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**ON A STRONG GRAPHICAL LAW OF LARGE NUMBERS FOR RANDOM SEMICONTINUOUS MAPPINGS** [1]

# 1 Introduction

From the fundamental LLN (Law of Large Numbers) of Artstein and Vitale (1975) [1], Lyashenko (1979) [2] and Artstein and Hart (1981) [3] one can immediately derive a strong pointwise LLN for osc random mappings; *osc = outer semicontinuous*, i.e., mappings with closed graphs. The (rich) potential applications to a variety of variational problems, however demand an *a.s.*-graphical LLN and not a pointwise one. More specifically, to be able to claim that the solutions of an inclusion, equivalently a generalized equation, of the type $\mathbb{E}\{S(\xi,\,\cdot\,)\} = \bar{S}(\,\cdot\,) \ni 0$ can be approximated by the solutions of approximating inclusions $S^\nu(\xi,\,\cdot\,) \ni 0$, a minimal condition is that almost surely the mappings $S^\nu(\xi,\,\cdot\,)$ converge graphically[2] to $\bar{S}$!

This article is concerned with such a graphical LLN for (osc) random set-valued mappings, namely to provide conditions under which the graphs of their associated SAA-mappings, *'Sample Average Approximating' mappings*, set-converge, i.e., in the Painlevé-Kuratowski [4] sense, with probability one to the graph of the *expectation mapping*. Mostly, this study is a first step[3]

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[2]Other convergence notions, like pointwise, for example, either don't yield the convergence of the solutions or the more demanding convergence notions, such as uniform or continuous convergence, fail to be applicable except when resorting to supplementary conditions that often restrict inappropriately the range of applicability

[3]Only a first step, because we restrict our attention mostly, but no exclusively, to compact-valued mappings. We do this, in part, to make the presentation more accessible but also to elucidate the relationship with the limited existing literature.

in validating the so-called SAA-method for a variety of variational problems such as stochastic variational inequalities, equilibrium problems in a stochastic environment (related to the GEI-model in economics), uncertainty quantification and so on, see [4, §5.F], for example.

As mentioned earlier, the first LLN [1, 2] where obtained for integrably bounded random sets (in $I\!R^m$), later generalized [3] to simply 'integrable' random sets, i.e., admitting an integrable selection, but not necessarily bounded; *a.s.*-convergence has to be understood as set-convergence to the closure of the convex hull of the Aumann's [5, 6] expectation of the random set. These results were extended to infinite dimensions, dependable and fuzzy random sets, cf. reviews by Taylor and Inoue (1996) [7], Molchanov (2005) [8] and Li and Yang (2010) [9]. The extension from random sets to random mappings, i.e., depending on parameters, is a qualitatively new problem because one has to select a new topology to analyze the convergence of not necessarily continuous mappings.

There are only a couple of papers that attempt to deal with this problem: Shapiro and Xu (2007) [10] and Terán (2008) [11] studied LLN of bounded-valued, integrably bounded, random set-valued mappings with respect to the uniform norm. Shapiro and Xu (2007) [10] proved the uniform convergence of SAA mappings to a certain fattened expectation mapping, but a genuine uniform LLN in their setting only holds for the case when the expectation mapping is continuous and single valued. Terán (2008) [11] treats sets as elements of the so-called convex combination metric space, equips set-valued mappings with the uniform metric and then applies the LLN due to Terán and Molchanov (2006) [12]. He derives a uniform LLN under the important assumption that the essential range of the random mapping is separable with respect to the uniform metric which renders it only applicable in quite restrictive settings.

We proceed as follows: we first prove, under the existence of a uniform integrable bound, that the graphs of SAA-mappings *a.s.*-converge to the graph of the expectation mapping; this convergence is equivalent to the convergence of graphs with respect to the Pompeiu-Hausdorff distance. Next, we show that the pseudo-uniform LLN by Shapiro and Xu (2007) [10], is in fact equivalent to the graphical LLN when restricting ourselves to their framework[4]. As already indicated earlier, applications of the graphical LLN

[4]However, it should be noted that these results are not indiscriminately applicable to unbounded random mappings, e.g., to random cone-valued mappings, although some easy, simple, extensions are possible; for example when the random osc-mapping is the sum of a compact-valued osc mapping (with a uniform integral bound) and a constant-valued, possibly unbounded, osc-mapping. A graphical LLN for an important, but a very

are mostly aimed at obtaining approximating solutions of stochastic generalized equations, stochastic variational inequalities and stochastic optimization problems with equilibrium constraints all involving, usually, unbounded mappings except when specific, if not artificial, restrictions are introduced, see, e.g., Shapiro and Xu (2008) [17], Xu and Meng (2007) [18], Ralph and Xu (2011) [19].

Section 2 introduces notation, concepts and some basic facts concerning set-valued mappings. Section 3 reviews, for reference purposes and later use, some known results about the law of large numbers for random sets and mappings. In Section 4, we prove the graphical LLN for random mappings uniformly bounded by an integrable function and bring to the fore the limitations of the pseudo-uniform LLN of Shapiro and Xu [10].

# 2 Notation, definitions and preliminaries

Our terminology and notation is pretty consistent with that of [4],

## 2.1 Set-valued mappings

Let $X$ be a closed subset of the complete separable metric space $H$ (e.g., $\mathbb{R}^n$ or more generally, a separable Banach space) with distance $\operatorname{dist}(\cdot\,,\cdot)$, $\mathbb{R}^m$ be $m$-dimensional Euclidean space with inner product $\langle \cdot\,,\cdot \rangle$ and Euclidean norm $|\cdot|$. Denote by cpct-sets($\mathbb{R}^m$) the hyperspace of compact subsets of $\mathbb{R}^m$ and cl-sets($\mathbb{R}^m$) the hyperspace of closed subsets of $\mathbb{R}^m$. Introduce the *distance* from a point $x$ to a set $A$ and the *excess* of the set $A$ on $B$ as

$$d_A(x) = \operatorname{dist}(x, A) = \inf_{x' \in A} \operatorname{dist}(x', x), \qquad \mathbb{e}(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$$

and $d\!\!l_\infty(A, B)$ the Pompeiu-Hausdorff distance between the sets $A$ and $B$,

$$d\!\!l_\infty(A, B) = \max\left\{\mathbb{e}(A, B), \mathbb{e}(B, A)\right\}.$$

Denote by $\mathbb{B}_\rho(x) \subset \mathbb{R}^m$ the ball centered at $x$ with radius $\rho$, $\mathbb{B}$ the (closed) unit ball and $\|A\| = \sup_{a \in A} |a|$.

specific class of unbounded random mappings, namely for *epigraphical* random mappings, was proved under a variety of assumption by Attouch and Wets (1990) [13], King and Wets (1991) [14], Artstein and Wets [15] and Hess [16]. These latter results state that epigraphs of SAA-lsc (lower semicontinuous) functions a.s. converge to the epigraph of the expectation functional.

For a set $A \in \mathbb{R}^m$ define its support (= Minkowski's) functional as $\sigma_A(u) = \sup_{a \in A} \langle a, u \rangle$. For sets $\{S_i \subset \mathbb{R}^m, i = 1, \dots, \nu\}$ define their (Minkowski's) *average* by

$$S^\nu = \nu^{-1} \sum_{i=1}^{\nu} S_i = \left\{ s = \nu^{-1} \sum_{i=1}^{\nu} s_i, s_i \in S_i \right\}$$

and for mappings $\{S_i : H \rightrightarrows \mathbb{R}^m, i = 1, \dots, \nu\}$ define their (Minkowski's) *pointwise averaged mapping* $S^\nu$: for $x \in X$, $x \mapsto S^\nu(x)$

$$S^\nu(x) = \nu^{-1} \sum_{i=1}^{\nu} S_i(x) = \left\{ s = \nu^{-1} \sum_{i=1}^{\nu} s_i,\ s_i \in S_i(x), i = 1, \dots, \nu \right\}.$$

**2.1 Definition** (set convergence, [4, Definition 4.1]). *Define the inner and outer limits of a sequence of sets $S^\nu \subset H$,*

$$\operatorname{Liminf}_\nu S^\nu = \left\{ x \in H \,\middle|\, \exists\, x^\nu \in S^\nu, x^\nu \to x \right\},$$

$$\operatorname{Limsup}_\nu S^\nu = \left\{ x \in H \,\middle|\, \exists \{\nu_k\} \subset \mathbb{N}, x^k \in S^{\nu_k} \text{ and } x^k \to x \right\}.$$

*A sequence of sets $S^\nu$ converges to a set $S = \operatorname{Lim}_\nu S^\nu$ if*

$$\operatorname{Liminf}_\nu S^\nu = \operatorname{Limsup}_\nu S^\nu = S.$$

**2.2 Definition** (osc and $\mathbb{e}$-osc). *A mapping $S : X \to$ cl-sets$(\mathbb{R}^m)$ is called* outer-semicontinuous (osc) *at $x$ relative to $X$ if for any $\rho > 0$ and any $\varepsilon > 0$ there exists a neighborhood $\mathbb{B}_{\delta(\varepsilon,\rho)}(x) = \{x' \in H \mid \operatorname{dist}(x', x) \le \delta(\varepsilon, \rho)\}$ of $x$ such that for all $x' \in \mathbb{B}_{\delta(\varepsilon,\rho)}(x) \cap X$*

$$S(x') \cap \mathbb{B}_\rho \subset S(x) + \mathbb{B}_\varepsilon.$$

*Furthermore, it's $\mathbb{e}$-osc at $x$ relative to $X$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x' \in \mathbb{B}_\delta(x) \cap X$, $\mathbb{e}(S(x), S(x')) \le \varepsilon$ or equivalently, $S(x') \subset S(x) + \varepsilon \mathbb{B}$.*

*Finally, $S$ is osc or $\mathbb{e}$-osc on $X$ if it's osc or $\mathbb{e}$-osc at every $x \in X$.*

**2.3 Definition** (graphical limits of mappings [4, Definition 5.32]). *The mappings $S^\nu : X \to$ cl-sets$(\mathbb{R}^m)$ defined on a subset $X \subset H$ are said to* converge graphically *to a mapping $S$ relative to $X$, denoted $S^\nu \xrightarrow{g} S$ or $S = \text{g-Lim}_\nu S^\nu$, if graphs of $S^\nu$, as sets, converge to the graph of $S$ in the product space $X \times \mathbb{R}^m$, i.e.*

$$\operatorname{gph} S^\nu = \{(x, s) \in X \times \mathbb{R}^m \mid s \in S^\nu(x)\} \to \operatorname{gph} S.$$

Note, that the limiting mapping $S = \text{g-Lim}_\nu S^\nu$ always has closed graph and, consequently is osc; also, a constant sequence of osc mappings $S^\nu \equiv S$ graphically converges to itself.

For the product space $X \times I\!R^m$ define the distance between $z' = (x', y')$ and $z = (x, y)$ by $\mathrm{Dist}(z', z) = \mathrm{dist}(x', x) + |y' - y|$ with the corresponding Pompeiu-Hausdorff distance between sets. When $\mathrm{gph}\, S^\nu$ and $\mathrm{gph}\, S$ are compact subsets of a bounded region in this product space, then graph-convergence is equivalent to their convergence with respect to the Pompeiu-Hausdorff distance, but that's definitely not the case in general.

By [4, Proposition 5.33] graphical convergence $S^\nu \xrightarrow{g} S$ on $X$ is equivalent to the inclusions[5]

$$\bigcup_{\{x^\nu \to x\}} \mathrm{Limsup}_\nu\, S^\nu(x^\nu) \subset S(x) \subset \bigcup_{\{x^\nu \to x\}} \mathrm{Liminf}_\nu\, S^\nu(x^\nu). \tag{1}$$

at all $x \in X$, where unions $\cup_{\{x^\nu \to x\}}$ are taken over all sequences $\{x^\nu \to x\} \subset X$. As already mentioned earlier, outer semicontinuity of the limit mapping $S$ is an immediate consequence of graphical convergence, see [4, Definition 5.32].

## 2.2 Random sets and random set-valued mappings

Let $X$ be a closed subset of $(H, \mathrm{dist})$, a complete separable metric space, $\mathcal{B}_X$ be the Borel $\sigma$-algebra of subsets of $X$, $(\Xi, \Sigma_\Xi, P)$ be a $P$-complete probability space. One refers to convergence with probability one in this space also as almost sure ($a.s.$-convergence); for more about random sets and measurable mappings, refer to [4, Ch. 14], [8].

**2.4 Definition** (random sets). *A mapping* $S : \Xi \to \text{cl-sets}(I\!R^m)$ *is a* random set *if it is measurable, i.e. for any open subset* $O \subset I\!R^m$ *one has*

$$S^{-1}(O) = \{\xi \in \Xi \,|\, S(\xi) \cap O \neq \emptyset\} \in \Sigma_\Xi.$$

**2.5 Definition** (random mappings). *A set-valued mapping* $S : \Xi \times X \to \text{cl-sets}(I\!R^m)$ *is called a random mapping, if its graph,* $\mathrm{gph}\, S$, *is a random set in the space* $X \times I\!R^m$ *equipped with Borel $\sigma$-algebra* $\mathcal{B}_X \times \mathcal{B}^m_R$.

**2.6 Definition** (iid random sets and mappings). *Random sets* $\{S_i : \Xi \to \text{cl-sets}(I\!R^m), i = 1, 2, \dots\}$ *are* independent identically distributed (iid) *if*

[5] Refer to the proof of this proposition to observe that these inclusions remain valid when $X$ is the subset of a Polish space.

*the induced $\sigma$-algebras $\{S_i^{-1}(\mathcal{B}_{\mathbb{R}^m})\}$ are independent and have the same (induced) distribution.*

*Random mappings $\{S_i : \Xi \times X \to \text{cl-sets}(\mathbb{R}^m), i = 1, 2, \dots\}$ are iid, if their graphs $\{\operatorname{gph} S_i\}$ are iid random sets in $X \times \mathbb{R}^m$.*

A typical construction of iid random mappings is the following: Let $\zeta_i : \Xi \to \mathbb{R}^l$ be an iid sequence of random variables, and $S : \mathbb{R}^l \times X \to \text{cl-sets}\, \mathbb{R}^m$ a $\mathcal{B}_{\mathbb{R}^l} \times \mathcal{B}_X$-measurable set valued mapping, i.e., for any open subset $O \in \mathcal{B}_{\mathbb{R}^m}$, $S^{-1}(O) \in \mathcal{B}_{\mathbb{R}^l} \times \mathcal{B}_X$. Then $\{S_i(\xi, \cdot\,) = S(\zeta_i(\xi), \cdot\,), i = 1, \dots\}$ are iid random mappings.

**2.7 Definition** (Aumann's expectation/integral [5, 6]). *The expectation of a random set $S : \Xi \to \text{cl-sets}(\mathbb{R}^m)$ consists of the expectations of all $P$-summable selections $s \in S$ a.s.. The expectation $\mathbb{E}S(\xi, \cdot\,)$ of a random mapping $S : \Xi \times X \to \text{cl-sets}(\mathbb{R}^m)$ is a deterministic mapping $ES$ whose values are, for each $x$, the expectations of the random sets $S(\,\cdot\,, x) : \Xi \to \text{cl-sets}(\mathbb{R}^m)$ for $x \in X$.*

# 3 LLNs for random sets and mappings

We review, for reference purposes, the law of large numbers (LLN) for random sets by Artstein and Hart (1981) [3], the epigraphical LLN of Attouch and Wets (1990) [13] and a pseudo-uniform LLN for random mappings due to Shapiro and Xu (2007) [10].

**3.1 Theorem** (LLN: unbounded closed random sets, [3]). *Let $\{S, S_i, i = 1, \dots$ be iid closed random sets in $\mathbb{R}^n$ with $\mathbb{E}S \neq \emptyset$. Then, for the averaged sets $S^\nu = \nu^{-1} \sum_{i=1}^{\nu} S_i$, one has,*

$$\operatorname{Lim}_\nu S^\nu = \operatorname{cl} \operatorname{con} \mathbb{E}S \quad a.s.,$$

*where* cl con *denotes a closure of the convex hull.*

For compact sets, this LLN goes back to Artstein and Vitale (1975) [1].

**3.2 Theorem** (an epigraphical LLN, [13]). *Suppose $H$ is a separable Banach space, $(\Xi, \Sigma_\Xi, P)$ is a probability space and $\{f_i : \Xi \times H \to (-\infty, +\infty]\}$ is a sequence of pairwise iid random lsc functions, bounded below $P$-almost sure by a polynomial minorant, $f_i(\xi, x) \geq -\alpha_0 \|x - x_0\|^p - \alpha_1(\xi)$ with $p \in [1, \infty)$, $x_0 \in H$, $\alpha_0 \in \mathbb{R}_+$ and $\alpha_1$ integrable. Then,*

$$\text{for } P\text{-almost all } \xi: \qquad \mathbb{E}f_1(\xi, \cdot\,) = \operatorname*{e-lim}_\nu \frac{1}{\nu} \sum_{i=1}^{\nu} f_i(\xi, \cdot\,),$$

The epigraphical limit, e-lim, means convergence of the epigraphs of corresponding functions. Epi-convergence, in particular, yields ([4, Proposition 7.2])

$$\operatorname{liminf}_\nu \left( \frac{1}{\nu} \sum_{i=1}^{\nu} f_i(\xi, x^\nu) \right) \geq I\!\!E f_1(\xi, x)$$

for all $x^\nu \to x \in H$ a.s.

**3.3 Theorem** *(*pseudo-uniform LLN, compact-valued mappings [10]). *Assume*

(a) *the metric space* $(X, d)$ *is compact;*

(b) $\{S_i(\xi, x), i = 1, \dots\}$ *is an iid sequence of realizations of the random mapping* $S : \Xi \times X \to \text{cpct-sets}(I\!\!R^m)$ *and* $S^\nu(\xi, x) = \nu^{-1} \sum_{i=1}^{\nu} S_i(\xi, x)$*;*

(c) *there exists a* $P$*-integrable function* $k : \Xi \to I\!\!R^1$ *such that*

$$\|S(\xi, x)\| \leq k(\xi), \quad \forall (\xi, x) \in \Xi \times X;$$

(d) *for* $P$*-almost all* $\xi$ *the mapping* $S(\xi, \cdot\,)$ *is* $\mathbb{e}$*-osc.*

*Then, the expectation mapping* $ES = I\!\!E\{\operatorname{con} S(\xi, \cdot\,)\}$ *is well-defined and the compact-valued mapping* $ES$ *is itself* $\mathbb{e}$*-osc. For any* $\rho > 0$*, one has a double 'one-sided uniform' convergence for* $P$*-almost all* $\xi$*, namely*

$$\lim_\nu \left[ \sup_{x \in X} \mathbb{e}(S^\nu(\xi, x), ES_\rho(x)) \right] = 0 = \lim_\nu \left[ \sup_{x \in X} \mathbb{e}(ES(x), S^\nu_\rho(\xi, x)) \right], \quad (2)$$

*where the fattened-up mappings*

$$ES_\rho(x) = \cup_{y \in \boldsymbol{B}_\rho(x)} ES(y), \quad S^\nu_\rho(\xi, x) = \cup_{y \in \boldsymbol{B}_\rho(x)} S^\nu(\xi, y).$$

The left inclusion of (2), for random matrices, was also proved in Xu and Meng [18, Lemma 3.2].

Terán [11], by relying on an abstract LLN due to Terán and Molchanov [12] (for convex combination metric spaces), obtained a strong uniform LLN for a random mapping $S(\xi, x)$, however under the essential assumption of separability of 'a' range of $S$, namely, for some measurable subset $\Xi'$ of $\Xi$ of $P$-measure 1, the set '$\operatorname{rge}' S = \cup_{\xi \in \Xi'} S(\xi, \cdot\,)$ is separable with respect to the $\text{dist}^\infty$-metric in the space of set-valued mappings. This metric is defined as follows: for mappings $S_1$ and $S_2$, $\text{dist}^\infty(S_1, S_2) = \sup_{x \in X} d\!\!l_\infty(S_1(x), S_2(x))$. Unfortunately, this assumption is not fulfilled in very simple and natural situations as confirmed by the example below.

**3.4 Example** (dist$^\infty$ range separability fails). *Define a set-valued mapping*

$$S(\xi, x) = \begin{cases} 0, & 0 \le x < \xi, \\ [0,1]\,, & x = \xi, \\ 1, & \xi < x \le 1, \end{cases}$$

*where $\xi$ is a random variable uniformly distributed on $[0,1]$, $x \in [0,1]$. Then any essential range of $S(\xi, x)$ (a subset of $[0,1]^2$) is not separable with respect to the uniform* dist$^\infty$*-metric, so Terán's (2008) [11] uniform LLN would not be applicable in this case.*

So, essentially this leaves us with only one 'genuine' LLN by Shapiro and Xu [10] when $\rho = 0$, in other words, when $ES$ is signle valued[6].

# 4 A strong graphical LLN for random mappings: Uniformly bounded case

The aim of this section is to establish a graphical LLN for random set-valued mappings and show that the pseudo-uniform law of large numbers for uniformly bounded (by an integrable function) random set-valued mappings due to Shapiro and Xu [10] is, for an appropriately restricted class of random mappings, a graphical LLN. This fact allows to substitute sample average approximations $\nu^{-1}\sum_{i=1}^{\nu} S_i(\xi,\,\cdot\,) = S^\nu(\xi,\,\cdot\,) \ni 0$ for a inclusion $I\!ES(\xi,\,\cdot\,) \ni 0$, where $\{S_i, i = 1,\ldots,\nu\}$ are independent identically distributed versions of a random osc mapping $S(\xi,\,\cdot\,)$. Suppose $I\!E\{S(\xi,x)\} = \mathrm{cl\,con}\, I\!E\{S(\xi,x)\}$, as is the case for convex-, compact-valued bounded mappings, then $a.s.$-graphical convergence (LLN) $S^\nu(\xi,\,\cdot\,) \xrightarrow{g} I\!ES(\xi,\,\cdot\,)$ implies by Theorem [4, Theorem 5.37] convergence of the solutions of the associated inclusions; again the proof of the referenced theorem applies without modifications to the case when $H$ is a Polish space.

**4.1 Theorem** ($a.s.$-graphical-LLN for compact-valued random mappings). *Assume*

(a) *$X$ is a closed subset of a separable Banach space $H$, $\mathcal{B}_H$ is the Borel $\sigma$-algebra and $(\Xi, \Sigma_\Xi, P)$ is $P$-complete;*

(b) *the mappings $S(\xi, x) = S_0(\xi, x)$, $S_i(\xi, x) : \Xi \times X \to$* cpct-sets$(I\!R^m)$, *$i = 0, 1, \ldots$ are nonempty-valued, $\Sigma_\Xi \times \mathcal{B}_H$-jointly measurable with respect*

[6] Let's observe that Terán and Molchanov (2006) [12] obtain a somewhat related result but this time for a different notion of expectation of random sets, i.e., not of the Aumann's type.

*to* $(\xi, x)$ *and osc in* $x \in X$ *for* $P$*-almost all* $\xi \in \Xi$*, i.e. the graphs* $\operatorname{gph} S_i(\xi, \cdot\,)$ *are closed random sets in* $X \times I\!R^m$*;*

(c) *the random graphs* $\operatorname{gph} S_i$ *are iid with the same distribution as* $\operatorname{gph} S$*;*

(d) *there is an integrable function* $\kappa(\xi)$ *such that*

$$\sup\{\,|s| \bigm| s \in S_i(\xi, x)\} \le \kappa(\xi), \quad \forall\, i,\ \forall\, (\xi, x) \in \Xi \times X.$$

*Let* $S^\nu(\xi, x) = \nu^{-1} \sum_{i=1}^{\nu} S_i(\xi, x)$ *and* $\bar S$*, the mapping whose graph,* $\operatorname{gph} \bar S$*, is the graph of* $I\!E \operatorname{con} S(\xi, \cdot\,)$*. Then* $\bar S$ *is osc and* $S^\nu \xrightarrow{g} \bar S$ *a.s. on* $X$*.*

**Proof.** Let's prove graphical convergence $S^\nu \xrightarrow{g} \bar S$ a.s. on $X$ by checking criterion (1). First let's prove the left inclusion by relying on Theorem 3.2. The right inclusion will be proved in the subsequent Lemma 4.2.

Let $D$ be a countable dense subset of $I\!R^m$. For $d \in I\!R^m$, define the support functions:

$$\begin{aligned}
&\sigma(\xi, x; d) = \sup_{y \in S(\xi, x)} \langle y, d\rangle, \quad \sigma_i(\xi, x; d) = \sup_{y \in S_i(\xi, x)} \langle y, d\rangle,\\
&\sigma^\nu(\xi, x; d) = \sup_{y \in S^\nu(\xi, x)} \langle y, d\rangle = \frac{1}{\nu}\sum_{i=1}^{\nu} \sigma_i(\xi, x; d),\\
&\bar\sigma(x; d) = \sup\big\{\langle y, d\rangle \bigm| y \in I\!E\{\operatorname{con} S(\,\cdot\,, x)\}\big\}.
\end{aligned}$$

Let us check applicability of the LLN of Theorem 3.2 to the random functions

$$\check\sigma_i(\xi, x; d) = \begin{cases} -\sigma_i(\xi, x; d), & x \in X;\\ +\infty, & x \in H \setminus X.\end{cases}$$

Under boundedness (d), the osc mappings $S(\xi, \cdot\,)$, $S_i(\xi, \cdot\,)$ are $\mathbb{e}$-osc for any fixed $\xi$ and hence their support functions $\sigma(\xi, \cdot\,; d)$, $\sigma_i(\xi, \cdot\,; d)$ are upper semicontinuous in $x \in X$ [21, §3.2, Proposition 2] and $\check\sigma_i(\xi, x; d)$ are lsc with respect to $x \in H$. For a fixed $d$, the functions $\check\sigma_i(\,\cdot\,, \cdot\,; d)$ are $\Sigma_\Xi \times \mathcal{B}_H$-measurable, indeed

$$\begin{aligned}
&\{(\xi, x) \in \Xi \times H \bigm| \check\sigma_i(\xi, x; d) > c\}\\
&\qquad = \Xi \times (H \setminus X) \cup \{(\xi, x) \in \Xi \times X \bigm| S_i(\xi, x) \cap B_d \ne \emptyset\} \in \Sigma_\Xi \times \mathcal{B}_H
\end{aligned}$$

by joint-measurability of $S_i$ where $B_d = \{(x, y) \in X \times I\!R^m \bigm| \langle y, d\rangle > c\} \in \mathcal{B}_H \times \mathcal{B}_{I\!R^m}$, $c \in I\!R$. Thus $\check\sigma_i(\,\cdot\,, \cdot\,; d)$ are random lsc functions ($\sim$ normal integrands) [4, Definition 14.27]. Note that by (d), $\check\sigma_i(\xi, x; d) \ge -|d|k(\xi)$ for all $x \in H$.

Let's now verify that $\{\check{\sigma}_i(\xi, \cdot\,; d)\}$ are iid. First show that random mappings $\{x \to S_i^d(\xi, x)\}$,

$$S_i^d(\xi, x) = \begin{cases} \{-\langle s, d\rangle \,\big|\, s \in S_i(\xi, x)\}, & x \in X, \\ +\infty, & x \in H \setminus X, \end{cases}$$

are iid, then the epigraphs

$$\Big\{\operatorname{epi} \check{\sigma}_i(\xi, \cdot\,; d) = \operatorname{gph} S_i^d(\xi, \cdot\,) + (\vec{0} \times I\!R_+)\Big\}$$

would be iid by [13, Lemma 1.2], where the zero vector $\vec{0} \in H$. Indeed for any $B_1 \in \mathcal{B}_H$, bounded $B_2 \in \mathcal{B}_{\boldsymbol{R}}$ one has

$$P\{\operatorname{gph} S_i^d(\xi, \cdot\,) \cap B_1 \times B_2 \neq \emptyset\} = P\{\operatorname{gph} S_i(\xi, \cdot\,) \cap (B_1 \cap X) \times B_2^d \neq \emptyset\},$$

where $B_2^d = \{y \in I\!R^m \,\big|\, -\langle y, d\rangle \in B_2\} \in \mathcal{B}_{\boldsymbol{R}^m}$. Hence, the mappings $\{S_i^d(\xi, \cdot\,)\}$ are identically distributed. For any $B_{i1} \in \mathcal{B}_H$, bounded $B_{i2} \in \mathcal{B}_{\boldsymbol{R}}$, $i \in I \subset \{0, 1, ...\}$,

$$\begin{aligned} & P\{\operatorname{gph} S_i^d(\xi, \cdot\,) \cap B_{i1} \times B_{i2} \neq \emptyset, i \in I\} \\ &= P\{\operatorname{gph} S_i(\xi, \cdot\,) \cap (B_{i1} \cap X) \times B_{i2}^d \neq \emptyset, i \in I\} \\ &= \prod_{i \in I} P\{\operatorname{gph} S_i(\xi, \cdot\,) \cap (B_{i1} \cap X) \times B_{i2}^d \neq \emptyset\} \\ &= \prod_{i \in I} P\{\operatorname{gph} S_i^d(\xi, \cdot\,) \cap B_{i1} \times B_{i2} \neq \emptyset\}, \end{aligned}$$

where $B_{i2}^d = \{y \in I\!R^m \,\big|\, -\langle y, d\rangle \in B_{i2}\} \in \mathcal{B}_{\boldsymbol{R}^m}$, hence the mappings $\{S_i^d(\xi, \cdot\,)\}$ are independent. Now applying Theorem 3.2 to $\{\check{\sigma}_i(\,\cdot\,, \cdot\,; d)\}$, one obtains

$$\operatorname*{e-lim}_{\nu} \left(\frac{1}{\nu} \sum_{i=1}^{\nu} \check{\sigma}_i(\xi', \cdot\,; d)\right) = I\!E\check{\sigma}(\xi, \cdot\,; d),$$

for all $\xi' \in \Xi \setminus \Xi_d$, $P\{\Xi_d\} = 0$. This, in particular, means that for any sequence $\{X \ni x^\nu \to x\}$ when $\xi' \in \Xi \setminus \Xi_d$,

$$\operatorname{limsup}_\nu \frac{1}{\nu} \sum_{i=1}^{\nu} \sigma_i(\xi', x^\nu; d) = \operatorname{limsup}_\nu \sigma^\nu(\xi', x^\nu; d) \leq I\!E\sigma(\xi, x; d).$$

This is also true for all $d \in D$ when $\xi' \in \Xi' = \Xi \setminus \cup_{d \in D} \Xi_d$, i.e., with probability $P\{\Xi'\} = 1$. Denote by $R(\xi, x; d) = \sup\{\langle s, d\rangle \,\big|\, s \in \operatorname{Limsup}_\nu S^\nu(\xi, x)\}$. Since for $\{X \ni x^\nu \to x\}$, $x \in X$ for all $d \in D$,

$$\operatorname{limsup}_\nu R(\xi', x^\nu; d) \leq \operatorname{limsup}_\nu \sigma^\nu(\xi', x^\nu; d) \leq I\!E\sigma(\xi, x; d) = \bar{\sigma}(x; d).$$

Taking into account $\mathrm{cl\,con}\, I\!\!E S(\xi,x) = I\!\!E\, \mathrm{cl\,con}\, S(\xi,x) = I\!\!E\, \mathrm{con}\, S(\xi,x)$ by [8, Theorem 1.17(iii)] and the fact that $S(\xi,x)$ is compact, this allows us to conclude

$$\mathrm{Limsup}_\nu\, S^\nu(\xi',x^\nu) \subset \mathrm{cl\,con}\, I\!\!E S(\xi,x) = I\!\!E\, \mathrm{con}\, S(\xi,x) = \bar{S}(x),$$

and hence the left inclusion (1) holds jointly for all $x \in X$ with probability one. For the converse inclusion in (1), see the next lemma. ☐

This lemma proves the converse inclusion (1) for the sample average mappings $S^\nu(\xi,x) = \nu^{-1}\sum_{i=1}^{\nu} S_i(\xi,x)$ for all $x \in X$ a.s. Note that in this lemma we do not assume boundedness of the random mappings. The proof exploits essentially the pointwise LLN of Theorem 3.1.

**4.2 Lemma** (Liminf inclusion). *Let's assume:*

(a) *$X$ is closed subset of a complete separable metric space and $(\Xi, \Sigma_\Xi, P)$ is $P$-complete;*

(b) *the mappings $S(\xi,x) = S_0(\xi,x)$, $S_i(\xi,x) \,\big|\, \Xi \times X \to I\!\!R^m$, $i = 0,1,\ldots$ are nonempty closed-valued, $\Sigma_\Xi \times \mathcal{B}_{\boldsymbol{R}^m}$-measurable in $(\xi,x)$, i.e., the graphs $\mathrm{gph}\, S_i(\xi,\,\cdot\,)$ are random closed sets in $X \times I\!\!R^m$;*

(c) *the random graphs $\{\mathrm{gph}\, S, (\mathrm{gph}\, S_i, i = 1,\ldots) \subset \Xi \times I\!\!R^m\}$ are iid;*

(d) *$I\!\!E S(\xi,x) \neq \emptyset$ for all $x \in X$.*

*Let $S^\nu(\xi,x) = \nu^{-1}\sum_{i=1}^{\nu} S_i(\xi,x)$ and $\hat{S}$ be the mapping whose graph, $\mathrm{gph}\,\hat{S}$, is the closure of the graph of $\mathrm{con}\, I\!\!E\{S(\xi,\,\cdot\,)\}$, $\mathrm{gph}\,\hat{S} = \mathrm{cl\,gph\,con}\, I\!\!E\{S(\xi,\,\cdot\,)\}$. Then, for $P$-almost all $\xi \in \Xi$,*

$$\mathrm{cl\,con}\, I\!\!E\{S(\xi,x)\} \subset \hat{S}(x) \subset \cup_{\{x^\nu \to x\}} \mathrm{Liminf}_\nu S^\nu(\xi,x^\nu).$$

**Proof.** Obviously, $\mathrm{cl\,con}\, I\!\!E\{S(\xi,x)\} \subset \hat{S}(x)$. Let's prove the second inclusion. Choose a countable dense subset $G$ in $\mathrm{gph}\,\hat{S}$ (any subset of a separable metric space, in our case $\mathrm{gph}\,\hat{S} \subset X \times I\!\!R^m$, is also separable [20, Section 16.7]) and denote by $X'$ its (countable) projection on $X$. For each $x' \in X'$ by the pointwise law of large numbers, Theorem 3.1, one has $S^\nu(\xi,x') \to \mathrm{cl\,con}\, I\!\!E S(\xi,x')$ a.s. Since $X'$ is countable, this is true for all $x \in X'$ jointly *a.s.*, i.e., for all $\xi \in \Xi'$ for some $\Xi'$ with $P\{\Xi'\} = 1$.

Now, fix $\xi' \in \Xi'$ and $z = (x,y) \in \mathrm{gph}\,\hat{S}$. We need to show that $\lim_\nu \mathrm{Dist}(z, \mathrm{gph}\, S^\nu(\xi',\,\cdot\,)) = 0$. Suppose, to the contrary, for some $\varepsilon > 0$ and some subsequence $\{\nu_k\}$, $\mathrm{Dist}(z, \mathrm{gph}\, S^{\nu_k}) \geq \varepsilon$. By definition of $G$, there exists $z'(\varepsilon) = (x',y') \in G$ with $x' \in X'$ such that $\mathrm{Dist}(z,z') \leq \varepsilon/3$. From the set convergence of $S^\nu(\xi',x') \to \mathrm{cl\,con}\, I\!\!E S(\xi,x')$, it follows [4, Proposition 5.33] that

$$\mathrm{cl\,con}\, I\!\!E S(\xi,x') \subset \mathrm{Liminf}_\nu\, S^\nu(\xi',x') \subset \mathrm{Liminf}_{k\to\infty}\, S^{\nu_k}(\xi',x').$$

Hence for the given $\varepsilon$ and $y' \in \mathrm{cl}\,\mathrm{con}\, I\!ES(\xi, x')$ one can find $\nu_{k'}$ and $y^{\nu_{k'}} \in S^{\nu_{k'}}(\xi', x')$ such that $|y' - y^{\nu_{k'}}| \le \varepsilon/3$. Then, for this subsequence $\nu_{k'}$,

$$\begin{aligned}\mathrm{Dist}(z, \mathrm{gph}\, S^{\nu_{k'}}(\xi', \cdot\,)) &\le \mathrm{Dist}(z, z') \; + \; \mathrm{Dist}(z', \mathrm{gph}\, S^{\nu_{k'}}(\xi', \cdot\,)) \\ &\le \mathrm{Dist}(z, z') + \mathrm{Dist}(z', (x', y^{\nu_{k'}})) \le \mathrm{Dist}(z, z') + |y' - y^{\nu_{k'}}| \le 2\varepsilon/3\end{aligned}$$

which contradicts the assumption that $\mathrm{Dist}(z, \mathrm{gph}\, S^{\nu_{k'}}(\xi', \cdot\,)) \ge \varepsilon$.

Thus, $\lim_\nu \mathrm{Dist}(z, \mathrm{gph}\, S^\nu(\xi', \cdot\,)) = 0$. Hence there is a sequence $z^\nu \in \mathrm{gph}\, S^\nu(\xi', \cdot\,)$ such that $z^\nu \to z \in \mathrm{gph}\, \hat{S}$. □

The next proposition shows that the "uniformity" statements in (2) are in fact equivalent to the graphical convergence of the involved mappings, $S^\nu \xrightarrow{g} S$.

**4.3 Proposition** (uniform characterization of graph-convergence). *Graphical convergence $S^\nu \xrightarrow{g} S$ of compact-valued mappings to an osc mapping $S : X \to$ cpct-sets$(I\!R^m)$ on a compact set $X \subset I\!R^n$ is equivalent to*

$$\lim_\nu \sup_{x \in X} \mathbb{e}(S^\nu(x), S_\rho(x)) = 0 = \lim_\nu \sup_{x \in X} \mathbb{e}(S(x), S^\nu_\rho(x)) \quad \forall \rho > 0, \tag{3}$$

*where*

$$S_\rho(x) = \cup_{y \in \boldsymbol{B}_\rho(x)} S(y), \quad S^\nu_\rho(x) = \cup_{y \in \boldsymbol{B}_\rho(x)} S^\nu(y).$$

**Proof.** Let $S^\nu \xrightarrow{g} S$ with $S$ osc on $X$ and let's prove (3). By [4, Exercise 5.34] for any $r > 0$ and $\varepsilon > 0$ for all $x \in X \cap I\!B_r$ and $\nu$ sufficiently large,

$$S^\nu(x) \cap I\!B_r \subset S(I\!B_\varepsilon(x)) + I\!B_\varepsilon = S_\varepsilon(x) + I\!B_\varepsilon,$$

$$S(x) \cap I\!B_r \subset S^\nu(I\!B_\varepsilon(x)) + I\!B_\varepsilon = S^\nu_\varepsilon(x) + I\!B_\varepsilon.$$

Fix any $\rho > 0$, set $d^\infty_X = \sup_{x \in X} \|x\| < +\infty$ and $M = \sup\{|y| \;\big|\; y \in S(x), x \in X\} < +\infty$ since $X$ is compact and $S$ is osc, compact-valued. For any $\varepsilon < \rho$, $r \ge \max\{d^\infty_X, M + \rho\}$, and any $x \in X$ the preceding inclusions become

$$S^\nu(x) \subset S_\varepsilon(x) + I\!B_\varepsilon, \quad S(x) \subset S^\nu_\varepsilon(x) + I\!B_\varepsilon.$$

From this, it follows

$$\begin{aligned}&\mathbb{e}(S^\nu(x), S_\rho(x)) \le \mathbb{e}(S^\nu(x), S_\varepsilon(x)) \le \varepsilon, \\ &\mathbb{e}(S(x), S^\nu_\rho(x)) \le \mathbb{e}(S(x), S^\nu_\varepsilon(x)) \le \varepsilon.\end{aligned}$$

Thus, for any $\varepsilon$ and sufficiently large $\nu$, one has $\sup_{x \in X} \mathbb{e}(S^\nu(x), S_\rho(x)) \le \varepsilon$ and $\sup_{x \in X} \mathbb{e}(S(x), S^\nu_\rho(x)) \le \varepsilon$ which is what we set out to prove.

Let's now concern ourselves with the converse, namely that (3) implies graphical convergence $S^\nu \xrightarrow{g} S$ on $X$. We begin by showing that the first identity of (3) implies $\mathbb{e}(\operatorname{gph} S^\nu, \operatorname{gph} S) \to 0$. For any $x$,

$$
\begin{aligned}
\mathbb{e}((x, S^\nu(x)), \operatorname{gph} S) &= \sup_{y \in S^\nu(x)} \operatorname{Dist}((x,y), \operatorname{gph} S) \\
&\leq \inf_{x' \in X} \Big( \operatorname{dist}(x, x') + \sup_{y \in S^\nu(x)} \operatorname{dist}(y, S(x')) \Big) \\
&= \inf_{x' \in X} \big( \operatorname{dist}(x, x') + \mathbb{e}(S^\nu(x), S(x')) \big) .
\end{aligned}
$$

The inequality $\sup_{x\in X} \mathbb{e}(S^\nu(x), S_\rho(x)) \leq \varepsilon$ means that for each $x$ there exists $x_\rho$ such that $\operatorname{dist}(x, x_\rho) \leq \rho$ and $\mathbb{e}(S^\nu(x), S(x_\rho)) \leq \varepsilon$, so

$$
\mathbb{e}((x, S^\nu(x)), \operatorname{gph} S) \leq \operatorname{dist}(x, x_\rho) + \mathbb{e}(S^\nu(x), S(x_\rho) \leq \rho + \varepsilon,
$$

and consequently $\mathbb{e}(\operatorname{gph} S^\nu, \operatorname{gph} S) \leq \rho + \varepsilon$. Since $\rho$ and $\varepsilon$ can be arbitrary small, it means that $\mathbb{e}(\operatorname{gph} S^\nu, \operatorname{gph} S) \to 0$ as $\nu \to \infty$.

Similarly, from $\sup_{x\in X} \mathbb{e}(S(x), S^\nu_\rho(x)) \to 0$, one obtains $\mathbb{e}(\operatorname{gph} S, \operatorname{gph} S^\nu) \to 0$ as $\nu \to \infty$. Hence, the Pompeiu-Hausdorff distance $d\!\!l_\infty(\operatorname{gph} S, \operatorname{gph} S^\nu) \to 0$ as $\nu \to \infty$. Under outer semicontinuity of $S$, recall it means that $\operatorname{gph} S$ is closed, this is equivalent to $S^\nu \xrightarrow{g} S$ and completes the proof. $\square$

**4.4 Example** (SAA of Clarke's subdifferential). *In [10], Shapiro and Xu consider sample average approximations of the expectation of the Clarke's subdifferential* $I\!\!E\{\bar\partial f(\xi, \cdot\}$ *of random Lipschitz functions which is interpreted to mean that for all* $\xi$, $x \mapsto f(\xi, x)$ *is Lipschitz continuous.*

**Detail.** Convergence of the SAA-versions is 'proved' under a reguarity assumption and the requirement that the Lipschitz constant be integrable. However, as seen from Theorem 4.1 to validate this approximation one needs the joint $(\xi, x)$-measurability of $\partial f(\xi, x)$; a proof of this property can be found in [22, Lemma 4]. $\square$

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**On a strong graphical law of large numbers for random semicontinuous mappings**

**Summary**. In the paper we establish a strong graphical law of large numbers (LLN) for random outer semicontinuous mappings, providing conditions when graphs of sample average mappings converge to the graph of the expectation mapping with probability one. This result extends a known LLN for compact valued random sets to random uniformly bounded (by an integrable function) set valued mappings. We give also an equivalent formulation for the graphical LLN by means of some fattened mappings. The study is motivated by applications of the set convergence and the graphical LLN in stochastic variational analysis, including approximation and solution of stochastic generalized equations, stochastic variational inequalities and stochastic optimization problems. The nature of these applications consists in sample average approximation of the inclusion mappings, application of

the graphical LLN and obtaining from here a graphical approximation of the set of solutions.

**Keywords:** random sets, random set-valued mappings, strong law of large numbers, graphical convergence.

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