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THE DUALITY BETWEEN ESTIMATION AND CONTROL FROM A VARIATIONAL VIEWPOINT: THE DISCRETE TIME CASE*

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The duality between estimation and control is shown to follow from *basic duality principles*. To do so we formulate the estimation problem in terms of a variational problem and rely on the duality for the convex optimization problem to obtain the associated control problem. The properties of the solution of this problem are exploited to obtain the recursive relations that yield the optimal estimators of a dynamical system.

Key words: Estimation, Filtering, Duality, Variational Principle.

1. Introduction

The duality between estimation and control, first exhibited by Kalman [4], is shown to follow from a basic variational principle. Earlier derivations rely on formal arguments, cf. for example [1; 3, Chapter V, Section 9]. We first show that the estimation problem can be embedded in a class of stochastic variational problems of the Bolza type, studied by Rockafellar and Wets [7–9]. The dual of this problem is a stochastic optimizaiton problem, which under the standard modeling assumption is equivalent to a deterministic control problem whose structure turns out to be that of the linear–quadratic regulator problem. In this context the duality between estimation and control takes on a precise meaning which until now was believed to be of a purely formal nature. In particular, we gain new insight into the two system-theoretic concepts of controllability and observability. They appear as the *same* property of two dual problems. A part of these results were sketched out in [5] relying on a variant of the arguments used here.

This derivation clearly exhibits those features of the problem that can easily be modified without impairing the main results. Also, since it relies on basic

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principles, one may hope that the insight so gained will be useful to study nonlinear filtering problems.

2. The one-step estimation problem

Let $(w_t, t = 0, 1, \dots, T)$ be a gaussian process defined on the probability space $(\Omega, \mathcal{A}, P)$ and with values in R^P. The dynamics of the state variables given by the finite difference equations: for $t = 1, \dots, T$

$$x_{t1}(\omega) = A_t x_t(\omega) + B_t w_t(\omega) \quad \text{a.s.},$$

with initial conditions

$$x_1(\omega) = B_0 w_0(\omega) \quad \text{a.s.},$$

where we write $t1$ for $t+1$, in particular $T1 = T+1$. The n-vector x represents the *state* of the system. The matrices A_t, B_t are $n \times n$ and $n \times p$. With

$$\Delta x_t = x_{t1} - x_t,$$

we can also express the dynamics by the relations

$$\Delta x_t(\omega) = (A_t - I)x_t(\omega) + B_t w_t(\omega)$$

with the same initial conditions. The vector-valued process $(x_t, t = 1, \dots, T1)$ is also gaussian, since for $t = 0, \dots, T$, we have that

$$x_{t1}(\omega) = \sum_{\tau=0}^{t} \left(\prod_{s=t}^{\tau 1} A_s \right) B_\tau w_\tau(\omega)$$

with the convention $\prod_{s=t}^{t1} A_s = I$.

Rather than observing the actual state x_t, we have only access to $y_t \in R^m$, a linear function of the state disturbed by an additive noise, specifically

$$y_t(\omega) = C_t x_t(\omega) + D_t w_t(\omega) \quad \text{a.s.}$$

The matrices C_t and D_t are $m \times n$ and $n \times p$, respectively. The information process $(y_t, t = 1, \dots, T)$ is also a gaussian process, since $t = 0, \dots, T$ we have that a.s.

$$y_{t1}(\omega) = C_{t1}\left[\sum_{\tau=0}^{t} \left(\prod_{s=t}^{\tau 1} A_s \right) B_\tau w_\tau(\omega) \right] + D_{t1} w_{t1}(\omega).^{1}$$

[1] Note that there is no loss of generality in the use of only one gaussian process to represent the measurements noise *and* the dynamics disturbances, in fact this model is more general in that it allows for arbitrary cross-correlation between the two noise processes. If (w_t^d) are the dynaimcs' disturbances and (w_t^n) the measurements' noise, and they are independent, simply set

$$w_t = \begin{pmatrix} w_t^d \\ w_t^n \end{pmatrix}, \qquad B_t = [I, 0], \qquad D_t = [0, I]$$

and we are in the framework of the proposed model.

Let $\mathcal{Y}_t = \sigma - (y_s, s \leq t)$ be the σ-fields induced by the information process, we simply write $\mathcal{Y}$ for $\mathcal{Y}_T$. A function which is $\mathcal{Y}_t$-measurable depends only on the information that can be collected up to time t. If the function is $\mathcal{Y}$-measurable it means that no more than the total information collected can be used in determining it value. We always have that $\mathcal{Y}_t \subset \mathcal{Y} \subset \mathcal{A}$.

The *one-step estimation (or prediction) problem* consists in finding a 'best' estimator γ of the final state x_{T1}, on the basis of the total information collected, in other words we seek a $\mathcal{Y}$-measurable function γ from Ω into R^n that minimizes

$$J(\gamma) = E\{\tfrac{1}{2}\|x_{T1}(\omega) - \gamma(\omega)\|^2\},$$

where $\|\cdot\|$ is the euclidean norm. An excellent, and more detailed, description of estimation in dynamical systems can be found in [2, Chapter 4]. Since $x_{T1} \in \mathcal{L}^2(\Omega, \mathcal{A}, P; R^n) = \mathcal{L}^2_n(\mathcal{A})$, it is natural to restrict γ to the same class of functions; besides the functional J might fail to be well defined otherwise. On the other hand, γ must be $\mathcal{Y}$-measurable, thus we must further restrict γ to $\mathcal{L}^2_n(\mathcal{Y})$, a closed linear subspace of $\mathcal{L}^2_n(\mathcal{A})$. The one-step estimation problem can then be formulated as follows:

EP Find $\gamma \in \mathcal{L}^2_n(\mathcal{Y}) \subset \mathcal{L}^2_n(\mathcal{A})$ such that $J(\gamma)$ is minimized.

This is an optimal recourse problem, the recourse function γ must satisfy the nonanticipativity constraint: $\mathcal{Y}$- rather than $\mathcal{A}$-measurability. The objective function is strictly convex; thus **EP** admits a unique solution γ^* that must satisfy the following conditions [7]: for almost all ω

$$\gamma^*(\omega) = \operatorname{argmin}[\tfrac{1}{2}\|x_{T1}(\omega) - \gamma\|^2 - \rho(\omega)' \cdot \gamma]$$

where $\rho \in \mathcal{L}^2_n(\mathcal{A})$ with $E^{\mathcal{Y}}\rho = 0$ a.s. or equivalently

$$\gamma^*(\omega) =_{\text{a.s}} (E^{\mathcal{Y}} x_{T1})(\omega),$$

where the components of ρ are the multipliers associated to the nonanticipativity constraints. The optimal estimator γ^* is the orthogonal projection of x_{T1} on $\mathcal{L}^2_n(\mathcal{Y})$ and thus belongs to the linear hull generated by the observations, i.e.,

$$\gamma^*(\omega) =_{\text{a.s.}} -\sum_{t=1}^{T} U_t^* y_t(\omega)$$

where for $t = 1, \ldots, T$, the U_t are $n \times n$ matrices. The minus sign is introduced for esthetic purposes that will come to light in the ensuing development. We can view these matrices as (estimator) *weights*; they are the weights required to construct the optimal estimator. Note that we can thus restrict the search for the optimal estimator to the class of *linear* estimators, i.e., those that are linear combinations of the observations.

3. A variational formulation

In view of the above, the original problem is equivalent to finding the *weights* $U_t^*, t = 1, \ldots, T$ that in turn, yield the optimal estimator. Each observation y_t contributing incrementally to the construction of this estimator. Define

$$\Delta\gamma_t(\omega) = -U_t(C_t x_t(\omega) + D_t w_t(\omega)), \quad t = 1, \ldots, T,$$

$$\gamma_1(\omega) = 0$$

with

$$\Delta\gamma_t = \gamma_{t1} - \gamma_t.$$

We can view these equations as the dynamics of the *estimation process*. Through U_t the available information is processed to yield γ_{T1}: an estimator of x_{T1}. The original problem **EP** has the equivalent formulation:

WP Find $U = (U_t, 1 \leq t \leq T)$ that minimizes $F(U)$

where

$$F(U) = \text{Inf } E[\Phi_{l,L}(\omega, x(\omega), \gamma(\omega); U) \,|\, (x, \gamma) \in \mathcal{L}_N^2(\mathcal{A}) \times \mathcal{L}_N^2(\mathcal{A})],$$

$$\Phi_{l,L}(\omega, x, \gamma; U) = l(\omega, x_1, \gamma_1, x_{T1}, \gamma_{T1}) + \sum_{t=1}^{T} L_t(\omega, x_t, \gamma_t, \Delta x_t, \Delta\gamma_t; U),$$

$$l(\omega, x_1, \gamma_1, x_{T1}, \gamma_{T1}) = \begin{cases} \frac{1}{2}\|x_{T1} - \gamma_{T1}\|^2, & \text{if } x_1 = B_0 w_0(\omega), \gamma_1 = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

$$L_t(\omega, x_t, \gamma_t, \Delta x_t, \Delta\gamma_t; U_t) = \begin{cases} 0, & \text{if } \Delta x_t = (A_t - I)x_t + B_t w_t(\omega), \\ & \quad \Delta\gamma_t = -U_t(C_t x_t + D_t w_t(\omega)), \\ +\infty, & \text{otherwise} \end{cases}$$

and $N = n \cdot T1$.

For each choice of weights U, the value of the function $F(U)$ is obtained by solving a variational problem of the Bolza type (discrete-time). Since there are nonanticipative restrictions on the choice of the decision variables, the functional $(\omega, (x, \gamma)) \to \Phi(\omega, x, \gamma; U)$ is a convex normal integrand and the space $\mathcal{L}^2(\mathcal{A})$ is decomposable, we have that

$$F(U) = Ef(\omega; U)$$

with

$$f(\omega; U) = \text{Inf}[\Phi_{l,L}(\omega, x, \gamma; U) \,|\, (x, \gamma) \in R^N \times R^N],$$

cf. [6, 7]. Given U, for each fixed ω, the value of $f(\omega; U)$ is obtained by solving a deterministic discrete-time problem of the Bolza type. The dual of this variational problem yields a 'dual' representation of f. It is in that form that we are able to exploit the specific properties of this problem.

4. The dual representation of f

Given U, and for fixed ω, we consider the (discrete time) Bolza problem:

VP $\qquad \operatorname{Inf}[\Phi_{l,L}(\omega, x, \gamma; U) \mid x \in R^N, \gamma \in R^N]$

and associate to **VP** the dual problem

VD $\qquad \operatorname{Inf}[\Phi_{m,M}(\omega, q, \alpha; U) \mid q \in R^N, \alpha \in R^N]$

with

$$m(\omega, q_0, \alpha_0, q_T, \alpha_T) = l^*(\omega, q_0, \alpha_0, -q_T, -\alpha_T),$$

$$M_t(\omega, q_t, \alpha_t, \Delta q_t, \Delta\alpha_t; U) = L_t^*(\omega, \Delta q_t, \Delta\alpha_t, q_t, \alpha_t; U),$$

$$\Phi_{m,M} = m + \sum_{t=1}^{T} M_t$$

where l^* and L_t^* denote the conjugates of l and L_T and

$$\Delta q_t = q_{t-1} \quad \text{and} \quad \Delta\alpha_t = \alpha_t - \alpha_{t-1}.$$

This dual problem is derived as follows: First embed **VP** in a class of Bolza problems, obtained from **VP** by submitting the state variables (x, γ) to (global) variations, viz.

VP$_{r,\eta}$ $\qquad \operatorname{Inf}_{x,\gamma}[l(\omega, x_1 + r_0, \gamma_1 + \eta_0, x_T, \gamma_T) + \sum_{t=1}^{T} L_t(\omega, x_t, \gamma_t, \Delta x_t + r_t, \Delta\gamma_t + \eta_t; U_t)].$

Let $\phi(\omega, r, \eta; U)$ be the infimum value; it is convex in (r, η). The *costate variables* (q, α) are paired with the variations through the bilinear form

$$\langle (q, \alpha), (r, \eta) \rangle = \sum_{t=0}^{T} (q_t' \cdot r_t + \alpha_t' \cdot \eta_t).$$

The problem **VD** is then obtained as the conjugate of ϕ:

$$\begin{aligned}
\Phi_{m,M}(\omega, q, \alpha; U) &= \operatorname{Sup}[\langle (q, \alpha), (r, \eta) \rangle - \phi(\omega, r, \eta; U)] \\
&= \operatorname{Sup}_{(r,\eta,x,\gamma)}\Big[q_0 \cdot r_0 + \alpha_0' \cdot \eta_0 \\
&\quad - l(\omega, x_1 + r_0, \gamma_1 + \eta_0, x_T, \gamma_T) \\
&\quad + \sum_{t=1}^{T} (q_t' \cdot r_t + \alpha_t' \cdot \eta_t \\
&\quad - L_t(\omega, x_t, \gamma_t, \Delta x_t + r_t, \Delta\gamma_t + \eta_t; U_t)) \\
&\quad + \sum_{t=1}^{T} (q_t' \cdot \Delta x_t + \alpha_t' \cdot \Delta\gamma_t) + \sum_{t=1}^{T} (\Delta q_t' \cdot x_t + \Delta\alpha_t' \cdot \gamma_t) \\
&\quad + q_0' \cdot x_1 + \alpha_0' \cdot \gamma_1 - q_T' \cdot x_{T1} - \alpha_T' \cdot \gamma_{T1} \Big].
\end{aligned}$$

Regrouping terms yields immediately the desired expression.

Calculating m and M_t, we get that

$$m(\omega, q_0, \alpha_0, q_T, \alpha_T) = \begin{cases} q_0' B_0 w_0(\omega) + \tfrac{1}{2}\|\alpha_T\|^2, & \text{if } q_T = -\alpha_T, \\ +\infty, & \text{otherwise} \end{cases}$$

and for $t = 1, \dots, T$,

$$M_t(\omega, q_t, \alpha_t, \Delta q_t, \Delta\alpha_t; U) = \begin{cases} (q_t' B_t - \alpha_t' U_t D_t) w_t(\omega), & \\ \qquad \text{if } -\Delta q_t' = q_t'(A_t - I) - \alpha_t' U_t C_t, & \\ \qquad\qquad -\Delta\alpha_t' = 0, & \\ +\infty, \quad \text{otherwise} & \end{cases}$$

Thus any feasible solution (q, α) to **VD** must satisfy for all t

$$\Delta\alpha_t = 0 \quad \text{and hence} \quad \alpha_t = \alpha_T.$$

Since also $-\alpha_T = q_T$, by substitution in the equations,

$$q_{t-1}' = q_t' A_t - \alpha_T' U_t C_t$$

we obtain by recursion that for $l = 0, 1, \dots, T$

$$q_{T-l}' = -\alpha_T' Q_{T-l}$$

where the matrices Q_t are defined by the relations

$$Q_{t-1} = Q_t A_t + U_t C_t \quad \text{and} \quad Q_T = I.$$

Now for $t = 0, 1, \dots, T$, set

$$Z_t = Q_t B_t + U_t D_t$$

with the understanding that $U_0 D_0 = 0$. We get the following version of **VD**:

Find $\alpha_T \in R^n$, $(Q_t, t = 0, \dots, T)$, $(Z_t, t = 0, \dots, T)$ such that

$\tfrac{1}{2}\|\alpha_T\|^2 - \alpha_T' \cdot \sum_{t=0}^{T} Z_t w_t(\omega)$ is minimized, and

VD′ $\quad Q_T = I,$

$$Q_{t-1} = Q_t A_t + U_t C_t, \quad t = 1, \dots, T,$$

$$Z_t = Q_t B_t + U_t D_t, \quad t = 0, \dots, T.$$

For given U, the problem **VD′** is solved by setting

$$Q_{T-l} = \sum_{s=0}^{l} U_{T1-s} C_{T1-s} \left(\prod_{\tau=T}^{T1-l+s} A_\tau \right)$$

with the conventions $U_{T1} C_{T1} = I$, and

$$\prod_{\tau=T}^{T1-l+s} A_\tau = I \quad \text{if } T1 - l + s > T, \quad \text{i.e., } s + 1 > l.$$

This in turn gives a similar expression for the Z_t; the problem is thus feasible.

The optimal solution is given by

$$\alpha_T^* = \sum_{t=0}^{T} Z_t w_t(\omega)$$

and the optimal value is

$$-\tfrac{1}{2}\left\|\sum_{t=0}^{T} Z_t w_t(\omega)\right\|^2.$$

Thus both **VD**′ (or **VD**) and **VP** are *solvable*; **VP** has only one feasible—and thus optimal—solution. It remains to observe that at the optimal the values are equal. This follows from the lower semicontinuity of

$$(r, \eta) \to \phi(\omega, r, \eta; U)$$

—the objective of **VP** is coercive—and the relations

$$\begin{aligned} f(\omega; U) &= \operatorname{Inf} \Phi_{l,L}(\omega, x, y; U) = \phi(\omega, 0, 0; U) \\ &= -\operatorname{Inf} \phi^*(\omega, q, \alpha; U) = -\operatorname{Inf} \Phi_{m,M}(\omega, q, \alpha; U) \\ &= \tfrac{1}{2}\left\|\sum_{t=0}^{T} Z_t w_t(\omega)\right\|^2 \end{aligned}$$

where the Z_t—and the Q_t—are defined by the dynamics of the (deterministic) problem **VD**′.

5. The linear–quadratic regulator problem associated to WP

The results of the previous section yield a new representation for $F(U)$ and consequently of the optimization problem **WP**, viz.

Find U that minimizes $F(U)$, where

$$F(U) = \tfrac{1}{2}E\left\|\sum_{t=0}^{T} Z_t w_t(\omega)\right\|^2,$$

LQR

$$\begin{aligned} Z_t &= Q_t B_t + U_t D_t, \quad && t = 0, \dots, T, \\ Q_{t-1} &= Q_t A_t + U_t C_t, \quad && t = 1, \dots, T, \\ Q_T &= I. \end{aligned}$$

This is a *deterministic* problem, in fact a matrix-version of the linear–quadratic regulator problem [3, Chapter II, Section 2]. The objective is quadratic, the coefficients of this quadratic form are determined by the covariances matrices of the random vectors (w_s, w_t). If the $(w_t, t = 0, \dots, T)$ are uncorrelated normalized centered random gaussian variables, then this problem takes on the form:

$$\text{Find } U \text{ that minimizes } \tfrac{1}{2}\,\text{trace}\Big[Q_0 P_0 Q_0' + \sum_{t=1}^{T} Z_t Z_t'\Big] \text{ with}$$

LQR′

$$Q_T = I,$$
$$Q_{t-1} = Q_t A_t + U_t C_t, \quad t = 1, \dots, T,$$
$$Z_t = Q_t B_t + U_t D_t,$$

where $P_0 = B_0 B_0'$. The optimal weights can thus be computed without recourse to the stochastics of the system. This derivation shows that the basic results do not depend on the fact that the w_t are uncorrelated gaussian random variables. In fact it shows that if the w_t are arbitrary random variables, correlated or not (but with finite second order moments), and the class of admissible estimators is restricted to those that are linear in the observations, then precisely **LQR** can be used to find the optimal weights; we then obtain the wide sense best estimator.

The linear–quadratic regulator problem **LQR** has an optimal solution, in feedback form of the type:

$$U_t = -\,Q_t K_t$$

for $t = 1, \dots, T$. This follows directly from the usual optimality conditions—the discrete version of Pontryagin's Maximum Principle—for example, cf. [3, Chapter II, Section 7]. Thus the search for an optimal set of weights U can be replaced by the search of the optimal (Kalman) *gains* $K = (K_t, t = 1, \dots, T)$, i.e.,

Find K that minimizes $G(K)$, where

GP

$$G(K) = \frac{1}{2} E\Big\|\sum_{t=0}^{T} Z_t w_t(\omega)\Big\|^2,$$
$$Z_t = Q_t(B_t - K_t D_t), \qquad t = 0, \dots, T,$$
$$Q_{t-1} = Q_t(A_t - K_T C_T), \quad t = 1, \dots, T,$$
$$Q_T = I.$$

If K^* solves **GP**, and U^* solves **LQR**, we have that

$$F(U^*) = G(K^*)$$

and in fact U^* may always be chosen so that $U_t^* = -\,Q_t^* K_t^*$. Thus if K^* solves **GP** and Q^* is the associate solution of the finite differences equations describing the dynamics of the problem, we have that the optimal estimator of x_{T1} is given by

$$\gamma^*(\omega) = \sum_{t=1}^{T} Q_t^* K_t^* y_t(\omega).$$

Problem **LQR′** simply becomes

Find K that minimizes $G(K)$, where

GP′

$$G(K)=\tfrac{1}{2}\operatorname{trace}\left[Q_0P_0Q_0'+\sum_{t=1}^{T}Z_TZ_t'\right],$$

$$\left.\begin{array}{l} Z_t=Q_t(B_t-K_tD_t),\\ Q_{t-1}=Q_t(A_t-K_tC_t),\end{array}\right. \quad t=1,\ldots,T,$$

$$Q_T=I$$

with the resulting simplifications in the derivations of the optimal estimator.

6. The Kalman filter

The characterization of the optimal gains (and hence weights) for the one-step estimator problem derived in the previous section allows us to obtain the optimal estimator at every time t, not just at some terminal time T. The optimal estimator γ^*_{t1} at time $t1$ being derived recursively from γ^*_t and the new information carried by the observation at time $t1$. We obtain this expression for γ^*_{t1} by relying once more on the duality for variational problems invoked in Section 4.

We have seen that there is no loss in generality in restricting the weights to the form $U_t=-Q_tK_t$, for $t=1,\ldots,T$. The optimal solution of **VD** will thus have $\alpha_tU_t=q_tK_t$. We can reformulate the original one-step optimization problem as follows:

DG

Find K that minimizes $G(K)=E[g(\omega;K)]$ where (see VD. and calculation of m, M_t)

$$g(\omega;K)=\mp\operatorname{Inf}[\Phi_{r,R}(\omega,q;K)\mid q\in R^N],$$

$$r(\omega,q_0,q_T)=q_0'B_0w_0(\omega)+\tfrac{1}{2}\|q_T\|^2,$$

$$R_t(\omega,q_t,\Delta q_t)=\begin{cases} q_t'(B_t-K_tD_t)w_t(\omega), & \text{if } -\Delta q_{t-1}'=q_t'(A_t-I-C_tK_t),\\ +\infty, & \text{otherwise.}\end{cases}$$

We rely here on the dual representation. From our preceding remarks we know that for almost all $\omega\in\Omega$

$$g(\omega;K^*)=f(\omega,U^*).$$

The value of $g(\omega;K)$ being defined as the infimum of a variational problem. By relying on the dual of this (deterministic) variational problem, we find a new representation for $g(\omega;K)$. The arguments are similar to those used in Section 4. We get (by redualization).

$$g(\omega;K)=\operatorname{Inf}[\Phi_{s,S}(\omega,e;K)\mid e\in R^N]$$

where

$$s(\omega,e_0,e_{T1})=r^*(\omega,e_0,-e_{T1})$$

$$=\begin{cases}\tfrac{1}{2}\|e_{T1}\|^2, & \text{if } e_1=B_0w_0(\omega),\\ +\infty, & \text{otherwise}\end{cases}$$

and

$$S_t(\omega, e_t, \Delta e_t; K) = R_t^*(\omega, \Delta e_t, e_t; K)$$
$$= \begin{cases} 0, & \text{if } (A_t - I - K_tC_t)e_t + (B_t - K_tD_t)w_t(\omega) = \Delta e_t, \\ +\infty, & \text{otherwise} \end{cases}$$

with $\Delta e_t = e_{t+1} - e_t$. Thus **DG** is equivalent to

PG

Find K that minimizes $G(K) = E[g(\omega; K)]$ where
$g(\omega; K) = \text{Inf } \frac{1}{2}\|e_{T+1}(\omega)\|^2,$
$e_{t+1}(\omega) = (A_t - K_tC_t)e_t(\omega) + (B_t - K_tD_t)w_t(\omega), \quad t = 1, \dots, T,$
$e_1(\omega) = B_0 w_0(\omega).$

For fixed K_t, the solution of the optimization problem defining $g(\omega; K)$ is unique and given by

$$e_{t+1}(\omega) = \sum_{\tau=0}^{t} \left(\prod_{s=t}^{\tau-1} \Gamma_s\right) \Lambda_\tau w_\tau(\omega)$$

for $t = 1, \dots, T$, with the usual convention that $\prod_{s=t}^{t-1} \Gamma_s = I$, where

$$\Gamma_t = A_t - K_tC_t \quad \text{and} \quad \Lambda_t = B_t - K_tD_t.$$

The process $(e_t, t = 0, \dots, T+1)$ is the *error process*, i.e., $e_t = x_t - \gamma_t$; for each t it yields the error between the actual state

$$x_{t+1}(\omega) = A_t x_t(\omega) + B_t w_t(\omega)$$

and the *estimated state*

$$\gamma_{t+1}(\omega) = A_t\gamma_t(\omega) + K_t(C_te_t(\omega) + D_tw_t(\omega))$$
$$= A_t\gamma_t(\omega) + K_t(y_t(\omega) - C_t\gamma_t),$$

K_t representing the weight attributed to the gain in information at time t. Note that K_t only affects the equation defining e_{t+1} and thus the functional $G(K)$ will be minimized if given e_t, each K_t is chosen so as to minimize $E[\frac{1}{2}\|e_{t+1}(\omega)\|^2]$, i.e., so that the incremental error is minimized.

The sequence K^* of optimal gains can now be found recursively in the following way: suppose that $K_1^*, \dots, K_{t-1}^*$ have already been obtained and $e_1^*, \dots, e_t^*$ are the corresponding values of the state variables. Let

$$\Sigma_t = E\{e_t^*(\omega) \cdot e_t^*(\omega)'\}$$

be the covariance of the state variable e_t^*. Then K_t^* must be chosen so that

$$E[\tfrac{1}{2}\|e_{t+1}(\omega)\|^2]$$

is minimized, or equivalently

$$\operatorname{trace}[\tfrac{1}{2}(A_t - K_tC_t)\Sigma_t(A_t - K_tC_t)' + \tfrac{1}{2}(B_t - K_tD_t)(B_t - K_tD_t)']$$

is minimized. The minimum is attained at

$$K_t^* = [A_t\Sigma_tC_t' + B_tD_t'][C_t\Sigma_tC_t' + D_tD_t']^{\#}$$

where # denotes the generalized inverse. Plugging this in the definition of e_{t1} and taking covariances on both sides, we get the following recursive scheme for the calculation of Σ_t:

$$\begin{aligned}\Sigma_{t1} &= A_t\Sigma_tA_t' + B_tB_t' \\ &= (A_t\Sigma_tC_t' + B_tD_t')(C_t\Sigma_tC_t' + D_tD_t')^{\#}(C_t\Sigma_tA_t' + D_t'B_t),\end{aligned}$$

$$\Sigma_1 = P_0.$$

This is the usual matrix Ricatti equation.

The process that yields the optimal estimator at every time t is given by the relations

$$\gamma_{t1}^*(\omega) =_{\text{a.s.}} A_t\gamma_t^*(\omega) + K_t^*[y_t(\omega) - C_t\gamma_t^*(\omega)],$$

$$\gamma_1^*(\omega) =_{\text{a.s.}} 0$$

where K_t^* is as defined above. The process $(y_t - C_t\gamma_t^*, t = 1, \dots, T)$ is called the *innovation process* and represents the new information contained in each observation.

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