandom, the problem (3.1) is said to have *fixed recourse.* When the random vector ξ_1 satisfies the condition

(3.4) For all i, j, k the random variables $q_i h_j$ and $q_i T_{jk}$ have finite first moments,

then ξ_1 is said to posses the *weak covariance property.* This is obviously satisfied if ξ_1 is square integrable. Let us define the *essential integrand* as

$$f(\xi, x) = c'x + Q(\xi, x) + \delta_{K_1}(x)$$

where $\delta_{K_1}(x) = +\infty$ if x is not in K_1 and zero otherwise. Clearly, minimizing (3.1) is equivalent to minimizing Ef and minimizing (3.3) is equivalent to minimizing $E^\nu f$. Epi-consistency can therefore be proved by showing that f satisfies the conditions of Theorem 2.3.

Theorem 3.1. *Suppose that the stochastic linear program (3.1) has fixed recourse and that the random elements satisfy the weak covariance condition (3.4). If there exists a point $\bar{x} \in K_1$ with $EQ(\bar{x})$ finite, then the $E^\nu f$ are epi-consistent with limit Ef.*

Proof. The function Q is normal, since it is the infimum of a normal integrand, and convex ([**11**] Proposition 7.5). Hence f is a convex normal integrand ([**10**] Proposition 2M) and we are in the setting of Theorem 2.3. We need only establish (2.6). By [**11**] Proposition 7.12, since $\bar{x} \in \operatorname{dom} Q(\cdot, \xi)$ for P-almost all ξ, there exists a measurable selection $\bar{\pi}(\xi)$ from the solutions to the dual of (3.2) at $\bar{x}$, as ξ varies over the support of P, and moreover, $\bar{\pi}(\xi)'T$ is a subgradient of $Q(\cdot, \xi)$ at $\bar{x}$ for P-almost all ξ. Thus

$$\bar{u}(\xi) = \bar{\pi}(\xi)'T + \bar{a}$$

is a measurable selection of $\partial f(\bar{x}, \xi)$, where $\bar{a}$ equals c' plus a fixed element from the normal cone to the contraint set K_1 at $\bar{x}$. This $\bar{u}(\xi)$ is integrable by the weak covariance assumption and Theorems 7.7 and 7.15 of [**11**]. $\square$

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