TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 266, Number 1, July 1981

ON THE CONVERGENCE OF CLOSED-VALUED MEASURABLE MULTIFUNCTIONS

BY

GABRIELLA SALINETTI AND ROGER J.-B. WETS[1]

ABSTRACT. In this paper we study the convergence almost everywhere and in measure of sequences of closed-valued multifunctions. We first give a number of criteria for the convergence of sequences of closed subsets. These results are used to obtain various characterizations for the convergence of measurable multifunctions. In particular we are interested in the convergence properties of (measurable) selections.

1. Introduction. Let $(\Omega, \mathcal{Q})$ be a measure space with $\mathcal{Q}$ the class of measurable sets and meas a nonnegative sigma-finite measure defined on $\mathcal{Q}$; (E, d) is the metric space obtained by equipping R^n with the metric d. A map Γ with domain Ω and whose values are subsets of E is called a *multifunction*; its *effective domain* is $\operatorname{dom}\Gamma = \{\omega \in \Omega | \Gamma(\omega) \neq \varnothing\}$. It is said to be *closed-* (*compact-*, *convex-* ...) *valued*, if its values are closed (compact, convex, ...) subsets of E. A closed-valued multifunction Γ is *measurable* if for all closed subsets F of E we have that

$$\Gamma^{-1}(F) = \{\omega \in \Omega | \Gamma(\omega) \cap F \neq \varnothing\} \in \mathcal{Q}. \tag{1.1}$$

We write $\Gamma^{-1}(F) \in \mathcal{Q}$ for all $F \in \mathcal{F}$ where $\mathcal{F}$ is the *hyperspace* of the closed subsets of E. Let $\mathcal{G}$ ($\mathcal{K}$ resp.) denote the hyperspace of open (compact resp.) subsets of E. It can be shown that when Γ is closed-valued, Γ is measurable if and only if any one of the following equivalent conditions is satisfied:

(i) $\Gamma^{-1}(G) \in \mathcal{Q}$ for all $G \in \mathcal{G}$;

(ii) $\Gamma^{-1}(K) \in \mathcal{Q}$ for all $K \in \mathcal{K}$;

(iii) $\Gamma^{-1}(B_\varepsilon^\circ(x)) \in \mathcal{Q}$ for all $\varepsilon > 0$, $x \in E$ where $B_\varepsilon^\circ(x)$ is the open ball of radius ε and center x;

(iv) $\Gamma^{-1}(B_\varepsilon(x)) \in \mathcal{Q}$ for all $\varepsilon > 0$, $x \in E$ where $B_\varepsilon(x)$ is the closed ball of radius ε and center x;

(v) Γ admits a *Castaing representation*, i.e. $\operatorname{dom}\Gamma \in \mathcal{Q}$ and there exists a countable collection $\{v_k\}_{k=1}^\infty$ of measurable functions from $\operatorname{dom}\Gamma$ to E such that for all ω in $\operatorname{dom}\Gamma$,

$$\operatorname{cl}\left[\bigcup_{k=1}^{\infty} v_k(\omega)\right] = \Gamma(\omega). \tag{1.2}$$

Received by the editors March 14, 1980 and, in revised form, June 18, 1980.

1980 *Mathematics Subject Classification*. Primary 28A20, 54C60, 54C65, 26E25; Secondary 90C15, 54B20.

Key words and phrases. Measurable multifunction, measurable selection, convergence in probability, convergence almost everywhere, hyperspaces, convergence in hyperspaces.

[1]Supported in part by C.N.R. (Gruppo Nazionale per Analisi Funzionale e le sue Applicazione) and the National Science Foundation under Grant ENG-7903731.

If $\mathcal{A}$ is complete then Γ is measurable if and only if its graph $\{(\omega, x)|x \in \Gamma(\omega)\}$ is an $(\mathcal{A} \otimes \mathcal{B})$-measurable subset of $\Omega \times E$ where $\mathcal{B}$ is the Borel algebra on E. For these and related results, cf. [1] or [2].

Every multifunction Γ can be identified with a function γ from Ω into $\mathcal{P} = \mathcal{P}(E)$, the power set of E. If the multifunction is closed- (compact- resp.) valued, then γ can be viewed as a function from Ω into $\mathcal{F}$ ($\mathcal{K}$ resp.). Let $\mathcal{T}$ be the topology on $\mathcal{F}$ generated by the subbase consisting of the families $\{\mathcal{F}^K, K \in \mathcal{K}\}$ and $\{F_G, G \in \mathcal{G}\}$ where

$$\mathcal{F}^K = \{F \in \mathcal{F}|F \cap K = \emptyset\} \quad \text{and} \quad \mathcal{F}_G = \{F \in \mathcal{F}|F \cap G \neq \emptyset\}.$$

The class of subsets of the form

$$\mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \cdots \cap \mathcal{F}_{G_n} \tag{1.3}$$

for $n \geqslant 0$ yields a base for the topology $\mathcal{T}$. The same topology $\mathcal{T}$ is also generated by the subbase consisting of the families

$$\{\mathcal{F}^{B_\varepsilon(x)}, \varepsilon > 0, x \in E\} \quad \text{and} \quad \{\mathcal{F}_{B^\circ_\varepsilon(x)}, \varepsilon > 0, x \in E\}. \tag{1.4}$$

This follows directly from the properties of E. The topological space $(\mathcal{F}, \mathcal{T})$ is compact, Hausdorff and second countable (cf. [3] and also [4]).[2] The choice of this topology for $\mathcal{F}$ is motivated by the fact that $\mathcal{T}$-convergence corresponds to the natural (standard) convergence of sequences of closed sets in E and also with this topology, the measurability of Γ corresponds to the measurability of the corresponding function γ.

Let $\mathcal{B}$ be the Borel algebra on $\mathcal{F}$ generated by the elements of the base of the topology $\mathcal{T}_{\mathcal{F}}$. Actually $\mathcal{B}_{\mathcal{F}}$ can be generated from the family of sets $\{\mathcal{F}_G, G \in \mathcal{G}\}$. To see this simply observe that the space $\{\mathcal{F}, \mathcal{T}\}$ is second countable and that the elements (1.3) of the base of $\mathcal{T}$ can be obtained as complements and countable intersections of the elements in $\{\mathcal{F}_G\}$. From the properties of E, it also follows that the Borel algebra $\mathcal{B}_{\mathcal{F}}$ can be generated by any one of the families $\{\mathcal{F}^K, K \in \mathcal{K}\}$, $\{\mathcal{F}_{B^\circ_\varepsilon(x)}, \varepsilon > 0, x \in E\}$ and $\{\mathcal{F}^{B_\varepsilon(x)}, \varepsilon > 0, x \in E\}$. A function γ from Ω to $\mathcal{F}$ is *measurable* if $\gamma^{-1}(D) \in \mathcal{A}$ for every D in $\mathcal{B}_{\mathcal{F}}$.

PROPOSITION 1.1. *Suppose that Γ is a closed-valued multifunction from Ω to E and γ is the associated function from Ω to $\mathcal{F}$. Then Γ is measurable if and only if γ is measurable.*

PROOF. For any open set $G \subset E$, we have that

$$\gamma^{-1}(\mathcal{F}_G) = \{\omega|\gamma(\omega) \in \mathcal{F}_G\} = \{\omega|\Gamma(\omega) \cap G \neq \emptyset\} = \Gamma^{-1}(G).$$

The measurability of γ implies that for all G in $\mathcal{G}$, $\gamma^{-1}(\mathcal{F}_G) \in \mathcal{A}$ and consequently $\Gamma^{-1}(G) \in \mathcal{A}$, which in turn implies the measurability of Γ. One argues the converse similarly. □

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[2]Professors Carl Eberhart (Kentucky) and James West (Cornell) pointed out that the space $(\mathcal{F}, \mathcal{T})$ is homeomorphic to the Hilbert cube. Let E_∞ be a one point compactification of E and $(\mathcal{F}_\infty, \mathcal{V})$ the hyperspace of closed subsets of E_∞ equipped with the "Vietoris finite topology." The map $F \to i_*(F) = F \cup \{\infty\}$ is an embedding from $\mathcal{F}$ into $\mathcal{F}_\infty$ with image $\{F \in \mathcal{F}_\infty|\mathcal{F} \supset \{\infty\}\} = \mathcal{K}$. The assertion now follows from the classical result of Curtis and Schori [5].

In [6] and [7] the measurability of a multifunction Γ is defined in terms of the measurability of the associated function γ when Γ is nonempty compact-valued. In this case the range of γ is $\mathcal{K}' = \mathcal{K} \setminus \{\emptyset\}$ and the topology $\mathcal{T}_h$ is generated by the Hausdorff distance. This topology is finer than the $\mathcal{T}$-relative topology on $\mathcal{K}'$; it is generated as follows: Let $\mathcal{T}_V$ be the Vietoris topology on $\mathcal{K}$, i.e. that generated by the subbase consisting of the families of sets $\{\mathcal{K}^F, F \in \mathcal{F}\}$ and $\{\mathcal{K}_G, G \in \mathcal{G}\}$. Then $\mathcal{T}_h$ is the $\mathcal{T}_V$-relative topology on $\mathcal{K}'$. The Borel algebra on $\mathcal{K}'$, consistent with $\mathcal{T}_h$ is denoted by $\mathcal{B}_h$; it can be generated by any one of the families $\{\mathcal{K}'^F\}$, $\{\mathcal{K}'_G\}$, $\{\mathcal{K}'^{B_\epsilon(x)}\}$ and $\{\mathcal{K}'_{B^\circ_\epsilon(x)}\}$. A function γ from Ω to $\mathcal{K}'$ is measurable if $\gamma^{-1}(D) \in \mathcal{A}$ for every $D \in \mathcal{B}_{\mathcal{K}'}$. A proof similar to that of Proposition 1.1 yields the following:

PROPOSITION 1.2. *Suppose that* Γ *is a nonempty compact-valued multifunction from* Ω *to* E *and* γ *is the associated function from* Ω *to* $\mathcal{K}'$. *Then* Γ *is measurable if and only if* γ *is* $\mathcal{B}_h$*-measurable.*

A different proof of this proposition appears in [2].

In this paper we are basically interested in studying the stochastic convergence of sequences of measurable multifunctions (set-valued random variables). We limit ourselves to almost everywhere (sure) convergence and convergence in measure (probability); convergence in distribution will be dealt with in a follow-up to this article. We are particularly interested in the convergence properties of (measurable) selections.

2. Convergence of sequences of closed sets. We already alluded to the relation between convergence of sequences in $(\mathcal{F}, \mathcal{T})$ and the classical notion of convergence for sequences of closed subsets of $\mathcal{F}$, due to Painlevé; the connection is made explicit in Theorem 2.2. Let N denote a countable index set (typically the natural numbers); we reserve M to denote an infinite (ordered) subset of N. A sequence of sets $\{C_n \subset E, n \in N\}$ *converges* to a (necessarily closed) set C, written $C = lim\ C_n$ if

$$li\ C_n = C = ls\ C_n \tag{2.1}$$

where

$$li\ C_n = \{x \in E | x = \lim x_n, x_n \in C_n \text{ for all } n > n_x\}$$

and

$$ls\ C_n = \{x \in E | x = \lim x_m, x_m \in C_m \text{ for all } m \in M_x\}.$$

The sets $li\ C_n$ and $ls\ C_n$ are clearly closed. Also note that, since $li\ C_n \subset ls\ C_n$, to prove convergence of the sequence $\{C_n, n \in N\}$ to C it always suffices to show that

$$ls\ C_n \subset C \subset li\ C_n. \tag{2.2}$$

We need also to consider the set-theoretic notions of lim inf and lim sup of a sequence of sets $\{C_n, n \in N\}$; we denote these by Li C_n and Ls C_n, respectively.

We write $C = \text{Lim } C_n$ if $\text{Ls } C_n = C = \text{Li } C_n$, where

$$\text{Li } C_n = \bigcup_{n=1}^{\infty} \bigcap_{m>n} C_m \tag{2.3}$$

and

$$\text{Ls } C_n = \bigcap_{n=1}^{\infty} \bigcup_{m>n} C_m. \tag{2.4}$$

The connections between the topological and set-theoretic notions of limits of sequences of sets is clarified by the following relations:

$$\begin{aligned} \text{Li } C_n \subset li\ C_n &= \bigcap_{k=1}^{\infty} \text{Li } k^{-1}C_n = \bigcap_{k=1}^{\infty} \text{Li}(\text{cl } k^{-1}C_n) \\ &= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m>n} \text{cl } k^{-1}C_n = \bigcap_{k=1}^{\infty} \text{cl}\left[\bigcup_{n=1}^{\infty} \bigcap_{m>n} \text{cl } k^{-1}C_n\right] \end{aligned} \tag{2.5}$$

and

$$\text{Ls } C_n \subset ls\ C_n = \bigcap_{n=1}^{\infty} \text{cl}\left(\bigcup_{m>n} C_m\right), \tag{2.6}$$

where by $\text{cl } A$ we denote the closure of the set $A \in E$ and εA is an open ε-neighborhood of the set A defined as follows: if A is nonempty then

$$\varepsilon A = \{x \in E | d(x, A) < \varepsilon\} \tag{2.7}$$

where $d(x, A) = \inf[d(x, y) | y \in A]$ and $\varepsilon\phi = E \setminus B_{\varepsilon^{-1}}(0)$.

We start with a characterization of convergence to the empty set that is exploited repeatedly in the proof of Theorem 2.2.

LEMMA 2.1. *Suppose that* $\{F_n, n \in N\}$ *is a sequence of closed subsets of E. Then* $lim\ F_n = \varnothing$ *if and only if to each K, there corresponds an index* n_K *such that* $F_n \cap K = \varnothing$ *for all* $n \geqslant n_K$. *Equivalently, if and only if to each* $\varepsilon > 0$ *and* $x \in E$, *there corresponds* $n(\varepsilon, x)$ *such that* $F_n \cap B_\varepsilon(x) = \varnothing$ *for all* $n \geqslant n(\varepsilon, x)$.

PROOF. The equivalence between these two assertions follows directly from the nature of E.

First, suppose that $lim\ F_n = \varnothing$ but there exists K such that $x_m \in F_m \cap K \neq \varnothing$ for all $m \in M \subset N$. The infinite sequence $\{x_m, m \in M\} \subset K$ admits a cluster point which belongs to $ls\ F_n$, contradicting the hypothesis that $\lim F_n = \varnothing$.

Since $li\ F_n \subset ls\ F_n$, to prove the only if part it will suffice to show that $ls\ F_n = \varnothing$. Suppose not and take $x \in ls\ F_n$ with $x = \lim\{x_m | x_m \in F_m, m \in M\}$. Now, let K be a compact neighborhood of x. Then $x_m \in K \cap F_m$ for m sufficiently large and hence there is no n_k such that $K \cap F_n = \varnothing$ for all $n \geqslant n_k$. □

THEOREM 2.2. *Suppose that* $\{F; F_n, n \in N\}$ *is a collection of closed subsets of* E. *Then* $F = \mathfrak{T}\text{-lim } F_n$ *if and only if both part* (a) *and part* (b), *of any one of the following statements, are satisfied*:

(i_a) *if* $F \cap G \neq \varnothing$ *then* $F_n \cap G \neq \varnothing$ *for all* $n \geqslant n_G$,
(i_b) *if* $F \cap K = \varnothing$ *then* $F_n \cap K = \varnothing$ *for all* $n \geqslant n_K$;
(ii_a) *if* $F \cap B_\varepsilon^\circ(x) \neq \varnothing$ *then* $F_n \cap B_\varepsilon^\circ(x) \neq \varnothing$ *for all* $n \geqslant n(\varepsilon, x)$,
(ii_b) *if* $F \cap B_\varepsilon(x) = \varnothing$ *then* $F \cap B_\varepsilon(x) = \varnothing$ *for all* $n \geqslant n'(\varepsilon, x)$;
(iii_a) *for all* x *in* E, $\limsup d(x, F_n) \leqslant d(x, F)$,
(iii_b) *for all* x *in* E, $d(x, F) \leqslant \liminf d(x, F)$;
(iv_a) $\textit{lim}(F \setminus \varepsilon F_n) = \varnothing$ *for all* $\varepsilon > 0$,
(iv_b) $\textit{lim}(F_n \setminus \varepsilon F) = \varnothing$ *for all* $\varepsilon > 0$;
(v_a) $F \cap B_r(x) \subset \varepsilon F_n$ *for any* $x \in E$ *and for all* $\varepsilon > 0$, $r > 0$ *with* $n \geqslant n(\varepsilon, r, x)$,
(v_b) $F_n \cap B_r(x) \subset \varepsilon F$ *for any* $x \in E$ *and for all* $\varepsilon > 0$, $r > 0$ *with* $n \geqslant n'(\varepsilon, r, x)$;
(vi_a) $F \subset \textit{lim}_{r\uparrow\infty}\, \textit{li}(F_n \cap B_r(x))$ *for any* $x \in E$,
(vi_b) $\textit{lim}_{r\uparrow\infty}\, \textit{ls}(F_n \cap B_r(x)) \subset F$ *for any* $x \in E$;
(vii_a) $F \subset \textit{li } F_n$,
(vii_b) $\textit{ls } F_n \subset F$.

PROOF. The equivalence between $\mathfrak{T}$-convergence in $\mathfrak{F}$ and (i) follows immediately from the base structure of $\mathfrak{T}$. Thus to prove the theorem it suffices to establish the equivalence between (i) and the other statements. We will assume that F is nonempty; if $F = \varnothing$ the equivalence is either trivial or requires a straightforward application of Lemma 2.1. We prove the rest in two parts, we show first that all (a) statements are equivalent; this is done by obtaining the following string of implications:

$$(\text{vii}_\text{a}) \Rightarrow (\text{vi}_\text{a}) \Rightarrow (\text{v}_\text{a}) \Rightarrow (\text{iv}_\text{a}) \Rightarrow (\text{iii}_\text{a}) \Rightarrow (\text{ii}_\text{a}) \Rightarrow (\text{i}_\text{a}) \Rightarrow (\text{vii}_\text{a}).$$

(vii_a) $\Rightarrow$ (vi_a). Take any $y \in F \subset li\ F_n$, i.e. $y = \lim\{y_n, n \in N | y_n \in F_n\}$. Now fix any x in E. For n sufficiently large and $r > d(y, x)$, $y_n \in F_n \cap B_r(x)$ and thus $y \in li(F_n \cap B_r(x)) \subset \lim_{r\uparrow\infty} li(F_n \cap B_r(x))$.

(vi_a) $\Rightarrow$ (v_a). Fix any x in F. If $y \in F \cap B_r(x)$ then with $s = r + \varepsilon$, $\varepsilon > 0$, there exists $y_n^s \in F_n \cap B_s(x)$ such that $y = \lim_{s\uparrow\infty} \lim_n y_n^s$. In particular this means that there exists $n(\varepsilon, s, x)$ such that $y_n^s \in F_n \cap B_s(x) \subset F_n$ and $d(y, y_n^s) < \varepsilon$ for all $n \geqslant n(\varepsilon, s, x)$, i.e. $y \in \varepsilon F_n$ for all $n \geqslant n(\varepsilon, s, x)$.

(v_a) $\Rightarrow$ (iv_a). Fix x in E. Now apply Lemma 2.1, more precisely the second version of the assertion, to the sequence $\{F \setminus \varepsilon F_n, n \in N\}$.

(iv_a) $\Rightarrow$ (iii_a). For the sake of the argument suppose that (iii_a) does not hold. Then there exists $x \in E$, $\varepsilon > 0$ and $M \subset N$ such that $d(x, F_m) > d(x, F) + 2\varepsilon$ for all $m \in M$ or, equivalently, $d(x, \varepsilon F_m) > d(x, F) + \varepsilon$. It follows that $\{y | d(x, y) = d(x, F)\} \subset ls(F \setminus \varepsilon F_n)$, which contradicts ($\text{iv}_\text{a}$).

(iii_a) $\Rightarrow$ (ii_a). Note that $\limsup d(x, F_n) \leqslant d(x, F)$ holds only if for all $\varepsilon > 0$ with $d(x, F) < \varepsilon$ we have that $d(x, F_n) < \varepsilon$ for n sufficiently large, or, equivalently, only if $F \cap B_\varepsilon^\circ(x) \neq \varnothing$ implies that $F_n \cap B_\varepsilon^\circ(x) = \varnothing$ for n sufficiently large.

(ii_a) ⇒ (i_a). Simply note that the properties of E allow us to write every open set as the countable union of open balls.

(i_a) ⇒ (vii_a). Take $x \in F$ and $\{G_i, i \in I\}$ a fundamental (nested) system of open neighborhoods of x with $G_1 \supset F$. Clearly, for all i in I, $G_i \cap F \neq \varnothing$. Now, ($i_a$) implies also that $G_i \cap F_n \neq \varnothing$ for $n \geqslant n_i$ and thus there exists $x_n \in F_n$ such that $x = \lim x_n$. Hence $F \subset li\, F_n$.

Next we prove that the (b) statements are equivalent. But this time we derive the sequence of implications in the opposite order, i.e.

$$(vii_b) \Rightarrow (i_b) \Rightarrow (ii_b) \Rightarrow (iii_b) \Rightarrow (iv_b) \Rightarrow (v_b) \Rightarrow (vi_b) \Rightarrow (vii_b).$$

(vii_b) ⇒ (i_b). Suppose not. Then there exists a compact K such that $F \cap K = \varnothing$ but for $F_m \cap K \neq \varnothing$ for all $m \in M \subset N$. Every sequence $\{x_m \in F_m \cap K\}$ admits a convergent subsequence, say to x. By definition $x \in K \cap ls\, F_n \subset K \cap F$, a contradiction.

(i_b) ⇒ (ii_b). Evident.

(ii_b) ⇒ (iii_b). Since $d(x, F) > \varepsilon$ if and only if $F \cap B_\varepsilon(x) = \varnothing$, which in view of ($ii_b$) implies that $F_n \cap B_\varepsilon(x) = \varnothing$, or equivalently, $d(x, F_n) > \varepsilon$ for n sufficiently large. From this (iii_b) follows directly.

(iii_b) ⇒ (iv_b). Suppose not. Then $ls(F_n \setminus \varepsilon F) \neq \varnothing$, i.e. there exists $\{x_m \in F_m \setminus \varepsilon F, m \in M\}$ such that $\lim x_m = x \in ls(F_n \setminus \varepsilon F)$. On one hand we have that for all $m \in M$, $\varepsilon < d(x_m, F) \leqslant d(x, F) + d(x, x_m)$ and thus $d(x, F) > \varepsilon - d(x, x_m)$; on the other hand $d(x, F_m) \leqslant d(x_m, F_m) + d(x, x_m)$. Via ($iii_b$), this implies that $0 = \liminf d(x, F_m) \geqslant d(x, F) > \varepsilon - 0$, a contradiction.

(iv_b) ⇒ (v_b). Apply Lemma 2.1 to the sequence $\{F_n \setminus \varepsilon F, n \in N\}$.

(v_b) ⇒ (vi_b). Fix any $x \in E$. Since for all $\varepsilon > 0$, $r > 0$ there exists $n(\varepsilon, r, x)$ such that for all $n \geqslant n(\varepsilon, r, x)$, $F_n \cap B_r(x) \subset \varepsilon F$, it follows that $ls(F_n \cap B_r(x)) \subset \varepsilon F$. This holds for every $\varepsilon > 0$, and since $ls(F_n \cap B_r(x))$ is closed, we also have that $ls(F_n \cap B_r(x)) \subset F$ from which the assertion follows directly.

(vi_b) ⇒ (vii_b). If $y \in ls\, F_n$ then there exists $\{y_m \in F_m, m \in M \subset N\}$ such that $y = \lim y_m$. Now fix any $x \in \mathfrak{F}$ and let $s > d(y, x)$; then $y_m \in (F_m \cap B_s(x))$ for m sufficiently large. Thus $y \in ls(F_m \cap B_s(x)) \subset \lim_{s\uparrow\infty} ls(F_m \cap B_s(x)) \subset F$. □

Parts of this theorem can be derived directly from the results of Choquet **[3]** and Michael **[4]**; cf. also **[8]**; it remains valid in a more general setting, viz. when E is locally compact, Hausdorff and second countable. More specialized results can be derived for sequences of closed *convex* sets; see **[9]**.

COROLLARY 2.3. *Suppose that* $\{F; F_n, n \in N\}$ *is a collection of closed subsets of E. Then the following are equivalent*:

(i) $F = \mathfrak{T}\text{-lim } F_n$,

(ii) $F = lim\, F_n$,

(iii) *for all* $x \in F$, $d(x, F) = \lim d(x, F_n)$,

(iv) $lim[(F \setminus \varepsilon F_n) \cup (F_n \setminus \varepsilon F)] = \varnothing$ *for all* $\varepsilon > 0$,

(v) $F = lim_{r\uparrow\infty}\, ls(F_n \cap B_r(x)) = lim_{r\uparrow\infty}\, li(F_n \cap B_r(x))$, *for any* $x \in E$.

PROOF. These are simply reformulations of some of the statements appearing in the theorem if we remember that for any sequence of sets we always have that $li \subset ls$ and for any sequence of numbers lim inf $\leqslant$ lim sup. □

3. Convergence almost everywhere (surely). A sequence of closed-valued measurable multifunctions $\{\Gamma_n, n \in N\}$ *converges almost everywhere* (a.e.) to a multifunction Γ if for almost all $\omega \in \Omega$, the closed sets $\Gamma_n(\omega)$ converge to the closed set $\Gamma(\omega)$, more precisely if $\text{meas}\{\omega \in \Omega | \lim \Gamma_n(\omega) \neq \Gamma(\omega)\} = 0$. We write $\Gamma_n \to \Gamma$ a.e. Note that meas is nonnegative but not necessarily bounded; if meas is a probability measure we write $\Gamma_n \to \Gamma$ a.s.

THEOREM 3.1. *Suppose that* $\{\Gamma; \Gamma_n, n \in N\}$ *is a collection of closed-valued measurable multifunctions. Then both* $\omega \mapsto (li\ \Gamma_n)(\omega)$ *and* $\omega \mapsto (ls\ \Gamma_n)(\omega)$ *are closed-valued measurable multifunctions.*

PROOF. They are clearly closed-valued; hence it suffices to show that they are measurable. In view of (2.5) for every ω we have

$$(li\ \Gamma_n)(\omega) = \bigcap_{k=1}^{\infty} \text{cl}\left[\bigcup_{n=1}^{\infty} \bigcap_{m>n} (\text{cl}\ k^{-1}\Gamma_m(\omega))\right].$$

Thus $(li\ \Gamma_n)$ is the countable union and intersection of multifunctions of the type $\text{cl}\ k^{-1}\Gamma_n$, and, hence, to prove that $li\ \Gamma_n$ is measurable it remains only to show that $\omega \mapsto \text{cl}\ k^{-1}\Gamma_n(\omega) = \{x | d(x, \Gamma_n(\omega)) \leqslant k^{-1}\}$ is a measurable multifunction. But this follows from the fact that $(x, \omega) \mapsto d(x, \Gamma_n(\omega))$ is continuous in x and measurable in ω, a so-called Carathéodory function, and consequently the multifunction $\omega \mapsto \Delta(\omega) = \{(x, \eta) \in E \times R | \eta \geqslant d(x, \Gamma_n(\omega))\}$ is closed-valued and measurable. Now simply note that for F any closed subset of E we have that

$$(\text{cl}\ k^{-1}\Gamma_n)^{-1}(F) = \Delta^{-1}(F \times [0, k^{-1}]) \in A.$$

To prove that $ls\ \Gamma_n$ is measurable, we show that for F, an arbitrary closed set, from (2.6) we have that

$$(ls\ \Gamma_n)^{-1}(F) = \bigcap_{k=1}^{\infty} \text{Ls}\ \Gamma_n^{-1}(k^{-1}F). \tag{3.1}$$

For $k = 1, 2, \ldots$ we always have that

$$(ls\ \Gamma_n)^{-1}(F) \subset \text{Ls}\ \Gamma_n^{-1}(k^{-1}F) \tag{3.2}$$

and thus (3.1) will be established if we show that

$$\bigcap_{k=1}^{\infty} \text{Ls}\ \Gamma_n^{-1}(k^{-1}F) \subset (ls\ \Gamma_n)^{-1}(F). \tag{3.3}$$

But this follows from the inclusion

$$\text{Ls}\ \Gamma_n^{-1}(k^{-1}F) \subset (ls\ \Gamma_n)^{-1}(\text{cl}\ k^{-1}F)$$

and the fact that for any closed-valued multifunction Γ,

$$\Gamma^{-1}(F) = \bigcap_{k=1}^{\infty} \Gamma^{-1}(\text{cl}\ k^{-1}F). \quad \square$$

COROLLARY 3.2. *Suppose that* $\{\Gamma_n, n \in N\}$ *is a sequence of closed-valued measurable multifunctions converging almost everywhere to a multifunction* Γ. *Then* Γ *is a closed-valued measurable multifunction on* $\Omega \setminus A$ *where* meas$(A = \{\omega | \Gamma_n(\omega) \nrightarrow \Gamma(\omega)\})$ $= 0$.

PROOF. For every $\omega \in \Omega \setminus A$, we have that $(ls\, \Gamma_n)(\omega) = \Gamma(\omega) = (li\, \Gamma_n)(\omega)$ and thus the assertion follows directly from the theorem. □

COROLLARY 3.3. *A closed-valued mutifunction* Γ *is measurable if and only if it is the limit of a sequence of simple closed-valued measurable multifunctions.*

PROOF. A closed-valued multifunction is *simple* if it takes on only a finite number of values. From Theorem 3.1 we know that the limit multifunction of any sequence of closed-valued measurable multifunctions is itself closed-valued and measurable. The only if part will be argued later when a stronger result is obtained (Proposition 4.4). □

In [**10**, Theorems 2.6, 2.7], M. Sion derives a related result for partitionable multifunctions defined on uniform spaces.

The next theorem yields various characterizations of almost sure convergence in terms of specific "test" families of subsets of E or Ω. Given $\{A_i, i \in I\}$ a collection of elements of $\mathcal{Q}$, we write $A_i \subset_0 A$ ($A \subset_0 A_i$ resp.) for all $i \in I$, if there exists a (fixed) set $A_0 \in \mathcal{Q}$ with meas $A_0 = 0$ such that $A_i \subset A \cup A_0$ ($A \subset A_i \cup A_0$ resp.) for all $i \in I$.

THEOREM 3.4. *Suppose that* $\{\Gamma; \Gamma_n, n \in N\}$ *is a collection of closed-valued measurable multifunctions from* Ω *to* E. *The following are equivalent statements*:

(i) $\Gamma_n \to \Gamma$ *a.e*;

(ii) *for any compact set* $K \subset E$ *and any open set* $G \subset K$,

$$\mathrm{Ls}\, \Gamma_n^{-1}(K) \subset_0 \Gamma^{-1}(K) \quad \textit{and} \quad \Gamma^{-1}(G) \subset_0 \mathrm{Li}\, \Gamma_n^{-1}(G);$$

(iii) *for any* $\varepsilon > 0$ *and* $x \in E$,

$$\Gamma^{-1}(B_\varepsilon^\circ(x)) \subset_0 \mathrm{Li}\, \Gamma_n^{-1}(B_\varepsilon^\circ(x)) \subset \mathrm{Ls}\, \Gamma_n^{-1}(B_\varepsilon(x)) \subset_0 \Gamma^{-1}(B_\varepsilon(x));$$

(iv) *for any* $\varepsilon > 0$, $r > 0$ *and* $x \in E$,

$$\lim \operatorname{meas}\left[\bigcup_{m > n} ((\Gamma_m \setminus \varepsilon\Gamma) \cup (\Gamma \setminus \varepsilon\Gamma_m))^{-1}(B_r(x))\right] = 0;$$

(v) *there exists* $A \in \mathcal{Q}$ *with* meas $A = 0$ *such that for all* $x \in E$ *and* $\omega \in \Omega \setminus A$,

$$\lim d(x, \Gamma_n(\omega)) = d(x, \Gamma(\omega));$$

(vi) *there exists* $A \in \mathcal{Q}$ *with* meas $A = 0$ *such that for all* $x \in E$ *and* $\omega \in \Omega \setminus A$,

$$\lim_{r \uparrow \infty} ls(\Gamma_n(\omega) \cap B_r(x)) \subset \Gamma(\omega) \subset \lim_{r \uparrow \infty} li(\Gamma_n(\omega) \cap B_r(x)).$$

PROOF. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii). Almost everywhere convergence implies the existence of a set A with meas $A = 0$ such that for every $\omega \in \Omega \setminus A$, $\Gamma(\omega) = \lim \Gamma_n(\omega)$. Restricting the Γ_n and Γ to $\Omega \setminus A$, we know from Theorem 2.2(i) that the closed sets $\Gamma_n(\omega)$ converge to $\Gamma(\omega)$ if and only if for every open set G and every compact set K: $\omega \in \Gamma^{-1}(G)$ implies that $\omega \in \Gamma_n^{-1}(G)$ for $n \geqslant n_G$ and $\omega \notin \Gamma^{-1}(K)$ implies that

$\omega \notin \Gamma_n^{-1}(K)$ for $n \geqslant n_K$, or, equivalently, if and only if the inclusions of (ii) are satisfied. The same argument, but relying this time on Theorem 2.2(ii), establishes the equivalence between (i) and (iii). The inclusion $\text{Li}\, \Gamma_n^{-1}(B_\varepsilon^\circ(x)) \subset \text{Ls}\, \Gamma_n^{-1}(B_\varepsilon(x))$ is always valid.

(i) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). These are clearly direct consequences of the equivalence between statements (ii), (iii) and (vi) of Corollary 2.3.

(i) $\Leftrightarrow$ (iv). In view of statements (vii) and (iv) of Theorem 2.2 and Lemma 2.1, the measurable multifunctions Γ_n converge almost everywhere to Γ if and only if there exists a set $A \in \mathcal{Q}$, meas $A = 0$, such that for every $\varepsilon > 0$, $r > 0$ and $x \in E$ there corresponds $n(\varepsilon, r, x)$ such that $[(\Gamma \setminus \varepsilon\Gamma_n) \cup (\Gamma_n \setminus \varepsilon\Gamma)](\omega) \cap B_r(x) = \varnothing$ for all $n > n(\varepsilon, r, x)$. Let

$$W_n(\varepsilon, r, x) = \left\{\omega \in \Omega \,\middle|\, \left[\bigcup_{m>n} ((\Gamma \setminus \varepsilon\Gamma_m) \cup (\Gamma_m \setminus \varepsilon\Gamma))(\omega)\right] \cap B_r(x) \neq \varnothing\right\}.$$

The sets $W_n(\varepsilon, r, x)$ are measurable and thus so is $W(\varepsilon, r, x) = \text{Lim}\, W_n(\varepsilon, r, x)$. Now, we have that $\Gamma = \mathit{lim}\, \Gamma_n$ a.e. if and only if for all $r > 0$, $\text{meas}(\bigcup_{k=1}^{\infty} \bigcup_{x \in D} W(k^{-1}, r, x)) = 0$ where D is a countable dense subset of E, or, equivalently, if and only if for all $r > 0$, $k = 1, \ldots$ and $x \in D$, $\text{meas}(W(k^{-1}, r, x)) = 0$ or still, if and only if for all $r > 0$, $\varepsilon > 0$ and $x \in E$, $\text{meas}(W(\varepsilon, r, x) = \text{Lim}\, W_n(\varepsilon, r, x)) = 0$, but this holds if and only if $\lim \text{meas}\, W_n(\varepsilon, r, x) = 0$. □

As can be easily gathered from the proof the somewhat weaker statement (iv′) also implies almost everywhere convergence:

(iv′) for any $\varepsilon > 0$ and $x \in E$ and some $r > 0$,

$$\lim \text{meas}\left[\bigcup_{m>n} ((\Gamma_m \setminus \varepsilon\Gamma) \cup (\Gamma \setminus \varepsilon\Gamma_m))^{-1}(B_r(x))\right] = 0.$$

COROLLARY 3.5. *Suppose that* $\{f; f_n, n \in N\}$ *is a family of measurable functions from* Ω *to* R. *Then* $f = \lim f_n$ *a.e. if and only if for every* $\eta \in R$ *we have that*

$$\{\omega \in \Omega | f(\omega) < \eta\} \subset_0 \text{Li}\{\omega \in \Omega | f_n(\omega) < \eta\} \tag{3.4}$$

and

$$\text{Ls}\{\omega \in \Omega | f_n(\omega) \leqslant \eta\} \subset_0 \{\omega \in \Omega | f(\omega) \leqslant \eta\}. \tag{3.5}$$

PROOF. Apply criterion (ii) of Theorem 3.4. □

4. Convergence of measurable selections. We turn next to finding conditions that will guarantee the convergence of measurable selections. A measurable function v from Ω to E is called a *measurable selection* of (the measurable multifunction) Γ if $v(\omega) \in \Gamma(\omega)$ for all $\omega \in \text{dom}\, \Gamma$. The basic theorem on measurable selections, already referred to in the introduction, asserts that a multifunction Γ is measurable if and only if Γ admits a Castaing representation, i.e. $\text{dom}\, \Gamma \in A$ and there exists a countable collection of measurable selections $\{v_k, k \in N'\}$ such that for all $\omega \in \text{dom}\, \Gamma$, $\text{cl}\{\bigcup_k v_k(\omega)\} = \Gamma(\omega)$.

THEOREM 4.1. *Suppose that* $\{\Gamma_n, n \in N\}$ *is a collection of closed-valued measurable multifunctions from* Ω *to* E *converging almost everywhere to the closed-valued measurable multifunction* Γ *such that for all* n, $\operatorname{dom}\Gamma_n = \operatorname{dom}\Gamma$. *Then there exist Castaing representations* $\{v_n^k, k \in N'\}$ *of the* Γ_n *such that for each* k *in* N', $v^k = \lim v_n^k$ *a.e. on* $\operatorname{dom}\Gamma$ *and* $\{v^k, k \in N'\}$ *is a Castaing representation of* Γ.

PROOF. The theorem is trivially true if $\operatorname{meas}(\operatorname{dom}\Gamma) = 0$. Let us thus assume–without loss of generality–that $\operatorname{dom}\Gamma = \operatorname{dom}\Gamma_n = \Omega$, $\operatorname{meas}\Omega > 0$ and that $\Gamma(\omega) = \lim \Gamma_n(\omega)$ for every ω in Ω. Since the topology on (E, d) is metric invariant we may also assume that d is the Euclidean distance.

Let Q^p be the points with rational coordinates in E where p is the dimension of E. Let

$$A = \left\{a_k = \left(a_k^1, a_k^2, \dots, a_k^{p+1}\right), k \in N' | a_k^i \in Q^p, \right.$$
$$\left. \left(a_k^1, \dots, a_k^{p+1}\right) \text{ affinely independent}\right\},$$

and for any closed set $F \subset E$, define

$$\text{"proj"}_F\, a_k = M_{p+1},$$

where for $i = 1, \dots, p$,

$$M_{i+1} = \left\{x \in M_i | d\left(a_k^i, x\right) = d\left(a_k^i, M\right)\right\}$$

and $M_1 = F$. Then "proj"$_F\, a_k$ is a singleton unless F is empty, in which case it is also empty. Note also that $\{\text{"proj"}_F\, a_k, k \in N'\}$ is dense in F.

Let

$$v_{n,k}(\omega) = \text{"proj"}_{\Gamma_n(\omega)} a_k, \qquad k \in N', \quad \text{and} \quad v_k(\omega) = \text{"proj"}_{\Gamma(\omega)} a_k.$$

By construction, the $\{v_{n,k}, k \in N'\}$ $[\{v_k, k \in N'\}]$ are Castaing representations of the Γ_n [Γ resp.] provided the functions $\omega \mapsto v_{n,k}(\omega)$ $[\omega \mapsto v_k(\omega)]$ are measurable. But this follows from the repeated application, $p + 1$ times, of the fact that if Λ is a closed-valued measurable multifunction from Ω to E and $y \in E$, then the multifunction

$$\omega \mapsto \operatorname{proj}_{\Lambda(\omega)} y = \{x \in \Lambda(\omega) | d(y, x) = d(y, \Lambda(\omega))\}$$

is closed-valued and measurable. To see this, simply note that the function $(\omega, x) \mapsto d(x, y) - d(y, \Lambda(\omega)) = q(\omega, x)$ is measurable in ω and continuous in x; thus the multifunction $\omega \mapsto \{x | q(\omega, x) = 0\}$ is measurable and so is $\operatorname{proj}_{\Lambda(\omega)} y = \Lambda(\omega) \cap \{x | q(\omega, x) = 0\}$. To complete the proof simply observe that for all $k \in N'$ and for all $\omega \in \Omega$, the sequence $\{v_{n,k}(\omega), n \in N\}$ converges to $v_k(\omega)$. This is a direct consequence of Theorem 3.4(v) since (almost everywhere) convergence of the Γ_n to Γ implies that (almost everywhere) the sequence $\{d(a_k^1, \Gamma_n(\omega)), n \in N\}$ converges to $d(a_k^1, \Gamma(\omega))$, and by construction, $d(a_k^1, \Gamma_n(\omega)) = d(a_k^1, v_{n,k}(\omega))$ and $d(a_k^1, v_k(\omega)) = d(a_k^1, \Gamma(\omega))$. □

COROLLARY 4.2. *Suppose that* $\{\Gamma_n, n \in N\}$ *is a sequence of closed-valued measurable multifunctions converging almost surely to a closed-valued measurable multifunction* Γ. *Then there exist measurable selections of* Γ_n *converging to a measurable selection of* Γ.

From the convergence of Castaing representations of the Γ_n to a countable collection of measurable functions $\{v_k,\ k \in N'\}$ it does not follow that the Γ_n converge to the closed-valued measurable multifunction Γ' where $\Gamma'(\omega) = \text{cl}\{\bigcup_k v_k(\omega)\}$, not even when the Γ_n and Γ' are convex-valued. To see this, consider the following simple example: Let $\Gamma_n(\omega) = [0, 1]$ for all $\omega \in \Omega$ and all n. Take a Castaing representation of Γ_n with the following properties: $\omega \mapsto v_{n,k}(\omega)$ piecewise constant with $v_{n,k}(\omega) \leqslant \frac{1}{2}$ if $k \leqslant n$. Let $v_k = \lim v_{n,k}$, necessarily $0 \leqslant v_k \leqslant \frac{1}{2}$. Thus $\Gamma' = \text{cl}[\bigcup_k v_k] \subset [0, \frac{1}{2}]$, whereas $\lim \Gamma_n(\omega) = [0, 1]$ for all ω in Ω.

Note that $\Gamma' \subset \Gamma$, since $v_k(\omega) = \lim v_{n,k}(\omega) \subset li\ \text{cl}\{\bigcup_k v_{n,k}(\omega)\} = li\ \Gamma_n(\omega) \subset \Gamma(\omega)$. The reverse inclusion would be valid if $ls\ \text{cl}\{\bigcup_k v_{n,k}(\omega)\} \subset \text{cl}\{\bigcup_k v_k(\omega)\}$. In order for this to hold, i.e. to derive a converse Theorem 4.1, we need at least the uniform convergence of the Castaing representations. But it is not always possible to find selections that exhibit uniform convergence, as is evident from the following example: Let $\Omega =]0, 1]$, $\Gamma_n(\omega) = (n\omega)^{-1}$; then $v_n(\omega) = \Gamma_n(\omega)$ is the only selection and $\lim v_n = v = 0 = \Gamma = \lim \Gamma_n$. Clearly there are no measurable selections converging uniformly to 0. However any measurable selection of the limit multifunction Γ can be obtained as the limit of a sequence of measurable selections of the Γ_n; this is the content of the next theorem.

THEOREM 4.3. *Suppose that* $\{\Gamma_n,\ n \in N\}$ *is a sequence of closed-valued measurable multifunctions from* Ω *to* E *converging almost everywhere to the closed-valued measurable multifunction* Γ. *Suppose also that* v *is a measurable selection of* Γ; *then there exist measurable selections* $\{v_n,\ n \in N\}$ *of the multifunctions* $\{\Gamma_n,\ n \in N\}$ *such that almost everywhere* $v = \lim v_n$.

PROOF. Let $\Gamma'_n(\omega) = \text{proj}_{\Gamma_n(\omega)} v(\omega)$. The multifunctions Γ'_n are closed-valued, measurable and nonempty-valued whenever $\Gamma_n(\omega)$ is nonempty; cf. proof of Theorem 4.1. Any sequence of measurable selections v_n of Γ'_n, $n \in N$, has the desired characteristics. This follows from Theorem 3.4(v) and the fact that $d(v(\omega), \Gamma(\omega)) = 0$. □

This theorem allows us to give an enlightening proof of Corollary 3.3. We are obviously only concerned with the only if part of the statement. Suppose that Γ is a closed-valued measurable multifunction and for $n = 1, 2, \ldots,$ let $\Gamma_n = \Gamma \cap B_n(0)$. Each Γ_n is a uniformly bounded compact-valued measurable multifunction and $\Gamma = \mathit{lim}\ \Gamma_n$. As in the proof of Theorem 4.1, for each Γ_n we build a Castaing representation $\{u_{nk},\ k \in N'\}$. Each u_{nk} is measurable and necessarily bounded. Let $\{v_{nk},\ k \in N'_n\} = \bigcup_{m \leqslant n} \{u_{mk},\ k \in N'\}$. This is also a Castaing representation of Γ_n with the following properties:

(i) $\{v_{n_1 k},\ k \in N'_{n_1}\} \subset \{v_{n_2 k},\ k \in N'_{n_2}\}$ if $n_1 \leqslant n_2$, and

(ii) the multifunctions $\{\bigcup_{j \leqslant k} v_{nj},\ k \in N'_n\}$ converge uniformly–with respect to the Hausdorff distance h–to the uniformly bounded multifunction Γ_n.

Each bounded measurable selection v_{nk} is in turn the uniform limit of a sequence of measurable simple functions, say $\{v_{nkl},\ l \in L_{nk}\}$. Let $\Delta_{nkl} = \bigcup_{j \leqslant k} v_{njl}$. This is a simple *finite-valued* measurable multifunction and, obviously, for each $\omega \in \Omega$ we have that $\Gamma(\omega) = \lim_n \lim_k \lim_l \Delta_{nkl}(\omega)$. (By finite-valued we mean that the range of the multifunction, i.e. $\{\bigcup_\omega \Delta_{nkl}(\omega)\}$, consists of a finite number of points, which is a stronger restriction than having $\Gamma(\omega)$ of finite cardinality for each $\omega \in \Omega$.) Since the $\{\Delta_{nkl}\}$ converge uniformly to $\{\bigcup_{j<k} v_{nj}\}$ which in turn converge uniformly to Γ_n, we can rely on the standard diagonalization argument to find a sequence of multifunctions $\{\Delta_{nk_nl_n}, n \in N\}$ that converge to Γ. This argument allows us to state a stronger version of Corollary 3.3.

PROPOSITION 4.4. *A closed-valued multifunction is measurable if and only if it is the limit of a sequence of simple finite-valued measurable multifunctions.*

5. Almost uniform convergence and convergence in probability. Henceforth we shall assume that meas is a probability measure. Let $\{\Gamma; \Gamma_n, n \in N\}$ be a collection of closed-valued multifunctions from Ω to E. We say that the Γ_n *converge uniformly* to Γ on a set $A \in \mathcal{A}$ if to every $x \in E$ and every pair $\varepsilon' > \varepsilon > 0$ there corresponds an index $n(\varepsilon', \varepsilon, x)$ such that for all $n \geq n(\varepsilon', \varepsilon, x)$,

$$\Gamma^{-1}(B^\circ_\varepsilon(x)) \cap A \subset \Gamma_n^{-1}(B^\circ_{\varepsilon'}(x)) \cap A \tag{5.1}$$

and

$$A \setminus \Gamma_n^{-1}(B_{\varepsilon'}(x)) \subset A \setminus \Gamma^{-1}(B_\varepsilon(x)). \tag{5.2}$$

This definition is motivated by criterion (ii) of Theorem 2.2. In particular, it follows that the sequence $\{\Gamma_n, n \in N\}$ converges uniformly to $\Gamma \equiv \varnothing$ on A if to every $\varepsilon > 0$ and $x \in E$ there corresponds $\bar{n}(\varepsilon, x)$ such that $\Gamma_n^{-1}(B_\varepsilon(x)) = \varnothing$ for all $n \geq \bar{n}(\varepsilon, x)$.

If, in addition, the Γ_n and Γ are measurable, we say that the Γ_n converge *almost uniformly* if given $\delta > 0$, there is a set $\Omega_\delta \in \mathcal{A}$ with $\text{meas}(\Omega_\delta) > 1 - \delta$ such that the Γ_n converge uniformly to Γ on Ω_δ. We then write $\Gamma_n \to \Gamma$ a.u.

THEOREM 5.1. *Suppose that* meas *is a probability measure and* $\{\Gamma; \Gamma_n, n \in N\}$ *is a collection of closed-valued measurable multifunctions. Then* $\Gamma_n \to \Gamma$ *a.s. if and only if* $\Gamma_n \to \Gamma$ *a.u.*

PROOF. Clearly almost uniform convergence implies almost sure convergence. By definition, the sequence Γ_n converges (uniformly) on $\Omega_0 = \bigcup_{k=1}^\infty \Omega_{k^{-1}}$ with meas $\Omega_{k^{-1}} > 1 - k^{-1}$ and meas $\Omega_0 \geq \text{meas}[\text{Lim}_k(\bigcup_{i=1}^k \Omega_{i^{-1}})] = 1$. To prove the other direction we proceed as follows: As in the proof of Theorem 3.4, for $\varepsilon > 0$, $r > 0$, $x \in E$ let

$$W_n(\varepsilon, r, x) = \left\{\omega \in \Omega \,\middle|\, \left[\bigcup_{m>n} ((\Gamma \setminus \varepsilon\Gamma_m) \cup (\Gamma_m \setminus \varepsilon\Gamma))(\omega)\right] \cap B_r(x) \neq \varnothing\right\}.$$

From Theorem 3.4(iv) we know that $\Gamma_n \to \Gamma$ a.s. if and only if for all $\varepsilon > 0$, $r > 0$ and $x \in E$, lim meas $W_n(\varepsilon, r, x) = 0$. In particular, if k is any positive integer we have that lim meas $W_n(k^{-1}, r, x) = 0$, i.e. given $\delta > 0$, there exists $\bar{n}(\delta, k^{-1}, r, x)$ such that meas $W_n(k^{-1}, r, x) < \delta/2^k$ for all $n > \bar{n}$. Let

$$V(r, x) = \bigcup_{k=1}^\infty W_{\bar{n}}(k^{-1}, r, x).$$

Then meas $V(r, x) < \delta$ and define $\Omega_\delta = \Omega \setminus V(r, x)$. If $\bar{\omega} \in \Omega_\delta$ then

$$\bar{\omega} \in \bigcap_{k=1}^{\infty} (\Omega \setminus W_{\bar{n}}(k^{-1}, r, x)),$$

i.e. for all $n \geqslant \bar{n}(\delta, k^{-1}, r, x)$, $\bar{\omega} \in \Omega \setminus W_n(k^{-1}, r, x)$, since the sequence $W_n(k^{-1}, r, x)$ is monotone nonincreasing (in n). This means that for all $n > \bar{n}$,

$$(\Gamma_n \setminus k^{-1}\Gamma)(\bar{\omega}) \cap B_r(x) = \varnothing \tag{5.3}$$

and

$$(\Gamma \setminus k^{-1}\Gamma_n)(\bar{\omega}) \cap B_r(x) = \varnothing. \tag{5.4}$$

Now assume that $\omega \in \Gamma^{-1}(B_r^\circ(x)) \cap \Omega_\delta$ and for the sake of the argument, let us assume that for some k, there is no $\hat{n}$ $(r' = r + k^{-1}, r, x)$ such that $\omega \in \Gamma_n^{-1}(B_{r'}^\circ(x)) \cap \Omega_\delta$ for all $n \geqslant \hat{n}$. This means that there exists $M \subset N$–determining a subsequence–such that $\omega \notin \Gamma_m^{-1}(B_{r'}^\circ(x))$ for all $m \in M$ or equivalently, for all $m \in M$, $k^{-1}\Gamma_m(\omega) \cap B_r(x) = \varnothing$, from which it follows that for all $m \in M$,

$$(\Gamma \setminus k^{-1}\Gamma_m)(\omega) \cap B_r(x) = \Gamma(\omega) \cap B_r(x) \neq \varnothing,$$

which contradicts (5.4).

Similarly, if $\omega \in \Omega_\delta \setminus \Gamma^{-1}(B_{r'}(x))$, and for all $m \in M \subset N$, $\omega \notin \Omega_\delta \setminus \Gamma_m^{-1}(B_r(x))$, we have that

$$(\Gamma_m \setminus k^{-1}\Gamma)(\omega) \cap B_r(x) = \Gamma_m(\omega) \cap B_r(x) \neq \varnothing,$$

and then (5.3) is contradicted. □

Theorem 5.1 is a version of Egorov's theorem; it raises an interesting question. The relationship between the closed-valued measurable multifunctions from Ω to E and the measurable functions from Ω to $\mathfrak{F}$ suggests yet another approach to the derivation of Egorov-type results. Again let ρ be a metric on $\mathfrak{F}$ compatible with $\mathfrak{T}$, the measurable functions γ_n converge almost uniformly to the measurable function γ if given $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \in \mathcal{A}$ with meas$(\Omega_\varepsilon) > 1 - \varepsilon$ and on Ω_ε, the γ_n converge uniformly (with respect to the metric ρ) to the function γ. With this definition, we can then rely on the standard version of Egorov's theorem to obtain the equivalence of almost sure and almost uniform convergence. Passing to the associated multifunctions, via Proposition 1.1, would yield an Egorov theorem for multifunctions. Such an approach is of interest only if we have a "concrete" version of the metric ρ. Although criteria (v) and (vi) of Theorem 2.2 provide us with quantities that can be associated with the notion of proximity (for two closed sets), there is, at present, no satisfactory representation of any metric compatible with $\mathfrak{T}$. Theorem 5.1 and the preceding considerations inform us that the definition of uniform convergence for sequences of multifunctions, introduced at the beginning of this section, provides a characterization of ρ-uniform convergence for functions with values in $\mathfrak{F}$. Is this the best possible, in the sense that a minimal class of "test"-sets are involved in the definition?

Finally, let us observe that the above lead us to a natural definition of convergence in probability. Let

$$\Delta_{\varepsilon,n} = (\Gamma_n \setminus \varepsilon\Gamma) \cup (\Gamma \setminus \varepsilon\Gamma_n).$$

From Corollary 2.3(iv) we know that for any fixed ω,

$$\Gamma(\omega) = lim\ \Gamma_n(\omega) \text{ if and only if for all } \varepsilon > 0,\ lim\ \Delta_{\varepsilon,n}(\omega) = \varnothing.$$

This, in terms of the criterion for convergence to the empty set provided by the Lemma 2.1, can be reexpressed as follows: given $r > 0$ and $x \in E$, there exists $\hat{n}(r, x)$ such that $\Delta_{\varepsilon,n}(\omega) \cap B_r(x) = \varnothing$ for all $n \geqslant \hat{n}$. In view of this, we may define convergence in probability as follows: As usual, let $\{\Gamma; \Gamma_n, n \in N\}$ be a collection of closed-valued measurable multifunctions; then the Γ_n *converge in probability* to Γ if for all $\varepsilon > 0$ and any $r > 0$, $x \in E$, $\lim \text{meas}(\Delta_{\varepsilon,n}^{-1}(B_r(x))) = 0$. To see that almost sure convergence implies convergence in probability, one proceeds as follows: If the Γ_n converge almost surely to Γ then for all $\varepsilon > 0$, the $\Delta_{\varepsilon,n}$ converge almost surely to $\varnothing$. In view of the preceding theorem this implies that for all $\varepsilon > 0$, the $\Delta_{\varepsilon,n}$ converge almost uniformly to $\varnothing$. This means that given any $\delta > 0$, there corresponds $\bar{\Omega}_\delta \in \mathcal{A}$ with meas $\bar{\Omega}_\delta < \delta$ and the $\Delta_{\varepsilon,n}$ converge uniformly to $\varnothing$ on $\Omega \setminus \bar{\Omega}_\delta$. In turn this yields, given $r > 0$, $x \in E$, there exists $\hat{n}_\delta(r, x)$ such that $(\Omega \setminus \bar{\Omega}_\delta) \cap \Delta_{\varepsilon,n}^{-1}(B_r(x)) = \varnothing$ for all $n \geqslant \hat{n}$, i.e. for all $n \geqslant \hat{n}$, $\text{meas}(\Delta_{\varepsilon,n}^{-1}(B_r(x))) \leqslant$ meas $\bar{\Omega}_\delta < \delta$, and thus $\lim \text{meas}\ \Delta_{\varepsilon,n}^{-1}(B_r(x)) = 0$ for all $\varepsilon > 0$, $r > 0$ and $x \in E$. This ends the argument.

Naturally, if the multifunctions Γ_n are *compact*-valued then it is possible to rely on the Hausdorff distance to find a satisfactory definition of convergence in probability. In view of the relations between the $\mathcal{T}$-topology and the Hausdorff metric, discussed in §1, it is easy to see that both definitions must coincide when the multifunctions are uniformly bounded. If the multifunctions Γ_n, and also Γ, are *convex*-valued it is possible to characterize convergence in probability in terms of the r-distance [9]. Let F_1, F_2 be two closed subsets of E; for $r > 0$ the r-distance is, by definition,

$$h_r(F_1, F_2) = \begin{cases} 0 & \text{if } F_1^r = F_2^r = \varnothing, \\ +\infty & \text{if } F_1^r = \varnothing \text{ or } F_2^r = \varnothing \text{ but } F_1^r \neq F_2^r, \\ h(F_1^r, F_2^r), & \end{cases}$$

where $h(F_1^r, F_2^r)$ is the Hausdorff distance between F_1^r and F_2^r and for any $D \subset E$, $D^r = D \cap B_r(0)$. With this definition we have that [9, Theorem 4]: Suppose that $\{C; C_n, n \in N\}$ is a collection of closed convex subsets of E. Then $C = lim\ C_n$ if and only if there exists $r_0 > 0$ such that for all $r \geqslant r_0$, $\lim h_r(C, C_n) = 0$. This statement clearly implies Theorem 2.2(vi) but convexity is necessary to get the converse. It is possible to exploit this result to find, in the convex-valued case, a criterion for the convergence in probability that does not involve all x in E (or all x in a dense subset of E). Let $\omega \mapsto r_0(\omega)$ be a measurable function from Ω to R such that $r_0(\omega) > d(0, \Gamma(\omega))$. Such a function exists (and is finite) whenever Γ is nonempty-valued since $\omega \mapsto d(0, \Gamma(\omega))$ is measurable. In view of the equivalence of (vi) and (iv) in Theorem 2.2, and that in the convex case (vi) can be expressed in terms of the r-distance, when the Γ_n and Γ are nonempty convex-valued, we have that the Γ_n converge in probability to Γ if and only if there exists a positive measurable function r_0 such that for all measurable functions $\omega \mapsto r(\omega)$ with $r(\omega) \geqslant r_0(\omega)$ and all $\varepsilon > 0$, $\lim \text{meas}\{\omega | h_{r(\omega)}(\Gamma_n(\omega), (\omega)) > \varepsilon\} = 0$. This can be further refined when

$d(0, \Gamma(\cdot))$ is bounded. It then suffices to check the above for every measurable function $\omega \mapsto r(\omega)$ such that for all ω, $r(\omega) > r_0$, where $r_0 > \sup d(0, \Gamma(\omega))$. But every measurable function is itself the limit of simple functions; thus in this latter case (Γ_n and Γ convex-, nonempty-valued and $d(0, \Gamma(\cdot))$ bounded) the Γ_n converge to Γ in probability if and only if there exists $r_0 > 0$ such that for all $r \geqslant r_0$ and all $\varepsilon > 0$, $\lim \operatorname{meas}\{\omega | h_r(\Gamma_n(\omega), \Gamma(\omega)) > \varepsilon\} = 0$.

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Istituto di Calcolo delle Probabilita, Citta Universitaria, 00100 Roma, Italy

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506