**ETH-Zürich, Mini-Course, May 2012**

# Six Lectures on Stochastic Variational Analysis

*Roger J-B Wets*, Mathematics, University of California, Davis

"Stochastic Variational Analysis" emerged in response to the need of solving (generalized) equations systems, optimization and variational problems whose parameters are, in part, stochastic. Problems of this type arise in stochastic optimization, stochastic equilibrium problems, uncertainty quantification, statistical estimation problems that turn up in a broad variety of engineering, economics, finance, energy networks, signal processing, ecology and biological problems.

These lectures will be introductory in nature, and as much as time will allow, will concentrate on applications. Because the solutions to such systems aren't generally unique, one can't rely on classical probabilistic techniques to either describe their solutions or find (probabilistic) approximations that might, in turn, be based on standard laws of large numbers and associated asymptotic analysis.

The foundations of the theory lies in an understanding of the geometry and the analytic (topological) properties of random sets, including a suitable translation to a functional setting, coupled with both appropriate laws of large numbers, i.e., what can be learned from large samples, and fundamental inequalities, i.e., what can be learned from small samples. This dual approach is fundamental to the potential applications of the theory in practical settings.

Prerequisites: An open mind and a reasonable level of mathematical maturity.

Schedule: 2 lectures per week, starting ...
Time & Location: TBA

# Stochastic Variational Analysis

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$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
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$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample $P^\nu$ (random) empirical measure

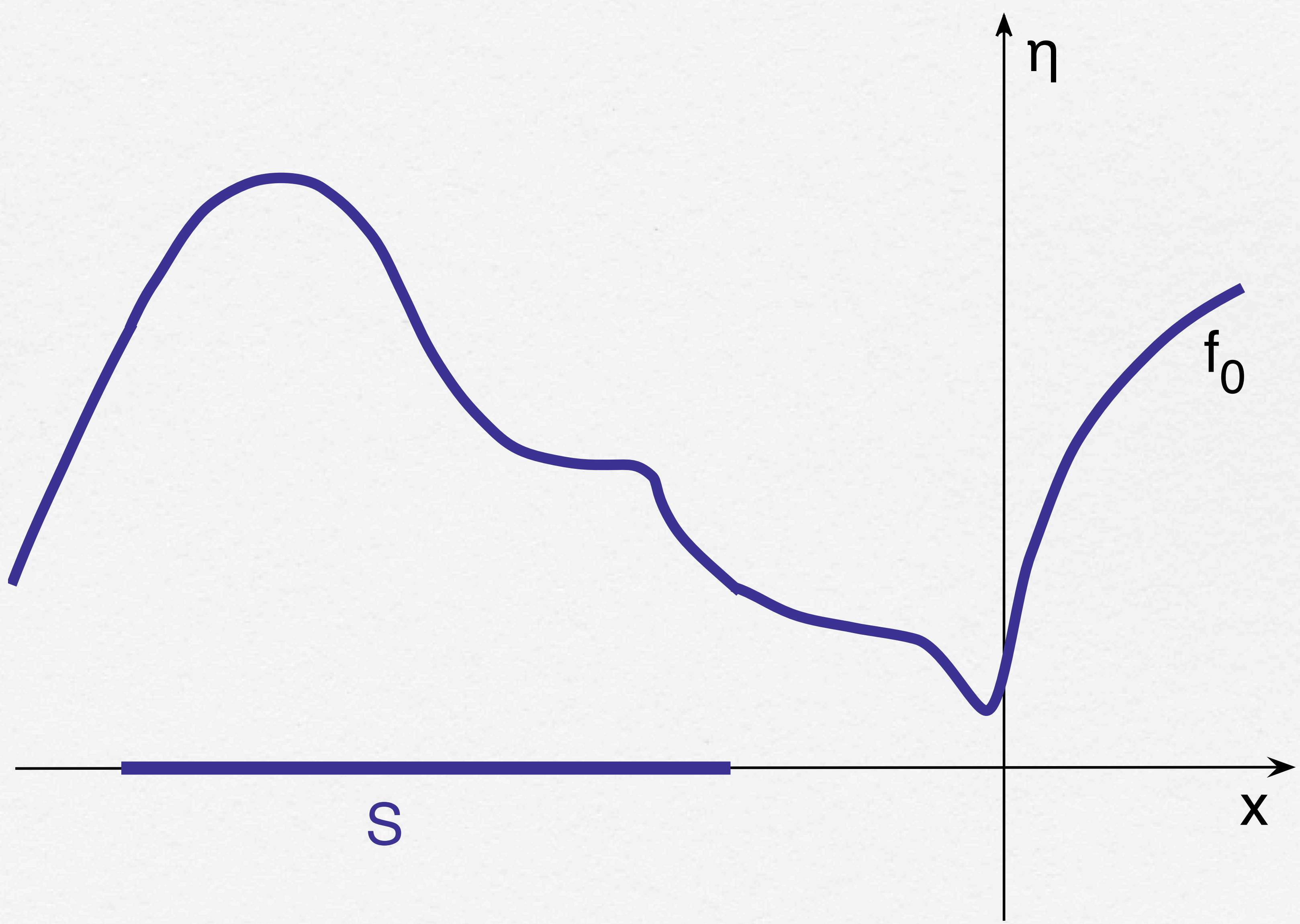
approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

# Preliminaries (unavoidable)

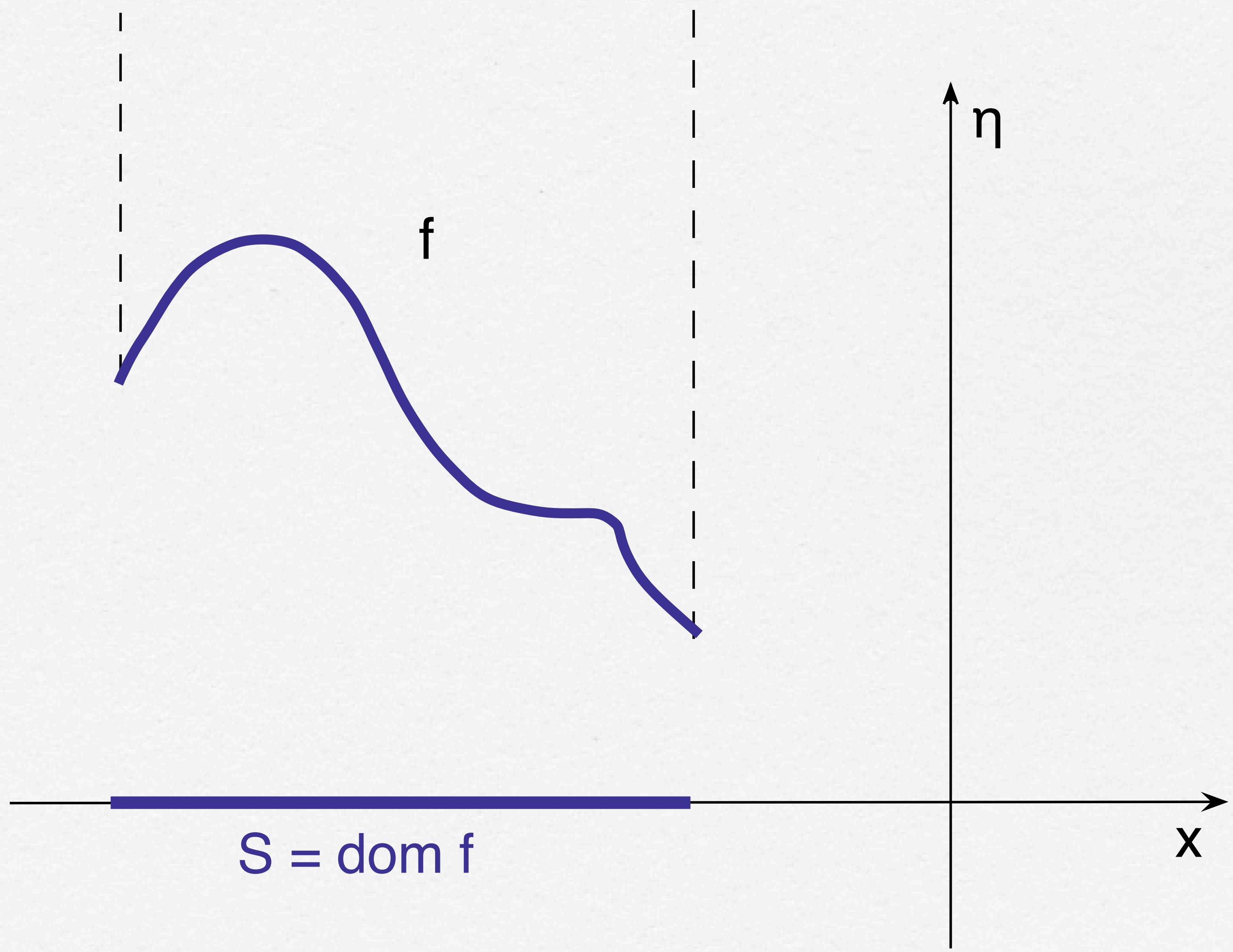
# Optimization problem

$$\min f_0(x),\ x \in S,$$

$$S = \left\{ x \in \mathbb{R}^n \middle| f_i(x) \le 0,\ i = 1 \to s,\ f_i(x) = 0,\ i = s+1 \to m \right\}$$



$$\min f \text{ on } E, \quad f = f_0 + \iota_S(x), \quad \iota_S \text{ indicator function of } S$$



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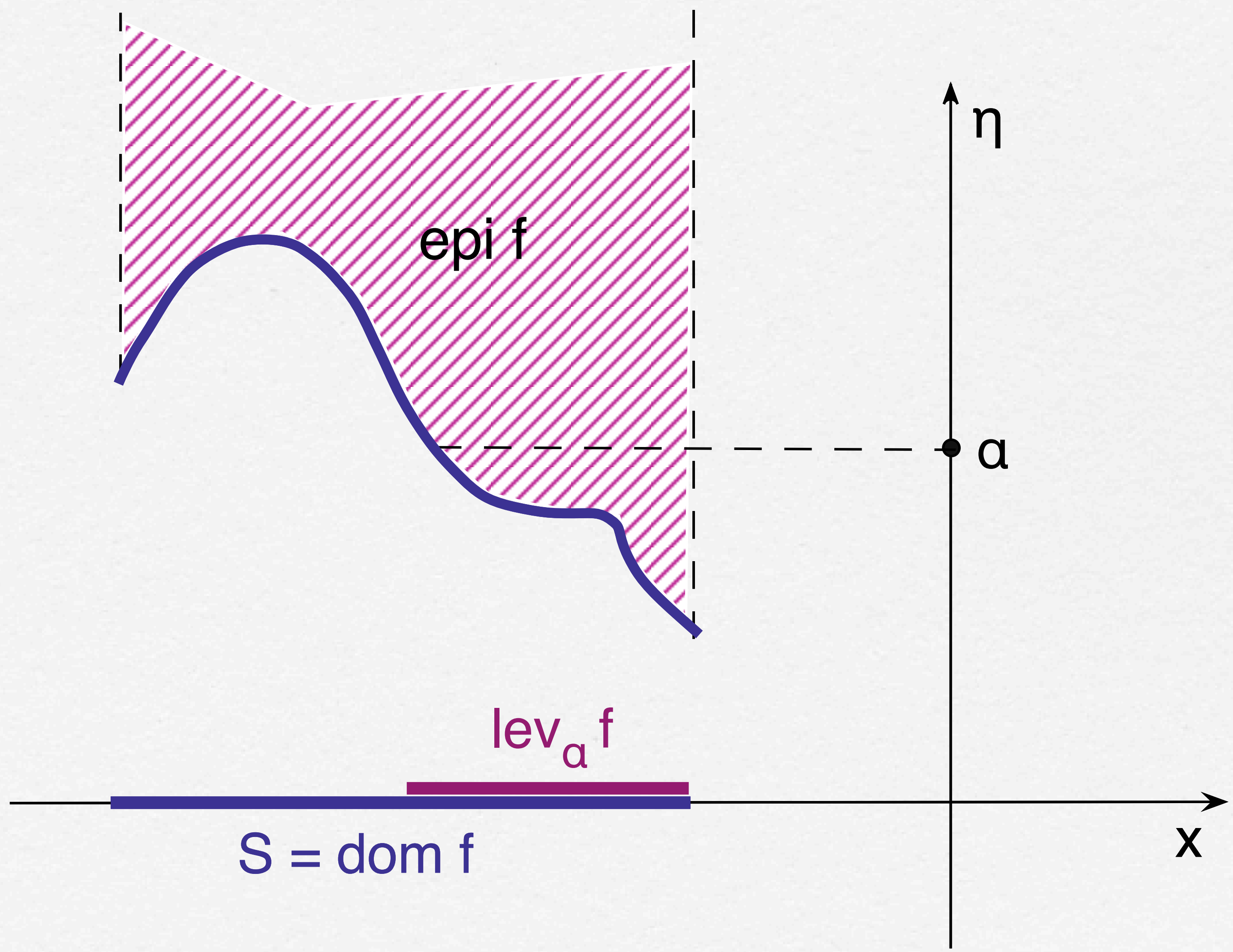
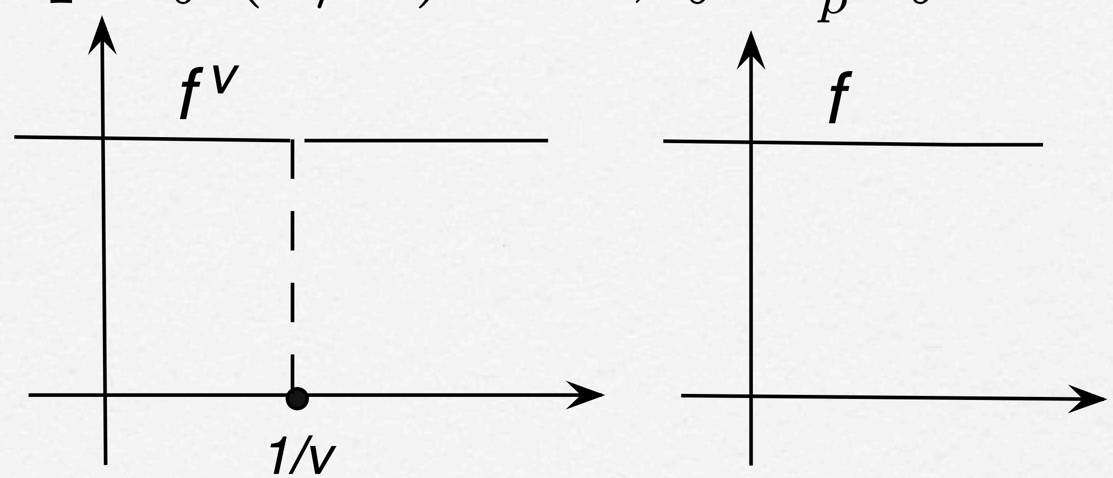
$$\operatorname{epi} f = \{(x,\alpha) \in E \times R \mid f(x) \le \alpha\}, \quad \operatorname{lev}_\alpha f = \{x \in E \mid f(x) \le \alpha\}$$



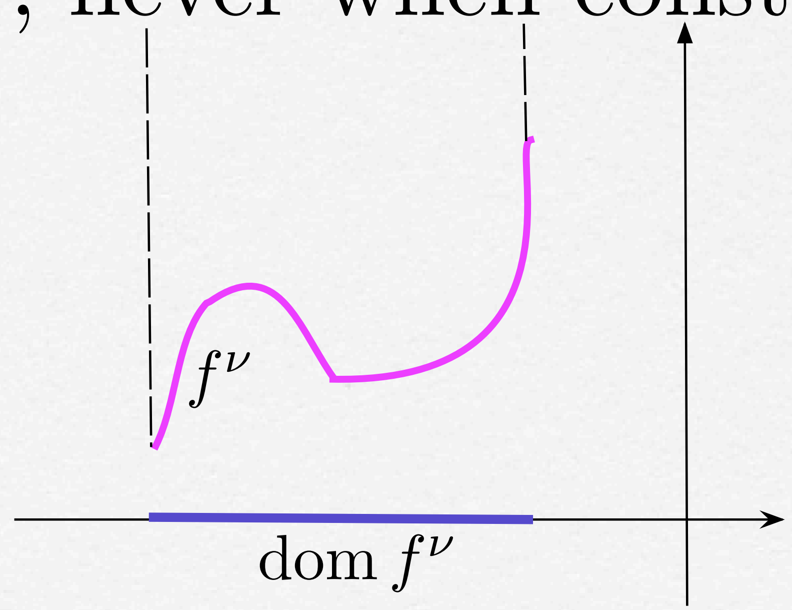
1. pointwise convergence $\not\Rightarrow$ convergence of minimizers

$f^\nu \equiv 1$ except $f(1/\nu) = 0$, $f^\nu \underset{p}{\to} f \equiv 1$




2. uniform convergence implies convergence of minimizers

but applies rarely, never when constraints depend on $\nu$



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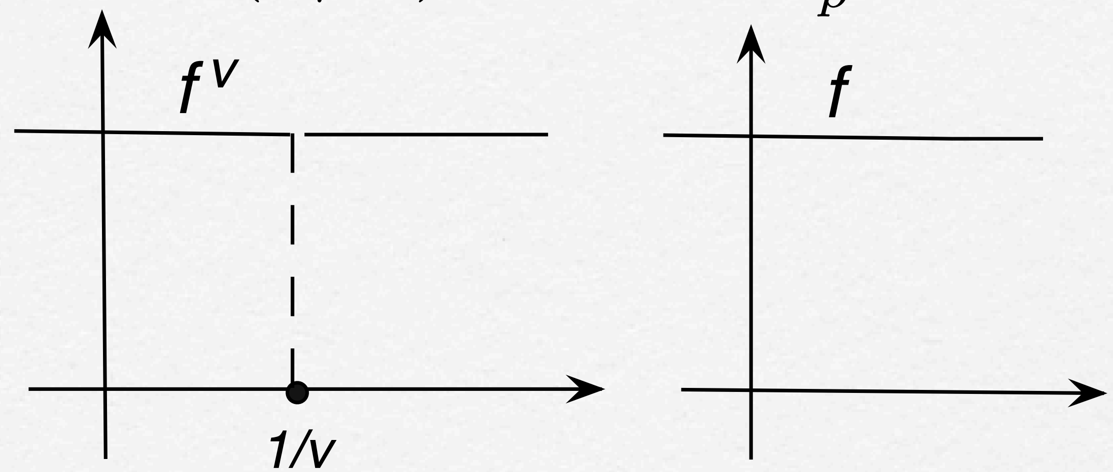


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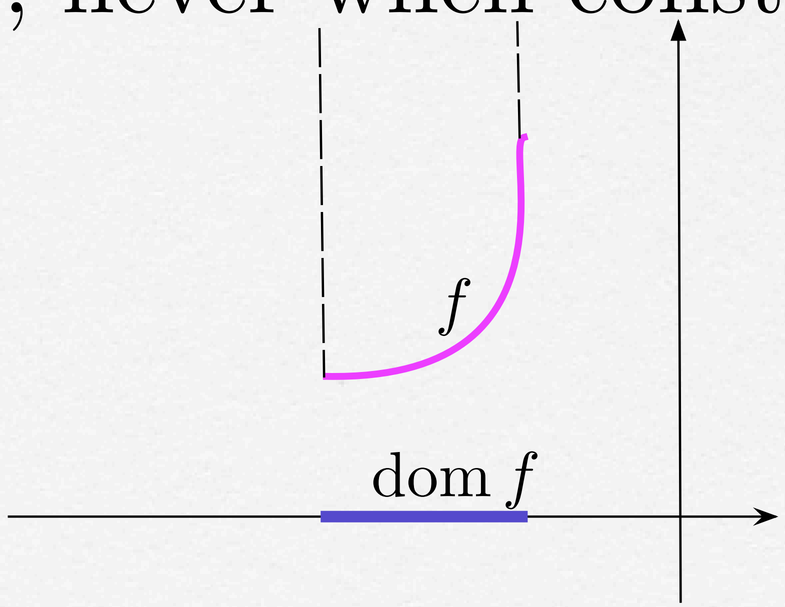

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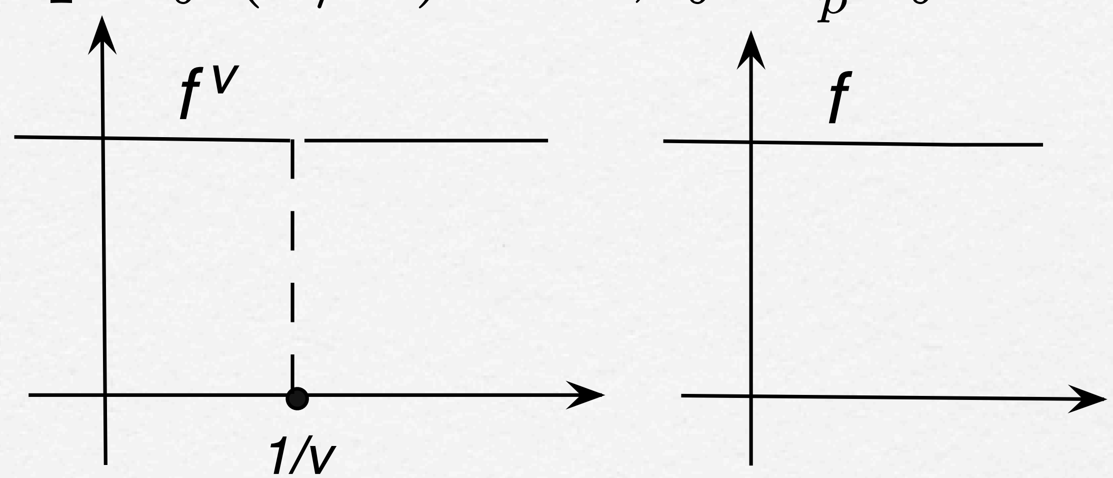


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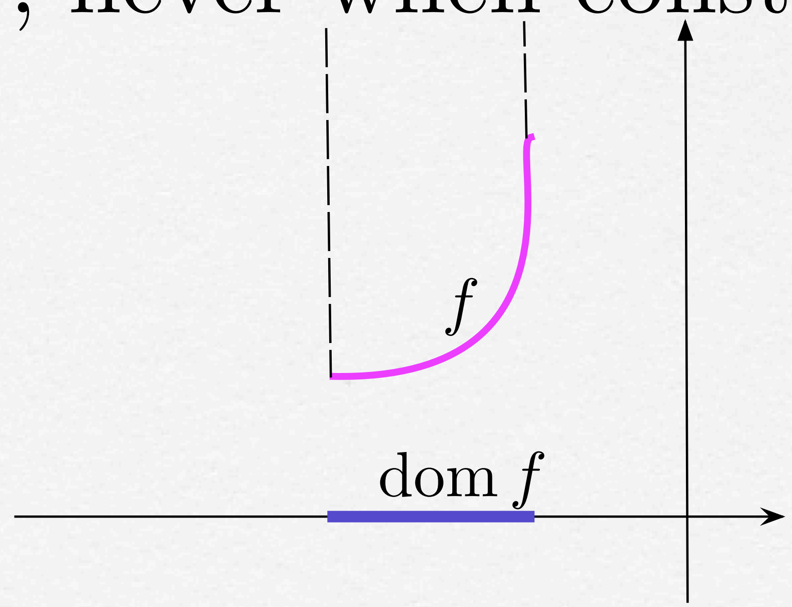


One-sided uniform convergence

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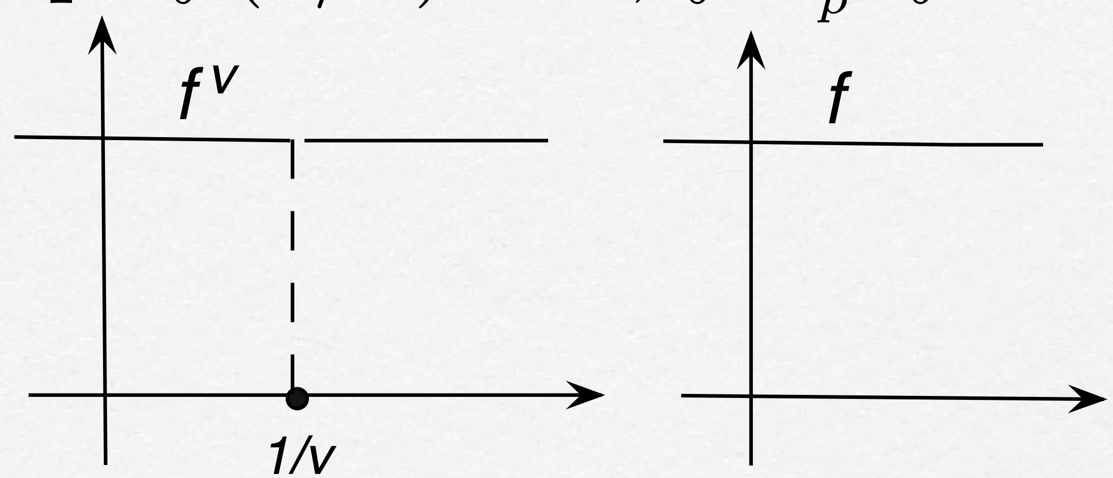


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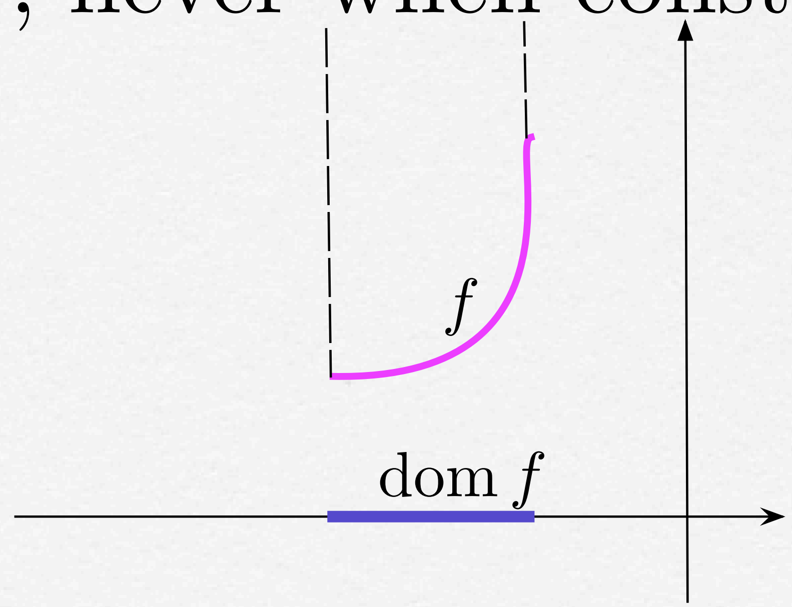


# Epi-Convergence

$f^\nu \xrightarrow{e} f$ if for all $x \in E$,

1. $\forall\, x^\nu \to x,\ \liminf_\nu f^\nu(x^\nu) \geq f(x)$

2. $\exists\, x^\nu \to x,\ \limsup_\nu f^\nu(x^\nu) \leq f(x)$

"Geometrically": $\text{epi}\, f^\nu \to \text{epi}\, f$ (later)

Pointwise:

$$\liminf_\nu f^\nu(x) \geq f(x), \qquad \limsup_\nu f^\nu(x) \leq f(x)$$

Continuous: $\forall x^\nu \to x$,

$$\liminf_\nu f^\nu(x^\nu) \geq f(x), \qquad \limsup_\nu f^\nu(x^\nu) \leq f(x)$$

# Epi-Convergence $\Rightarrow$

$A^\nu = \arg\min f^\nu$, $\varepsilon$-$A^\nu$ : $\varepsilon > 0$ approximate minimizers,

$A = \arg\min f$ of limit problem, $\varepsilon$-$A$ approx. minimizers

$A^\nu$ v-converges to $A$, written $A^\nu \Rightarrow_v A$, if

a) $\bar{x} \in$ cluster-points$\left\{x^\nu \in A^\nu\right\} \Rightarrow \bar{x} \in A$

b) $\bar{x} \in A \Rightarrow \exists\ \varepsilon_\nu \searrow 0, \mathrm{x}^\nu \in \varepsilon_\nu\text{-}A^\nu \to \bar{x}$

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$f^\nu \xrightarrow{e} f$ implies $\varepsilon$-$A^\nu \Rightarrow_v \varepsilon$-$A$, $\forall \varepsilon \geq 0$
$A$ unique minimizer, $\varepsilon^\nu$-$A^\nu \rightrightarrows A$ as $\varepsilon^\nu \searrow 0$.

($\inf f > -\infty$)

# Stochastic Optimization

1. **Stochastic Programming** (recourse model)

$$f(\xi, x) = \begin{cases} f_{01}(x) + Q(\xi, x) & \text{if } x \in C_1 \\ \infty & \text{otherwise} \end{cases}$$

$$Q(\xi, x) = \inf_y \left\{ f_{02}(\xi, y) \,\middle|\, y \in C_2(\xi, x) \right\}$$

$$\min Ef(x) = \mathbb{E}\{f(\xi, x)\},$$

SAA-problem: $\min f^\nu(\vec{\xi}^{\,\nu}, x) = \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x)$

2. **Statistical Estimation** (fusion of hard & soft information)

$$L(\xi, h) = \begin{cases} -\ln h(\xi) & \text{if } h \geq 0, \int h = 1, h \in A^{\text{soft}} \subset E \\ \infty & \text{otherwise} \end{cases}$$

$EL(h) = \mathbb{E}\{L(\xi, h)\}$, $h^{\text{true}} = \text{argmin}_E \, \mathbb{E}\{L(\boldsymbol{\xi}, h)\}$

estimate: $h^\nu \in \text{argmin}_E \, \mathbb{E}^\nu\{L(\boldsymbol{\xi}, h)\} = \frac{1}{\nu} \sum_{l=1}^{\nu} L(\xi^l, h)$

$A^{\text{soft}}$ : constraints on support, moments, shape, smoothness, ...

# Pricing financial instruments

3. A contingent claim: environment process: $\left\{\boldsymbol{\xi}^t \in \mathbb{R}^d\right\}_{t=0}^T$

history: $\overset{\rightarrow t}{\xi}$, $\xi = \xi^T$, price process: $S^t(\overset{\rightarrow t}{\xi}) \in \mathbb{R}^n$; numéraire (risk-free): $S_1^t \equiv 1$

claims: $\left\{G^t(\overset{\rightarrow t}{\xi})\right\}_{t=1}^T$; $i$-strategy: $\left\{X^t(\overset{\rightarrow t}{\xi})\right\}_{t=0}^T$; value @ $t$ : $\langle S^t(\overset{\rightarrow t}{\xi}), X^t(\overset{\rightarrow t}{\xi})\rangle$

*Instruments*: T-bonds, options, swaps, insurace contracts, mortgages, ...

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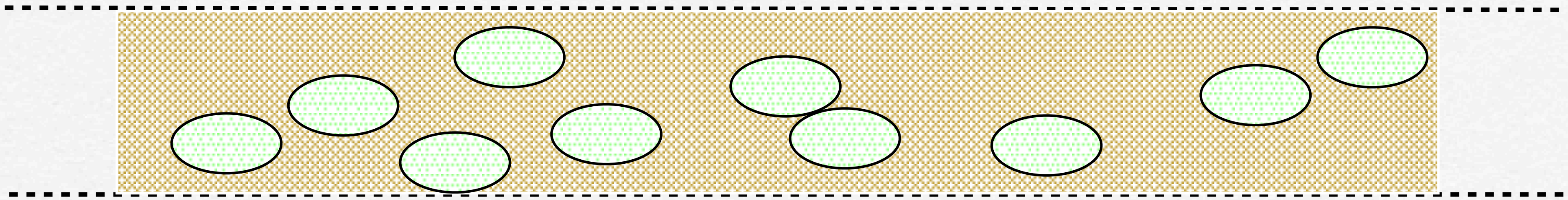
$$\max \mathbb{E}\left\{\langle S^T, X^T\rangle\right\} \text{ such that } \langle S^t, X^t\rangle \le G^t + \langle S^t, X^{t-1}\rangle,\ t = 1 \to T$$

$$\langle S^0, X^0\rangle \le G^0,\ \langle S^T, X^T\rangle \le G^T \text{ a.s.}$$

feasible if $G^0 + \cdots + G^T \ge 0\ \ \forall \xi$; arbitrage $\Rightarrow$ unbounded

prob$[\boldsymbol{\xi} = \xi] = p_\xi$ (finite sample?): $\max \sum_{\xi \in \Xi} p_\xi \langle S^T(\xi), X^T(\xi)\rangle \ldots$

# 4. Stochastic homogenization, ...



$$-\nabla\cdot\big(a(\boldsymbol{\xi},x)\nabla u(\boldsymbol{\xi},x)\big) = h(x) \text{ for } x\in\Omega, \quad u(\boldsymbol{\xi},x)=0 \text{ on bdry } \Omega$$

Variational formulation: $\forall\,\xi,\quad g(\xi,u) := \frac{1}{2}\int_\Omega a(\xi,x)|\nabla u|^2 dx - \langle h,u\rangle$

find $u(\xi,x)\in \operatorname{argmin}_{u\in H_0^1(\Omega)} g(\xi,u), \quad g(\xi,\cdot): L^2\to(-\infty,\infty]$. convex

$\mathbb{E}\{u(\boldsymbol{\xi},x)\}\in \operatorname{argmin}_{u\in H_0^1(\Omega)} G(u)$ where $\operatorname{epi} G = \mathbb{E}\{\operatorname{epi} g(\boldsymbol{\xi},\cdot)\}$

$G(u) = \inf_z \big\{\mathbb{E}\{g\big(\boldsymbol{\xi},z(\boldsymbol{\xi})\big)\,\big|\,\mathbb{E}\{z(\boldsymbol{\xi})\}=u\big\}$

$G^* = \mathbb{E}\{g^*(\boldsymbol{\xi},\cdot)\},\quad g^*(\xi,v)=\sup_u\big\{\langle v,u\rangle - g(\xi,u)\big\}$, conjugate fcn

$\xi^1,\xi^2,\ldots$ stationary, use Ergodic Theorem for random lsc functions

$$G = g^{\text{hom}} = \big(\operatorname{epi}_{\text{w}}\text{-}\lim_\nu \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} g^*(\boldsymbol{\xi}^l,\cdot)\big)^* \quad\Longrightarrow\quad \text{values of } a^{\text{hom}}(x)$$

# Expectation Functionals

# $Ef = \mathbb{E}\{f(\boldsymbol{\xi},\cdot)\}$

$f:\Xi\times E\to\overline{\mathbb{R}}$, random lsc function, $f(\xi,x)=f_0(\xi,x)$ when $x\in C(\xi)$

$E\subset\mathcal{M}(\Xi,\mathcal{A};\mathbb{R}^n):\mathcal{L}^p(\Xi,\mathcal{A},P;\mathbb{R}^n),\ldots$

others: $C\big((\Xi,\tau);\mathbb{R}^n\big)$,Orlicz, Sobolev, lsc-fcns($E$)

$$Ef(x)=\int_\Xi f(\xi,x(\xi))P(d\xi)=\mathbb{E}\{f(\boldsymbol{\xi},x(\boldsymbol{\xi}))\}$$

$$=\infty \text{ whenever } \int_\Xi f_+(\xi,x(\xi))P(d\xi)=\infty$$

$Ef:E\to\overline{\mathbb{R}}$ always defined

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Regression: ($E$ is not a linear space)

$$\min\left\{\int_{y\in\mathbb{R}}\int_{x\in[0,1]^n}\phi(y-h(x))P(dx,dy)\,\middle|\, h\in \text{ lsc-fcns}(\mathbb{R}^n)\cap\mathcal{H}\right\}$$

$\mathcal{H}$ shape restrictions (convex, unimodal, ...)

# Random lsc functions

$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, $\xi$ values in $(\Xi, \mathcal{A}, P)$

(a) lsc (lower semicontinuous) in $x$, $(\forall \xi \in \Xi)$

(b) $(\xi, x)$-measurable $\quad (\mathcal{A} \times B_E)$-measurable

recall: $f(\xi, x) = f_0(\xi, x)$ when $x \in C(\xi)$ -- stochastic constraints

$$f^\nu(\xi, x) = \begin{cases} \frac{1}{\nu} \sum_{l=1}^{\nu} \big( f(\xi^l, x) \text{ if } x \in C(\xi^l) \big) & \text{(typically)} \\ \infty \text{ otherwise} & (\sim \text{SAA of optimisation problems}) \end{cases}$$

Question: Do the $f^\nu(\xi, \cdot)$ epi-converge to $\mathbb{E}\{f(\xi, h)\}$ $P$-a.s.?

does $x^\nu \in \arg\min f^\nu \Rightarrow_\nu x^* \in \arg\min \mathbb{E}\{f(\xi, x)\}$ $P$-a.s.?

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Law of Large Numbers for random lsc functions

$\sim$ LLN for Stochastic Optimization Problems.

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$$E^\nu f \xrightarrow{e} Ef \ a.s., \quad E^\nu f(x) = \frac{1}{\nu} \sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, x)$$

# Random lsc functions

(via inf-projections)

$D$ countable dense subset of $E$

$f : E \to \overline{\mathbb{R}}$, lsc fcn completely identified by

$\left\{ o_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$, countable

or $\left\{ c_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}$

$$f(\bar{x}) = \sup_{V \in \mathcal{N}(\bar{x})} \left[ \inf_{x \in V} f(x) \right], \quad f \text{ lsc}, \; f(\bar{x}) = \liminf_{x \to \bar{x}} f(x)$$

$$= \sup_{V \in \mathcal{Q}(\bar{x})} \left[ \inf_{x \in V} f(x) \right], \quad E \text{ separable} \quad \text{(Polish)}$$

$$\mathcal{Q}(\bar{x}) = \left\{ \mathbb{B}^o(x,\delta) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+, \bar{x} \in \mathbb{B}^o(x,\delta) \right\}$$

$$= \sup_{\delta \in \mathbb{Q}_+} \inf_{\{x \,|\, \mathbb{B}^o(x\delta) \in \mathcal{Q}(\bar{x})\}} o_{x,\delta}$$

$\{c_{x,\delta}\}$ same argument

# Epi-convergence

(via inf-projections)

$$f^\nu : \mathbb{R}^n \to \overline{\mathbb{R}},\ f^\nu \xrightarrow{e} f,\ f \text{ lsc},\ \iff \forall \delta \in \mathbb{Q}_+,\ x \in D$$

$$\text{limsup}_\nu\, c^\nu_{x\delta} \le c_{x\delta}, \quad \text{liminf}_\nu\, o^\nu_{x\delta} \ge o_{x\delta}$$

for $x \in D, \delta \in \mathbb{Q}_+$: $\;o^\nu_{x\delta} = \inf_{\mathbb{B}^o(x,\delta)} f^\nu,\;\; c^\nu_{x\delta} = \inf_{\mathbb{B}(x,\delta)} f^\nu$

(fundamental) **Theorem.** $f^\nu : E \to \overline{\mathbb{R}}$ & $f$ lsc (necessarily)

1. $\text{e-}\liminf_\nu f^\nu \iff \liminf_\nu (\inf_B f^\nu) \ge \inf_B f$ for all compact $B$
2. $\text{e-}\limsup_\nu f^\nu \iff \limsup_\nu (\inf_O f^\nu) \le \inf_O f$ for all open $O$

□ Hit-and-miss topology on the space of epigraphs, (later?). □

# Scalarization of random lsc fcns

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$$\left\{ o_{x\delta}(\xi) = \inf_{\mathbb{B}^o(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}, \text{ countable}$$

$$\text{or } \left\{ c_{x\delta}(\xi) = \inf_{\mathbb{B}(x,\delta)} f(\xi,\cdot) \,\middle|\, x \in D, \delta \in \mathbb{Q}_+ \right\}, \quad \forall \xi \in \Xi$$

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COUNTABLE

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COUNTABLE

$\forall\, x \in \mathbb{R}^n, \delta > 0$

$\xi \mapsto o_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,

$o_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable

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$\xi \mapsto c_{x\delta} : \Xi \to \overline{\mathbb{R}}$ are measurable,

$c_{x\delta}(\boldsymbol{\xi})$ extended real-valued random variable.

- $f$ random lsc fcn $\Rightarrow f + \iota_{\mathbb{B}(x,\delta)}$ random lsc fcn
- $f$ random lsc fcn $\Rightarrow \xi \mapsto \alpha(\xi) = \inf_x f(x, \xi)$ measurable

# Probabilistic properties

$f$ random lsc fcn: $\{f(\boldsymbol{\xi}^\nu,\cdot)\}_{\nu\in\mathbb{N}}$ iid whenever $\{\boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid

$$\text{Effös field on lsc-fcns}(E) = \sigma\text{-}\big\{f\in\text{lsc-fcns}(E)\,\big|\,\inf_O<\alpha\big\},\ \ O \text{ open}, \alpha\in\mathbb{R}$$
$$=\mathcal{B}\big(\text{lsc-fcns}(E)\big),\ E \text{ Polish}$$

1. $\{f(\boldsymbol{\xi}^\nu,\cdot)\}_{\nu\in\mathbb{N}}$ "i" $\iff$ $\{o_{x\delta}(\boldsymbol{\xi}^\nu),\,\nu\in\mathbb{N}\}$ "i", $\forall x\in\mathbb{Q}^n,\delta\in\mathbb{Q}_+$
2. $f(\boldsymbol{\xi}^1,\cdot),f(\boldsymbol{\xi}^2,\cdot)$ "id" $\iff$ $o_{x\delta}(\boldsymbol{\xi}^1),o_{x\delta}(\boldsymbol{\xi}^2)$ "id", $\forall x\in\mathbb{Q}^n,\delta\in\mathbb{Q}_+$

the same holds for $\{c_{x\delta}(\cdot)\}$

# Summary

$f : \Xi \times E \to \overline{\mathbb{R}}$ random lsc fcn,
$\boldsymbol{\xi}, \{\boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid
$\Rightarrow f(\boldsymbol{\xi}, \cdot), \{f(\boldsymbol{\xi}^\nu, \cdot)\}_{\nu\in\mathbb{N}}$ iid

$\Rightarrow \{o_{x\delta}(\boldsymbol{\xi}^\nu) : \Xi \to \overline{\mathbb{R}}\}_{\nu\in\mathbb{N}}$, iid,
countable, identify $f(\boldsymbol{\xi}^\nu \cdot)$

$\Rightarrow \{c_{x\delta}(\boldsymbol{\xi}^\nu) : \Xi \to \overline{\mathbb{R}}\}_{\nu\in\mathbb{N}}$, iid,
countable, identify $f(\boldsymbol{\xi}^\nu \cdot)$

# Countable ⇒ a.s.

**Lemma.** $f, g : E \to \overline{\mathbb{R}}$, lsc. $D = \mathrm{prj}_E$ countable dense subset of $\mathrm{epi}\, f$.

$$f \le g \text{ on } D \implies f \le g \text{ on } E.$$

**Proof.** $f \le g$ on $D$ only if $\{(x,\alpha) \mid \alpha \ge g(x), x \in R\} \subset \mathrm{epi}\, f$.

Taking closure on both sides $\implies$ $\mathrm{epi}\, g \subset \mathrm{epi}\, f$. $\square$

**Implication**. To check $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ *a.s.* on $E$ only needs $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ *a.s.* on $D$ a countable dense subset of $E$.
Restrict $\boldsymbol{\xi}$ to a set of $P$-measure 1, say $\Xi$ itself (from now on), and $f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on $D \implies f(\boldsymbol{\xi}, \cdot) \le g(\boldsymbol{\xi}, \cdot)$ on $E$.

# LLN: random lsc functions?

$\forall\, x \in D,\ \delta \in \mathbb{Q}_+$

1. $\frac{1}{\nu} \sum_{l=1}^{\nu} o_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{o_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$
2. $\frac{1}{\nu} \sum_{l=1}^{\nu} c_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{c_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$

# LLN: random lsc functions?

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2. $\frac{1}{\nu}\sum_{l=1}^{\nu} c_{x\delta}(\boldsymbol{\xi}^l) \to \mathbb{E}\{c_{x\delta}(\boldsymbol{\xi})\},\ (P^\infty\text{-}a.s.)$

$\not\Longrightarrow \sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l,\cdot\,) \xrightarrow{e} \mathbb{E}\{f(\boldsymbol{\xi},\cdot\,)$ because

$$\min\left\{\mathbb{E}\{f(\boldsymbol{\xi},z)\}\,\middle|\, z \in \mathbb{B}(x,\delta)\right\} \neq \mathbb{E}\left\{\min\{f(\boldsymbol{\xi},z)\}\,\middle|\, z\in\mathbb{B}(x,\delta)\right\}$$

in general

# Law of Large Numbers: Random lsc functions

# LLN: Proof

1. $\exists\, x^\nu \to x : \limsup_\nu E^\nu f \le Ef$
   for any $x \in E$ and any sample $\xi^\infty = (\xi^1, \xi^2, \ldots)$
   $\lim_\nu \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x) \;\sim \lim_\nu \mathbb{E}^\nu\{f(\boldsymbol{\xi}^\infty, x)\} = Ef(x).$

2. $\forall x^\nu \to x, \liminf_\nu E^\nu f \ge Ef$
   for any $x \in E$ and any $\xi^\infty = (\xi^1, \xi^2, \ldots) \in \Xi^\infty$

$$\text{e-}\liminf_{\nu\to\infty} f^\nu(\xi^\infty, x) = \sup_{\delta \searrow 0} \liminf_{\nu\to\infty} \inf_{\mathbb{B}^o(x,\delta)} E^\nu f \ge \sup_{\delta^l \searrow 0} \liminf_{\nu\to\infty} \frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l \delta^l}(\xi^l)$$

where $x^l \in D \to x$, $\delta^l \in \mathbb{Q}_+ \searrow 0$: $\quad x \in \mathbb{B}^o(x^l, \delta^l)$ & $\{\mathbb{B}^o(x^l, \delta^l)\} \searrow$
$\frac{1}{\nu} \sum_{l=1}^{\nu} o^l_{x^l \delta^l}(\xi^l) \to \mathbb{E}\{o^l_{x^l \delta^l}(\boldsymbol{\xi})\}$ & $\mathbb{E}\{o^l_{x^l \delta^l}(\boldsymbol{\xi})\} \nearrow Ef(x)$

$$\Longrightarrow \quad \text{e-}\liminf_{\nu\to\infty} E^\nu f(x) \ge Ef(x) \quad \Box$$

# Theorem

$f : \Xi \times E \to \overline{\mathbb{R}}$, locally inf-integrable random lsc function
$\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \ldots, \}$ are iid $\Xi$-valued random variables. Then,

$$E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot\,) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot\,) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$$

which means $\varepsilon$-argmin $E^\nu f \rightrightarrows_v \varepsilon$-argmin $Ef$, $\forall\, \varepsilon \geq 0$

$Ef$ unique minimizer, $\varepsilon^\nu$-argmin $E^\nu f \rightrightarrows$ argmin $Ef$ as $\varepsilon^\nu \searrow 0$.

SAA-applies without ‘any’ restrictions

loc.inf-integrable: $\int \inf\{f(\xi, \cdot) \,\big|\, \mathbb{B}(x,\delta)\} > \infty$ for some $\delta > 0$,
irrelevant in applications

# Ergodic Theorem

$(E, d)$ Polish, $(\Xi, \mathcal{A}, P)$ & $\mathcal{A}$ $P$-complete
$f : \Xi \times E \to \overline{\mathbb{R}}$ a random lsc function, locally inf-integrable
$\varphi : \Xi \to \Xi$ ergodic measure preserving transformation. Then,

$$\frac{1}{\nu} \sum_{l=1}^{\nu} f(\varphi^l(\boldsymbol{\xi}, \cdot\,) \xrightarrow{e} Ef \quad a.s.$$

allows for stationary rather than iid.

Application: "samples" coming from dynamic systems,
time series, SDE, etc.

# Random Sets

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Partial support: Army Research Office W911NF-10-1-0246

May 2012, ETH-Zürich

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \dots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

$G : \Xi \times D \rightrightarrows E$, set-valued $G(\xi, x) \subset E$, inclusion $\mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$

$\xi^1, \dots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \dots \xi^\nu$ sample $P^\nu$ (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

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$f$ on $\Xi \times E$, random lsc fcn (loc. inf-$\int$), $\{\boldsymbol{\xi}, \boldsymbol{\xi}^1, \dots, \}$ iid

Then $E^\nu f = \mathbb{E}^\nu\{f(\boldsymbol{\xi}, \cdot) = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l, \cdot) \xrightarrow{e} Ef = \mathbb{E}\{f(\boldsymbol{\xi}, \cdot\}$

$\varepsilon$-argmin $E^\nu f \Rightarrow_v \varepsilon$-argmin $Ef$, $\forall\, \varepsilon \geq 0$

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**Stochastic Programming** (with recourse)

$f(\xi, x) = f_{01}(x) + Q(\xi, x), \; Q(\xi, x) = \inf_y \left\{ f_{02}(\xi, y) \,\middle|\, y \in C_2(\xi, x) \right\}$

SAA-problem: $\min \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x) \xrightarrow{e} Ef(x) = \mathbb{E}\{f(\boldsymbol{\xi}, x)\}$

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

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$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

**Statistical Estimation** (fusion of hard & soft information)

$L(\xi, h) = -\ln h(\xi)$ if $h \geq 0, \int h = 1, h \in A^{\text{soft}} \subset E$

Then, estimate $h^\nu \in \operatorname{argmin}_E \mathbb{E}^\nu\{L(\boldsymbol{\xi}, h)\} \to h^{\text{true}} = \operatorname{argmin} \mathbb{E}\{L(\boldsymbol{\xi}, h)\}$

# example: Normal density

mean = (0,0) ... data samples correlated

covariance: $MDM^T$, $D=$ diag(4,1), $M=\begin{pmatrix} \cos(\pi/6) & \cos(2\pi/3) \\ \sin(\pi/6) & \sin(2\pi/3) \end{pmatrix}$

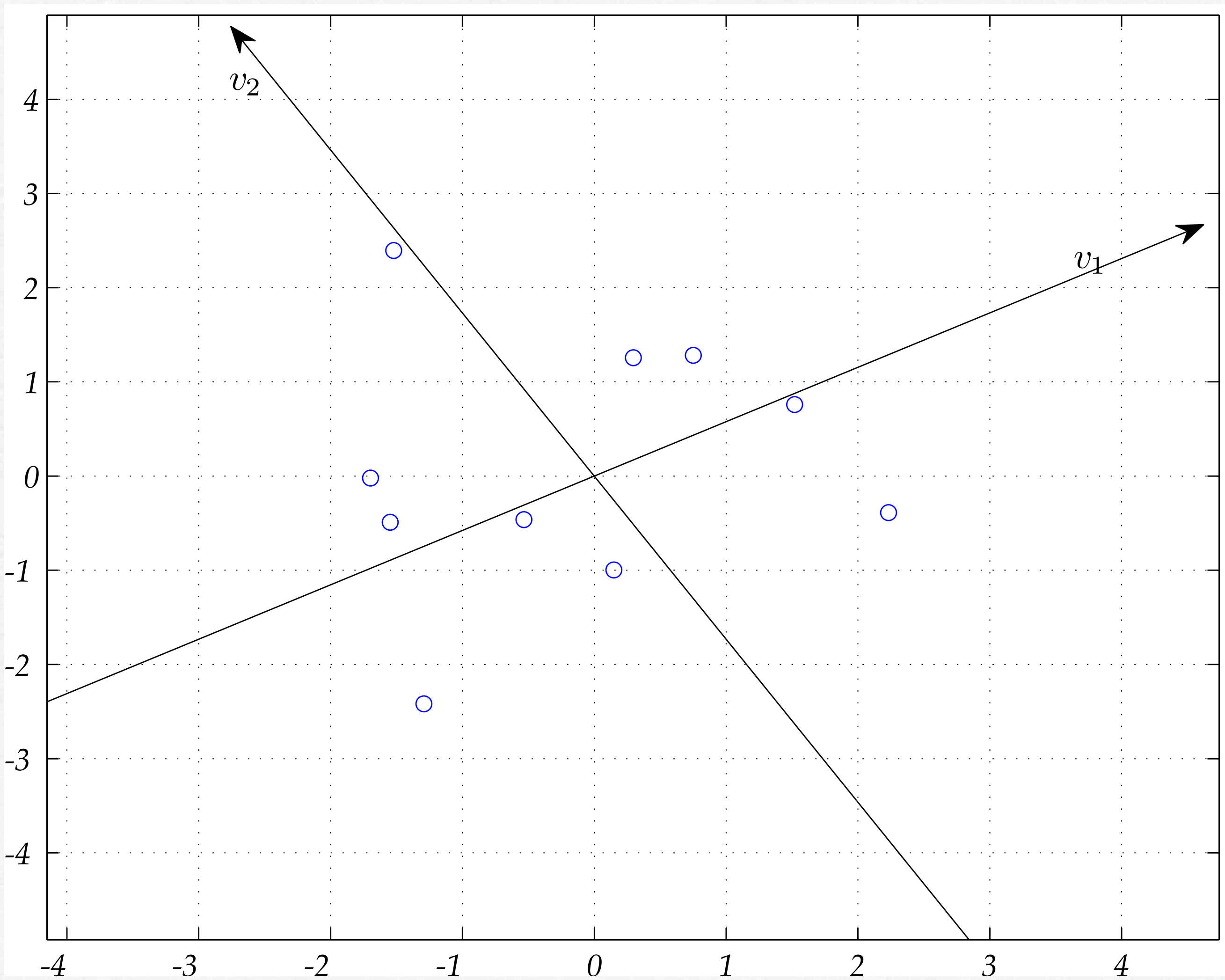
# samples: $\nu = 10$,

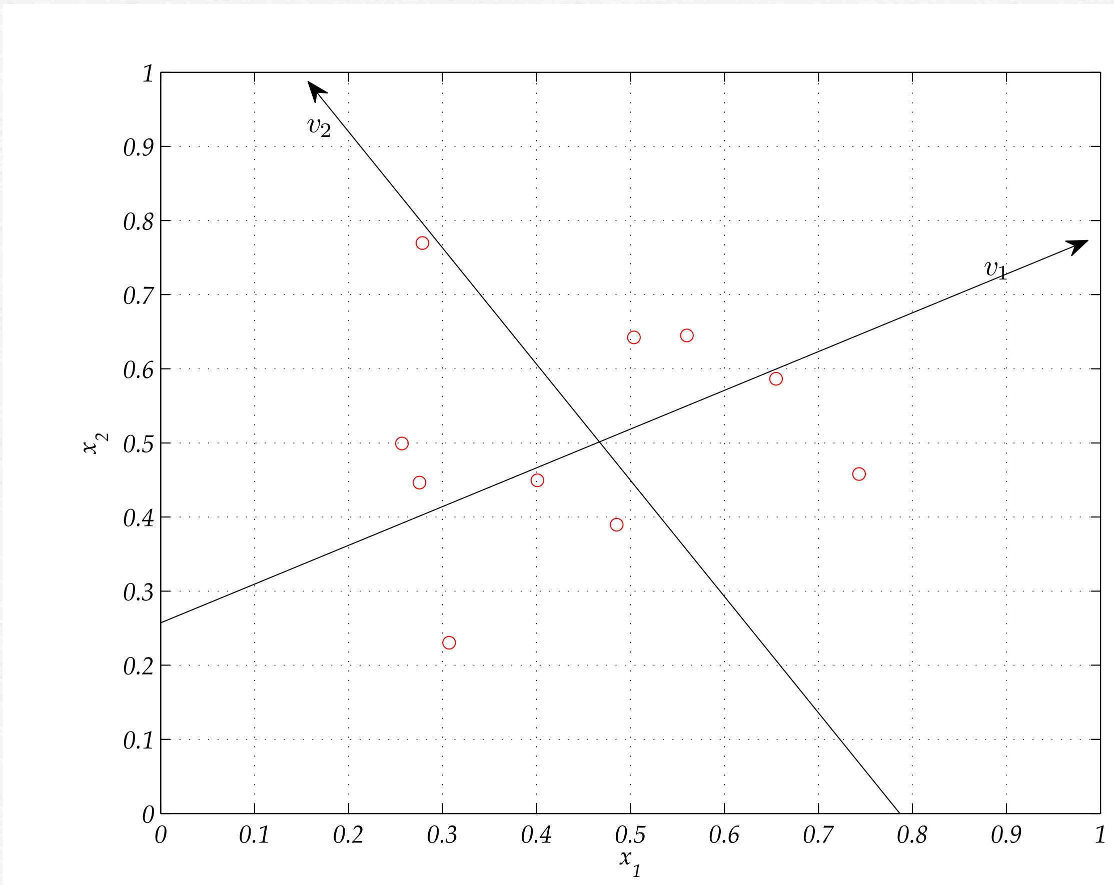
"soft" information: $h$ unimodal

Results:

$$\left\|h^{true} - h^{est}\right\|_2^2 = 0.028, \quad \left\|h^{true} - h^{est}\right\|_\infty = 0.006$$

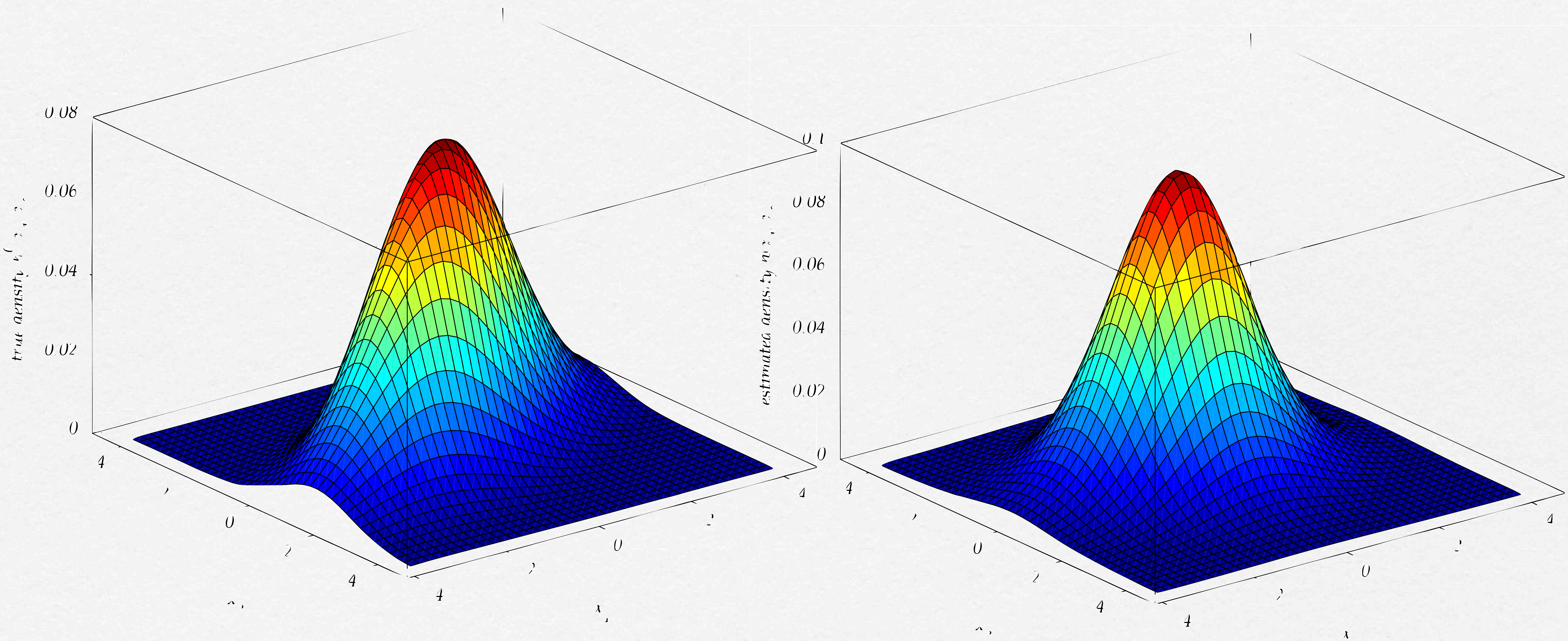
# Sampled data



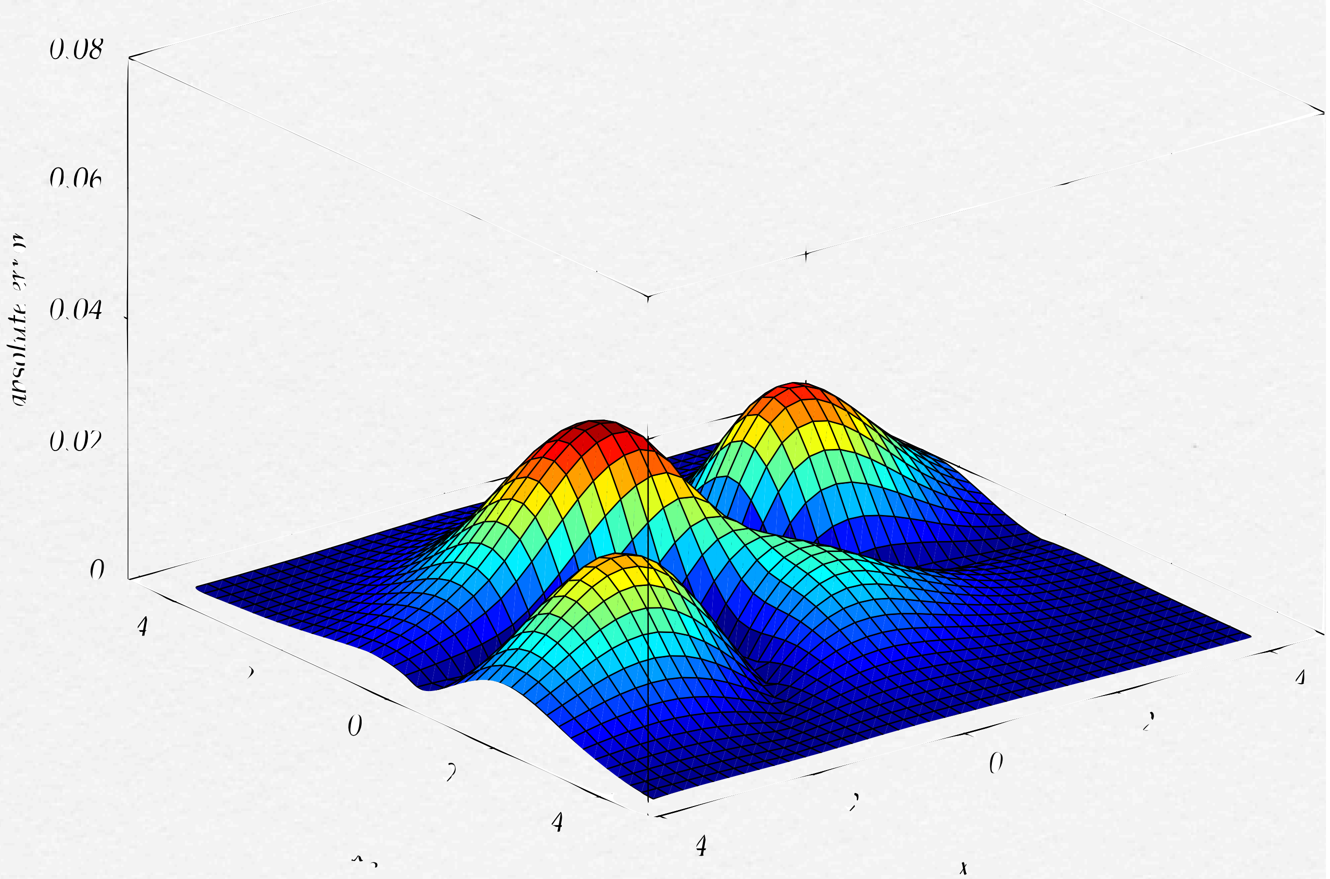



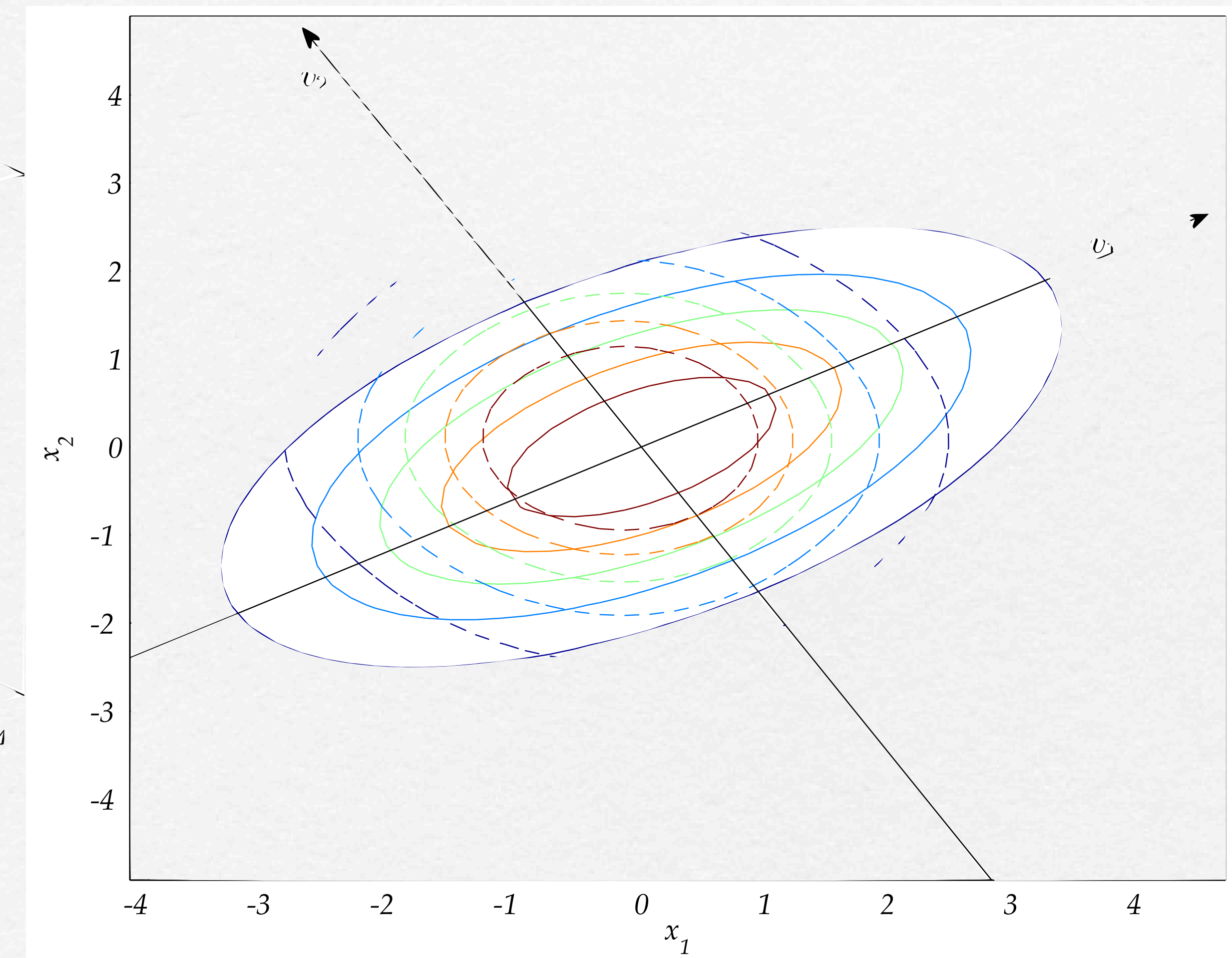

*normalized*

# True & Estimated density



# Measurement Errors




Absolute Error

*Level curves: true & estimate*

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$$\xi^1, \dots, \xi^\nu \text{ sample}, \;\; G^\nu(x) = \tfrac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$$

$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
$$\xi^1, \dots, \xi^\nu \;\; \text{sample, approximation } \tfrac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$$

**Pricing contingent claims**

claims $\left\{G^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}$, instrum. prices $\left\{S^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}_t$, invest. $\left\{X^t(\vec{\boldsymbol{\xi}}^{\,t})\right\}$

$\max \mathbb{E}\{\langle S^T, X^T\rangle\}$ s.t. $\langle S^t, X^{t-1}\rangle \le G^t + \langle S^t, X^{t-1}\rangle$ + end conditions.

Use 'improved estimation' & sampling: $\max \sum p_\xi \langle S^T(\xi), X^T(\xi)\rangle$

Correct pricing = well regulated market??

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

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$\xi^1, \ldots, \xi^\nu$ sample, approximation $\frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$

**Stochastic homogenization**: Variational formulation

given $u(\xi, x) \in \operatorname{argmin}_{H_0^1(\Omega)} g(\xi, u) = \frac{1}{2}\int_\Omega a(\xi, x)|\nabla u|^2\, dx - \langle h, u\rangle$

find $g^{\text{hom}}$ such that $\mathbb{E}\{u(\boldsymbol{\xi}, \cdot)\} \in \operatorname{argmin} g^{\text{hom}}$

via Ergodic Thm: $g^{\text{hom}} = \Big(\text{epi}_w\text{ - lim})\nu\frac{1}{\nu}\sum l = 1^\nu g^*(\boldsymbol{\xi}^l, \cdot)\Big)^*$

Wednesday, September 5, 2012

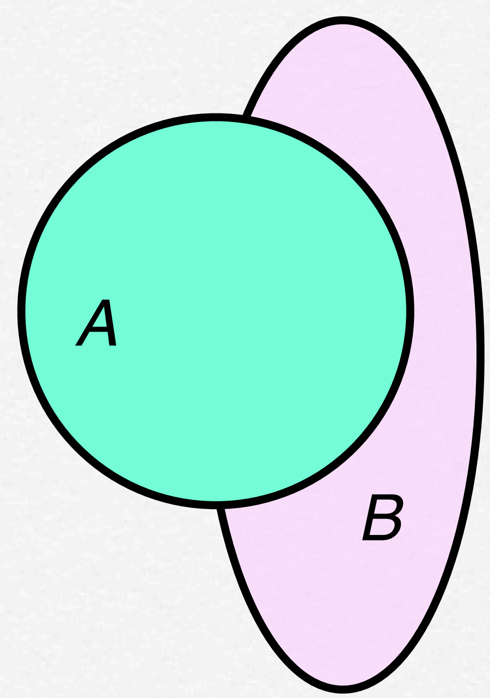
# Topology of Hyperspaces

Painlevé, Pompeiu, Zoretti
Zarankiewicz, Hausdorff, Lubben, Moore
Choquet, Vietoris, Fell, Attouch-Wets, Beer, ...

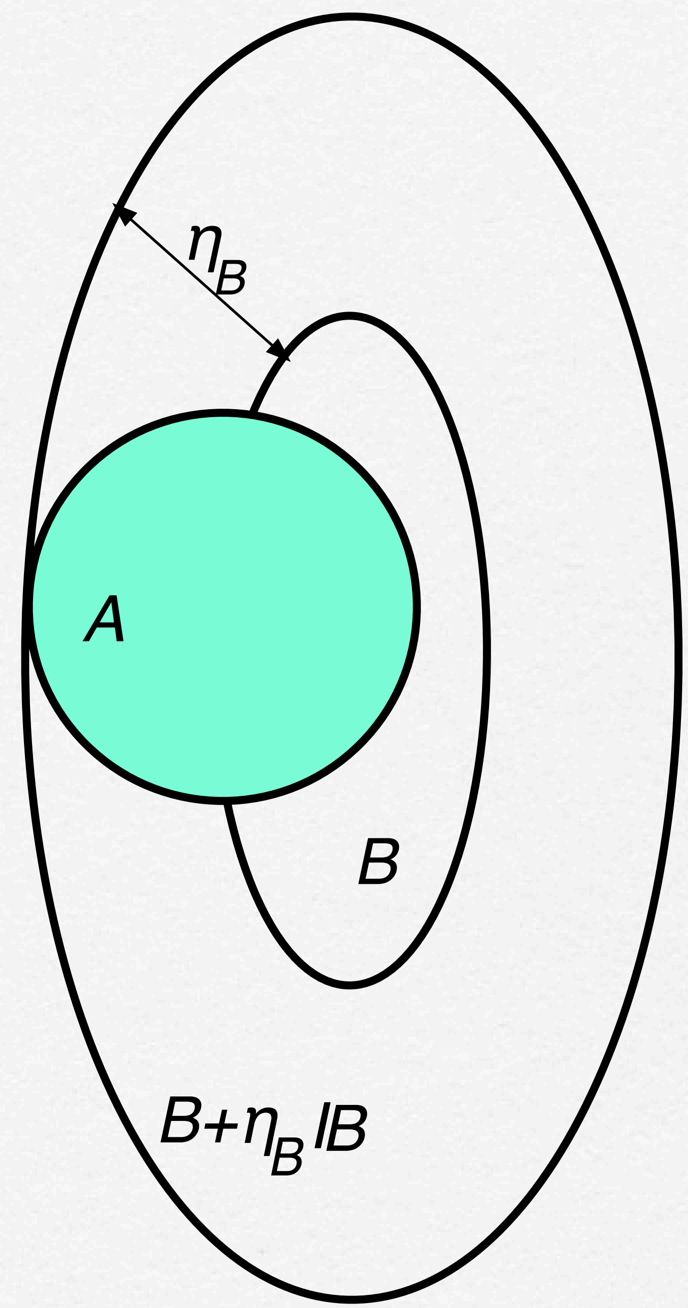
# Hyperspace: *sets(E)*

- $(E,d)$ always a Polish space
- $C \subset E,\ d(x,C) = \inf\{d(z,x) \mid z \in C\},\quad d(x,\emptyset) = \infty$
- $\text{cl-sets}(E) = \{\text{all closed subsets of } E\},\quad \emptyset, E \in \text{cl-sets}(E)$
- $dl(A,B) =$ distance between $A$ & $B$, metric(?) on cl-sets($E$)
- $(\text{cl-sets}(E), dl)$ Polish space = complete separable metric ??
- $dl(C^\nu, C) \to 0$ means $C^\nu \to C$ (set-convergence)

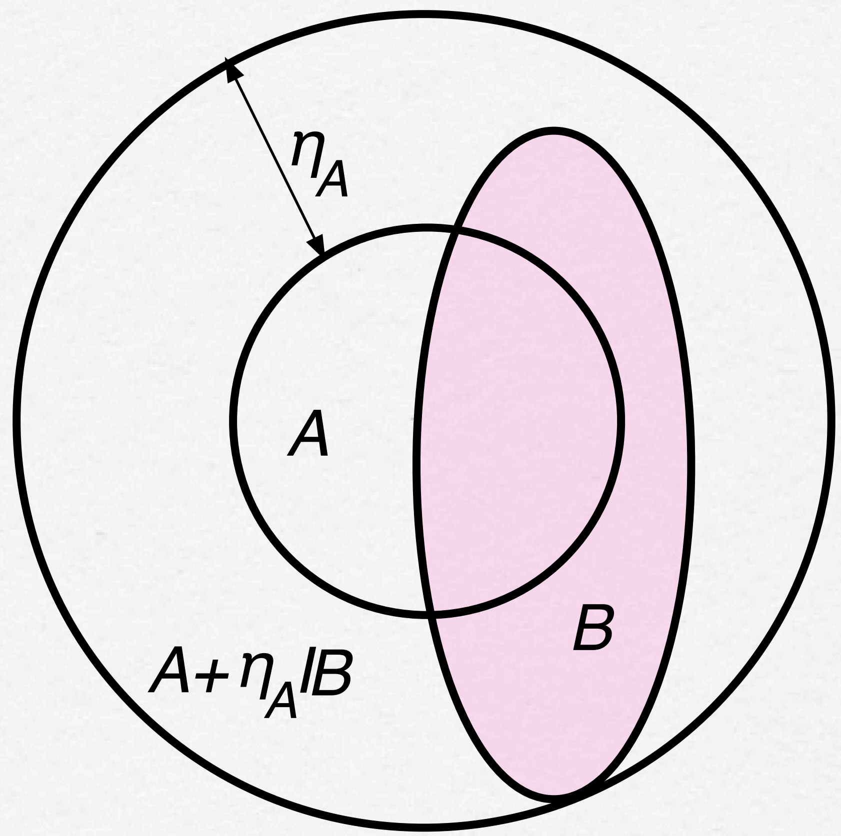
# *Pompeiu-Hausdorff* distance



# *Pompeiu-Hausdorff* distance

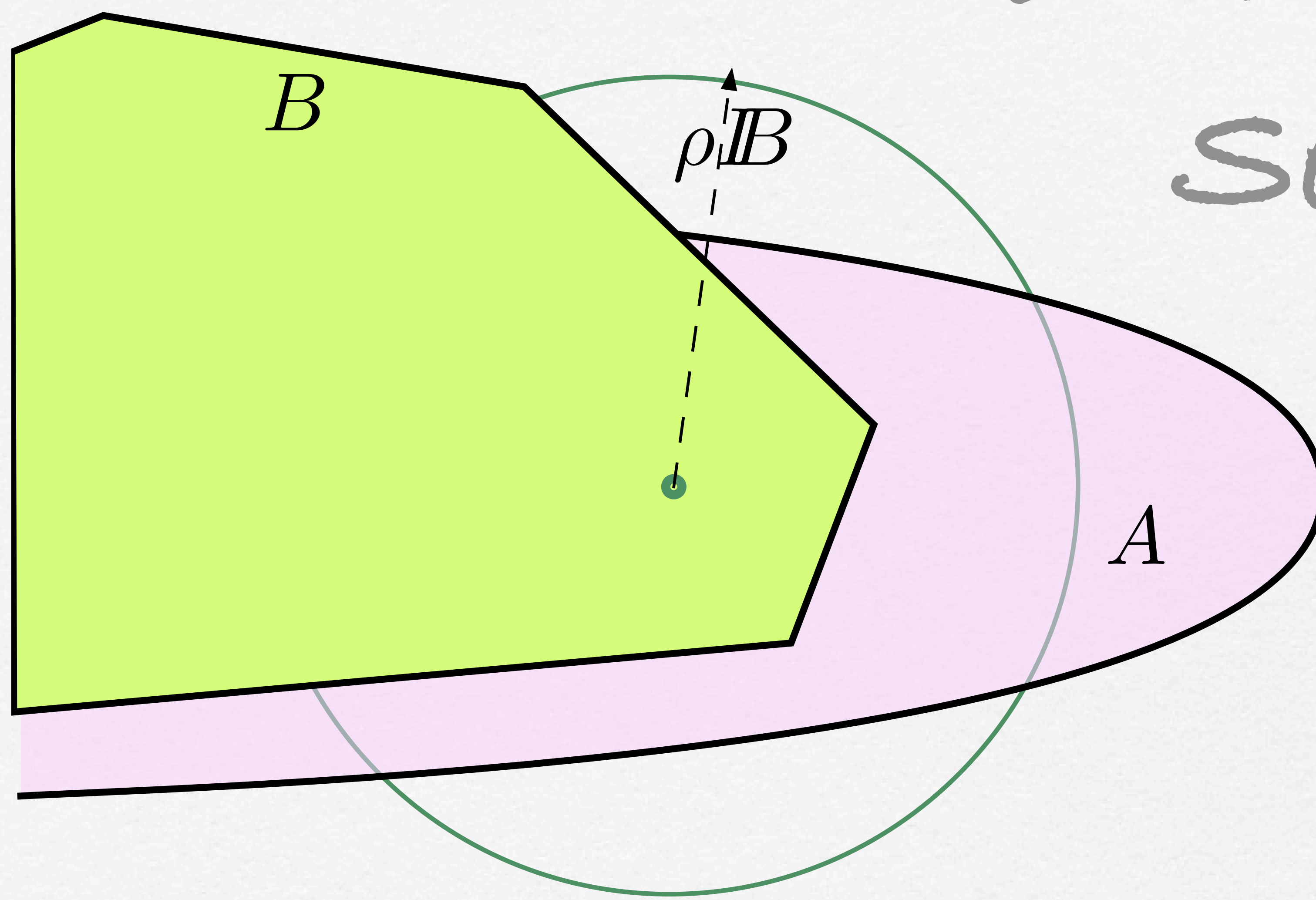


# *Pompeiu-Hausdorff* distance

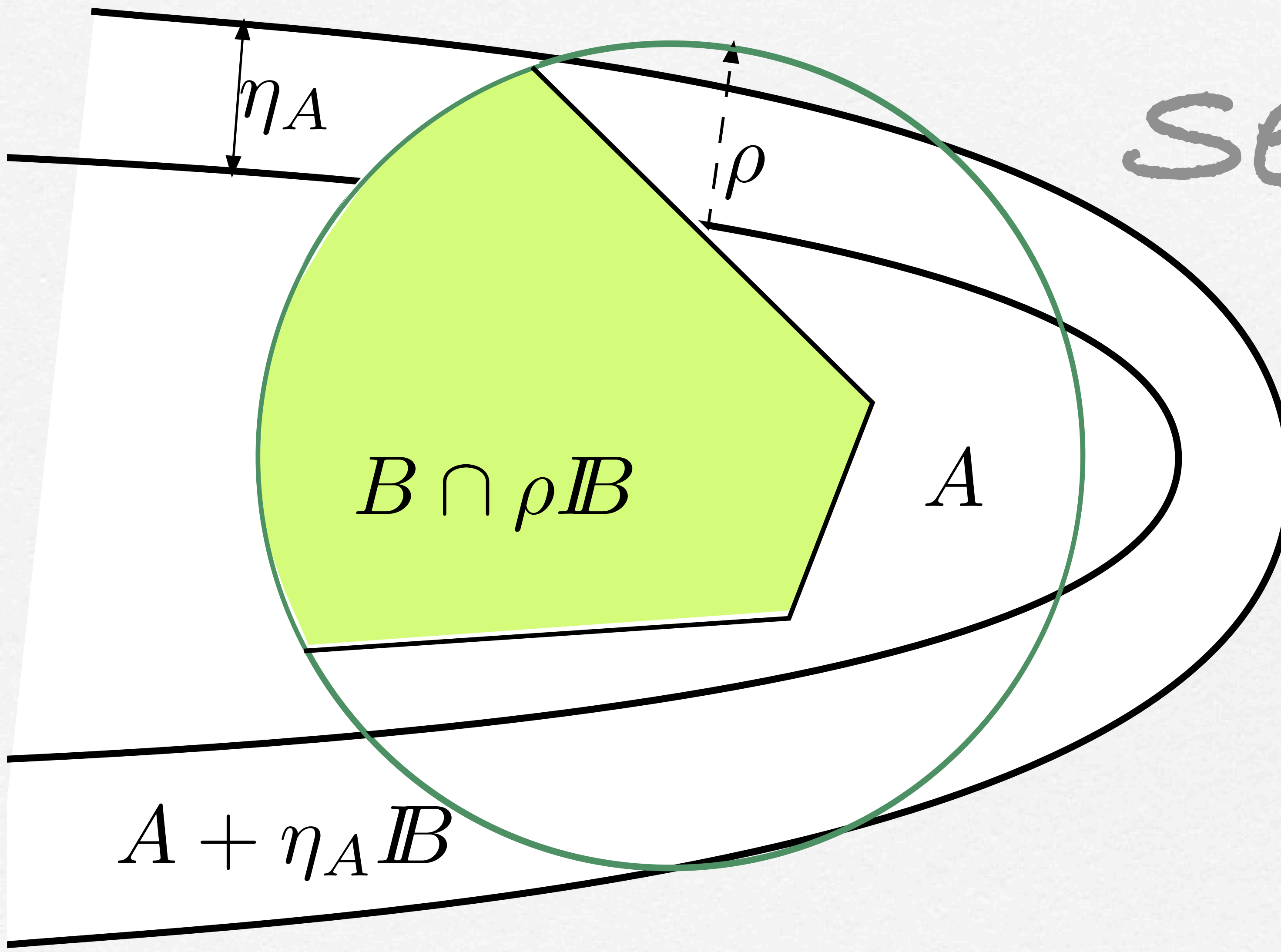



$$\hat{d\!l}(A,B) = \max\,[\,\eta_A, \eta_B\,]$$
$$= d\!l_\infty(A,B)$$

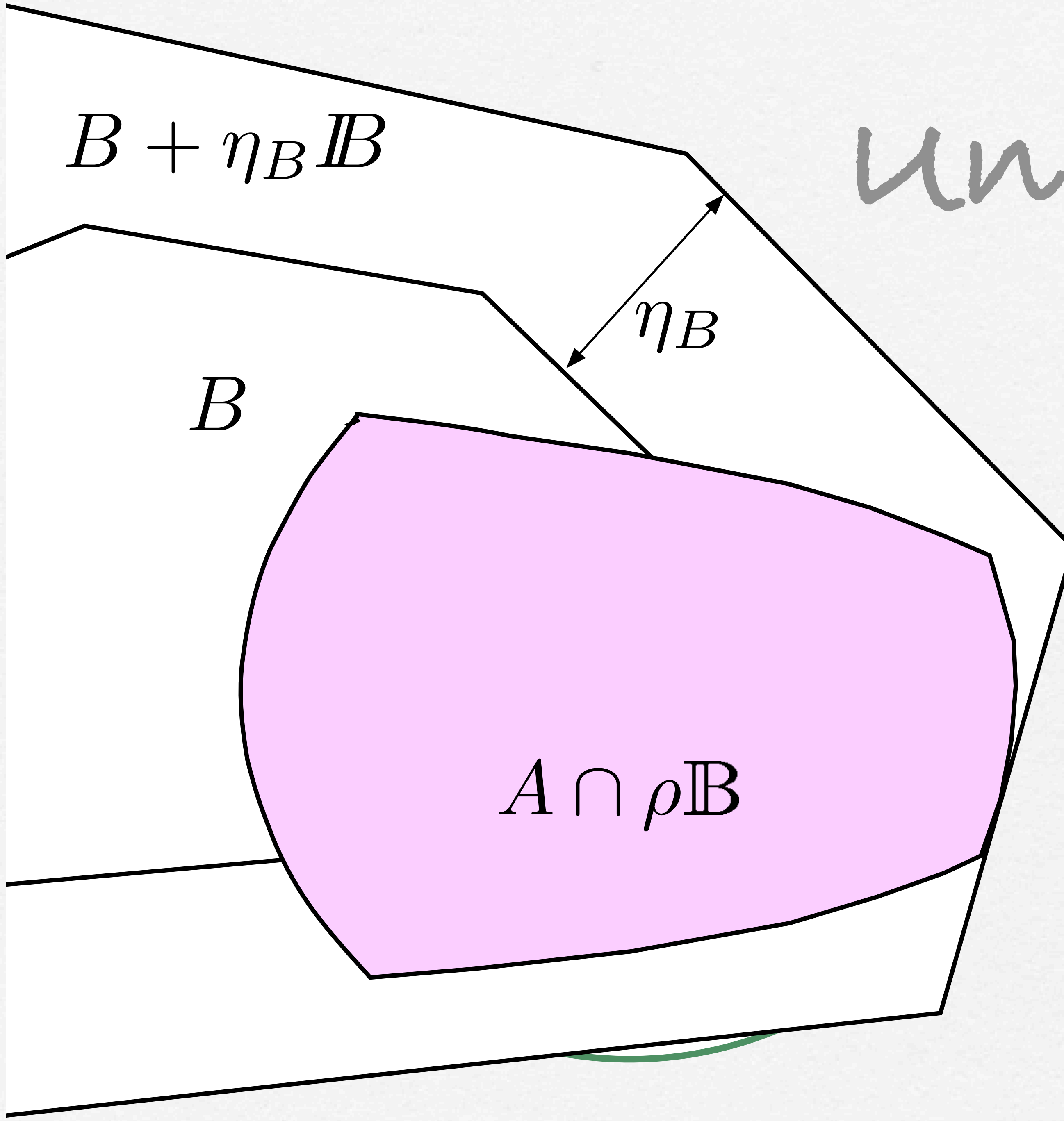
# Unbounded Sets



# Unbounded Sets



Unbounded Sets




$$\hat{d\!l}_\rho(A, B) = \max\left[\,\eta_A, \eta_B\,\right]$$

# set distance (~Attouch-Wets)

$\tau_{aw}$ topology

- $\hat{d\!l}_\rho(A,B) \geq 0,\ \hat{d\!l}(A,A) = 0, \triangle$ inequality
- but $\hat{d\!l}_\rho(A,B) = 0$ possibly when $A \neq B$
-
-
-
- $\hat{d\!l}_\rho(A,B) \leq d\!l_\rho(A,B) \leq \hat{d\!l}_{\rho'}(A,B) \quad \rho' \geq 2\rho + d_0$

# set distance (~Attouch-Wets)

$\tau_{aw}$ topology

- $\hat{d\!l}_\rho(A,B) \geq 0,\ \hat{d\!l}(A,A) = 0, \triangle$ inequality
- but $\hat{d\!l}_\rho(A,B) = 0$ possibly when $A \neq B$
- $d\!l_\rho(A,B) = \sup_{x \in \rho\boldsymbol{B}} \big[\, d(x,A),\, d(x,B) \,\big]$
- for all $\rho \geq 0,\ d\!l_\rho$ is a pseudo-metric
- $d\!l(A,B) = \int_{\rho \geq 0} d\!l_\rho(A,B) e^{-\rho}\, d\rho$, set-metric
- $\hat{d\!l}_\rho(A,B) \leq d\!l_\rho(A,B) \leq \hat{d\!l}_{\rho'}(A,B) \quad \rho' \geq 2\rho + d_0$

# Properties of the set-distance

$$C^\nu \to C \text{ if } d\!l(C^\nu, C)) \to 0 \iff \text{for any } \bar\rho \geq 0,$$

$$\begin{cases} d\!l_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \\ \hat{d\!l}_\rho(C^\nu, C) \to 0 & \text{for all } \rho \geq \bar\rho \end{cases}$$

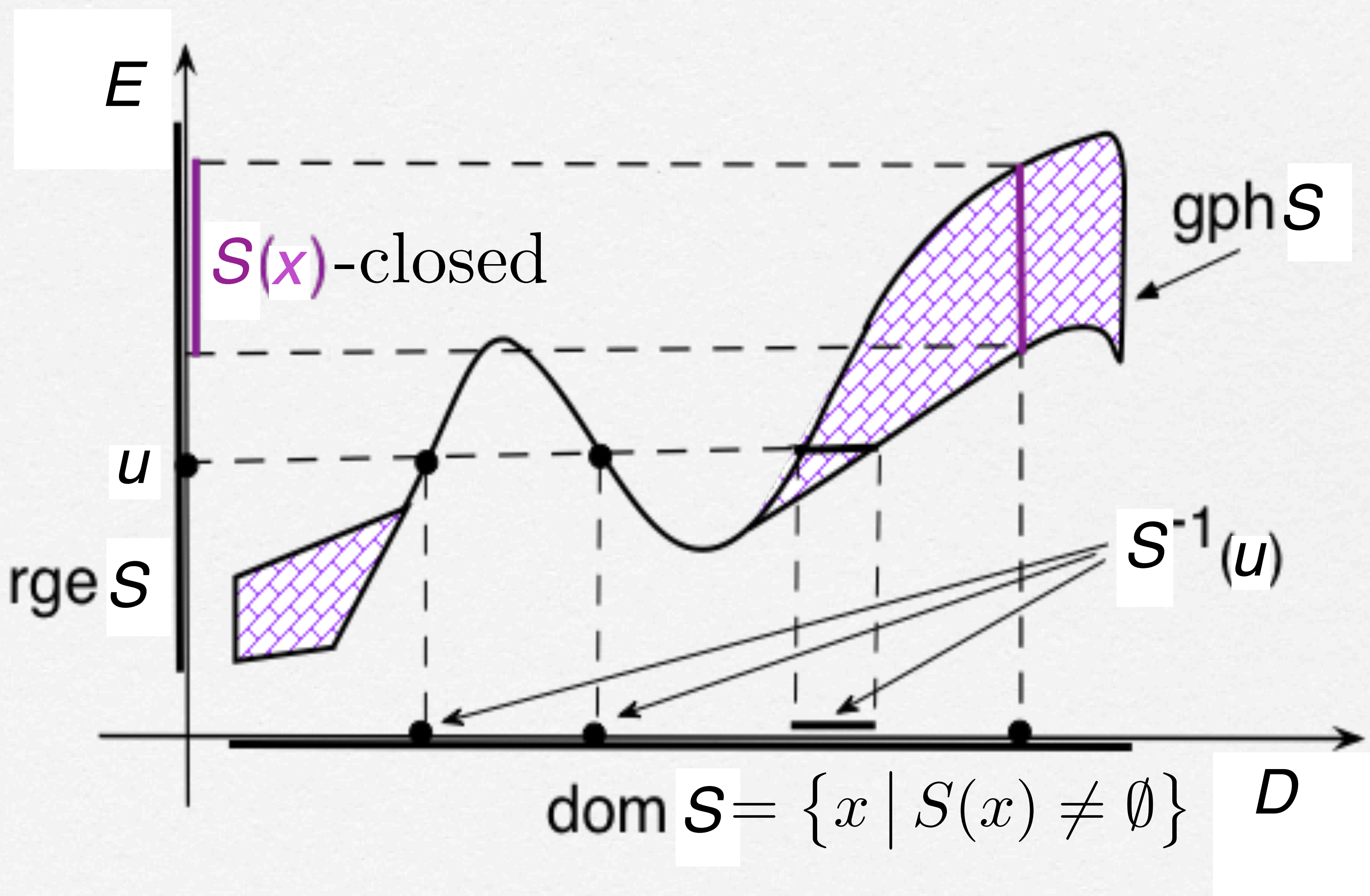
$$(E, d) \text{ Polish} \implies (\text{cl-sets}(E), d\!l) \text{ complete, metric space}$$

$$(\text{cl-sets}(E), d\!l) \text{ Polish} \iff E = \mathbb{R}^n$$

# space of *osc-mappings*

outer semicontinuous

$$S : D \rightrightarrows E \text{ osc} \iff \text{gph}\, S \subset D \times E \text{ closed}$$
$$\text{gph}\, S = \{(x,u) \,|\, u \in S(x), x \in E\}$$



# space of osc-mappings

outer semicontinuous

$$\mathbb{B} = \mathbb{B}_D \times \mathbb{B}_E \text{ (or } \mathbb{B}_{E\times D})$$

$$d\!l(R,S) = d\!l(\text{gph}\, R, \text{gph}\, S), \quad d\!l_\rho,\ \hat{d\!l}_\rho$$

$(\text{osc-maps}(D,E), d\!l)$ complete metric, Polish: $D = \mathbb{R}^n, E = \mathbb{R}^m$

$S : D \to E$ (single-valued) continuous $\Longrightarrow$ osc, …

$$d\!l(f^\nu, f) \to 0 \Longrightarrow \text{argmin}\, f^\nu \rightrightarrows_v \text{argmin}\, f$$

# space of osc-mappings

outer semicontinuous

$$\mathbb{B} = \mathbb{B}_D \times \mathbb{B}_E \text{ (or } \mathbb{B}_{E\times D})$$

$$d\!l(R,S) = d\!l(\operatorname{gph} R, \operatorname{gph} S), \quad d\!l_\rho,\ \hat{d\!l}_\rho$$

$(\text{osc-maps}(D,E), d\!l)$ complete metric, Polish: $D = \mathbb{R}^n, E = \mathbb{R}^m$

$S : D \to E$ (single-valued) continuous $\Longrightarrow$ osc, ...

$$d\!l(f^\nu, f) \to 0 \Longrightarrow \operatorname{argmin} f^\nu \rightrightarrows_v \operatorname{argmin} f$$

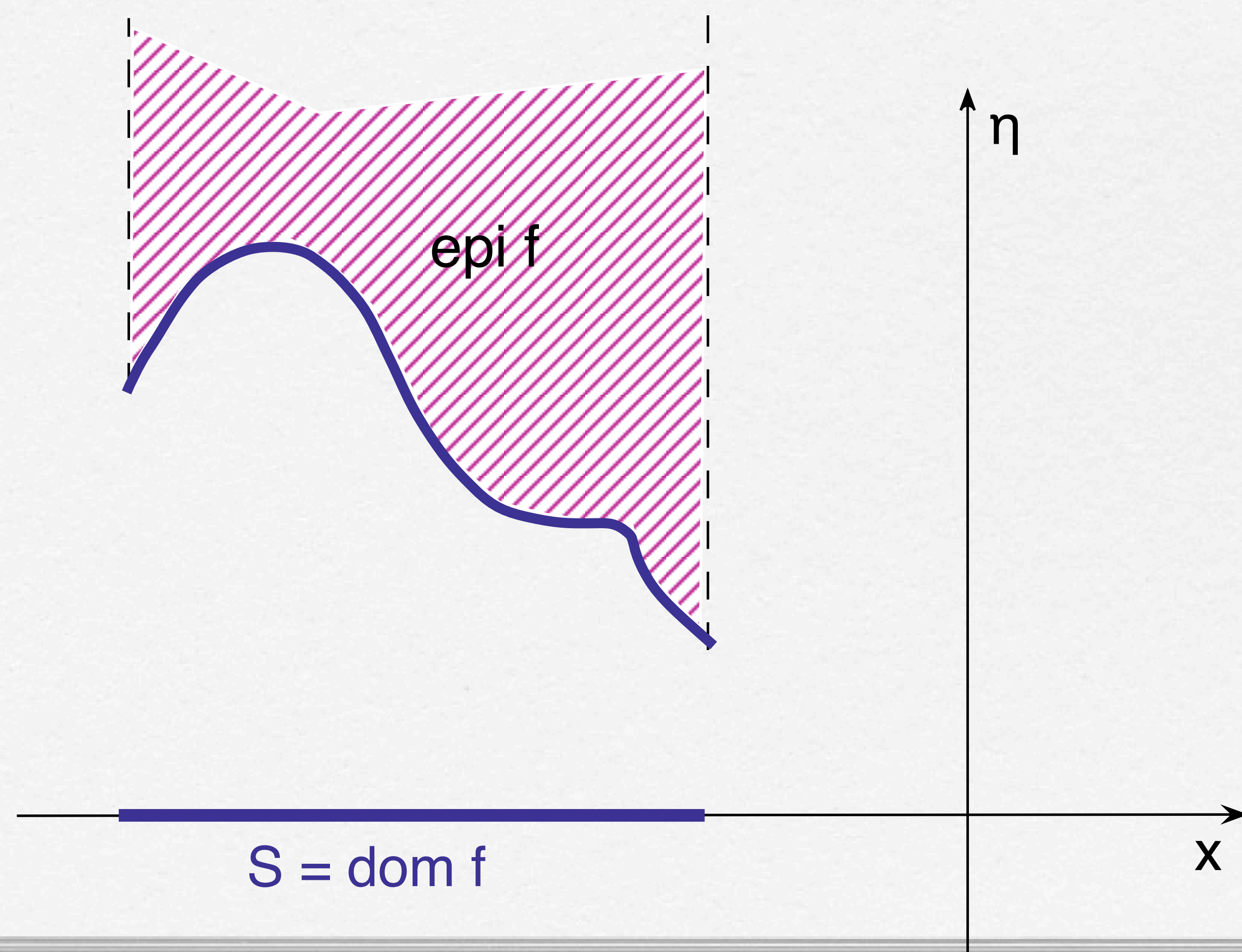
$S^{-1}(0) =$ sol'ns of $S(x) \ni 0$

$S^\nu \to S$ uniformly $\Rightarrow d\!l(S^\nu, S) \to 0$

# space of *lsc-fcns(E)*

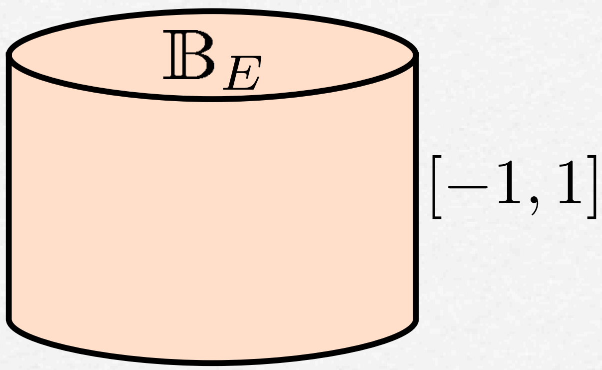
*lower semicontinuous*

$$f : E \to \overline{\mathbb{R}} \text{ lsc} \iff \text{epi} f \subset E \times \mathbb{R} \text{ closed}$$
$$\text{epi} f = \{(x,\eta) \,|\, \eta \geq f(x)\}$$



# space of *lsc-fcns(E)*

lower semicontinuous




unit ball $\mathbb{B} = \mathbb{B}_E \times [-1,1]$

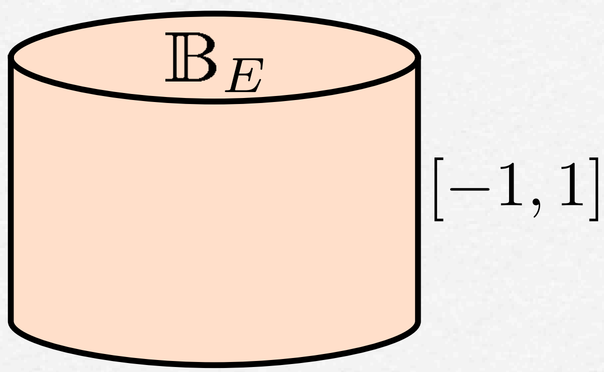
$$dl(f,g) = dl(\text{epi}\, f, \text{epi}\, g) \qquad dl_\rho,\ \hat{dl}_\rho$$

$(\text{lsc-fcns}(E), dl)$ complete metric, Polish $E = \mathbb{R}^n$

$$dl(f^\nu, f) \to 0 \Longrightarrow \text{argmin}\, f^\nu \rightrightarrows_v \text{argmin}\, f$$

# space of *lsc-fcns(E)*

lower semicontinuous



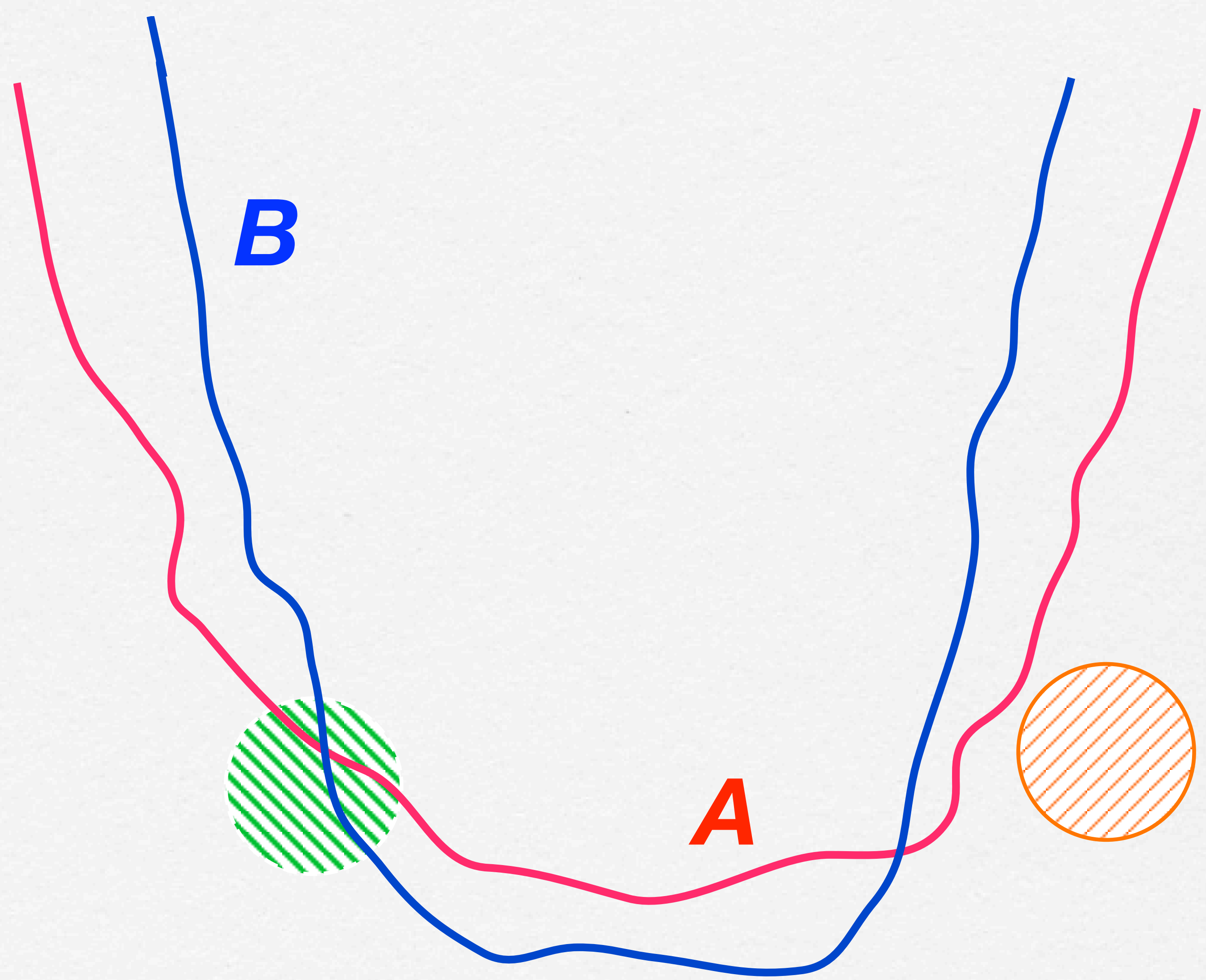

unit ball $\mathbb{B} = \mathbb{B}_E \times [-1,1]$

$$d\!l(f,g) = d\!l(\text{epi}\, f, \text{epi}\, g) \qquad d\!l_\rho,\ \hat{d\!l}_\rho$$

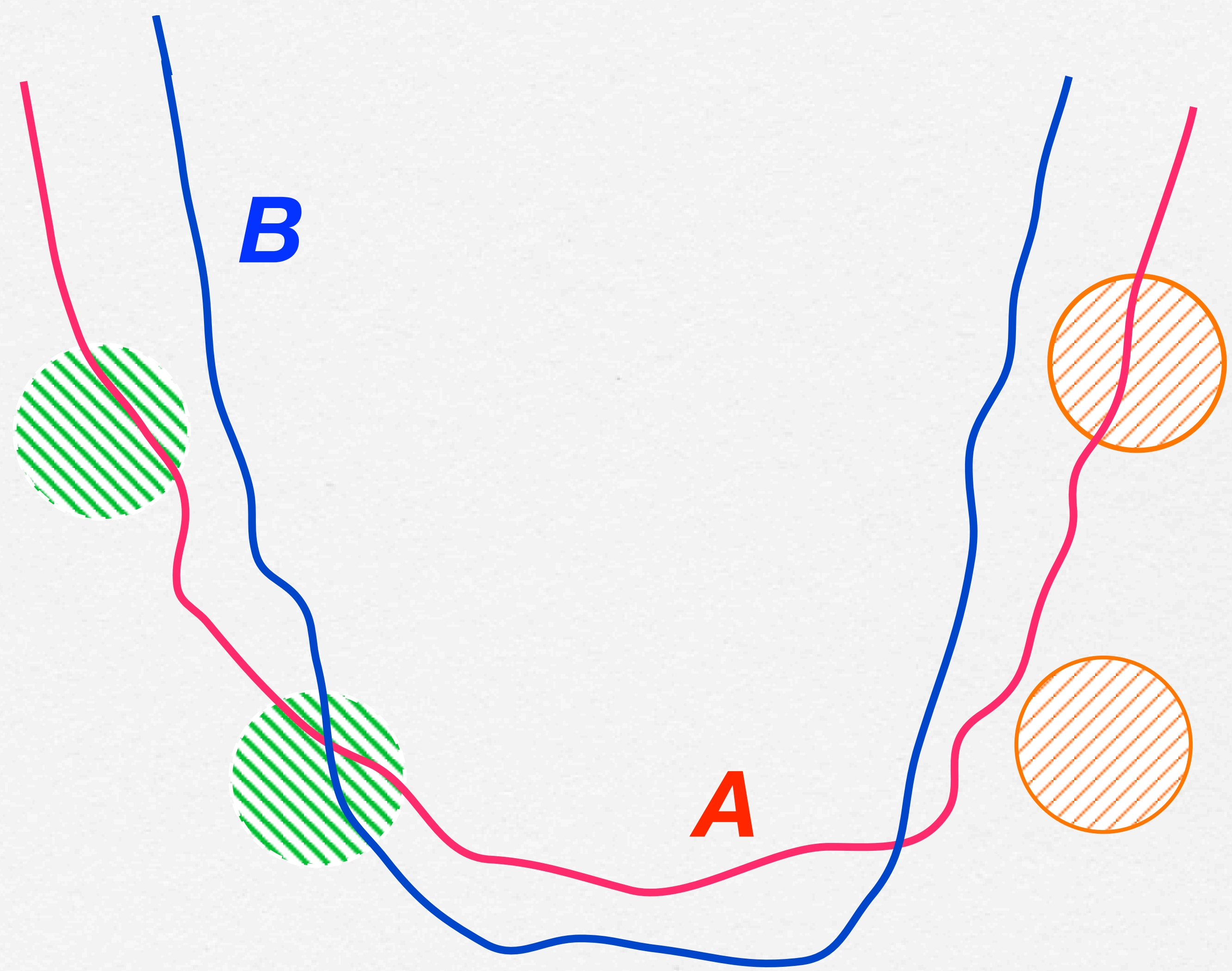
$$(\text{lsc-fcns}(E), d\!l) \text{ complete metric,} \quad \text{Polish } E = \mathbb{R}^n$$

$$d\!l(f^\nu, f) \to 0 \Longrightarrow \text{argmin}\, f^\nu \rightrightarrows_v \text{argmin}\, f$$

# Hit-Open & Miss-Compact Sets



# Hit-Open & Miss-Compact Sets



# $\mathbb{R}^n$: Set-convergence ($\tau_{aw} = \tau_f$) topology

$\mathcal{F}$ = cl-sets($\mathbb{R}^n$), all closed subsets of $\mathbb{R}^n$

$\mathcal{F}^D$ = subsets $\mathbb{R}^n$ that *miss* $D = \{F \cap D = \emptyset\}$

$\mathcal{F}_D$ = subsets $\mathbb{R}^n$ that *hit* $D = \{F \cap D \neq \emptyset\}$

Hit-and-miss topology (= $\tau_f$ Fell topology)

subbase: $\{\mathcal{F}^K \mid K \text{ compact}\}$ & $\{\mathcal{F}_O \mid O \text{ open}\}$

$\mathbb{B}(x,\rho)$ closed ball, center $x$ radius $\rho$, $\mathbb{B}^o(x,\rho)$ open

a subbase $\left\{\mathcal{F}^{\mathbb{B}(x,\rho)}, \mathcal{F}_{\mathbb{B}^o(x,\rho)} \mid x \in \mathbb{Q}^d, \rho \in \mathbb{Q}_{++}\right\}$

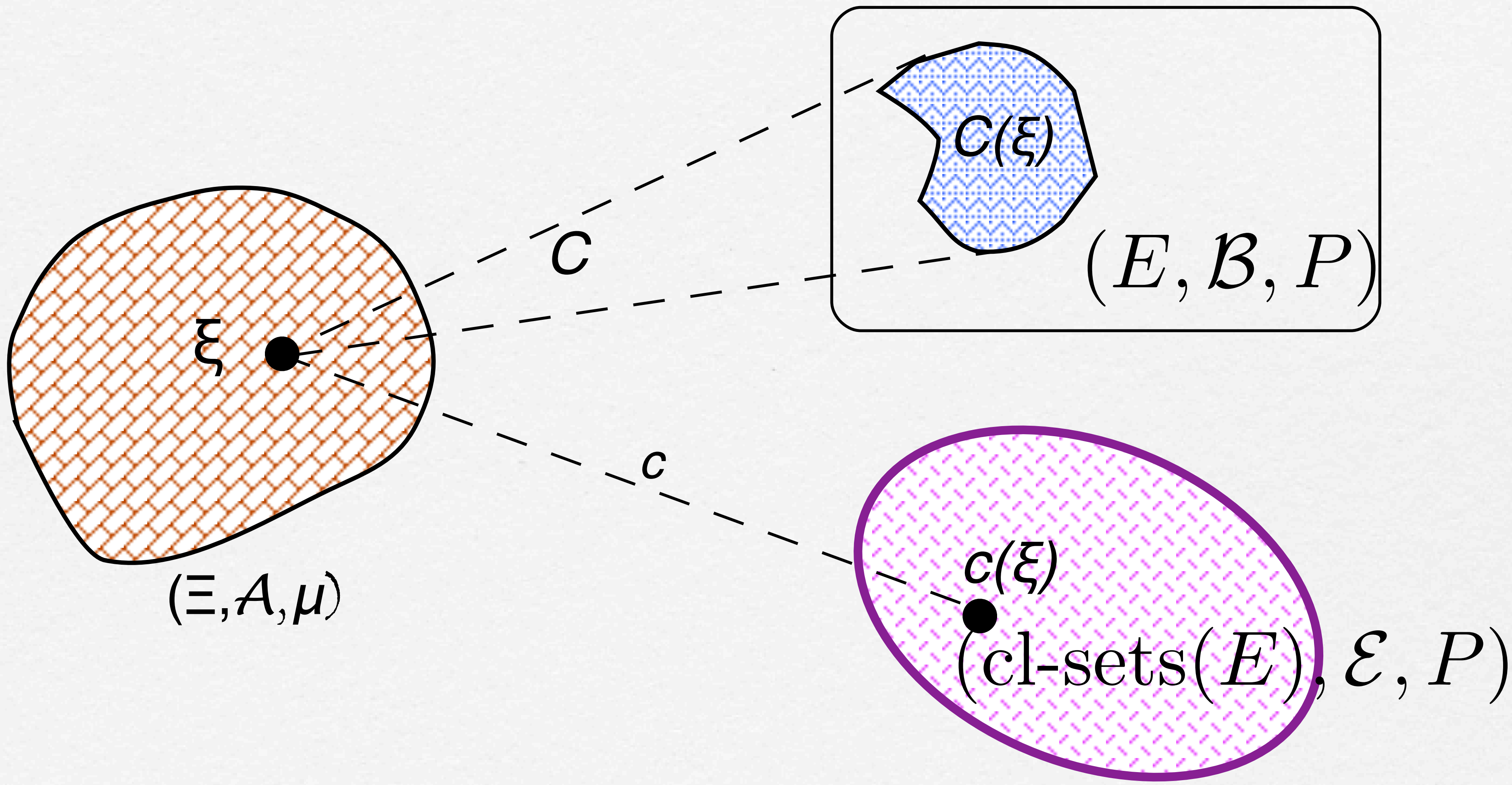
countable base: $\left\{\mathcal{F}^{\mathbb{B}(x^1,\rho_1)\cup\ldots\cup\mathbb{B}(x^r,\rho_r)} \cap \mathcal{F}_{\mathbb{B}^o(x^1,\rho_1)\cup\ldots\cup\mathbb{B}^o(x^s,\rho_s)}\right\}$

(cl-sets($\mathbb{R}^n$), $\tau_{aw}$) Polish space (separable, complete metric)

# Random Sets

Mattheron, Choquet
Salinetti-Wets, Castaing, Valadier, Hess, Stoyan, ...

# Random sets



# Random Closed Sets

$(\Xi, \mathcal{A}, P), \quad \Xi \subset \mathbb{R}^N$ & $E$ Polish, for example $\mathbb{R}^n$

$C : \Xi \rightrightarrows E,\ C(\xi) \subset E$ closed set for all $\xi \in \Xi$

& $C^{-1}(O) = \left\{ \xi \,\middle|\, C(\xi) \cap O \neq \emptyset \right\} \in \mathcal{A},\ \forall O \subset E$, open

$\Rightarrow$ dom $C = C^{-1}(E) \in \mathcal{A}$, measurability ~ hit open sets

$=================================$

# Random Closed Sets

$(\Xi, \mathcal{A}, P)$, $\quad \Xi \subset \mathbb{R}^N$ & $E$ Polish, for example $\mathbb{R}^n$

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$\Rightarrow$ dom $C = C^{-1}(E) \in \mathcal{A}$, measurability ~ hit open sets

==================================================

$c : \Xi \to$ cl-sets$(E)$, $c(\xi) \sim C(\xi)$, $\mathcal{F}_O = \{F \subset E \text{ closed} \,|\, F \cap O \neq \emptyset\}$

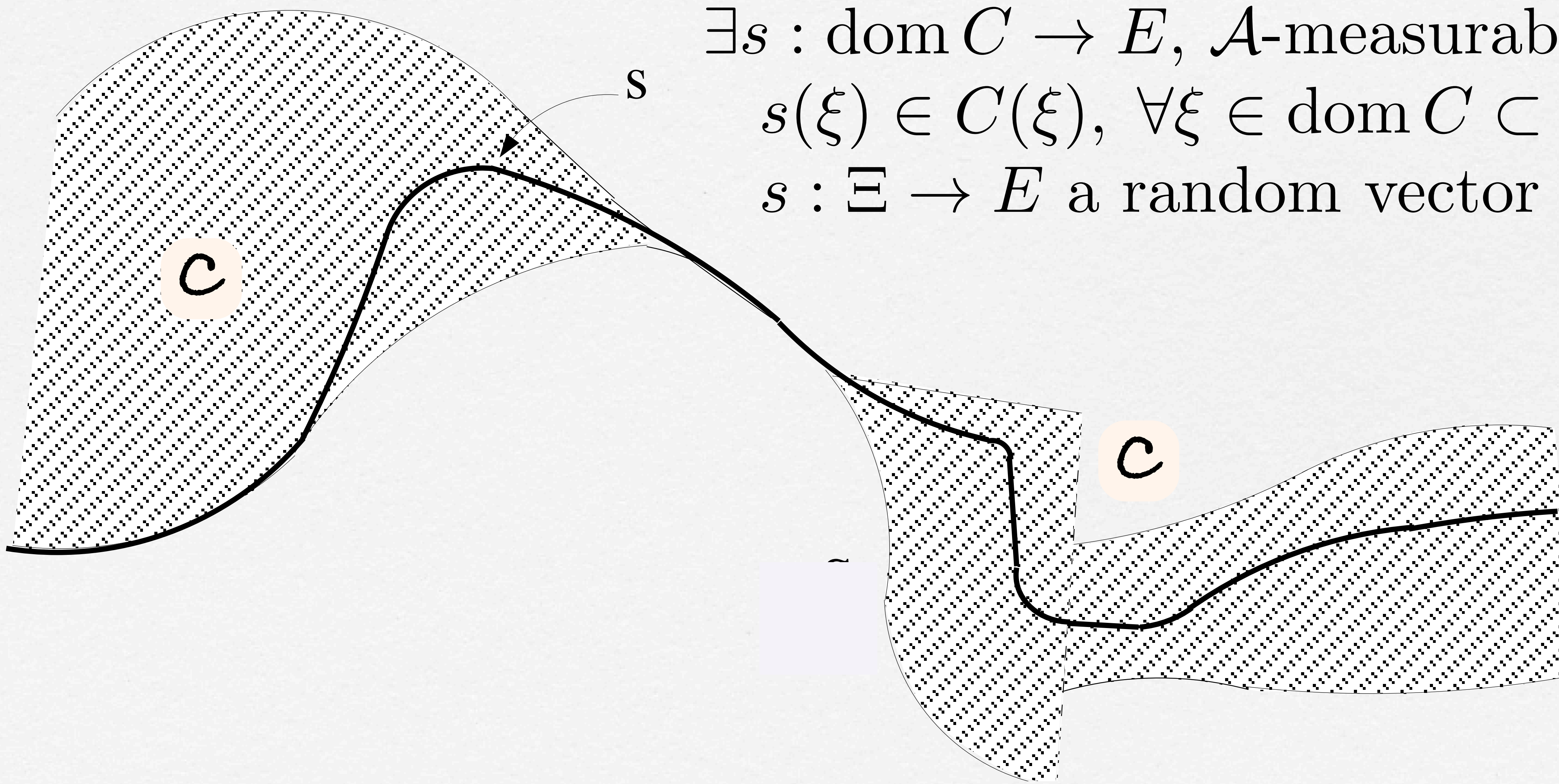
$(\text{sets}(E), \mathcal{E})$, $\mathcal{E}$ Effrös field $= \sigma$-$\{\mathcal{F}_O \in \text{sets}(\mathbb{R}^n), O \text{ open}\}$,

$C$ measurable $\Leftrightarrow c$ measurable $[c^{-1}(\mathcal{F}_O) \in \mathcal{A}]$

$\mathcal{E} = \mathcal{B}$ Borel field when $E$ Polish (complete separable metric space)

# Measurable selection

- a random closed set $C$ always admits a measurable selection!




$$\exists s : \operatorname{dom} C \to E,\ \mathcal{A}\text{-measurable},$$
$$s(\xi) \in C(\xi),\ \forall \xi \in \operatorname{dom} C \subset \Xi$$
$$s : \Xi \to E \text{ a random vector}$$

# Castaing Representation

- C is a random closed set (& dom C measurable) ⇔ it admits a Castaing representation: ∃ a countable family

$$\left\{s^{\nu} : \text{ dom } C \to E, \text{ meas.-selections}\right\}$$

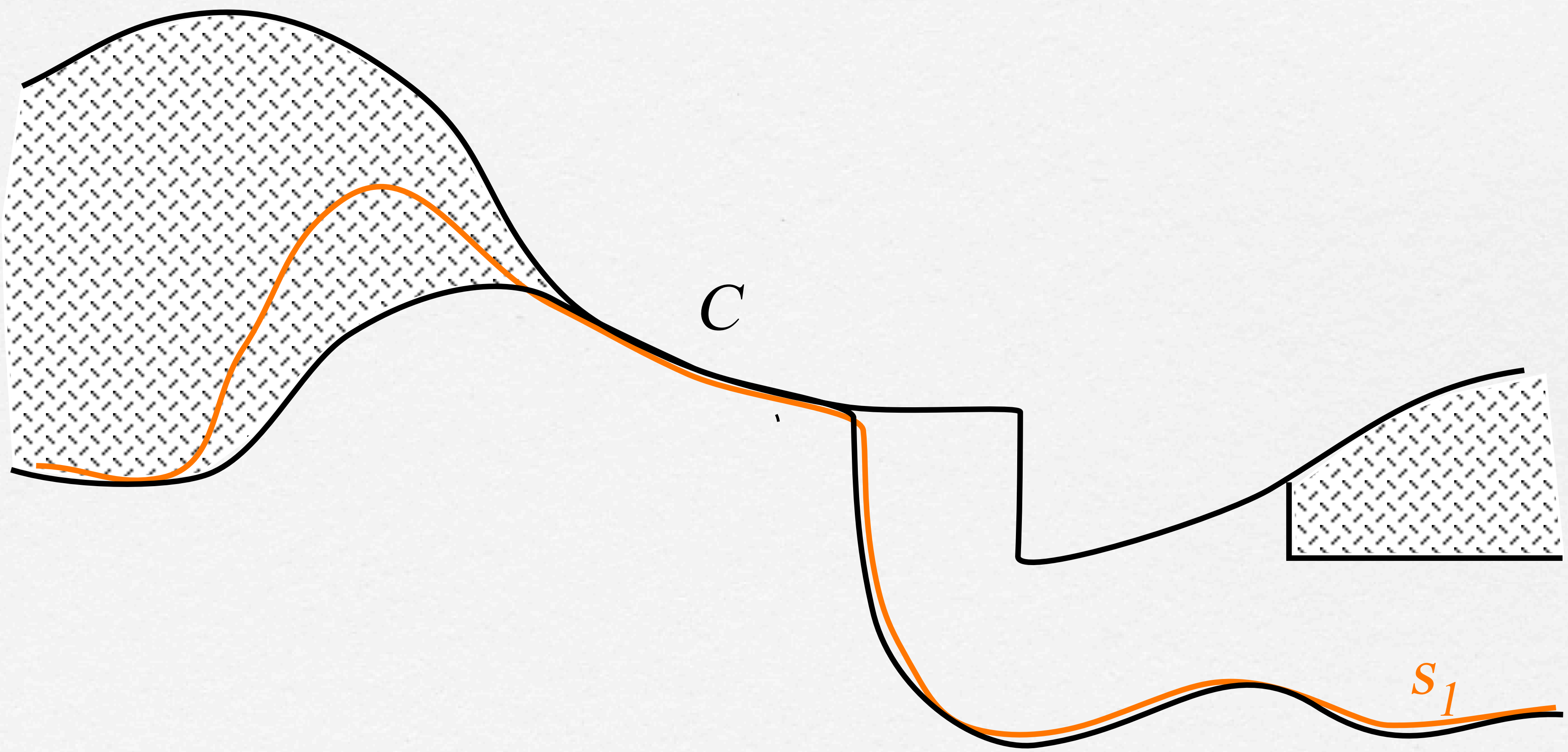
$$\text{cl} \bigcup_{\nu \in \mathbb{N}} s^{\nu}(\xi) = C(\xi), \forall \xi \in \text{ dom } C \subset \Xi$$

- Graph measurability

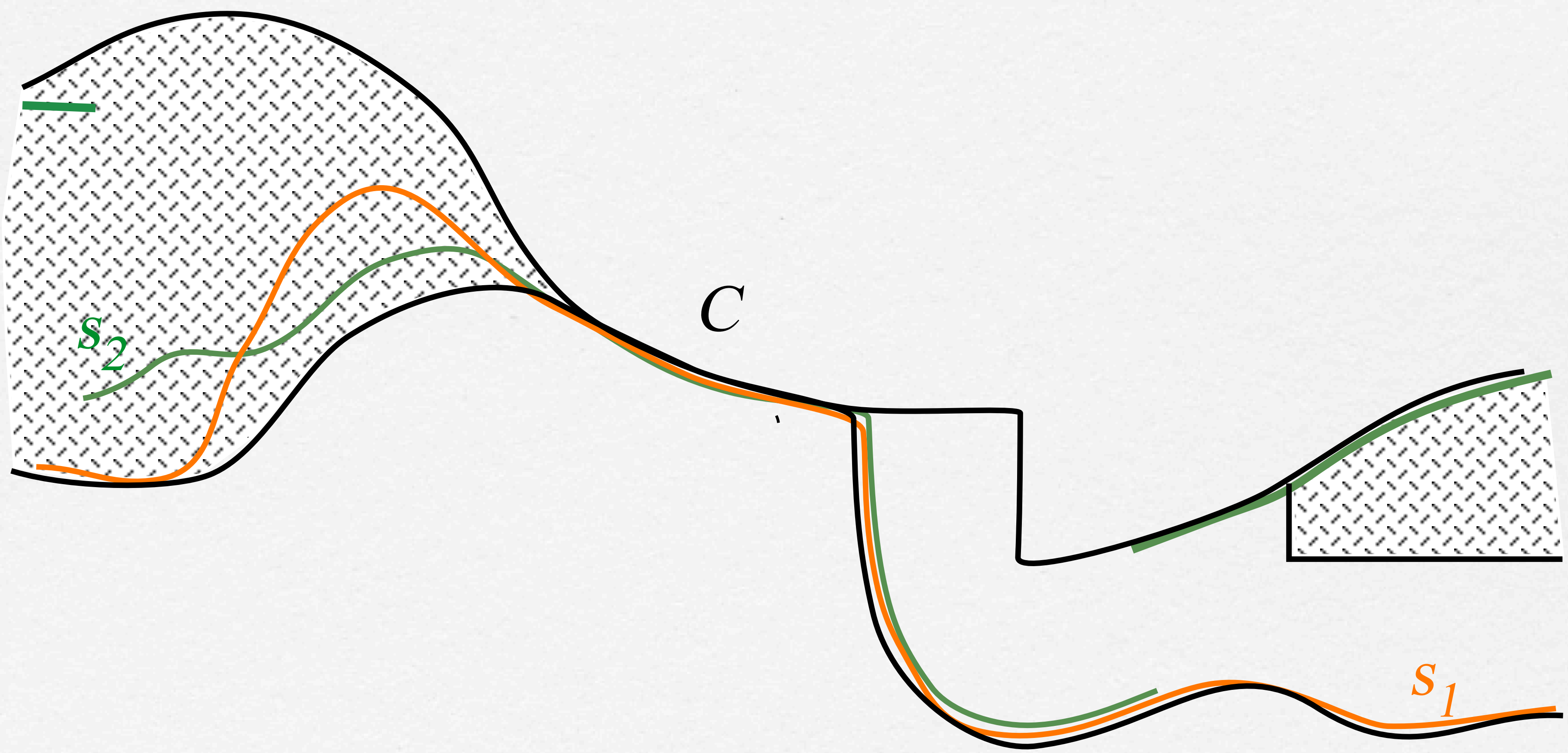
$(\Xi, \mathcal{A})$ $P$-complete for some $P$, (negligible sets are $P$-measurable)

$C$ random set ⇔ gph $C$ $\mathcal{A} \otimes \mathcal{B}_n$-measurable

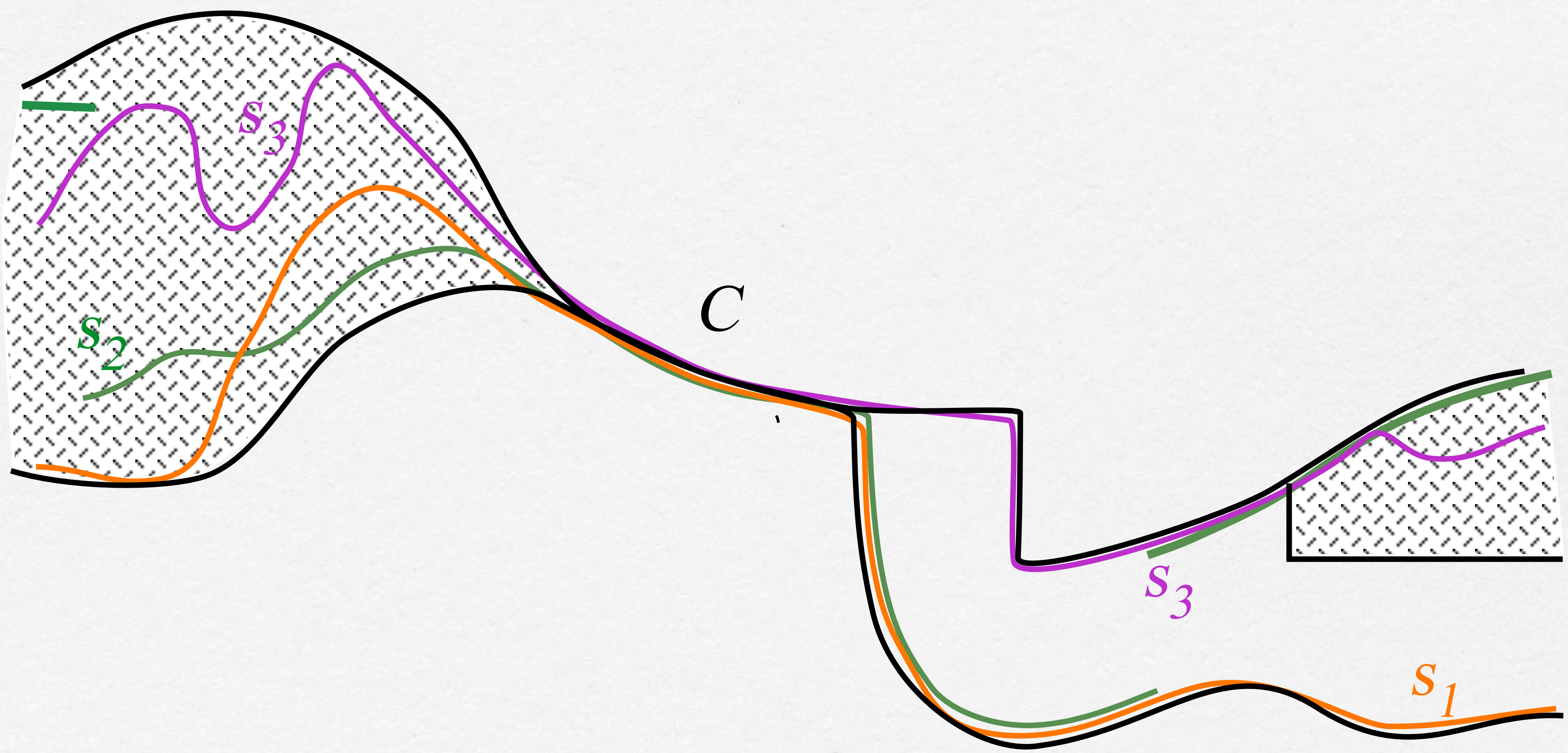
# Castaing Representation



# Castaing Representation



# Castaing Representation



# Random Elements: Convergence *(review)*

$$\boldsymbol{\xi} : (\Omega, \mathcal{F}, \mu) \to (\Xi, \mathcal{A}, P), \quad \boldsymbol{\xi}^\nu \xrightarrow{\star} \boldsymbol{\xi}$$

- a.s. (almost sure) convergence:

$$P\{\xi \mid \lim_\nu \boldsymbol{\xi}^\nu(\omega) = \xi \neq \boldsymbol{\xi}(\omega), \omega \in \Omega\} = 0$$

- convergence in probability:

$$P(|\boldsymbol{\xi}^\nu - \boldsymbol{\xi}| > \varepsilon) \to 0 \text{ for all } \varepsilon > 0$$

- convergence in distribution: $P^\nu \xrightarrow{\mathcal{D}} P$

# a.s.-Convergence

- $\left\{C^\nu : \Xi \rightrightarrows \mathbb{R}^d, \nu \in \mathbb{N}\right\}$ random closed sets
- a.s. convergence: $dl\left(C^\nu(\xi), C(\xi)\right) \to 0$ for $P$-almost all $\xi \in \Xi$

  $C^\nu \to C$ a.s. $\Rightarrow C$ random closed set on $\Xi_0, \mu(\Xi_0) = 1$
- $C^\nu \to C$ $P$-a.s. and dom $C^\nu =$ dom $C$. Then,

  $\exists$ Castaing representations of $C^\nu \to$ a Castaing representation of $C$

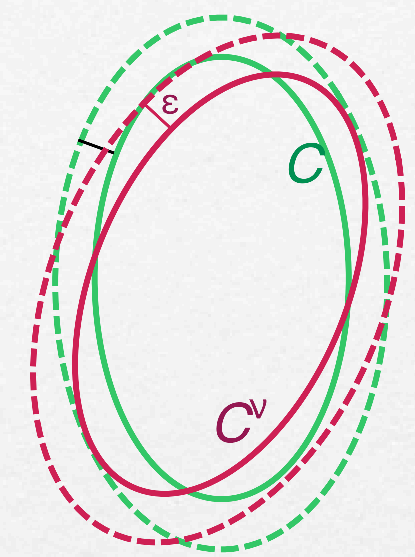
  If $s : \Xi \to E$ is a measurable selection of $C$, then

  $\exists\, s^\nu : \Xi \to E$ selections of $C^\nu$ converging $P$-a.s. to $s$
- ('Egorov's Theorem': $C^\nu \to C$ $\mu$-a.s. $\Leftrightarrow C^\nu \to C$ almost uniformly)

# Convergence in probability

Let $\varepsilon^o C = \left\{ x \in \mathbb{R}^m \,\middle|\, d(x,C) < \varepsilon \right\}$, $C^\nu, C$ random sets

$$\Delta_{\varepsilon,\nu} = \left(C^\nu \setminus \varepsilon^o C\right) \cup \left(C \setminus \varepsilon^o C^\nu\right)$$




$\mu$-a.s. convergence: $\mu\left\{ \xi \,\middle|\, C^\nu(\xi) \to C(\xi) \right\} = 1$

in probability: $P\left[\Delta^{-1}_{\varepsilon,\nu}(K)\right] \to 0, \forall \varepsilon > 0, K \in \mathcal{K} =$ cpct-sets

$C^\nu$ converges to $C$ in probability

$\Leftrightarrow P(dl(C^\nu, C) > \varepsilon) \to 0$ for all $\varepsilon > 0$

$\Leftrightarrow$ every subsequence of $\{C^\nu\}_{\nu \in \mathbb{N}}$

contains a sub-subsequence converging $\mu$-a.s to $C$

i.e., in probability $\Rightarrow$ in distribution $\left[\int h(\xi)\, dl(C^\nu(\xi), C(\xi))\, P(d\xi) \to 0\right]$

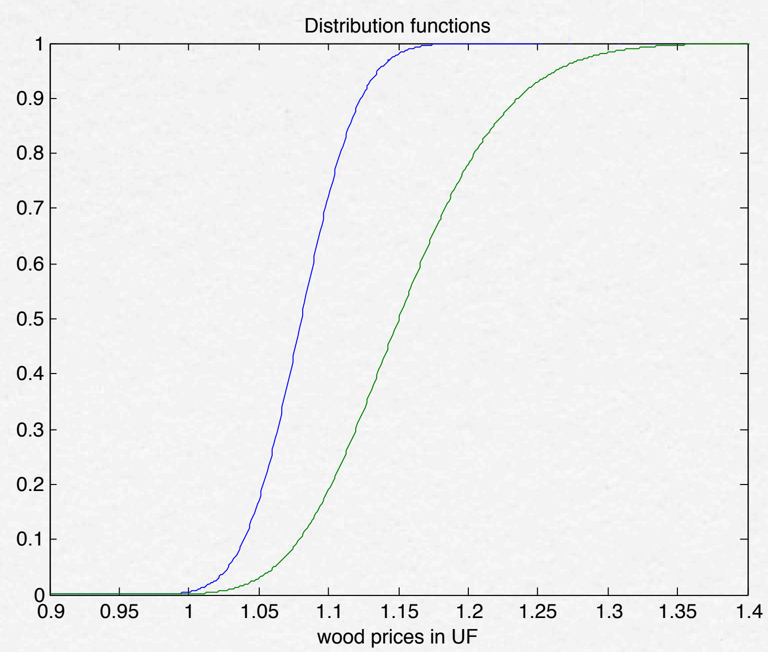
# $P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

$P^\nu, P$ defined on $(\mathbb{R}, \mathcal{B})$

$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\ \forall h$ continuous

$F^\nu(z) = P^\nu\big((-\infty, z)\big), \quad F(z) = P\big((-\infty, z)\big)$, cumulative distributions

$P^\nu \xrightarrow{\mathcal{D}} P \iff F^\nu \xrightarrow{p} F$ on cont $F = \{$ all continuity points of $F\}$




$\xrightarrow{h}$ : hypo-convergence

# $P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

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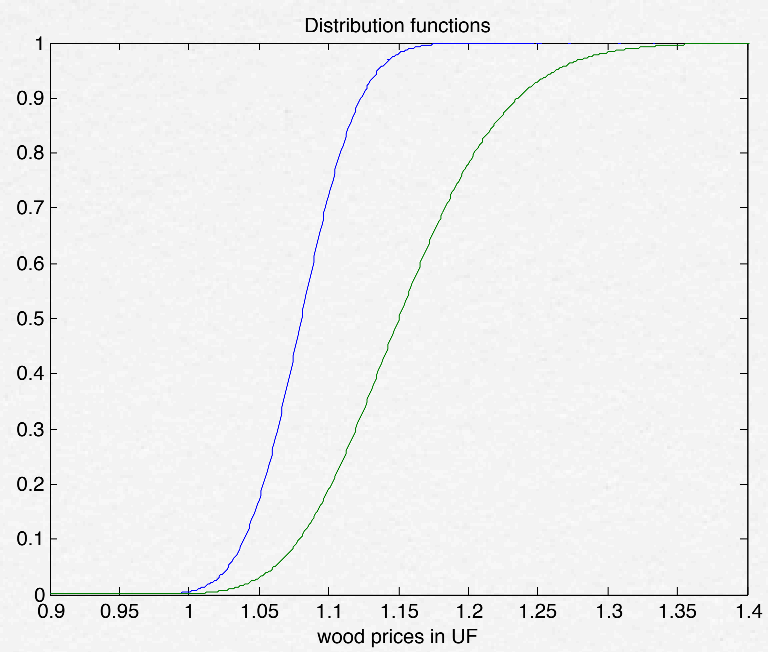



$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

$(F^\nu \xrightarrow{h} F,\ F$ usc $= -$lsc $)$

$\xrightarrow{h}$ : hypo-convergence

# $P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

$P^\nu, P$ defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

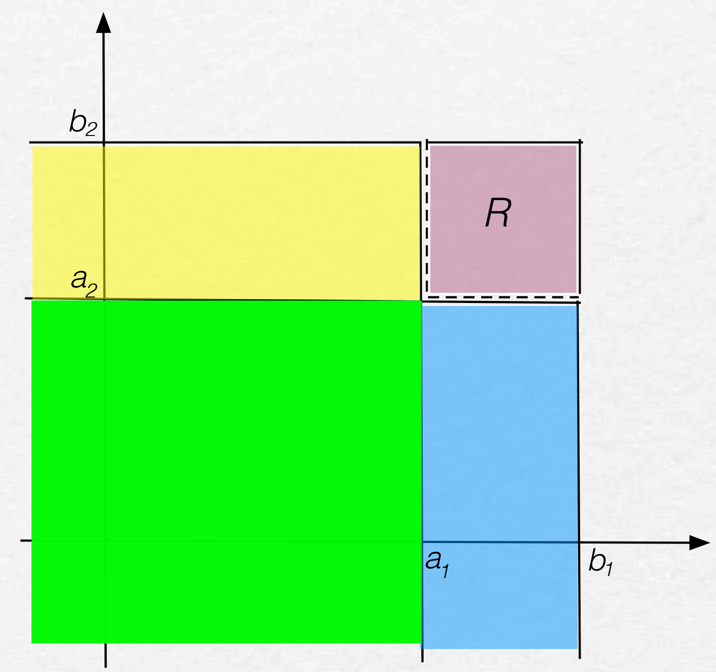
$P^\nu \xrightarrow{\mathcal{D}} P \iff \int h(\xi)\, P^\nu(d\xi) \to \int h(\xi)\, P(d\xi)\;\; \forall h$ continuous

$F^\nu(z) = P^\nu(\xi_i \leq z_i, i = 1, \ldots, n), \quad F(z) = P(\xi_i \leq z_i, i = 1, \ldots, n)$

1. $z \leq \tilde{z} \implies F(z) \leq F(\tilde{z})$ "increasing"

2. $\lim_{z \to \infty} F(z) = 1, \;\; \lim_{z_j \to -\infty} F(z) \to 0,$

3. $F$ is usc (upper sc) $\limsup_{z' \to z} F(z') \leq F(z),$




4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of $R$
   $\forall R \subset \mathbb{R}^n,\; P(\boldsymbol{\xi} \in R) = \sum_{v \in V} \text{sgn}(v) F(v), \quad \text{sgn}(v \in V) = (-1)^{\# a \text{ in } v}$

# $P^\nu \xrightarrow{\mathcal{D}} P \;\sim\;$ distribution fcns converge

$P^\nu, P$ defined on $(\mathbb{R}^n, \mathcal{B}_n)$ random vectors $\boldsymbol{\xi}^\nu, \boldsymbol{\xi}$

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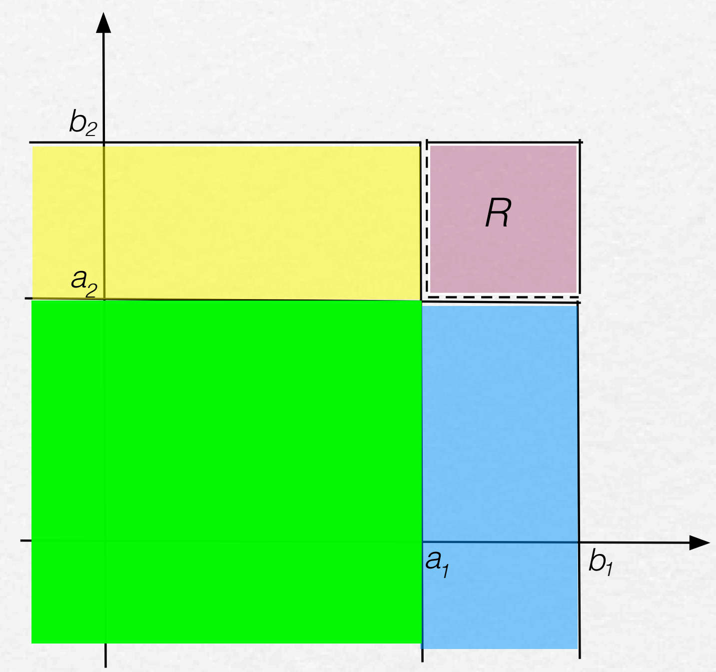



4. $R = (a_1, b_1] \times \cdots \times (a_n, b_n], \quad V = \{a_1, b_1\} \times \cdots \times \{a_n, b_n\}$ vertices of $R$
   $\forall R \subset \mathbb{R}^n,\ P(\boldsymbol{\xi} \in R) = \sum_{v\in V} \mathrm{sgn}(v) F(v), \quad \mathrm{sgn}(v \in V) = (-1)^{\#a \text{ in } v}$

$$P^\nu \xrightarrow{\mathcal{D}} P \iff -F^\nu \xrightarrow{e} -F$$

# Distribution of a random set

Borel $\sigma$-field: $\mathcal{B} = \sigma\text{-}\{\mathcal{F}^K \,|\, K \text{ compact}\}$ or $\sigma\text{-}\{\mathcal{F}_O \,|\, O \text{ open}\}\ldots$

Distribution $(P, \mathcal{B})$ regular, $\mathcal{K}$ compact subsets $E$

determined by values on $\{\mathcal{F}^K \,|\, K \in \mathcal{K}\}$ or $\{\mathcal{F}_K \,|\, K \in \mathcal{K}\}$

Distribution function (Choquet capacity):

$T : \mathcal{K} \to [0,1]$, (2) $T(\varnothing) = 0$ and $\forall \{K^\nu, \nu \in \{0\} \cup \mathbb{N}\} \subset \mathcal{K}$: (1,3)

a) $T(K^\nu) \searrow T(K)$ when $K^\nu \searrow K$ (~ usc on $\mathbb{R}^n$)

b) $\{D_\nu : \mathcal{K} \to [0,1]\}_{\nu \in \mathbb{N}}$ where $D_0(K^0) = 1 - T(K^0)$

(4) $D_1(K^0; K^1) = D_0(K^0) - D_0(K^0 \cup K^1)$ and for $\nu = 2, \ldots$

$D_\nu(K^0; K^1, \ldots, K^\nu) = D_{\nu-1}(K^0; K^1, \ldots, K^{\nu-1}) - D_{\nu-1}(K^0 \cup K^\nu; K^1, \ldots, K^{\nu-1})$

(~ rectangle condition on $\mathbb{R}^n$)

Wednesday, September 5, 2012

# Existence-Uniqueness T

$P$ on $\mathcal{B}$ determines a unique distribution function $T$ on $\mathcal{K}$

$$T(K) = P(\mathcal{F}_K)$$

$$D_\nu(K^0; K^1, \ldots, K^\nu) = P(\mathcal{F}^{K^0} \cap \mathcal{F}_{K^1} \cap \cdots \cap \mathcal{F}_{K^\nu})$$

$T$ on $\mathcal{K}$ determines a unique probability measure $P$.

Proof. via Choquet Capacity Theorem ***(Matheron)***

(refined) via probabilistic arguments ***(Salinetti-Wets)***

$C : \Xi \rightrightarrows \mathbb{R}^d$ a random closed set

$(P, \mathcal{B})$ induced probability measure:

$$P(\mathcal{F}_G) = P\left[C^{-1}(G)\right] \quad \forall G \in \mathcal{B}, \quad T(K) = P\left[C^{-1}(K)\right] \quad \forall K \in \mathcal{K}$$

# Convergence in Distribution

random sets $C^\nu$ converge in distribution to $C$ when

induced $P^\nu$ narrow-converge to $P: P^\nu \to_n P$ $= P^\nu \overset{\mathcal{D}}{\to} P$

$\Leftrightarrow T^\nu \to_p T$ on $\mathcal{K}_{T\text{-cont}}$ (convergence of distribution functions)

$\mathcal{K}_{T\text{-cont}}$ ?

a) $\forall C^\nu, \nu \in N, \exists$ converging subsequence (pre-compact)

b) $K^\nu \nearrow K =$ cl $\bigcup_\nu K^\nu$ regularly if int $K \subset \bigcup_\nu K^\nu$

c) distribution (fcn) continuity: $\lim_\nu T(K^\nu) = T(\text{cl} \bigcup_\nu K^\nu)$

d) convergence $T^\nu \to_p T$ on $C_T$ continuity set $\Rightarrow P^\nu \to_n P$

e) $P^\nu \to_n P \Leftrightarrow T^\nu \to_p T$ on $C_T^{ub} = C_T \cap \mathcal{K}^{ub}$

$\mathcal{K}^{ub} =$ finite union of rational ball, positive radius

f) $\varepsilon \mapsto T(K + \varepsilon\mathbb{B})$: countable number of discontinuities

# a detour about rates

$T^\nu \to_p T$ on $C_T \Leftrightarrow P^\nu \to_n P$ (Polish space: $E, d$)

$P^\nu, P$ defined on $\mathcal{B}$

probability sc-measures on cl-sets(E): $\lambda$

(i) $\lambda \geq 0$, (ii) $\lambda \nearrow$ $\lambda(C^1) \leq \lambda(C^2)$ if $C^1 \subset C^2$

(iii) $\lambda$ is $\tau_f$-usc on cl-sets($E$), (iv) $\lambda(\emptyset) = 0, \lambda(E) = 1$

(v) $\lambda$ modular: $\lambda(C^1) + \lambda(C^2) = \lambda(C^1 \cup C^2) + \lambda(C^1 \cap C^2)$

$P$ and $\lambda = P_{\text{cl-sets}}$ define each other uniquely ($E$ complete $\Rightarrow$ tight)

$\{P^\nu, \nu \in \mathbb{N}\}$ tight: $P^\nu \to_n P \Leftrightarrow \lambda^\nu \to_h \lambda$ ($\sim -\lambda^\nu \to_e -\lambda$) on cl-sets($E$)

tightness ~ equi-usc of $\{\lambda^\nu\}_{\nu \in \mathbb{N}}$ at $\emptyset$

rates: $dl(\lambda^\nu, \lambda) \to 0$ (for $\mathbb{R}$-valued r.v., "~" Skorohod distance)

# Random Sets Convergence & Expectation

Artstein-Vitale-Hart-Wets,
Cressis, Hiai, Weyl, ...

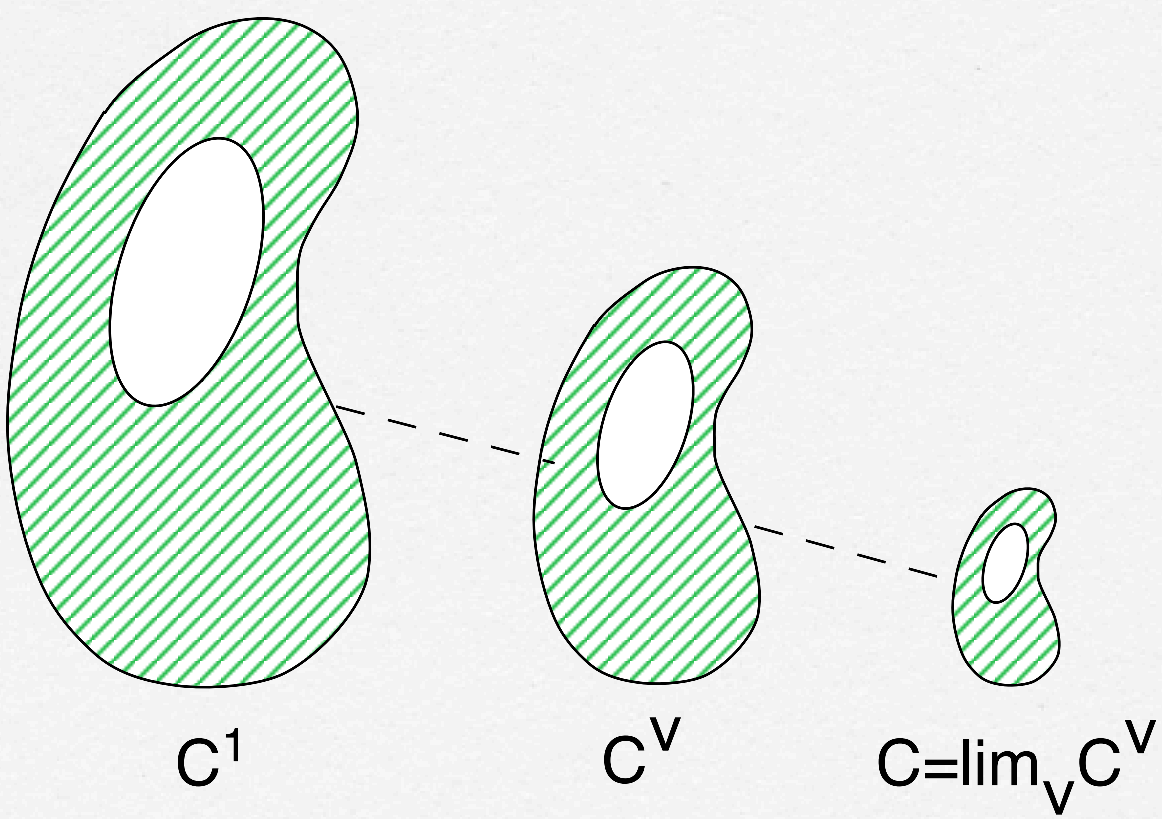
# Outer/Inner Limits

outer limit: $\text{Lo}_\nu C^\nu = \{x \in \text{ cluster-points}\{x^\nu\}, x^\nu \in C^\nu\} = \text{Ls}_\nu C^\nu$

inner limit: $\text{Li}_\nu C^\nu = \{x = \lim_\nu x^\nu, x^\nu \in C^\nu \subset \mathbb{R}^n\} \subset \text{Lo}_\nu C^\nu$

limit: $C^\nu \to C$ if $C = \text{Li}_\nu C^\nu = \text{Lo}_\nu C^\nu$ (Painlevé - Kuratowski)

All limit sets are closed



# Outer/Inner Limits

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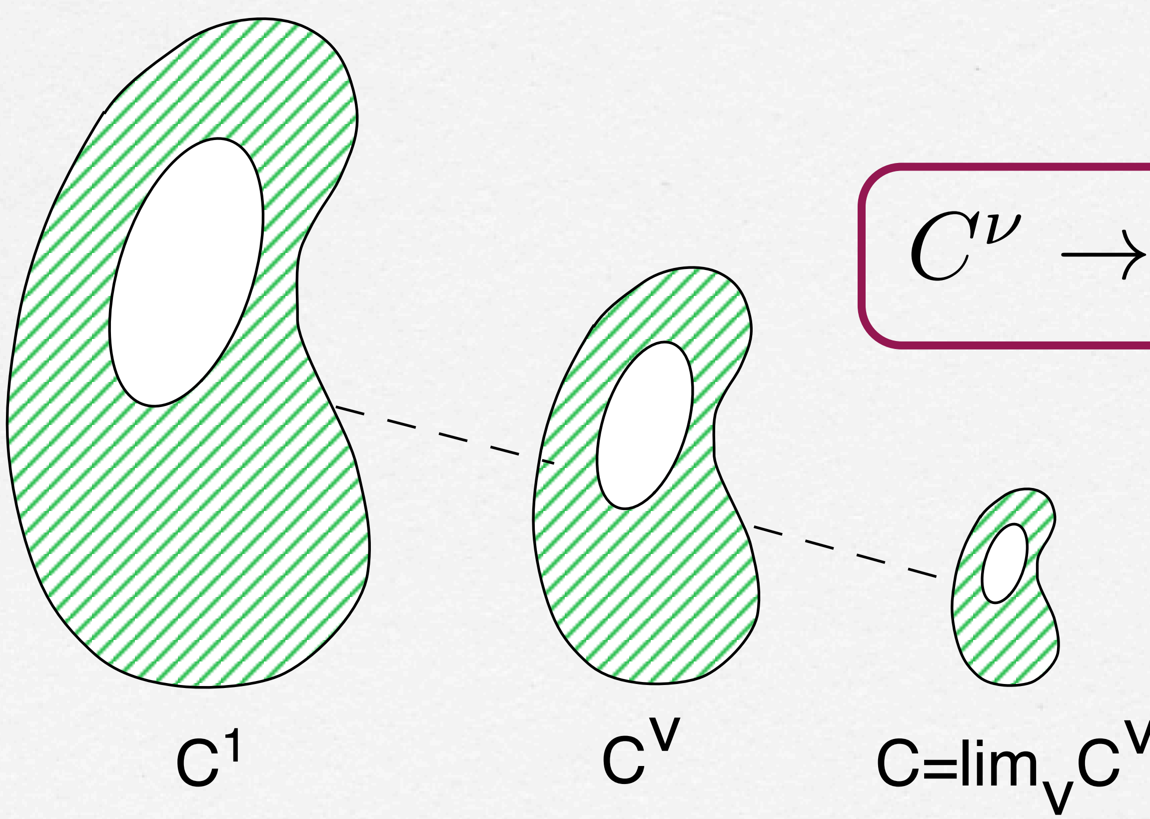
All limit sets are closed




$$C^\nu \to C \iff d\!l(C^\nu, C) \to 0$$

# Characterizing a.s. convergence

$\Big\{C; C^\nu : \Xi \rightrightarrows \mathbb{R}^n,\ \nu \in \mathbb{N}\Big\}$ random closed sets. Then,

1. $C^\nu \to C$ $a.s.$, $d\!l(C^\nu, C) \to 0$ $a.s.$, $\mathrm{Lo}_\nu(C^\nu) \subset C \subset \mathrm{Li}_\nu(C^\nu)$ $a.s.$,

2. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$, $d(x, C^\nu(\xi)) \to d(x, C(\xi))$,

3. $\forall x \in \mathbb{R}^n$ and $\xi \in \Xi_1$ with $P(\Xi_1) = 1$,

$$\lim_{\rho \nearrow \infty} \mathrm{Lo}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\big) \subset C(\xi) \subset \lim_{\rho \nearrow \infty} \mathrm{Li}_\nu \big(C^\nu(\xi) \cap \mathbb{B}(x, \rho)\big).$$

# "Proof 1. ⇔ 2."

$$C^\nu \to C \iff \forall x \in \mathbb{R}^n,\, d(x, C^\nu) \to d(x, C) \text{ provided } E = \mathbb{R}^n.$$

$C^\nu \to C$ if and only if the hit-miss criterion is satisfied

$C$ hits $\mathbb{B}^o(x, \rho)$ then $C^\nu$ hits $\mathbb{B}^o(x, \rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \subset \mathrm{Li}_\nu\, C^\nu \iff d(x, C) \geq \mathrm{limsup}_\nu\, d(x, C^\nu), \forall x$

$C$ misses $\mathbb{B}(x, \rho)$ then $C^\nu$ misses $\mathbb{B}(x, \rho)$ for $\nu \geq \nu_{x,\rho}$
so, $C \supset \mathrm{Lo}_\nu\, C^\nu \iff d(x, C) \geq \mathrm{liminf}_\nu\, d(x, C^\nu), \forall x$

# Building Castaing representations

$C : \Xi \rightrightarrows \mathbb{R}^n$, a random closed set. Let

$$A = \left\{ a_k = (a_k^1, \dots, a_k^n, a_k^{n+1}) \,\middle|\, a_k^i \in \mathbb{Q}^n \text{ \& aff. independent } \right\}$$

for $\emptyset \neq D = D^0$ closed, define $\operatorname{prj}_D a_k = \operatorname{prj}_{D^n} a_k^{n+1}$
where $D^l = \operatorname{prj}_{D^{l-1}} a_k^l$ for $l = 1, \dots, n$

$\operatorname{prj}_D a_k$ is a singleton: intersection of n+1 "aff. independent" spheres.
Moreover, $\{ \operatorname{prj}_D a_k, a_k \in A \}$ also dense in $D$

---

$s_k : \Xi \to \mathbb{R}^n$ with $s_k(\xi) = \operatorname{prj}_{C(\xi)} a_k$ is a measurable selection of $C$

$\square$ When $D$ is a random closed set, so is $\xi \mapsto \operatorname{prj}_{D(\xi)} a$, $a \in \mathbb{R}^n$
repeat the argument $n+1$ times to obtain $s_k$ measurable. $\square$

# Converging Castaing representations

$C^\nu : \Xi \rightrightarrows \mathbb{R}^n$ random closed sets converging $P$-$a.s.$ to $C$, $\operatorname{dom} C^\nu = \operatorname{dom} C$.
Then, $\exists \{ s_k^\nu, k \in \mathbb{N} \}$ Castaing representations of $C^\nu$ converging for each $k$ to a Castaing representation $\{ s_k, k \in \mathbb{N} \}$ of $C$.

$\square$ All Castaing representations are built via our earlier "projections".
Then, $\forall \xi \in \Xi_1, s_k^\nu(\xi) \to s_k(\xi)$, $P(\Xi_1) = 1$ the set of $a.s.$-convergence.
Since, $P$-$a.s.$ convergence of $C^\nu \to C \Longrightarrow$ (rely on 2. earlier)

$$d(a_k^1, s_k^\nu(\xi)) = d(a_k^1, C^\nu(\xi)) \to d(a_k^1, C(\xi)) = d(a_k^1, s_k(\xi)), \forall \xi \in \Xi_1. \quad \square$$

(a) Convergence of Castaing representations $\not\Rightarrow$ convergence of random sets!
(b) $v$ meas-selection of $C \Rightarrow \exists v^\nu$ meas-selection of $C^\nu$ converging $a.s.$ to $v$.

# “Simple” random sets

$C : \Xi \rightrightarrows \mathbb{R}^n$ is a *simple* random set if $\operatorname{rge} C$ is finite.
$C$ is a closed random set $\iff$ $C = P$-$a.s.$ limit of simple random sets.

$\square$ $\Leftarrow$: the limit of a sequence of random sets is a random set
$\Rightarrow$: let $C^\nu = C \cap \nu\mathbb{B}$, unif. bounded closed random set, $C = \operatorname{Lm}_\nu C^\nu$
build (via "prj") Castaing representations $\{r_k^\nu\}_{k\in\mathbb{N}}$ of the $C^\nu$
let $\{s_k^\nu\}_{k\in\mathbb{N}'} = \bigcup_{\upsilon\le\nu} \{r_k^\upsilon\}_{\upsilon\in\mathbb{N}}$, also Castaing for $C^\nu$
$D_k^\nu = \bigcup_{j\le k} s_j^\nu$ $d\!l$-converge uniformly to $C^\nu$ as $k\to\infty$
since each $s_k^\nu = \lim_{l\to\infty} s_{kl}^\nu$ uniformly, $s_{kl}^\nu$ simple random variables
$\Delta_{kl}^\nu = \bigcup_{j\le k} s_{jl}^\nu$ is a simple random set, $C(\xi) = \operatorname{Lm}_\nu \operatorname{Lm}_k \operatorname{Lm}_l \Delta_{kl}^\nu(\xi)$

$\Delta_{kl}^\nu \overset{u}{\to} D_k^\nu \overset{u}{\to} C^\nu$ allows diagonalization to find $\Delta_{k^\nu l^\nu}^\nu \to C$. $\square$

# Sierpiński-Lyapunov Theorems

$(\Xi, \mathcal{A})$ a measure space

**Sierpiński** (1922). Suppose $P$ is an atomless probability measure.
Given $A_0, A_1 \in \mathcal{A}$ with $0 \le P(A_0) \le P(A_1) \le 1$, then

$\forall \lambda \in [0,1]$, $\exists$ $A_\lambda \in \mathcal{A}$ such that $P(A_\lambda) = (1-\lambda)P(A_0) + \lambda P(A_1)$.

In particular, it implies $\forall\, \lambda \in [0,1]$, $\exists$ $A \in \mathcal{A}$ such that $P(A) = \lambda$;

choose $A_0 = \emptyset$ and $A_1 = \Xi$.

**Lyapunov** (1940) $\mu : \mathcal{A} \to \mathbb{R}^n$ atomless, $\sigma$-additive measure.
For $A \in \mathcal{A}$, define $\operatorname{rge}\mu(A) = \left\{\mu(B) \,\middle|\, B \subset A \cap \mathcal{A}\right\}$. Then,

$\operatorname{rge}\mu(\Xi) \subset \mathbb{R}^n$ is convex and if $\mu$ is also bounded, it's compact.

# Expectation: simple random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a simple random set, i.e., $\operatorname{rge} C = \left\{ z^k \in \mathbb{R}^n \,\middle|\, k \in K, |K| \text{ finite } \right\}$
Given $\bar{r}, \bar{s} \in EC = \mathbb{E}\{C(\boldsymbol{\xi})\} \implies$

$\exists$ simple selections $r, s : \Xi \to \mathbb{R}^n$ with $\mathbb{E}\{r(\boldsymbol{\xi})\} = \bar{r}, \mathbb{E}\{s(\boldsymbol{\xi})\} = \bar{s}$.

Let $\lambda \in [0,1]$. Define $v : \Xi \to \mathbb{R}^n$ as follows:

1. partition $\Xi$ into subsets $A_=$ and $\mathcal{A}_{\neq}$
2. $A_= = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = s(\xi) \right\} \in \mathcal{A}$
3. $A = \left\{ \xi \in \Xi \,\middle|\, r(\xi) = z_k, s(\xi) = z_l, k \neq l \right\} \in \mathcal{A}_{\neq}$, a finite collection
4. split each $A \in \mathcal{A}_{\neq}$, $P(A_r) = \lambda P(A)$ & $A_s = A \setminus A_r$ (Sierpiński)

$$\text{set } v(\xi) = \begin{cases} r(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_r \;\cup\; A_= \\ s(\xi) & \text{on } \bigcup_{A \in \mathcal{A}_{\neq}} A_s \end{cases}$$

then $\bar{v} = \mathbb{E}\{v(\xi)\} = \lambda \bar{r} + (1-\lambda)\bar{s} \implies EC$ convex.

Clearly $EC$ is bounded and it's easy to show it's also closed $\implies$ compact.

# Expectation of random set

$C : \Xi \rightrightarrows \mathbb{R}^n$ a closed random set

$\Longrightarrow C = P\text{-}a.s.$ limit of simple random sets,

say $C^\nu \underset{a.s.}{\to} C$ with $C^\nu \nearrow$ w.l.o.g

$EC^\nu = \mathbb{E}\{C^\nu(\boldsymbol{\xi})\} \nearrow$ are convex, compact $\Rightarrow$

$EC = \mathbb{E}\{C(\boldsymbol{\xi})\} = \bigcup_\nu EC^\nu$

$\Longrightarrow EC$ convex

$\Longrightarrow EC$ closed if $C$ is integrably bounded

$\Longrightarrow$ compact if $\operatorname{rge} C$ is bounded

# Random Mappings

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May 2012, ETH-Zürich

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) = 0$, approximations?

$$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0 \quad \text{"approximated" by } G^\nu(x) = 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample, } \quad G^\nu(x) = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x)$$

$$G : \Xi \times D \rightrightarrows E, \text{ set-valued } G(\xi, x) \subset E, \text{ inclusion } \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \ni 0$$
$$\xi^1, \dots, \xi^\nu \text{ sample, approximation } \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} G(\xi^l, x) \ni 0$$

## An appendix: more about solution bounds

$$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\}, \; x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \textstyle\int_{\Xi} f(\xi, x)\, P(d\xi)$$
$$\xi^1, \dots \xi^\nu \text{ sample } P^\nu \text{ (random) empirical measure}$$
$$\text{approx.: } \min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\xi^l, x), \; x \in C$$

# ε-Solutions Estimates

$f, g : E \to \overline{\mathbb{R}}$ lsc, convex & $\operatorname{argmin} f \cap \bar\rho\mathbb{B} \neq \emptyset \neq \operatorname{argmin} g \cap \bar\rho\mathbb{B}$

$\min f \geq -\bar\rho, \quad \min g \geq -\bar\rho$

with $\rho > \bar\rho$, $\varepsilon > 0$, $\quad \bar\eta = \hat{d\!l}_\rho(f,g)$:

$$\begin{aligned} \hat{d\!l}_\rho\big(\varepsilon\text{- argmin}\, f,\ \varepsilon\text{- argmin}\, g\big) &\leq \bar\eta\Big(1 + \frac{2\rho}{\bar\eta + \varepsilon/2}\Big) \\ &\leq \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(f,g) \end{aligned}$$

# Epi-distance alternative

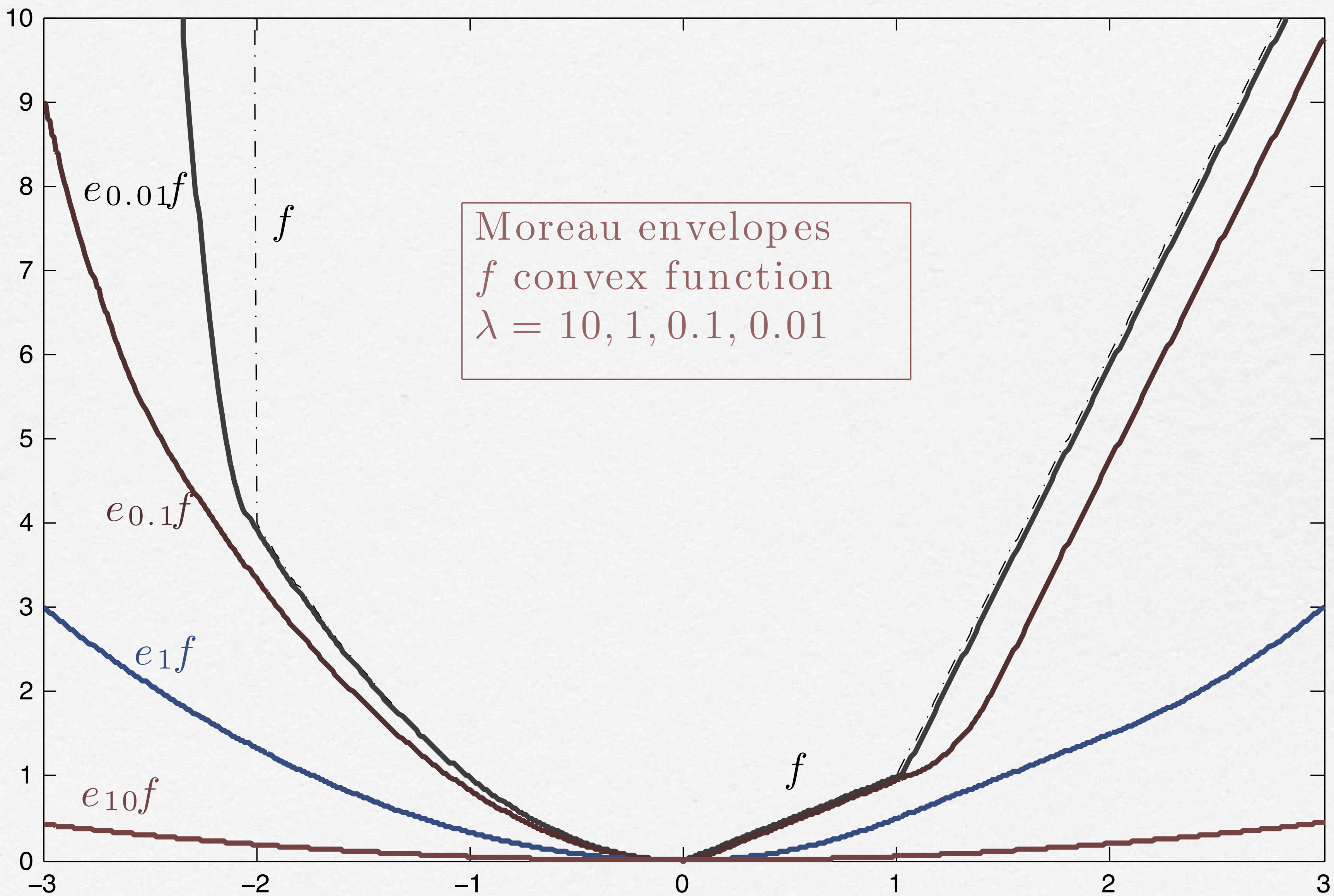
$$\check{d\!l}_{\lambda,\rho}(f,g)$$

**same topology: $\tau_{aw}$**

# Moreau envelopes

***epi-sums*** *~ sum of epigraphs*

$$(f \# g)(x) = \inf_u \{ f(u) + g(u-x) \}, \quad e_\lambda f(x) \text{ with } g = \frac{1}{2\lambda} |\cdot|^2$$



# Alternative epi-distance

$$\check{d\!l}_{\lambda,\rho}(f,g) = \sup\left\{|f_\lambda(x) - g_\lambda(x)| \,\Big|\, x \in \rho\mathbb{B}\right\}$$

$$f, g \quad \text{majorizing} \quad -\alpha_1|\cdot|^p - \alpha_0$$

1. $\forall\, \lambda \geq 0,\ \check{d\!l}_{\lambda,\rho}(f,g) \leq \beta(\lambda,\rho)\hat{d\!l}_{\gamma(\lambda,\rho)}(f,g)$

2. $\hat{d\!l}_\rho(f_\lambda, g_\lambda) \leq \check{d\!l}_{\lambda,\rho}(f,g)$
   $\hat{d\!l}_\rho(f,g) \leq \check{d\!l}_{\lambda,9\rho}(f,g) + \kappa(\lambda,\alpha_1,\alpha_0,p)$

# "Quantitative" LLN-a.s.

$E$ separable Banach space, $f$ random lsc function, $\{\boldsymbol{\xi}, \boldsymbol{\xi}^\nu\}_{\nu\in\mathbb{N}}$ iid

1. $\{f(\xi,\,\cdot\,), \xi \in \Xi\}$ separable subspace (lsc-fcns($E$), $\tau_{aw}$)
2. $P$-$a.s.$, $\forall\,\theta > 0, \rho \geq 0, \nu$ :

   $\check{d\!l}_{\theta,\lambda}\Big(\frac{1}{\nu}\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l,\,\cdot\,), \frac{1}{\nu}\sum_{l=1}^{\nu} f_\lambda(\boldsymbol{\xi}^l,\,\cdot\,)\Big) \leq \varepsilon_{\theta,\rho}(\lambda)$

   with $\varepsilon_{\theta,\rho}(\lambda) \to 0$ as $\lambda \searrow 0$
3. $\forall\,\theta > 0, \rho \geq 0, \check{d\!l}_{\theta,\rho}\big(Ef_\lambda, Ef\big) \searrow 0$ as $\lambda \searrow 0$.

Then, $d\!l(E^\nu f, Ef) \to 0 \quad P^\infty$-$a.s.$

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convex

$x \mapsto f(\xi, x)$ convex $\implies$ conditions 2 & 3.

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$\hat{d\!l}_\rho\big(\varepsilon\text{- argmin}\, E^\nu f,\ \varepsilon\text{- argmin}\, Ef\big) \leq \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(E^\nu f, Ef)$

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   $$\check{d\!l}_{\theta,\lambda}\left(\tfrac{1}{\nu}\textstyle\sum_{l=1}^{\nu} f(\boldsymbol{\xi}^l,\cdot\,), \tfrac{1}{\nu}\sum_{l=1}^{\nu} f_\lambda(\boldsymbol{\xi}^l,\cdot\,)\right) \leq \varepsilon_{\theta,\rho}(\lambda)$$

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$$\hat{d\!l}_\rho\big(\varepsilon\text{-}\operatorname{argmin} E^\nu f,\ \varepsilon\text{-}\operatorname{argmin} Ef\big) \leq \big(1 + 4\rho\varepsilon^{-1}\big)\hat{d\!l}_\rho(E^\nu f, Ef)$$

$E$ reflexive, $E^\nu f \xrightarrow[s,w]{} Ef \Longrightarrow d\!l(E^\nu f, Ef) \to 0\ \ a.s.$

# Approximating Mappings

# Why?

$G : E \to \mathbb{R}^d, \quad G^{-1}(0)$ soln's of $G(x) \ni 0$, approximations?

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} = 0$ "approximated" by $G^\nu(x) = 0$

$\xi^1, \ldots, \xi^\nu$ sample, $G^\nu(x) = \frac{1}{\nu}\sum_{l=1}^{\nu} G(\xi^l, x)$

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$\min \mathbb{E}\{f(\boldsymbol{\xi}, x)\},\ x \in C, \quad \mathbb{E}\{f(\boldsymbol{\xi}, x)\} = Ef(x) = \int_\Xi f(\xi, x)\, P(d\xi)$

$\xi^1, \ldots \xi^\nu$ sample $P^\nu$ (random) empirical measure

approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu}\sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

# Examples:

$\min f = f_0 + \iota_C$, optimality: "$0 \in \partial f(\bar{x}) = S(x)$" $\sim 0 = \nabla f(\bar{x})$)

generally, $\partial(f+g) \neq \partial f + \partial g$

$\mathbb{C}.\mathbb{Q}.$ (Constraint Qualification): $-N_C(\bar{x}) \cap \partial^\infty f_0(\bar{x}) = \{0\}$

$v \in \partial^\infty f_0(\bar{x}) =$ horizon subgradient if

$\exists\, x^\nu \to \bar{x}$ with $f(x^\nu) \to f(\bar{x}), v^\nu \in \hat{\partial} f(x^\nu), \lambda_\nu \searrow 0$ & $\lambda_\nu v^\nu \to v$

with $\mathbb{C}.\mathbb{Q}.$ $\bar{x}$ locally optimal $\Rightarrow \partial f_0(\bar{x}) + N_C(\bar{x}) = S(\bar{x}) \ni 0$

$f$ convex ($\Rightarrow$ regular), $\partial f_0(\bar{x}) + N_C(\bar{x}) \ni 0$

$\Rightarrow$ globally optimal (without $\mathbb{C}.\mathbb{Q}$)

When $f_0, C$ are convex: $-\partial f_0(\bar{x}) \in N_C(\bar{x})$,

a functional variational inequality

# “Variational” Approximations

$(E, d)$ Polish, in paricular $E = \mathbb{R}^n$

$(\text{cl-sets}(E), d\!l)$ complete metric space; Polish if $E = \mathbb{R}^n$

$d\!l(C^\nu, C) \to 0 \iff C^\nu \to C$

osc-mappings = closed graph

$(\text{osc-maps}(S), d\!l)$ complete, metric space;

Polish if $\text{dom} \subset \mathbb{R}^n$, $\text{rge} \subset \mathbb{R}^m$

Convergence:

$S^\nu \xrightarrow{g} S$ if $d\!l(\text{gph}\, S^\nu, \text{gph}\, S) \to 0 \implies (S^\nu)^{-1}(0) \Rightarrow_v S^-1(0)$

# Why?

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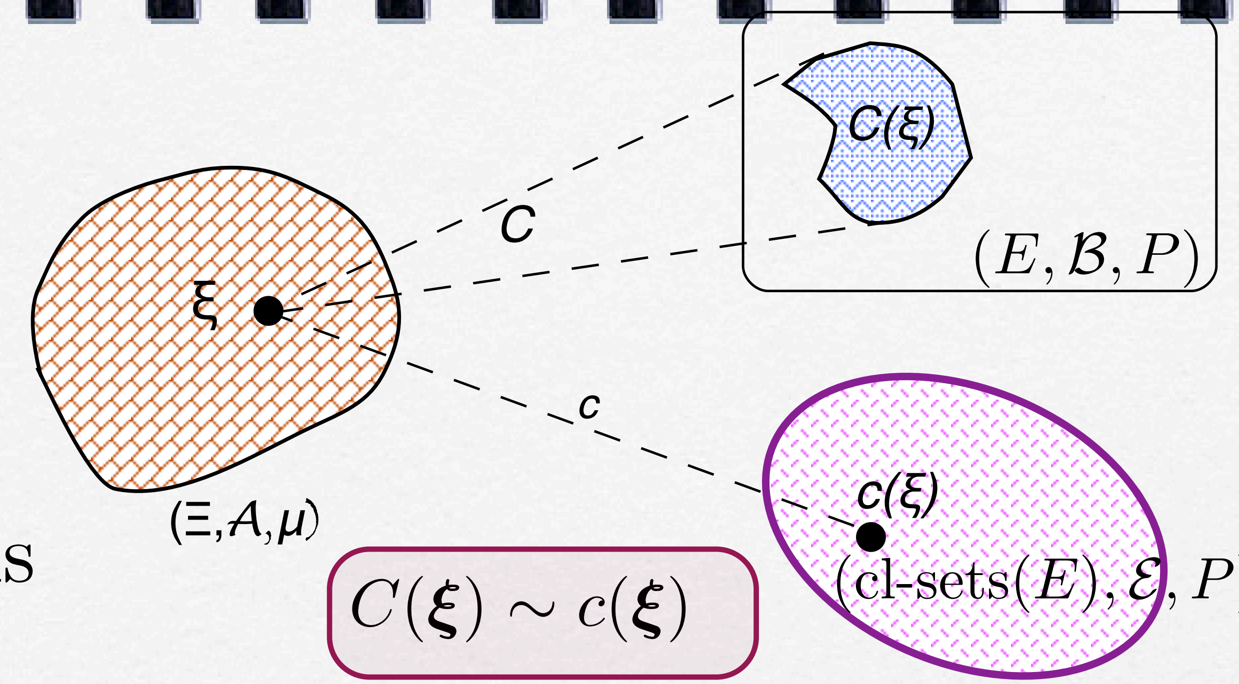
approx.: $\min \mathbb{E}^\nu\{f(\boldsymbol{\xi}, x)\} = \frac{1}{\nu} \sum_{l=1}^{\nu} f(\xi^l, x),\ x \in C$

# Random sets




$C$ 'covered' by countable selections
*Castaing represnetation*

a.s convergence: $P\{\xi \,|\, d\!l(C^\nu(\xi), C(\xi)) \to 0\} = 0$

$\Rightarrow$ in probability: $\forall \varepsilon > 0,\ P\{\xi \,|\, d\!l(C^\nu(\xi), C(\xi)) > \varepsilon\} \to 0$

$\Rightarrow$ in distribution $T$ : cpct-sets$(E) \to [0,1],\ T(\emptyset) = 0,$
(a) $T(K^\nu) \searrow T(K)$ for $K^\nu) \searrow K$, (b) 'rectangle cond'n'
$P^\nu \xrightarrow{\mathcal{D}} P \iff T^\nu \to T$ on cpct-sets$(\mathbb{R}^n)$
or, even, on finite union of closed rational balls.

# Random Sets: Expectation

# Random set: Expectation

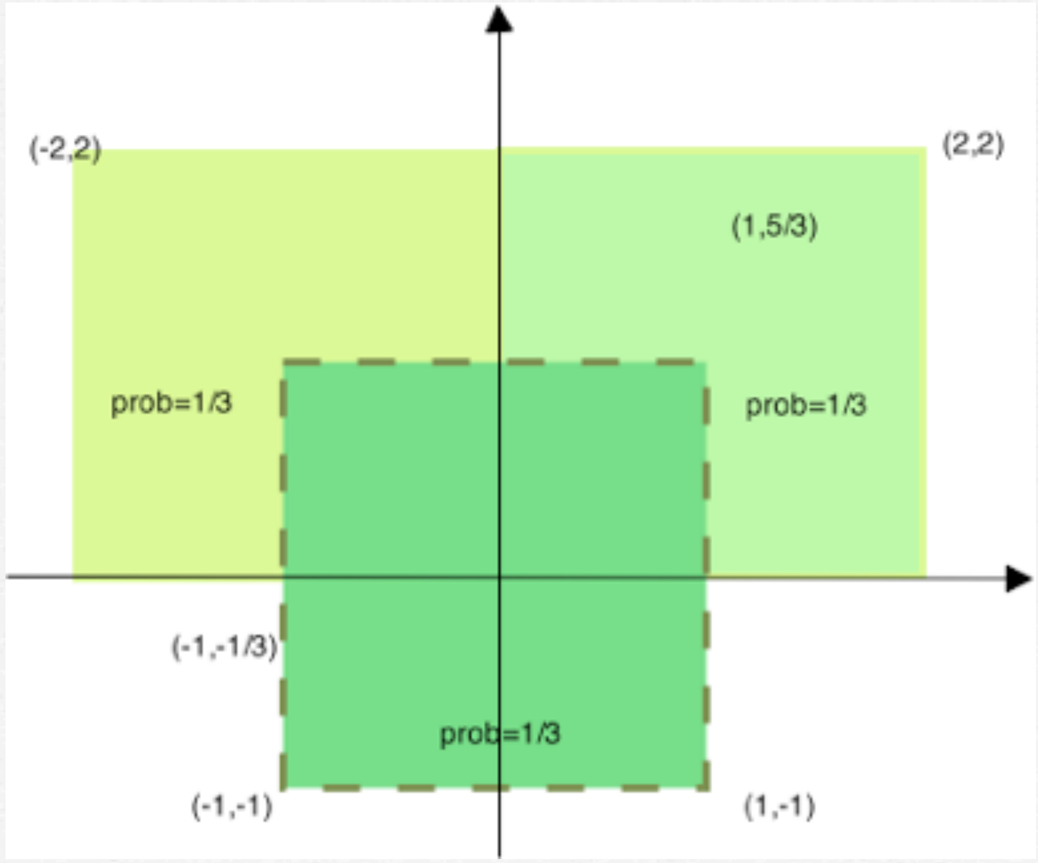
$$EC = \mathbb{E}\{C(\xi)\} = \left\{ \int_\Xi s(\xi)\,P(d\xi) \,\middle\|\, s(\bullet)\ P\text{-summable selection} \right\}$$

..not necessarily closed even when $C$ is closed-valued

Convexity:

$C$ $P$-atom convex $\Rightarrow EC$ is convex

(certainly when $P$ is atomless).



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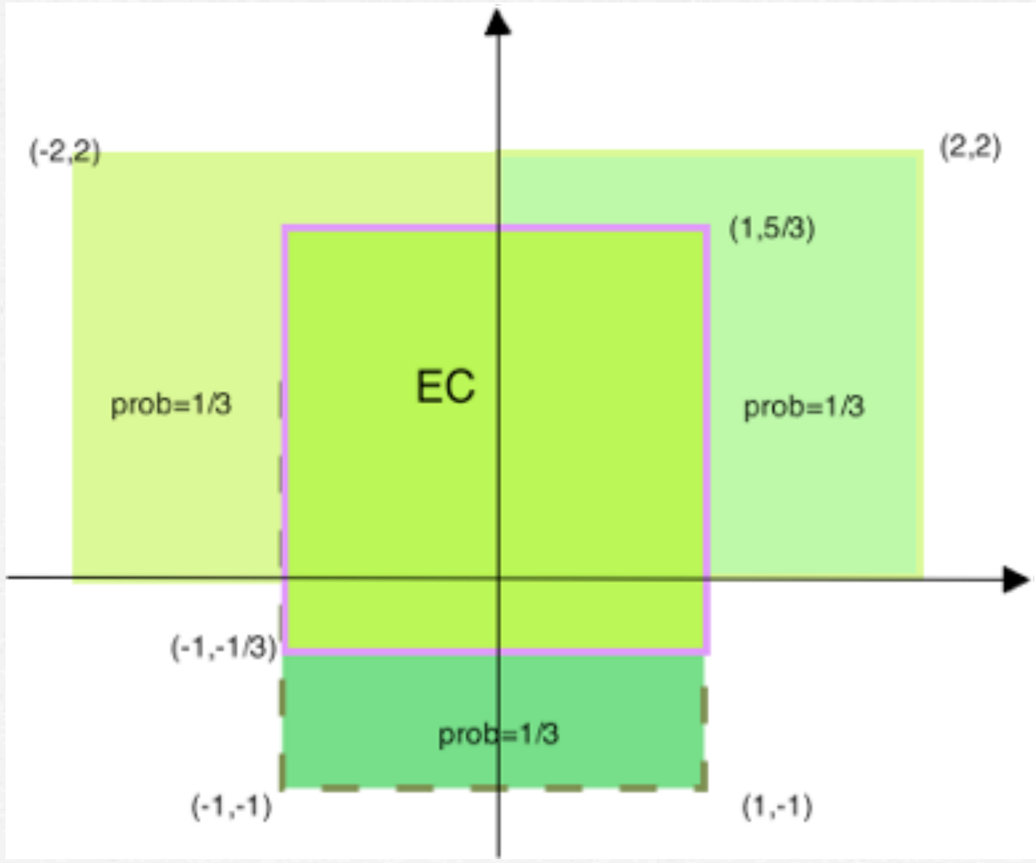
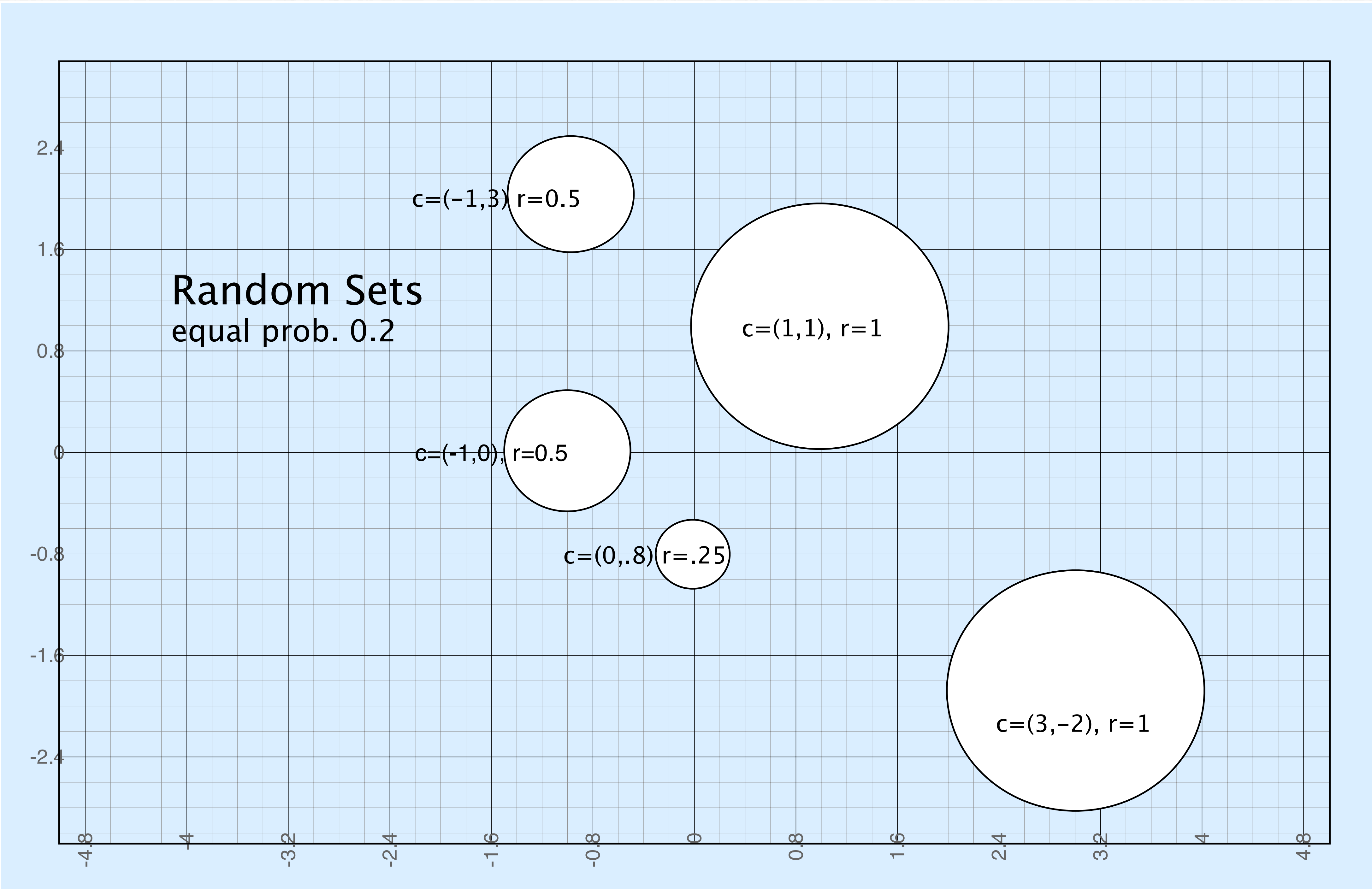
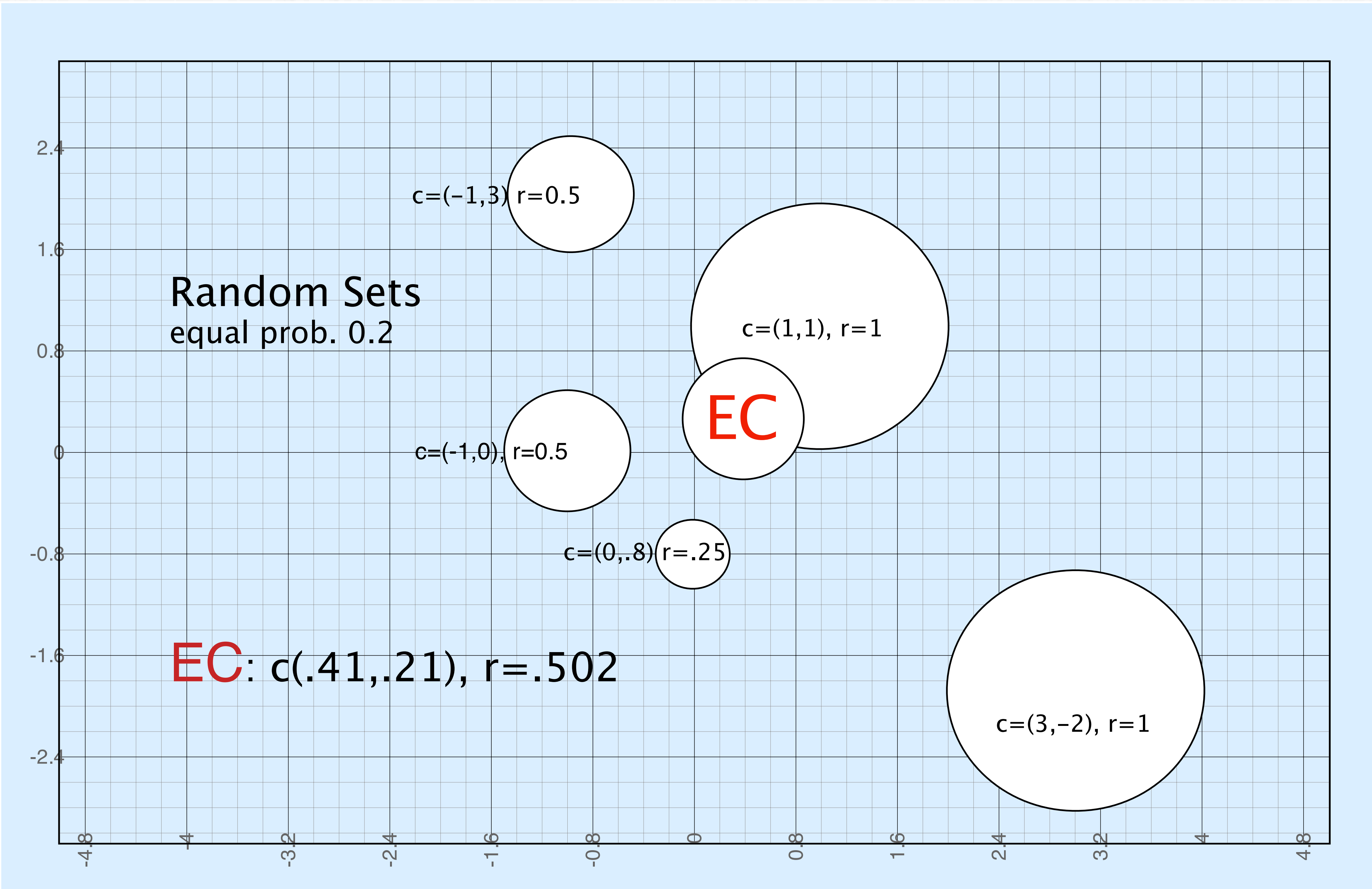


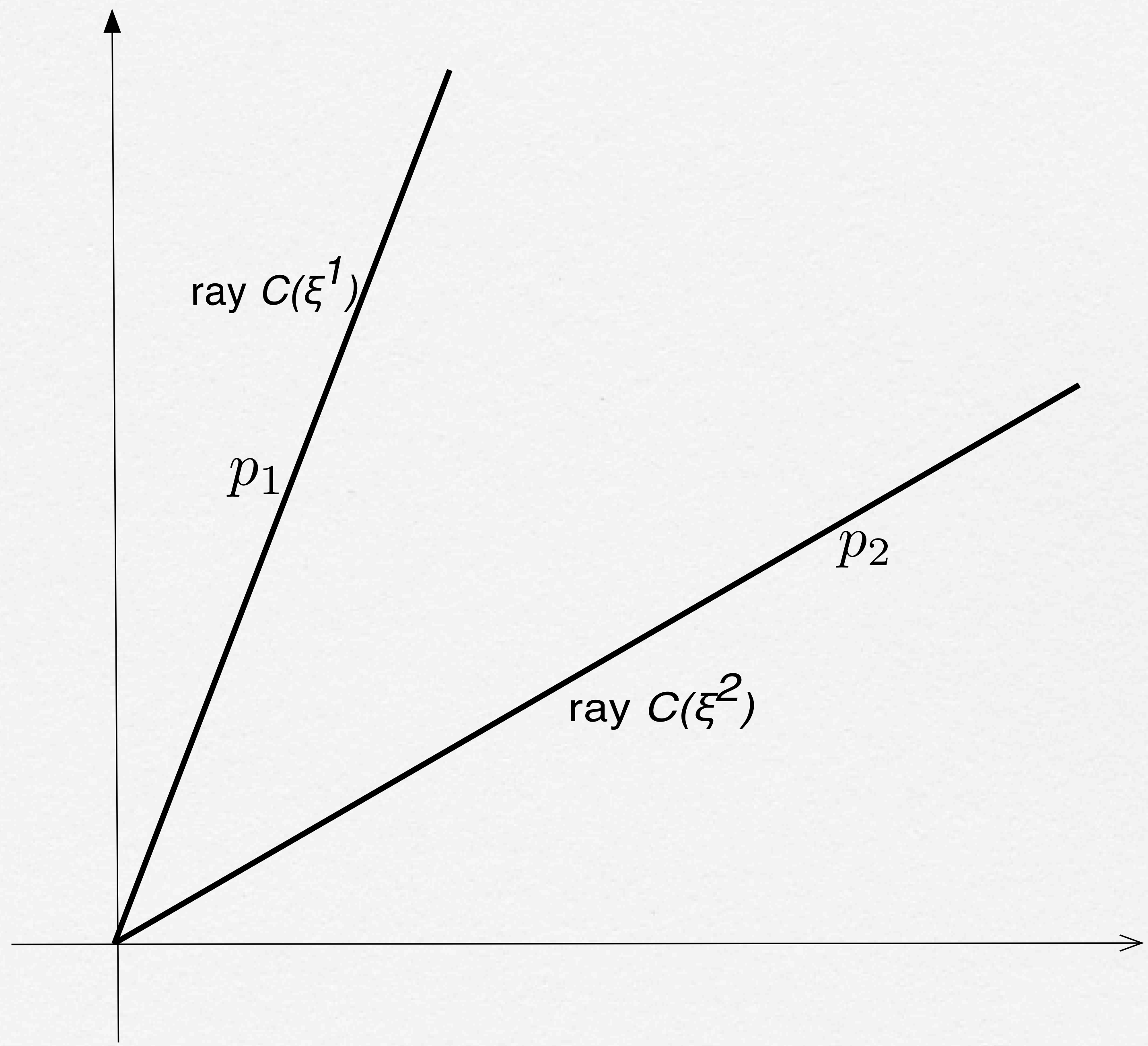
# Bounded random set



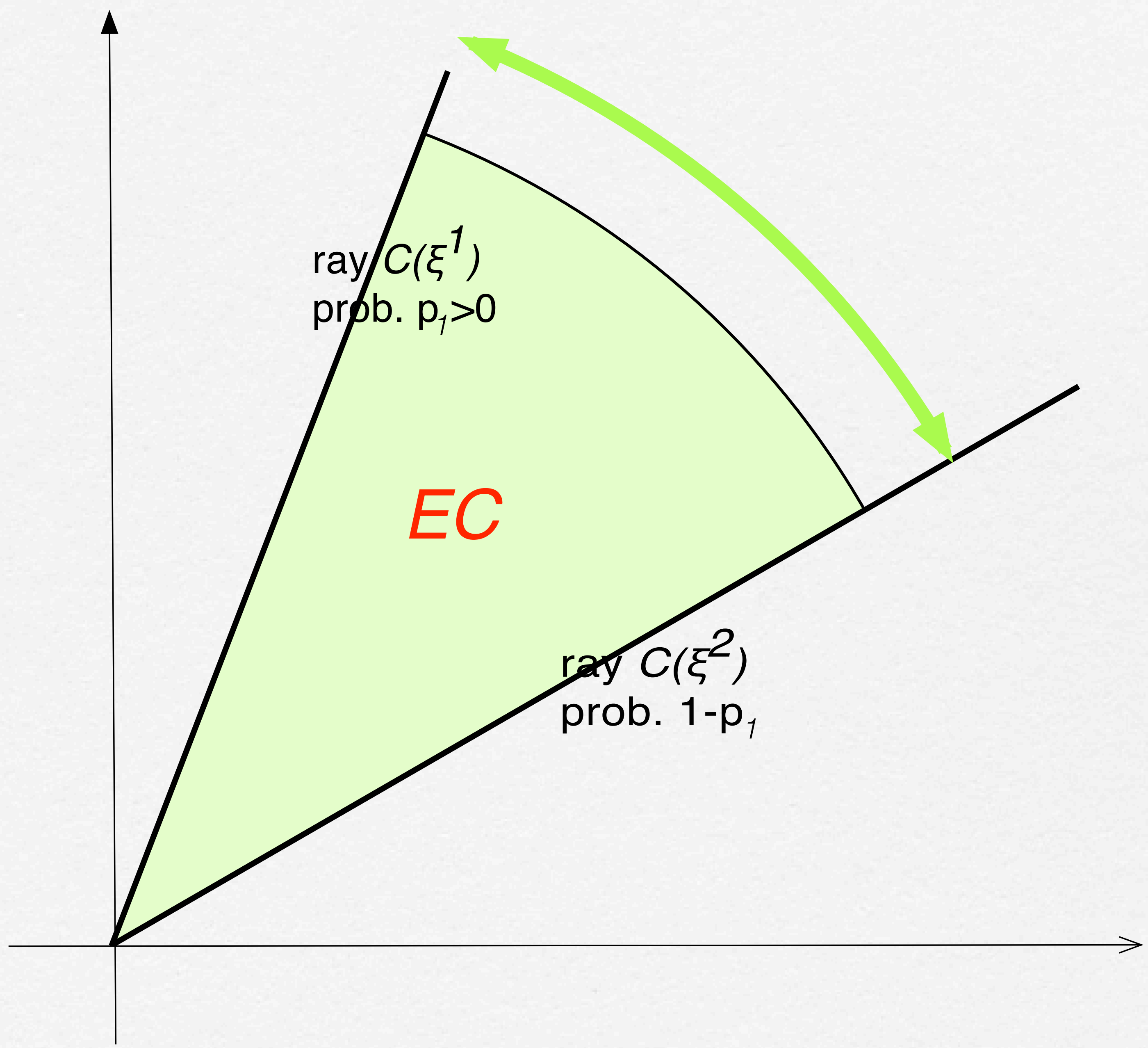
# Expectation: Bounded r. set



# Expectation: Unbounded r. sets



# Expectation: Unbounded r. sets



# Some properties: $\mathbb{E}\{C(\boldsymbol{\xi})\}$

- measure $P$ atomless, then $EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ is convex (Richter, Lyapounov,...)
- $P$ is $P$-atom convex $\Longrightarrow$ $EC$ is convex; [an atom contains no (measurable) subset of positive probability]
- $C$ a random set, $\emptyset \neq EC = \mathbb{E}\{C(\boldsymbol{\xi})\}$ contains no line, then

  $$\operatorname{con} EC = \mathbb{E}\{\operatorname{con} X(\boldsymbol{\xi})\}$$

  this essentially requires that $C(\xi) \subset$ a pointed cone
- in general, the expectation of a (closed-valued) random set is *not* closed
- if $|C| = \mathbb{E}\big\{\sup\left[\,|s(\xi)|\,\big|\, s(\xi) \in C(\xi)\,\right]\big\} < \infty$ then $EC$ is closed; $C$ is then *integrably bounded.*

# Strong law of large numbers

## for random sets *(Artstein-Hart)*

$C : \Xi \rightrightarrows E$ measurable, $\{\xi^\nu, \nu \in \mathbb{N}\}$ iid $\Xi$-valued random variables

$C(\xi^\nu)$ iid random sets (i.e. induced $P^\nu$ independent and identical)

$EC = \mathbb{E}\{C(\bullet)\} = \left\{ \int_\Xi s(\xi) P(d\xi) \,\middle|\, s : P\text{-summable } C(\xi)\text{-selection} \right\}$

independence $\Rightarrow$ all (measurable) selections are independent

$\{C(\xi^\nu) : \Xi \rightrightarrows \mathbb{R}^m \; \nu \in \mathbb{N}\}$ iid with $EC \neq \emptyset$. Then, with

$$C^\nu(\xi^\infty) = \nu^{-1}\left(\sum_{k=1}^{\nu} C(\xi^k)\right) \to \bar{C} = \text{cl con } EC \;\; P^\infty\text{-a.s.}$$

$\mathrm{Lo}_\nu C^\nu(\xi^\infty) \subset \bar{C} \;\Leftrightarrow\; \limsup_\nu \sigma_{C^\nu} \leq \sigma_{\bar{C}}$ support functions

$\mathrm{Li}_\nu C^\nu(\xi^\infty) \supset \bar{C}$ relies on LLN for (vector-valued) selections

***Proof: time allowing***

# Random mappings

$S : \Xi \times E \rightrightarrows \mathbb{R}^m, \quad E \subset \mathbb{R}^n$

$\mathcal{A} \otimes \mathcal{B}^n$-jointly measurable: $S^{-1}(O) \in \mathcal{A} \otimes \mathcal{B}^n$, $O$ open

$\Rightarrow \forall\, x : \xi \mapsto S(\xi, x)$ a random set

random closed set when $S$ is closed-valued

$ES : E \rightrightarrows \mathbb{R}^m$ with $ES(x) = \mathbb{E}\{S(\xi, x)\}$ expected mapping

$ES$ convex-valued when $\xi \mapsto S(\xi, \cdot)$ $P$-atom convex

Law of Large Numbers for random sets

applies pointwise

# Sample Average Approximation (SAA)

stochastic variational problem: $\bar{S}(x) = \mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

$S : \Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ random set-valued mapping

$\boldsymbol{\xi}$ random vector with values $\xi \in \Xi \subset \mathbb{R}^N$

solution (a 'stationary point') $\bar{x} \in \bar{S}^{-1}(0)$

—o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o —o

sample $\overset{\rightarrow\nu}{\xi} = (\xi^1, \ldots, \xi^\nu)$ of $\boldsymbol{\xi}$

$\frac{1}{\nu}\left(\sum_{k=1}^{\nu} S(\xi^k, x)\right) = S^\nu(\overset{\rightarrow\nu}{\xi}, x) \ni 0$, approximating system?

i.e., $\left(S^\nu\right)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$ a.s. ???

# So far ...

➪ *generalized equations*

$S : \Xi \times D \rightrightarrows E$, set-valued $S(\xi, x) \subset E$, inclusion $\mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

iid-sample $\vec{\xi}^{\,\nu} = \xi^1, \ldots, \xi^\nu$ and $x \mapsto S(\xi, x)$ osc

SAA-mapping $S^\nu : \Xi^\infty \times D \rightrightarrows E$, random osc mappings

$$S^\nu(\xi, x) = \frac{1}{\nu}\sum_{k=1}^{\nu} S(\xi^k, x) \eqsim S^\nu(\vec{\xi}^{\,\nu}, x),\ \forall \xi \in \Xi^\infty$$

# So far ... ➪ *generalized equations*

$S : \Xi \times D \rightrightarrows E$, set-valued $S(\xi, x) \subset E$, inclusion $\mathbb{E}\{S(\boldsymbol{\xi}, x)\} \ni 0$

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SAA-mapping $S^\nu : \Xi^\infty \times D \rightrightarrows E$, random osc mappings

$S^\nu(\xi, x) = \frac{1}{\nu} \sum_{k=1}^{\nu} S(\xi^k, x) \approx S^\nu(\vec{\xi}^{\,\nu}, x)$, $\forall \xi \in \Xi^\infty$

$\forall x \in D$, $S(\cdot, x)$, closed random set,

let $\bar{S} = \mathrm{cl}\,\mathrm{con}\, ES$, $\quad ES(x) = \mathbb{E}\{S(x, \boldsymbol{\xi})\}$

Artstein-Hart LLN applies: $S^\nu \xrightarrow{p} \bar{S}$ $a.s.$ when $E = \mathbb{R}^m$

but $\xrightarrow{p}$ $\not\Rightarrow$ $(S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

# So far ...

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but $\xrightarrow{p}$ $\not\Rightarrow$ $(S^\nu)^{-1}(0) \rightrightarrows \bar{S}^{-1}(0)$. Needed $S^\nu \xrightarrow{g} \bar{S}$

recall: $\bar{S}(x) = \operatorname{cl} ES(x)$ when $P$-atom convex, $ES(x)$ closed if $\xi \mapsto S(\xi, x)$ is integrably bounded and compact if $\operatorname{rge} S(\cdot, x)$ is bounded.

# Consistent approximations?

$$S^\nu(\xi,\cdot) \to_p \bar{S} \;\; P^\infty\text{-a.s.} \Rightarrow ? \;\; S^\nu(\xi,\cdot)^{-1}(0) \Rightarrow_\nu \bar{S}^{-1}(0)$$

sometimes!

graphical rather than pointwise convergence is required

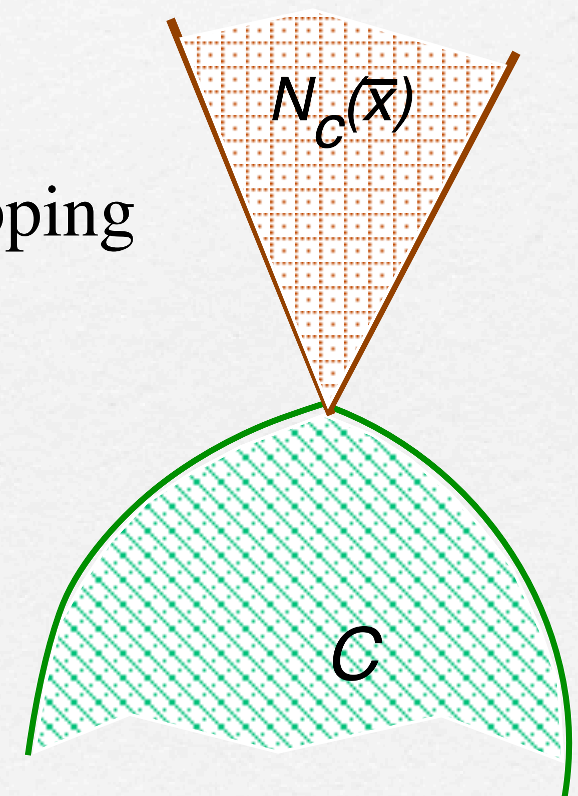
$$S^\nu(\xi,\cdot) \underset{\text{gph}}{\longrightarrow} \bar{S} \;\; P^\infty\text{-a.s. is needed}$$

relationship between graphical and pointwise convergence?

# Some Examples

# Stochastic VI, Variational Inequality

*Network flow equilibrium with stochastic demand and link capacities*
*Economic equilibrium in a stochastic environment*




$\xi = (\xi^1, \xi^2, \ldots),\ \ G^\nu(\cdot, x)\ \ \sigma\text{-}(\xi^1, \ldots \xi^\nu)$ measurable

$-G^\nu(\xi, x) \in N_C(x),\ \ C$ compact, convex

$N_{C(x)} + G^\nu(\xi, x) = S^\nu(x) \ni 0, \quad S^\nu$ closed set-valued mapping

$G^\nu(\xi, \cdot) \to^{?} G(\xi, \cdot)$

$x^\nu(\xi)$ solution of $-G^\nu(\xi, x) \in N_C(x)$ for sample $\xi \approx \overset{\to\nu}{\xi}$

does $x^\nu(\xi) \ \to$ a solution of $-G(\xi, x) \in N_C(x)$? a.s.

what if $C$ depends on $(\xi, \nu)$: sequence of random sets $C^\nu(\xi)$?

# “static” Walras Equilibrium

agent's problem: $a \in \mathcal{A}$, $|\mathcal{A}|$ finite, possibly "large"

$\bar{x}_a \in \arg\max u_a(x_a)$ so that $\langle p, x_a \rangle \le \langle p, e_a \rangle$, $x_a \in X_a$

$e_a$: endowment of agent a, $e_a \in \operatorname{int} X_a$

$u_a$: utility of agent $a$, concave, usc

$u_a : X_a \to \mathbb{R}$, $X_a \subset \mathbb{R}^n$ (survival set) convex

market clearing: $s(p) = \sum_{a \in \mathcal{A}} (e_a - \bar{x}_a)$ excess supply

equilibirum price: $\bar{p} \in \Delta$ such that $s(\bar{p}) \ge 0$, $\Delta$ unit simplex

# Walras: a Variational Inequality

$$c_a = \arg\max_x u_a(x) \text{ so that } \langle p, x \rangle \le \langle p, e \rangle,\ x \in C_a$$

$$\sum_a (e_a - c_a) = s(p) \ge 0.$$

$$\Updownarrow$$

$$N_D(\bar{z}) = \left\{ v \middle| \langle v, z - \bar{z} \rangle \le 0, \forall z \in D \right\}$$

$$G(p,(x_a),(\lambda_a)) = \left[ \sum_a (e_a - x_a); (\lambda_a p - \nabla u_a(x_a)); \langle p, e_a - x_a \rangle \right]$$

$$D = \Delta \times \left( \prod_a C_a \right) \times \left( \prod_a \mathbb{R}_+ \right)$$

$$-G(\bar{p},(\bar{x}_a),(\bar{\lambda}_a)) \in N_D(\bar{p},(\bar{x}_a),(\bar{\lambda}_a))$$

$$D \text{ unbounded} \to \hat{D} \text{ bounded}$$

# Equilibrium: stochastic environment

$$(c^1_a, y_a, c^2_{a,\xi}) = \arg\max_{x^1, y\in\mathbb{R}^L, x^2_\bullet \in \mathcal{M}} u^1_a(x^1) + \mathbb{E}^a\left\{u^2_a(\boldsymbol{\xi}, x^2(\boldsymbol{\xi}))\right\}$$

$$\text{such that } \left\langle p^1, x^1_a + T^1_a y\right\rangle \le \left\langle p^1, e^1_a\right\rangle$$

$$\left\langle p^2_\xi, x^2_{a,\xi}\right\rangle \le \left\langle p^2_\xi, e^2_{a,\xi} + T^2_{a,\xi} y\right\rangle, \ \forall \xi \in \Xi$$

$$x^1_a \in X^1_a, \ x^2_{a,\xi} \in X^2_{a,\xi}, \quad \forall \xi \in \Xi$$

$\mathbb{E}^a\left\{\bullet\right\}$ expecttion with respect to $a$-beliefs, $\Xi$ finite support

2-stage stochastic programs with recourse

solution procedures & approximation theory "well-estblished"

$T^1_a, T^2_{a,\xi}$ : input-output matrices (production, investments)

$e^1_a \in \operatorname{int} X^1_a, \ e^2_{a,\xi} \in \operatorname{int} X^2_{a,\xi}$ for all $\xi$

# Market Clearing ~Equilibrium

excess supply: agent-$a$: $\left( c_a^1, y_a^1, \left\{ c_{a,\xi}^2 \right\}_{\xi \in \Xi} \right)$

$$\sum_{a \in \mathcal{A}} \left( e_a^1 - (c_a^1 + T_a^1 y_a) \right) = s^1 \left( p^1, \{p_\xi^2\}_{\xi \in \Xi} \right) \geq 0$$

$$\forall \xi, \ \sum_{a \in \mathcal{A}} \left( (e_{a,\xi}^2 + T_{a,\xi}^2) - c_{a,\xi}^2 \right) = s_\xi^2 \left( p^1, \{p_\xi^2\}_{\xi \in \Xi} \right) \geq 0$$

Variational inequality: $-G(p,(x_a),(\lambda_a)) \in N_D(p,(x_a),(\lambda_a))$,

$$p = \left( p^1, \{p_\xi^2\}_{\xi \in \Xi} \right), x = \left( x^1, \{x_\xi^2\}_{\xi \in \Xi} \right), \lambda = \left( \lambda^1, \{\lambda_\xi^2\}_{\xi \in \Xi} \right)$$

$$S(\xi,(p,x,\lambda)) = G(\xi,(x,p,\lambda)) + N_{D(\xi)}(p,x,\lambda)),$$

$$\mathbb{E}\left\{ S(\xi,(p,x,\lambda)) \right\} \ni 0$$

# a.s. Congergence of SAA-mappings

# Graphical vs Pointwise convergence

$S, S^\nu : X \rightrightarrows \mathbb{R}^m$. Then, $S^\nu \underset{\text{point}}{\longrightarrow} S$ and $S^\nu \underset{\text{gph}}{\longrightarrow} S$ (at $x$)

$\Leftrightarrow \{C^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically) (at $x$)

~ Arzela-Ascoli Theorem for set-valued mappings

$S$ random mapping, $P^\infty$-a.s., $S^\nu(\xi, \cdot) \underset{\text{point}}{\longrightarrow} \operatorname{cl} \operatorname{con} ES = \bar{S}$

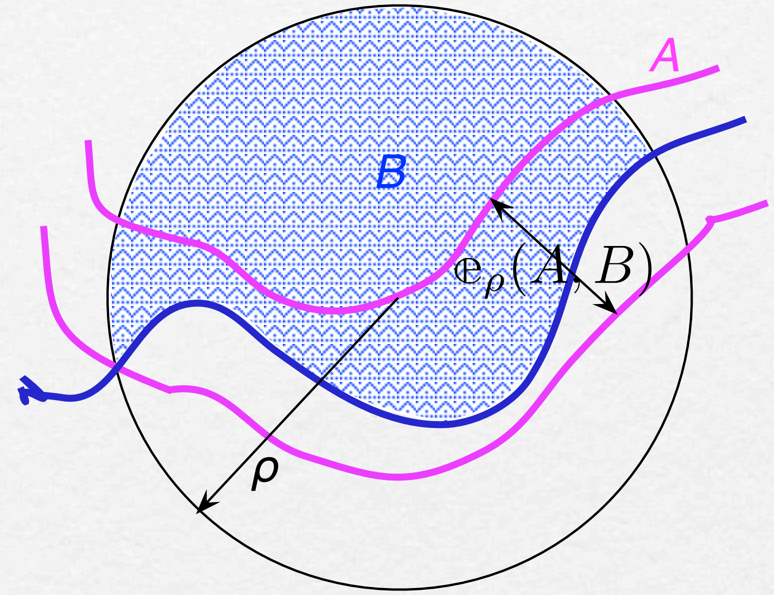
then $S^\nu \underset{\text{gph}}{\longrightarrow} \bar{S} \Leftrightarrow \{S^\nu, \nu \in \mathbb{N}\}$ are equi-osc (asymptotically)

# Semicontinuity: osc/isc

$S : D \rightrightarrows \mathbb{R}^m$ continuous at $\bar{x}$ if $\lim_{x^\nu \to \bar{x}} d\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$

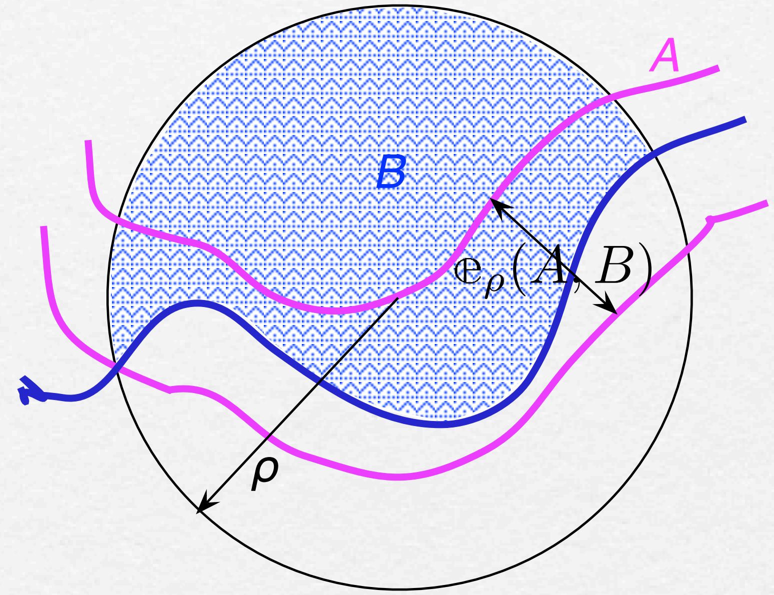
$$d\!l(S(x^\nu), S(\bar{x})) \to 0 \iff d\!l_\rho(S(x^\nu), S(\bar{x})) \to 0$$

$$\iff \hat{d\!l}_\rho(S(x^\nu), S(\bar{x})) \to 0. \forall\, \rho > \bar{\rho} \geq 0$$



# Semicontinuity: osc/isc

$$S : D \rightrightarrows \mathbb{R}^m \text{ continuous at } \bar{x} \text{ if } \lim_{x^\nu \to \bar{x}} d\!\!l\big(S(x^\nu), S(\bar{x})\big) \to 0$$




$$d\!\!l(S(x^\nu), S(\bar{x})) \to 0 \iff d\!\!l_\rho(S(x^\nu), S(\bar{x})) \to 0$$

$$\iff \hat{d\!\!l}_\rho(S(x^\nu), S(\bar{x})) \to 0. \forall\, \rho > \bar{\rho} \geq 0$$

$$\hat{d\!\!l}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\big[\, e_\rho\big(S(x^\nu), S(\bar{x})\big), e_\rho\big(S(\bar{x}), S(x^\nu)\big)\,\big]$$

# Semicontinuity: **osc**/isc

$$S : D \rightrightarrows \mathbb{R}^m \text{ continuous at } \bar{x} \text{ if } \lim_{x^\nu \to \bar{x}} \mathbb{d}\big(S(x^\nu), S(\bar{x})\big) \to 0$$

$$\mathbb{d}(S(x^\nu), S(\bar{x})) \to 0 \iff \mathbb{d}_\rho(S(x^\nu), S(\bar{x})) \to 0$$

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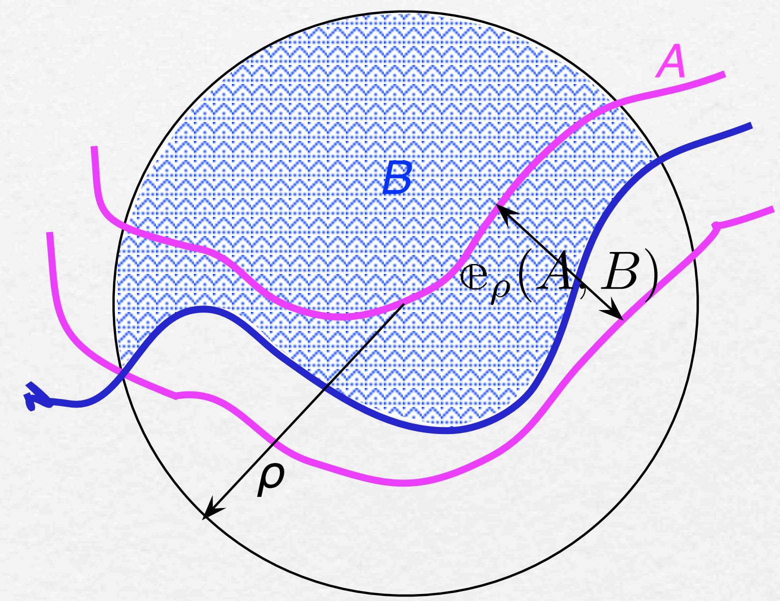


$$\hat{\mathbb{d}}_\rho\big(S(x^\nu), S(\bar{x})\big) = \max\big[\, \mathbb{e}_\rho\big(S(x^\nu), S(\bar{x})\big), \mathbb{e}_\rho\big(S(\bar{x}), S(x^\nu)\big) \,\big]$$

$S$ is osc (outer semicontinuous) at $\bar{x}$ if $\mathbb{e}_\rho\big(S(x^\nu), S(\bar{x})\big) \to 0$ as $x^\nu \to \bar{x}$

$S$ is isc (inner semicontinuous) at $\bar{x}$ if $\mathbb{e}_\rho\big(S(\bar{x}), S(x^\nu)\big) \to 0$ as $x^\nu \to \bar{x}$

# Equi-osc mappings

$S : D \rightrightarrows \mathbb{R}^m$, $D \subset \mathbb{R}^n$ is osc if gph $S$ is closed

osc at $\bar{x}$ : given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) : e_\rho\big(S(x), S(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$\{S^\nu : D \rightrightarrows \mathbb{R}^m\}$ are equi-osc at $\bar{x}$

given any $\rho > 0, \epsilon > 0$

$$\exists V \in \mathcal{N}(\bar{x}) : e_\rho\big(S^\nu(x), S^\nu(\bar{x})\big) < \varepsilon,\ \forall\, x \in V$$

$V = V(\rho, \epsilon)$ doesn't depend on $\nu$.

# G-convergence of SAA-mappings

$S: \Xi \times X \rightrightarrows \mathbb{R}^m$ random mapping, $(\Xi, \mathcal{A}, P)$

$P^\infty$-a.s.: $S^\nu(\boldsymbol{\xi}, \cdot) \underset{\text{gph}}{\longrightarrow} \bar{S}$ at $\bar{x} \Leftrightarrow$ SAA-mappings $\{S^\nu(\boldsymbol{\xi}, \cdot)\}$ equi-osc at $\bar{x}$

$\Rightarrow$ sol'ns of $S^\nu(\xi, \cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: $P^\infty$-a.s.

$S(\boldsymbol{\xi}, \cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\boldsymbol{\xi}, \cdot)\}$ equi-osc

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$\Rightarrow$ sol'ns of $S^\nu(\xi, \cdot) \ni 0 \Rightarrow_\nu$ sol'ns of $\bar{S}(\cdot) \ni 0$

Sufficient condition: $P^\infty$-a.s.

$S(\xi, \cdot)$ stably osc & steady under averaging $\Rightarrow$ $\{S^\nu(\xi, \cdot)\}$ equi-osc

Law of large Numbers for Random Mappings

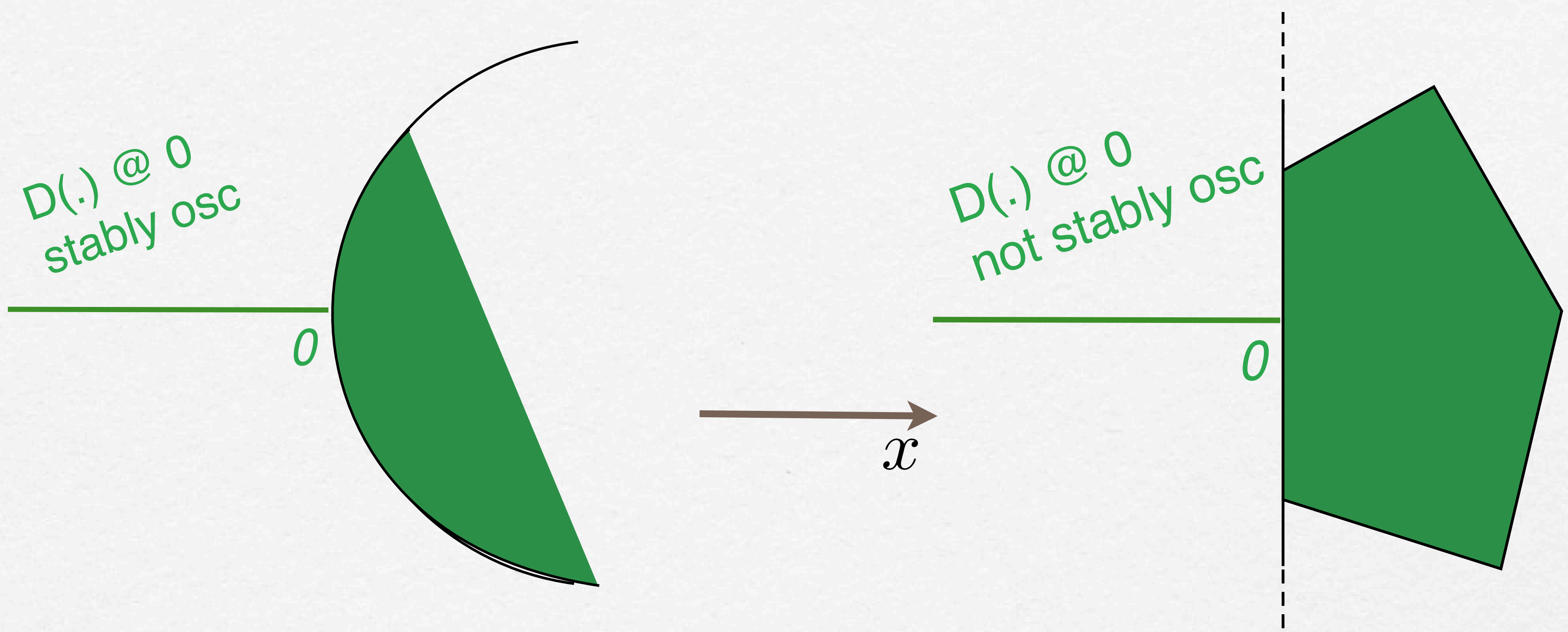
$S$ random osc mapping: $\Xi \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$

stably osc & steady under averaging

$\xi^1, \xi^2, \ldots,$ iid random variables (values in $\Xi$), distribution $P$

Then, $\nu^{-1} \sum_{k=1}^{\nu} S(\xi^k, \cdot) \to_{\text{gph}} \bar{S} = \operatorname{cl} \operatorname{con} E\{S(\xi^0, \cdot)\}$ $P^\infty$-a.s.

# Stably osc mapping

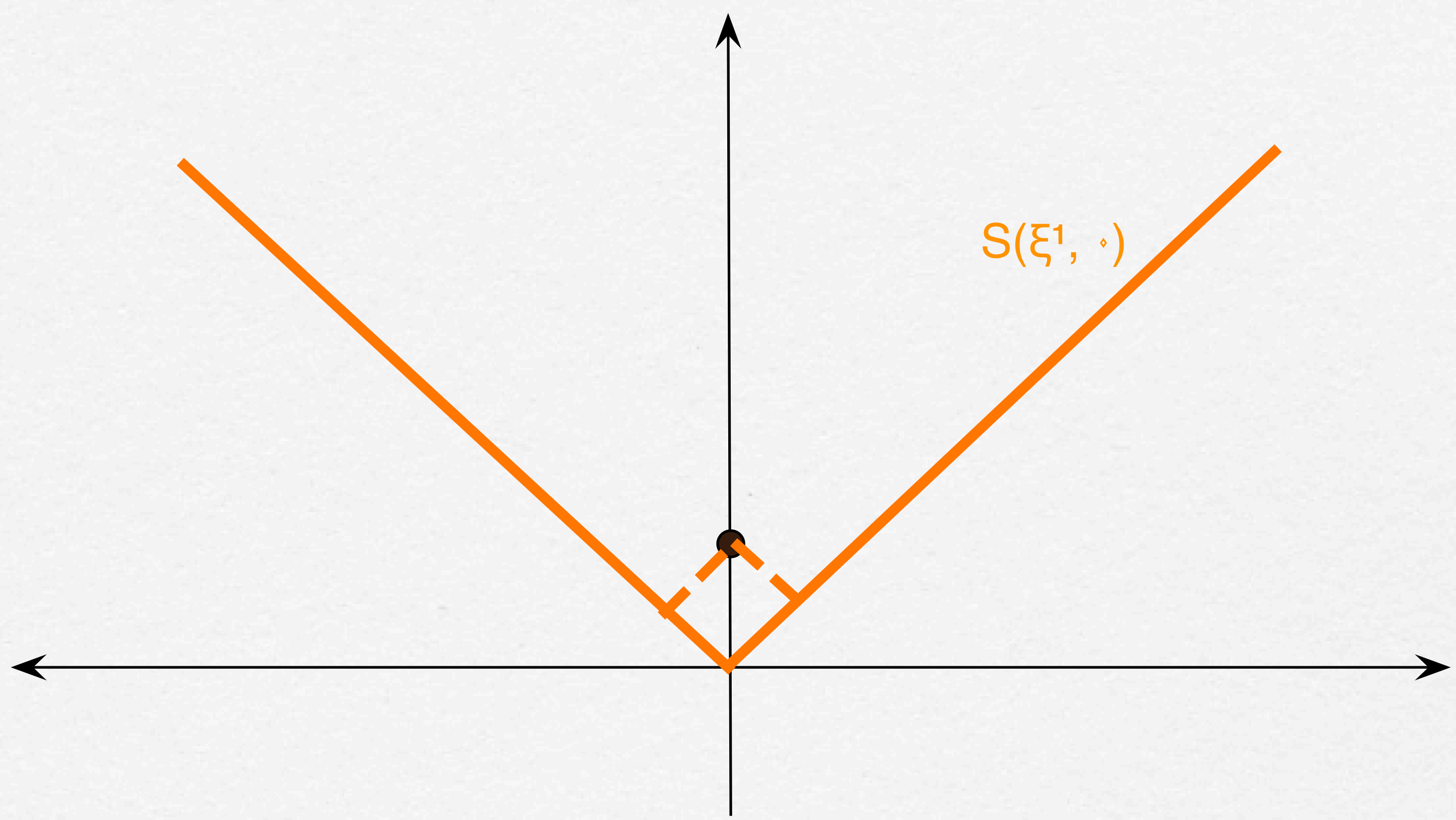



$S$ stably osc near $\bar{x}$ if $P$-a.s.,

$$\forall \rho > 0, \varepsilon > 0,\ \exists W \in \mathcal{N}(\bar{x})\ \&\ \eta\mathbb{B}\ (\eta > 0):$$

$$\mathrm{e}_\rho\big(S(\xi, x'), S(\xi, x)\big) < \varepsilon,\ \forall x' \in x + \eta\mathbb{B},\ x \in W$$

# Steady under averaging



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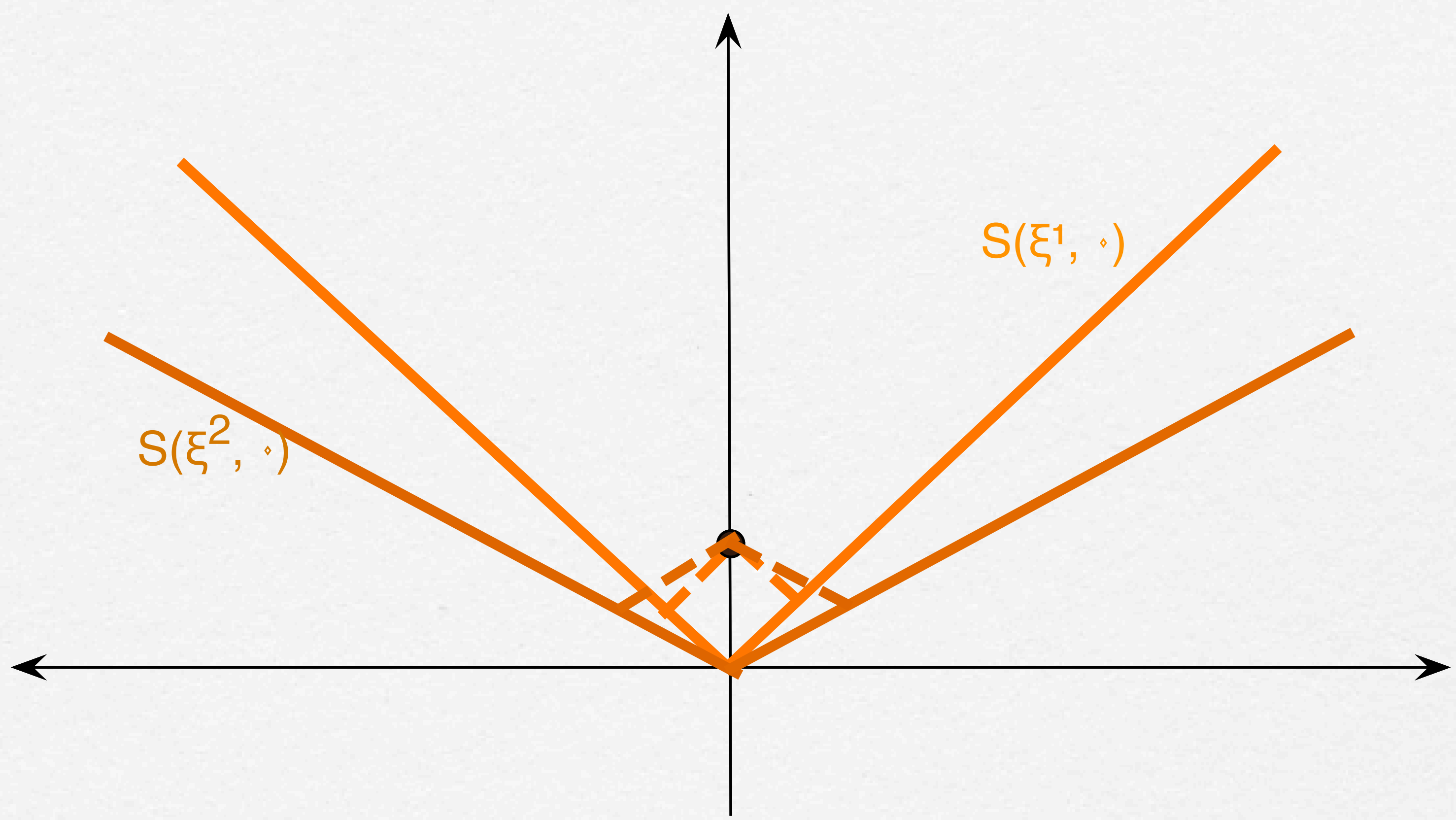
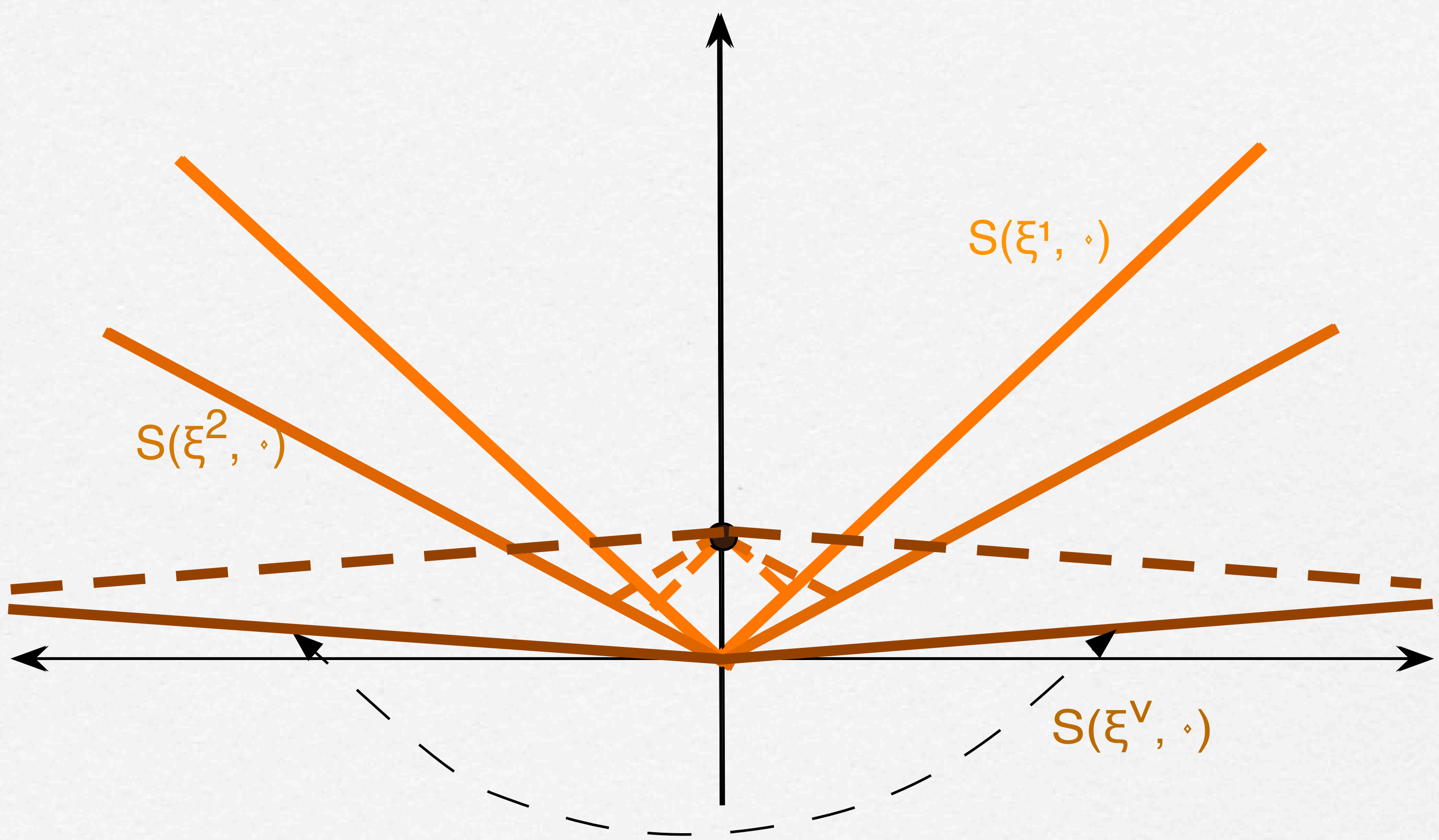
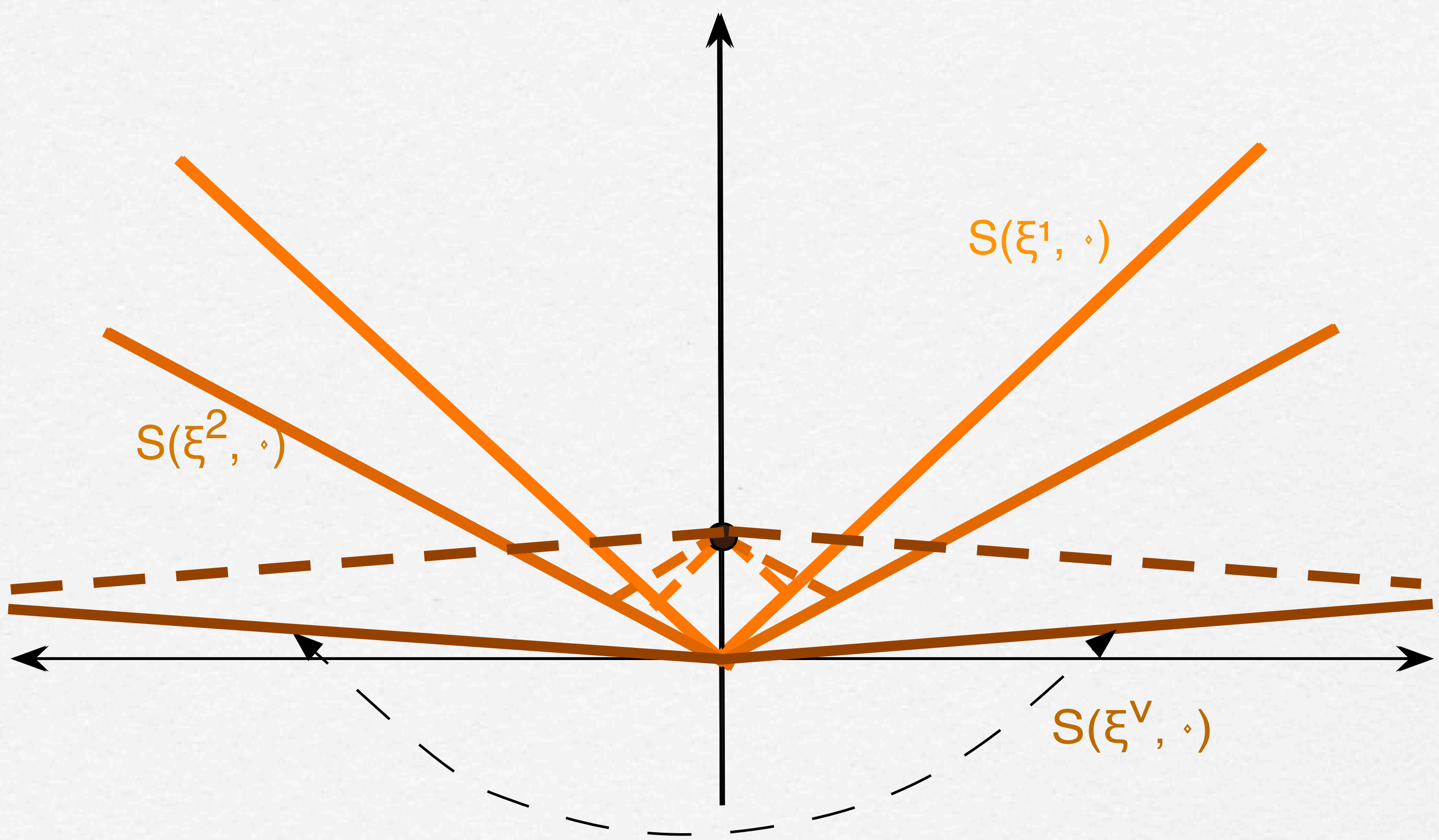


# Steady under averaging



# Steady under averaging




$$u \in S^\nu(\vec{\xi}^\nu, x) \cap \rho\mathbb{B} \Rightarrow \exists\, \hat{\rho} \geq \rho,\ u^k \in S(\xi^k, x) \cap \hat{\rho}\mathbb{B} \text{ such that}$$

$$u = \nu^{-1}(u^1 + \cdots + u^\nu);\ S^\nu(\vec{\xi}^\nu, x) \cap \rho\mathbb{B} \subset \frac{1}{\nu}\left[\sum_{k=1}^{\nu} S(\xi^k, x) \cap \hat{\rho}\mathbb{B}\right]$$

# Steady u. averaging & stably osc

rge $S \subset B$ bounded $\Rightarrow$ steady under averaging

$S$ cone-valued and rge $S \subset$ pointed cone $K$. Then,

$\bar{S} = ES$ and $\Rightarrow$ steady under averaging.

$S, R$ steady under averaging $\Rightarrow$ so is $S + R$

$R(\xi, x) = R(x) \Rightarrow R$ steady under averaging

rge $S$ bounded + $R$ constant $\Rightarrow$ steady under averaging

$G(\xi, x) + N_C(x) \Rightarrow$ steady under averaging (V.I.)

provided $G : \Xi \times X \to \mathbb{R}^n$ is bounded

---

$S, R$ stably osc $\Rightarrow S + R$ stably osc

although $D^1, D^2$ osc $\not\Rightarrow D^1 + D^2$osc

$\mathbb{B}$ closed, convex $\quad x \mapsto N_{\mathbb{B}}(x)$ osc

but not stably osc ($x^\nu \in$ int $\mathbb{B} \to \bar{x} \in$ bdry $\mathbb{B}$)

# Implementing SAA ** locally

$EG(x) = \mathbb{E}\{G(\boldsymbol{\xi}, x)\} \in R(x)$

(V.I.: $S = N_C$, applied to option pricing, ...)

$G^\nu(\vec{\xi}^\nu, \cdot) = \nu^{-1} \sum_{k=1}^{\nu} G(\xi^k, x)$. Assume $G^\nu(\vec{\xi}^\nu, \cdot)$, $EG \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}^n)$,

$\bar{x}$ strongly regular solution [Robinson] of $EG(x) \in R(x)$,

$\exists V \in \mathcal{N}(\bar{x}), \rho > 0$ such that $\forall z \in \rho\mathbb{B}$ :

$z + EG(\bar{x}) + \nabla EG(\bar{x})\big(x - \bar{x}\big) \in S(x)$

has a unique solution $\bar{x}(z) \in V$, Lipschitz continuous on $\rho\mathbb{B}$, and

$\left\| G^\nu(\vec{\xi}^\nu, \cdot) - EG \right\| \to 0$ $P$-a.s. Then, for $\nu$ sufficiently large

on a neighborhood of $\bar{x}$, $G^\nu(\vec{\xi}^\nu, \cdot) \in R(x)$ has a unique solution

$\bar{x}(\vec{\xi}^\nu) \to \bar{x} \quad P$-a.s.