CONVERGENCE OF SET-VALUED MAPPINGS: EQUI-OUTER SEMICONTINUITY†

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Abstract. The concept of equi-outer semicontinuity allows us to relate the pointwise and the graphical convergence of set-valued-mappings. One of the main results is a compactness criterion that extends the classical Arzelà-Ascolì Theorem for continuous functions to this new setting; it also leads to the exploration of the notion of continuous convergence. Equi-lower semicontinuity of functions is related to the outer semicontinuity of epigraphical mappings. Finally, some examples involving set-valued mappings are reexamined in terms of the concepts introduced here.

Keywords: set-valued mappings, epi-convergence, multifunction, equi-continuity, equi-semicontinuity, Arzelà-Ascolì Theorem, maximal monotone, operators, differential inclusions, closed convex processes, sublinear mappings, subgradient mappings

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Let's consider the following system:

$$A(x) = f, \quad x \in D,$$

or more generally,

$$A(x) \ni f, \quad x \in D$$

where A is an operator defined on a space X and whose values are either points in a space Y (in the first case) or subsets of Y (in the second case), $f \in Y$ and D is a subset of X. This operator A could be a (partial) differential operator, or simply represents the amalgamated version of a linear or nonlinear system of equations; such systems might represent the Karush-Kuhn-Tucker (KKT) conditions of an optimization problem involving constraints or the Euler equation of a variational problem. The spaces X and Y are finite or infinite dimensional normed linear spaces, or even more generally metric spaces. In many situations, we have to allow for multi-valuedness, that's why we need to also deal with systems of the second type. Examples include KKT-conditions, differential systems with turbulence, economic and biological models involving preference relations, etc.; for more about this, refer to [**4**] and the references therein.

There are three basic questions that must be answered about such systems (with equalities or inclusions): existence of a solution, uniqueness of the solution, and consistency of the approximations. Although the ensuing development can shed some light on existence and uniqueness, we are going to be concerned here with questions related to approximations. This will be dealt with in the following framework: let

$$S(x) = \begin{cases} A(x) - f & \text{if } x \in D; \\ \emptyset & \text{if } x \notin D. \end{cases}$$

The mapping $S : X \rightrightarrows Y$ is then a set-valued mapping with $\rightrightarrows$ indicating that it is not necessarily single-valued. The preceding systems can now be formulated:

$$\text{find } \ x \in X \ \text{ such that } \ S(x) \ni 0.$$

An approximating system would then take the form:

$$\text{find } \ x \in X \ \text{ such that } \ S^\nu(x) \ni 0,$$

where S^ν approximates S in some sense. We are going to be interested in approximating systems where S^ν is close to S in terms of their graphs. The *graph* of a set-valued mapping $S : X \rightrightarrows Y$ is the subset of $X \times Y$:

$$\operatorname{gph} S := \{ (x, y) \in X \times Y \mid y \in S(x) \}.$$

Convergence of S^ν to S would then be defined in terms of the convergence of their graphs. The basic reason for the interest in graph convergence is that it's the "weakest" convergence notion that will "guarantee" the convergence of solutions [**2, 5**].

However graph convergence of mappings, even when they are single-valued, isn't always easy to verify. This paper explores the relationship between graph convergence and some other convergence notions, in particular pointwise convergence.

After some preliminaries (§1) about set convergence and the continuity of set-valued mappings, the notion of equi-semicontinuity for collections of set-valued mappings is introduced in §3 to answer the question raised in §2 about the relationship between graph and pointwise convergence. Equi-outer semicontinuity with respect to the Choquet-Wijsman convergence of sets is explored in §4. Other convergence notions for set-valued mappings are explored in §5 that culminates with a compactness result akin to the Arzelà-Ascolì theorem. The relationship between the notions of equi-outer semicontinuity for mappings and equi-lower semi-continuity of extended real-valued functions is analyzed in §6. The relationship between Mosco-pointwise and Mosco-graph-convergence of mappings is covered in §7. The paper concludes with applications to the convergence of the subgradients of convex functions (§8), maximal monotone operators (§9), and differential inclusions (§10).

1. Preliminaries.

Let (Y, d_Y) be a metric space; a neighborhood system at a point y will be denoted by $\mathcal{N}(y)$ and $\mathbb{B}(y,\rho)$ and $\mathbb{B}^o(y,\rho)$ are the closed and open balls centered at y and of radius ρ.

Given a collection of sets $\{\, C^\nu \subset Y,\ \nu \in N \,\}$ with N an index space and $\mathcal{H}$ a filter on N, its *outer limit*, also called the limit superior, is the set

$$\operatorname{ls}_{\mathcal{H}} C^\nu = \operatorname{ls} C^\nu := \bigcap_{H\in\mathcal{H}} \operatorname{cl}\Big(\bigcup_{\nu\in H} C^\nu\Big), \tag{1.1}$$

and, its *inner limit*, also called the limit inferior, is the set

$$\operatorname{li}_{\mathcal{H}} C^\nu = \operatorname{li} C^\nu := \bigcap_{H\in\mathcal{H}^\#} \operatorname{cl}\Big(\bigcup_{\nu\in H} C^\nu\Big) \tag{1.2}$$

where $\mathcal{H}^\#$ is the *grill* associated with the filter $\mathcal{H}$, i.e., the family of subsets of N that meet all sets H in $\mathcal{H}$. The subscripts in $\operatorname{ls}_{\mathcal{H}}$ and $\operatorname{li}_{\mathcal{H}}$ will be used whenever the context might require it, otherwise they will simply be omitted. If $\operatorname{ls} C^\nu = \operatorname{li} C^\nu$, this set, denoted $\operatorname{lm} C^\nu$, is the *(Painlevé-Kuratowski or plain) limit* of the collection and one writes $C^\nu \to \operatorname{lm} C^\nu$ to indicate that the sets C^ν converge (in the Painlevé-Kuratowski sense) to $\operatorname{lm} C^\nu$. All these limit sets are closed as follows directly from the definitions. Moreover, since $\mathcal{H} \subset \mathcal{H}^\#$, one always has

$$\operatorname{li} C^\nu \subset \operatorname{ls} C^\nu.$$

Some typical examples of filters and their grills are:

(i) $\mathcal{H}$ is the (topological) neighborhood system of a point $\bar y \in N$, in which case $\mathcal{H}^\#$ is the collection of all sets that have $\bar y$ in their closure;

(ii) $N = I\!N$, $\mathcal{H}$ is the Fréchet filter consisting of all sets $H \subset I\!N$ that are co-finite, and $\mathcal{H}^{\#}$ consists of all sets H in $I\!N$ of infinite cardinality.

It will be assumed throughout that the filter $\mathcal{H}$ on N has *a countable base*; e.g., $\mathcal{H}$ is the Fréchet filter on $I\!N$ or $\mathcal{H}$ is the neighborhood system –possibly punctured– of a point $\nu \in N$ which has a countable base. Convergence to a point can then also be expressed in terms of sequential limits.

It will be convenient to have at our disposal the following equivalent expressions for the outer and inner limit sets:

$$\begin{aligned} \operatorname{ls} C^\nu &= \{\, y \,|\, \forall V \in \mathcal{N}(y), \exists\, H \in \mathcal{H}^{\#}, \forall \nu \in H : C^\nu \cap V \neq \emptyset \,\} \\ &= \{\, y \,|\, \exists\, H \in \mathcal{H}^{\#}, \exists\, y^\nu \in C^\nu (\nu \in H) \text{ with } y^\nu \underset{H}{\rightarrow} y \,\}, \\ \operatorname{li} C^\nu &= \{\, y \,|\, \forall V \in \mathcal{N}(y), \exists\, H \in \mathcal{H}, \forall \nu \in H : C^\nu \cap V \neq \emptyset \,\} \\ &= \{\, y \,|\, \exists\, H \in \mathcal{H}, \exists\, y^\nu \in C^\nu (\nu \in H) \text{ with } y^\nu \underset{H}{\rightarrow} y \,\}. \end{aligned} \tag{1.3}$$

The next propositions provides a possibly new characterization of set convergence that will serve as a stepping stone to the definition of equi-semicontinuity later on.

For $\varepsilon \geq 0$, the *ε-fattening* of a set $C \subset Y$ is defined by

$$\varepsilon C := \begin{cases} \{\, y \in Y \mid \inf_{z \in C} d(z,y) =: d(y,C) \leq \varepsilon \,\} & \text{if } C \neq \emptyset, \\ \emptyset & \text{if } C = \emptyset. \end{cases} \tag{1.4}$$

The ε-fattening of the empty set is sometimes defined as the complement of a "ball" of radius ε^{-1} and centered a some point in Y (usually the origin when Y is a linear space); this alternative definition will not be used here.

Proposition 1.1. *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets and $C \subset Y$ a closed set.*

(a) *$C \supset \operatorname{ls} C^\nu$ if and only if for all compact set B and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that $C^\nu \cap B \subset \varepsilon C$ for all $\nu \in H$;*

(b) *$C \subset \operatorname{li} C^\nu$ if and only if for every compact set B and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that $C \cap B \subset \varepsilon C^\nu$ for all $\nu \in H$.*

Proof. Sufficiency in (a): Let $\bar{y}$ be an arbitrary point in $\operatorname{ls} C^\nu$, i.e., there exist $H \in \mathcal{H}^{\#}$, H countable, $y^\nu \in C^\nu$ for all $\nu \in \mathcal{H}$ such that $y^\nu \underset{H}{\rightarrow} \bar{y}$; the restriction to countable H is possible because $\mathcal{H}$ has a countable base. The set $B = \{\, \bar{y}, y^\nu,\, \nu \in H \,\}$ is compact since it is closed and sequentially compact. Thus, from the inclusion in (a), it follows that for any $\varepsilon > 0$, one can find an index set $H_\varepsilon \in \mathcal{H}$ such that $y^\nu \in C^\nu \cap B \subset \varepsilon C$ for all $\nu \in H \cap H_\varepsilon$. This means that $\bar{y}$ is in εC for all $\varepsilon > 0$, which in turn implies that $\bar{y} \in C$ since $C = \cap_{\varepsilon > 0} \varepsilon C$ is closed.

Necessity in (a): Suppose to the contrary that one can find a compact set B, $\varepsilon > 0$ and $H \in \mathcal{H}^{\#}$ such that for all $\nu \in H$, there is $y^\nu \in \big[C^\nu \cap B\big] \setminus \varepsilon C$. Let $\bar{y}$ be a cluster point of the y^ν's. Then $\bar{y} \in \operatorname{ls} C^\nu$ and $\bar{y} \notin (\varepsilon/2)C$, and so $\operatorname{ls} C^\nu$ can't be included in C.

Sufficiency in (b): Consider $\bar{y} \in C$ and let $B = \{\bar{y}\}$. Then $\bar{y} \in C \cap B$, so for any $\varepsilon > 0$ there is an index set $H \in \mathcal{H}$ with $\bar{y} \in \varepsilon C^\nu$ for all $\nu \in H$, and this implies that $\bar{y} \in \operatorname{li} C^\nu$.

Necessity in (b): Suppose to the contrary that one can find a compact set B, $\varepsilon > 0$ and $H \in \mathcal{H}^\#$ such that for all $\nu \in H$, there exists $y^\nu \in \big[C \cap B\big] \setminus \varepsilon C^\nu$. Let $\bar{y}$ be a cluster point of the y^ν. Thus, there exists $H_2 \subset H, H_2 \in \mathcal{H}$ such that for all $\nu \in H_2$, one has $\varepsilon \leq d(y^\nu, C^\nu) \leq d(\bar{y}, C^\nu) + d(\bar{y}, y^\nu) \leq d(\bar{y}, C^\nu) + \varepsilon/2$. And consequently, $\varepsilon/2 \leq \limsup_{\nu \in H_2} d(\bar{y}, C^\nu)$. But this can't actually occur when $C \subset \operatorname{li} C^\nu$: Since $\bar{y} \in C$, from (1.3) it follows that there is an index set, say $H_4 \in \mathcal{H}$, such that $I\!B(\bar{y}, \varepsilon/4) \cap C^\nu \neq \emptyset$ for all $\nu \in H_4$ implying $\limsup_{\nu \in H_2} d(\bar{y}, C^\nu) \leq \varepsilon/4$. $\square$

Corollary 1.2. *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets. Then $C^\nu \to \emptyset$ if and only if for all compact B there exists $H \in \mathcal{H}$ such that $C^\nu \cap B = \emptyset$.*

Proof. Follows from (a) in the proposition after observing that $C^\nu \to \emptyset$ if and only if $\operatorname{ls} C^\nu = \emptyset$. $\square$

When $Y = I\!R^m$, or more generally Y is a linear space whose closed balls are compact, then rather than working with *all* compact sets, it suffices to check the inclusions in conditions (a) and (b) of proposition 1.1 when taking the intersection with all balls centered at the origin. The resulting (well-known) criteria when $Y = I\!R^m$ are recorded below.

Corollary 1.3. *Let $I\!B \subset I\!R^m$ denote the unit ball with respect to a metric d topologically equivalent to the euclidean metric. Let $\{C^\nu \subset I\!R^m,\, \nu \in N\}$ a filtered collection of sets and $C \subset I\!R^m$ a closed set.*

(a) *$C \supset \operatorname{ls} C^\nu$ if and only if for all $\rho \geq 0$ and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that $C^\nu \cap \rho I\!B \subset \varepsilon C$ for all $\nu \in H$;*

(b) *$C \subset \operatorname{li} C^\nu$ if and only if for all $\rho \geq 0$ and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that $C \cap \rho I\!B \subset \varepsilon C^\nu$ for all $\nu \in H$.*

Proof. The "if" implications follow from the "if" parts of the proposition, since for all $\rho \geq 0$, $\rho I\!B$ is compact. In the other direction, since every compact subset of $I\!R^m$ is contained in some ball $\rho I\!B$, one can appeal to the "only if" parts of the proposition. $\square$

Although ε-fattening of sets can be used effectively in the formulation of criteria for set convergence, in general, set convergence isn't preserved under ε-fattening. One actually has the following:

Proposition 1.4. *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets, and $C \subset Y$, a closed set. Then for all $\varepsilon \geq 0$,*

$$C \subset \operatorname{li} C^\nu \qquad \textit{implies} \qquad \varepsilon C \subset \operatorname{li} \varepsilon C^\nu,$$

but in general $C \supset \operatorname{ls} C^\nu$ does not imply $\varepsilon C \supset \operatorname{ls} \varepsilon C^\nu$.

Proof. This is certainly the case if C is empty. So, let's assume that C is nonempty. If $y \in \varepsilon C$, there exists $\hat{y} \in C$ such that $d(\hat{y}, y) \leq \varepsilon$, and since $\hat{y} \in \operatorname{li} C^\nu$, there exist $H \in \mathcal{H}$

and for $\nu \in H$, $\hat{y}^\nu \in C^\nu$ such that $\hat{y}^\nu \to \hat{y}$. The the points $y^\nu := \hat{y}^\nu + (y - \hat{y})$ belong to εC^ν for all $\nu \in H$ and the y^ν converge to y, i.e. $y \in \operatorname{li} \varepsilon C^\nu$ as follows from (1.3).

As an example of $\varepsilon C \supset \operatorname{ls} \varepsilon C^\nu$ not being implied by $C \supset \operatorname{ls} C^\nu$, let $(Y, d) = (\ell^1, \|\cdot\|_1)$, $N = I\!N$ with $\mathcal{H}$ the Fréchet filter, $C^\nu = \{e^\nu\}$ where e^ν is the unit vector having a 1 in the ν-th position. Then $C := \emptyset \supset \operatorname{ls} C^\nu$. With $\varepsilon = 1$, $\varepsilon C^\nu \ni 0$ for all ν, and consequently $\{0\} \in \operatorname{ls} \varepsilon C^\nu$ but $\varepsilon C = \emptyset$. $\square$

The situation is reversed when dealing with taking intersections.

Proposition 1.5. *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y, \nu \in N\}$ a filtered collection of sets, and $C, D \subset Y$ closed sets. For the outer limit, one has*

$$\operatorname{ls} C^\nu \subset C \qquad \Longrightarrow \qquad \operatorname{ls}(C^\nu \cap D) \subset C \cap D,$$

but in general, a similar implication isn't valid when dealing with the inner limit, i.e., $\operatorname{li} C^\nu \supset C$ *doesn't imply* $\operatorname{li} C^\nu \cap D \supset C \cap D$.

Proof. The assertion about the outer limit follows directly from the characterization of $\operatorname{ls}(C^\nu \cap D)$ and $\operatorname{ls} C^\nu$ provided by (1.3). For a case when $C^\nu \to C$ but $\operatorname{li}(C^\nu \cap D) \not\supset C \cap D$, simply let $(Y, d) = (I\!R, d)$ where d is the usual metric, $N = I\!N$ with the Fréchet filter and $C^\nu = \{1/\nu\}$ and $C = D = \{0\}$. $\square$

Now, let (X, τ) be a topological space, and $S : X \rightrightarrows Y$ a set valued-mapping that associates to each $x \in X$ a set, possibly empty, $S(x) \subset Y$. Continuity of such mappings is defined in terms of the set-limits just introduced. The mapping $S : X \rightrightarrows Y$ is *outer semicontinuous (osc) at* $\bar{x} \in X$ if

$$S(\bar{x}) \supset \operatorname{ls}_{\mathcal{N}(\bar{x})} S(x) = \bigcap_{V \in \mathcal{N}(\bar{x})} \operatorname{cl}\left(\bigcup_{x \in V} S(x)\right) \tag{1.5}$$

where $\mathcal{N}(\bar{x}) = \mathcal{N}_\tau(\bar{x})$ is the neighborhood system of $\bar{x}$, and thus convergence to $\bar{x}$ is with respect to the τ-topology. Similarly, S is *inner semicontinuous (isc) at* $\bar{x}$ if

$$S(\bar{x}) \subset \operatorname{li}_{\mathcal{N}(\bar{x})} S(x) = \bigcap_{V \in \mathcal{N}^\#(\bar{x})} \operatorname{cl}\left(\bigcup_{x \in V} S(x)\right). \tag{1.6}$$

It is *continuous at* $\bar{x}$ if it is both osc and isc at $\bar{x}$. The mapping S is *osc, isc or continuous* if the corresponding property holds at every point $\bar{x}$ in X.

One could refer to these properties as plain, or Painlevé-Kuratowski, outer and inner semicontinuity, but one usually just attaches modifiers to any other notions of continuity or semicontinuity when the convergence of the images is other than (plain) convergence.

Define the *ε-fattening* εS of a mapping S to be such that $(\varepsilon S)(x) := \varepsilon(S(x))$, i.e., for every x, the ε-fattening of the set $S(x) \subset Y$. The next proposition is then an immediate consequence of this definition and proposition 1.1.

Proposition 1.6. *Let (X,τ) be a topological space and (Y,d_Y) a metric space. A set-valued mapping $S : X \rightrightarrows Y$, closed-valued at $\bar{x}$, is*

(a) *osc at $\bar{x}$ if and only if for every compact set $B \subset Y$ and $\varepsilon > 0$, there exists a neighborhood V of $\bar{x}$ such that for all $x \in V$: $S(x) \cap B \subset \varepsilon S(\bar{x})$;*

(b) *isc at $\bar{x}$ if and only if for every compact set $B \subset Y$ and $\varepsilon > 0$, there exists a neighborhood V of $\bar{x}$ such that for all x in V: $S(\bar{x}) \cap B \subset \varepsilon S(x)$.*

The next (well-known) characterization of outer semicontinuity has sometimes led to the use of the term "closed" when referring to outer semicontinuous mappings.

Proposition 1.7. *Let (X,τ) be a topological space and (Y,d_Y) a metric space. A mapping $S : X \rightrightarrows Y$ is outer semicontinuous if and only if $\operatorname{gph} S \subset X \times Y$ is closed.*

Proof. Use the second identity in (1.3) for the outer limit. If $\bar{y} \in \operatorname{ls}_{\mathcal{N}(\bar{x})} S(x)$, there exists $V \in \mathcal{N}^{\#}(\bar{x})$, $y \in S(x)$ for $x \in V$ such that $y \to \bar{y}$ as $x \to \bar{x}$. If $\operatorname{gph} S$ is closed, this means that $(\bar{x},\bar{y}) \in \operatorname{gph} S$, and hence $\bar{y} \in S(\bar{x})$ implying $S(\bar{x}) \supset \operatorname{ls}_{\mathcal{N}(\bar{x})} S(x)$.

In the other direction, let $(x,y) \to (\bar{x},\bar{y})$ with $y \in S(x)$. This implies that $\bar{y} \in \operatorname{ls}_{\mathcal{N}(\bar{x})} S(x)$. Since this latter set is included in $S(\bar{x})$, it follows that $(\bar{x},\bar{y}) \in \operatorname{gph} S$ implying, in turn, that $\operatorname{gph} S$ is closed. $\square$

2. Pointwise and graph convergence

Let $\{\, S^\nu : X \rightrightarrows Y,\ \nu \in (N,\mathcal{H}) \,\}$ be a filtered collection of set-valued mappings where (X,τ) is a topological space and (Y,d_Y) is a metric space.

The mappings S^ν *pointwise converge* to S at $x \in X$ if $S^\nu(x) \to S(x)$, and they are said to *pointwise converge* if this hold at all $x \in X$. In general, the pointwise limit of a collection of mappings $\{\, S^\nu,\ \nu \in N \,\}$ isn't well defined, but one can always associate with any collection its *inner and outer pointwise limits*: for all x in X,

$$\text{p-li}\, S^\nu(x) := \operatorname{li} S^\nu(x), \qquad \text{p-ls}\, S^\nu(x) := \operatorname{ls} S^\nu(x). \tag{2.1}$$

When $\text{p-li}\, S^\nu = \text{p-ls}\, S^\nu$, this mapping is the *pointwise limit* of the collection and is denoted by $\text{p-lm}\, S^\nu$; one also writes $S^\nu \xrightarrow{p} S$ to indicate that the mappings S^ν pointwise converge to S. Note that these limit mappings are always *closed-valued*, i.e., $\text{p-li}\, S^\nu(x)$, $\text{p-ls}\, S^\nu(x)$ and $\text{p-lm}\, S^\nu(x)$ are closed sets for all $x \in X$ as follows from the definitions of inner and outer limits. Moreover, $\text{p-li}\, S^\nu \subset \text{p-ls}\, S^\nu$; given $S_1, S_2 : X \rightrightarrows Y$, one writes

$$S_1 \subset S_2 \ \text{ when } \ \operatorname{gph} S_1 \subset \operatorname{gph} S_2.$$

These definitions of inner and outer pointwise limits might be reminiscent of those for the upper and lower pointwise limits of (extended) real-valued functions. However, even when the mappings S^ν are actually (extended) real-valued, one can't identify the (usual) pointwise lower and upper limits with the inner and outer pointwise limits just introduced.

Consider the sequence of functions $\{ f^\nu : I\!R \to I\!R,\ \nu \in I\!N \}$ where $f^\nu = f$ if ν is odd, $f^\nu = -f$ if ν is even, and $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = -1$ otherwise. The lower and upper (pointwise) limits of this sequence are the constant functions -1 and 1. When these functions $\{f^\nu\}$ are viewed as set-valued mappings $x \mapsto \{f^\nu(x)\}$, then $\text{p-li}\{f^\nu\} \equiv \emptyset$ and $\text{p-ls}\{f^\nu\} \equiv \{-1, 1\}$. Let's also observe that in this example the mappings $\{f^\nu\}$ are single-valued but the outer limit mapping isn't.

Pointwise limits are not the only type of limit mappings that can be associated with a collection $\{ S^\nu : X \rightrightarrows Y,\ \nu \in N \}$. In fact, in terms of the potential applications mentioned in the introduction, graphical limits play a more pivotal role. The *inner and outer graphical limits* are the mappings $\text{g-li}\, S^\nu$ and $\text{g-ls}\, S^\nu$ defined by the identities:

$$\mathrm{gph}\,(\text{g-li}\, S^\nu) = \mathrm{li}\,(\mathrm{gph}\, S^\nu). \qquad \mathrm{gph}\,(g\,\mathrm{ls}\, S^\nu) = \mathrm{ls}\,(\mathrm{gph}\, S^\nu). \tag{2.2}$$

By making use of the identities (1.3) for the outer and inner limit sets, one obtains the following expressions for the outer and inner graphical limits:

$$\begin{aligned} \text{g-li}\, S^\nu(x) &= \{ y \in Y \mid \exists\, H \in \mathcal{H}, x^\nu \xrightarrow[H]{} x, y^\nu \xrightarrow[H]{} y, y^\nu \in S^\nu(x^\nu) \} \\ \text{g-ls}\, S^\nu(x) &= \{ y \in Y \mid \exists\, H \in \mathcal{H}^\#, x^\nu \xrightarrow[H]{} x, y^\nu \xrightarrow[H]{} y, y^\nu \in S^\nu(x^\nu) \} \end{aligned} \tag{2.3}$$

One always has $\text{g-li}\, S^\nu \subset \text{g-ls}\, S^\nu$. If $\text{g-li}\, S^\nu = \text{g-ls}\, S^\nu$, this mapping is called the *graph* or *graphical limit* of the collection $\{S^\nu, \nu \in N\}$ and is denoted by $\mathrm{lm}\, S^\nu$, and one writes $S^\nu \xrightarrow{g} \mathrm{lm}\, S^\nu$ to indicate that the collection *graph-* or *graphically converges* to $\mathrm{lm}\, S^\nu$. Thus,

$$S^\nu \xrightarrow{g} S \quad \Longleftrightarrow \quad \mathrm{gph}\, S^\nu \to \mathrm{gph}\, S.$$

All the graphical limit mappings are closed-valued. In fact, because their graphs are limit sets, their graphs are closed which not only implies that they are closed-valued, but also that they all are osc, cf. proposition 1.7. However, in general, these limit mappings are not isc, even when the mappings S^ν are isc.

One can reexpress the identities (2.3), as follows:

$$\begin{aligned} \big(\text{g-li}\, S^\nu\big)(\bar{x}) &= \bigcup_{\{x^\nu \to \bar{x}\}} \mathrm{li}\, S^\nu(x^\nu), \\ \big(\text{g-ls}\, S^\nu\big)(\bar{x}) &= \bigcup_{\{x^\nu \to \bar{x}\}} \mathrm{ls}\, S^\nu(x^\nu), \end{aligned}$$

where the unions are taken over all $x^\nu \to x$. Thus, S^ν converges graphically to S if and only if, at each point $\bar{x} \in X$, one has

$$\big(\text{g-ls}\, S^\nu\big)(\bar{x}) \subset S(\bar{x}) \subset \big(\text{g-li}\, S^\nu\big)(\bar{x}). \tag{2.4}$$

These inclusions lead to the notion of graphical convergence at a point: S^ν converges graphically to S if and only if it does so at every point. More generally, we say that S^ν

converges graphically to S *relative to a set* D if (2.4), with $x^\nu \to \bar{x}$ constrained to D, holds for every $\bar{x} \in D$; for closed D, this is the same as saying that the restrictions S^ν_D converge graphically to the restriction S_D.

With $S^{-1}(y) := \{ x \in X \,|\, y \in S(x) \}$, one has $\operatorname{gph} S^{-1} = \operatorname{gph} S$ from which it follows that

$$S^\nu \xrightarrow{g} S \quad \Longleftrightarrow \quad (S^\nu)^{-1} \xrightarrow{g} S^{-1}. \tag{2.5}$$

In general, neither graph nor pointwise convergence implies the other. In fact, certain collections can have graphical and pointwise limits that do not coincide.

Example 2.1. Consider the sequence $\{ S^\nu : I\!R \to I\!R,\ \nu \in I\!N \}$ with

$$S^\nu(x) = \begin{cases} -1 & \text{if } x \le -\nu^{-1}, \\ \nu x & \text{if } -\nu^{-1} < x < \nu^{-1}, \\ 1 & \text{if } x \ge \nu^{-1}. \end{cases}$$

The pointwise and graphical limits exist and, actually coincide for all $x \in I\!R \setminus \{0\}$, but

$$\text{p-lm}\, S^\nu(0) = \{0\}, \qquad \text{g-lm}\, S^\nu(0) = [-1, 1].$$

There are however certain basic relations between graph and pointwise convergence as recorded in the next proposition.

Proposition 2.2. *Let* $\{S^\nu : X \rightrightarrows Y,\ \nu \in \mathcal{N}\}$ *be a filtered collection of set-valued mappings with* (X, τ) *a topological space and* (Y, d_Y) *a metric space. Then*

$$\begin{array}{ccc} \text{p-ls}\, S^\nu & \subset & \text{g-ls}\, S^\nu \\ \cup & & \cup \\ \text{p-li}\, S^\nu & \subset & \text{g-li}\, S^\nu \end{array}$$

and so, if they exist, $\text{p-lm}\, S^\nu \subset \text{g-lm}\, S^\nu$.

Proof. We already observed that outer limits always contain inner limits, so it will suffice to check the "horizontal" inclusions. But those are also immediate. By (1.3), y belongs to the outer (resp. inner) pointwise limit at x, if there exist $H \in \mathcal{H}^\#$ (resp. $H \in \mathcal{H}$), $y^\nu \in S^\nu(x)$ and $y^\nu \underset{H}{\to} y$, hence the condition in (2.3) for y to belong to $\text{g-ls}\, S^\nu(x)$ (resp. $\text{g-li}\, S^\nu(x)$) is certainly fulfilled. □

3. Equi-outer semicontinuity

As suggested by Example 2.1, pointwise limits can be properly included in graphical limits. We are going to explore here the conditions under which equality will hold.

Definition 3.1. *Let (X, τ) be a topological space, (Y, d) a metric space and $(N, \mathcal{H})$ a filtered index set. A collection of set-valued mappings $\{ S^\nu : X \rightrightarrows Y, \nu \in N \}$ is (asymptotically) equi-outer semicontinuous, or equi-osc, at $\bar{x} \in X$ if for all compact set $B \subset Y$ and $\varepsilon > 0$ there exist $V \in \mathcal{N}_\tau(\bar{x})$ and $H \in \mathcal{H}$ such that*

$$\forall x \in V, \forall \nu \in H : \qquad S^\nu(x) \cap B \subset \varepsilon S^\nu(\bar{x}).$$

It is (asymptotically) equi-inner semicontinuous, or equi-isc, at $\bar{x} \in X$ if for all compact set $B \subset Y$ and $\varepsilon > 0$ there exist $V \in \mathcal{N}_\tau(\bar{x})$ and $H \in \mathcal{H}$ such that

$$\forall x \in V, \forall \nu \in H : \qquad S^\nu(\bar{x}) \cap B \subset \varepsilon S^\nu(x).$$

It is equi-continuous at $\bar{x}$ if it is both equi-osc and equi-isc at $\bar{x}$.

The mappings are said to be equi-osc, equi-isc or equi-continuous if these properties hold at every $\bar{x} \in X$.

Notice that the definition of equi-osc doesn't require that the mappings S^ν be themselves outer semicontinuous! They are only required to be outer semicontinuous "asymptotically." If the S^ν are outer semicontinuous, then equi-osc (as defined) would be more precisely designated as "*eventually* equi-osc". But in this paper it won't be necessary to appeal to these distinctions, and it will thus suffice to refer to the notion introduced as (plain) equi-osc. The same observations apply to the definitions of equi-isc and equi-continuity.

One of the implications of the equi-osc condition is the following: let B again be a compact set, for any $H \in \mathcal{H}^\#$ and any $x^\nu \underset{H}{\to} \bar{x}$ there exists $\varepsilon_\nu \searrow 0$ such that

$$S^\nu(x^\nu) \cap B \subset \varepsilon_\nu S^\nu(\bar{x}),$$

which, assuming that for all $\nu \in H$, $S^\nu(x^\nu) \cap B$ is nonempty, in turn implies that there exits always $\hat{y}^\nu \in S^\nu(x^\nu)$, $y^\nu \in S^\nu(\bar{x})$ such that $d(\hat{y}^\nu, y^\nu) \to 0$.

Dolecki [**9**, §3] introduced a notion related to equi-outer semicontinuity but significantly stronger: Let (X, τ) be a topological space, (Y, d) a normed linear space and $(N, \mathcal{H})$ a filtered index set. A collection of osc mappings $\{ S^\nu : X \rightrightarrows Y, \nu \in N \}$ is (eventually) *quasi equi semicontinous at $\bar{x}$*, if for every $\varepsilon > 0$, one can find $H \in \mathcal{H}$ and $V \in \mathcal{N}(\bar{x})$ such that

$$\forall x \in V, \forall \nu \in H : \qquad S^\nu(x) \subset \varepsilon S^\nu(\bar{x}).$$

This turns out to coincide with equi-outer semicontinuity at $\bar{x}$ if the ranges of the mappings S^ν are contained in a compact set D, i.e., $\operatorname{rge} S^\nu \subset D$ for all ν. In general, however quasi equi semicontinuity turns out to be too constringent for applications purposes —think of epigraphical or subgradient mappings— and it wouldn't yield the key theorem 3.3 below, except for a quite restricted class of mappings.

In definition 3.1, the compact sets play the role of "test" sets for checking equi-semicontinuity. When the metric space Y is actually $I\!R^m$, or more generally, a linear vector space whose closed balls are compact, the class of test sets can be restricted to that of the closed (compact) balls centered at the origin; when $Y = I\!R^m$, one can further restrict the test sets to the closed balls centered at origin with rational radius, we won't exploit this here.

When Y is a linear space, let

$$I\!B := I\!B(0,1) \text{ denote the unit ball in } Y.$$

Proposition 3.2. *Let (X,τ) be a topological space, (Y,d) a linear space whose closed balls are compact, and $(N,\mathcal{H})$ a filtered index set. A collection of set-valued mappings $\{\, S^\nu : X \rightrightarrows Y,\, \nu \in N \,\}$ is equi-outer semicontinuous (equi-osc) at $\bar{x} \in X$ if and only if for all $\rho \geq 0$ and $\varepsilon > 0$, there exist $V \in \mathcal{N}_\tau(\bar{x})$ and $H \in \mathcal{H}$ such that*

$$\forall\, x \in V,\ \forall\, \nu \in H: \qquad S^\nu(x) \cap \rho I\!B \subset \varepsilon S^\nu(\bar{x}).$$

It is equi-inner semicontinuous (equi-isc) at $\bar{x} \in X$ if and only if for all $\rho \geq 0$ and $\varepsilon > 0$ there exist $V \in \mathcal{N}_\tau(\bar{x})$ and $H \in \mathcal{H}$ such that

$$\forall\, x \in V,\ \forall\, \nu \in H: \qquad S^\nu(\bar{x}) \cap \rho I\!B \subset \varepsilon S^\nu(x).$$

Proof. Since the closed balls are compact sets, the "if" conditions are immediate. On the other hand, every compact set is contained in a sufficiently large ball $\rho I\!B$, and this is all that's needed to obtain the "only if" direction. $\square$

Theorem 3.3. *Let (X,τ) be a topological space, (Y,d) a metric space and $\{\, S^\nu : X \rightrightarrows Y,\, \nu \in N \,\}$ a filtered collection of closed-valued mappings. If this collection is equi-outer semicontinuous at $\bar{x}$, then*

$$(\text{g-ls}\, S^\nu)(\bar{x}) = (\text{p-ls}\, S^\nu)(\bar{x}),$$
$$(\text{g-li}\, S^\nu)(\bar{x}) = (\text{p-li}\, S^\nu)(\bar{x}).$$

Thus in particular, if the collection is equi-osc, one has

$$S^\nu \xrightarrow{g} S \text{ if and only if } S^\nu \xrightarrow{p} S.$$

More generally, for a set $D \subset X$ and a point $\bar{x} \in X$, any two of the following conditions implies the third:

(a) *the collection is equi-osc at $\bar{x}$ relative to D;*
(b) *S^ν converges graphically to S at $\bar{x}$ relative to D;*
(c) *S^ν converges pointwise to S at $\bar{x}$ relative to D.*

Proof. To obtain the first identity it will suffice, in view of proposition 2.2, to show that $(\text{g-ls}\, S^\nu)(\bar{x}) \subset (\text{p-ls}\, S^\nu)(\bar{x})$. For any $\bar{y} \in \text{g-ls}\, S^\nu(\bar{x})$ there exist an index set $H \in \mathcal{H}^\#$, and $(x^\nu, y^\nu) \underset{H}{\rightarrow} (\bar{x}, \bar{y})$ with $y^\nu \in S(x^\nu)$. With $B = \{\bar{y}, y^\nu, \nu \in H\}$, a compact subset of Y, and

$\varepsilon > 0$, equi-outer semicontinuity at $\bar{x}$ implies the existence of $V \in \mathcal{N}(\bar{x})$ and $H_0 \subset H$, $H_0 \in \mathcal{H}^{\#}$ such that $S^\nu(x^\nu) \cap B \subset \varepsilon S^\nu(\bar{x})$ when $x^\nu \in V$ and $\nu \in H_0$, or equivalently (since $x^\nu \to \bar{x}$), for all $\nu \in H_1 \subset H_0$ for some index set $H_1 \in \mathcal{H}^{\#}$ such that $x^\nu \in V$ when $\nu \in H_1$. This means that $y^\nu \in \varepsilon S^\nu(\bar{x})$ for all $\nu \in H_1$. Since such a $H_1 \in \mathcal{H}^{\#}$ exists for every $\varepsilon > 0$, from the identity (1.1) defining $\operatorname{ls} S^\nu(\bar{x})$, one has that $\bar{y} \in \text{p-ls}\, S^\nu(\bar{x})$.

The proof of the second assertion is identical, except H_0 and H_1 are taken to belong to $\mathcal{H}$ instead of $\mathcal{H}^{\#}$.

For the rest, we can redefine $S^\nu(x)$ to be empty outside of D if necessary in order to reduce without loss of generality to the case of $D = X$. Then the implication (a) and (b) $\Rightarrow$ (c) and the implication (a) and (c) $\Rightarrow$ (b) both follow directly from the identities just established. There remains only to show that (b) and (c) $\Rightarrow$ (a). Suppose this isn't true, i.e., that despite both (b) and (c) holding at $\bar{x} \in D$, the collection fails to be equi-osc at $\bar{x}$: This means that there exist $\varepsilon > 0$, a compact subset B of Y, $H \in \mathcal{H}^{\#}$ and $x^\nu \underset{H}{\to} \bar{x}$ such that

$$S^\nu(x^\nu) \cap B \not\subset \varepsilon S^\nu(\bar{x}) \text{ when } \nu \in H.$$

When this is the case, for each $\nu \in H$ we can find $y^\nu \in S^\nu(x^\nu) \cap B \setminus \varepsilon S^\nu(\bar{x})$. The collection $\{y^\nu\}_{\nu \in H}$ then has a cluster point $\bar{y} \in B$, which by virtue of (b) must belong to $S(\bar{x})$. Yet $d(y^\nu, S^\nu(\bar{x})) \geq \varepsilon$ for all $\nu \in H$ and hence, there exists $H' \subset H, H' \in \mathcal{H}^{\#}$ such that for all $\nu \in H'$, $d(\bar{y}, S^\nu(\bar{x})) \geq \varepsilon/2$. This in turn implies that $\limsup_{\mathcal{H}} d(\bar{y}, S^\nu(\bar{x})) \geq \varepsilon/2$ which, in view of proposition 4.2(b) , would lead to the conclusion that $d(\bar{y}, S(\bar{x})) \geq \varepsilon/2$ since $S^\nu(\bar{x}) \to S(\bar{x})$ by (c). This excludes the possibility that $\bar{y}$ belongs to $S(\bar{x})$, in contradiction with the earlier conclusion that $\bar{y} \in S(\bar{x})$. □

4. Choquet-Wijsman convergence

Choquet-Wijsman convergence, (cw-*convergence*) of a filtered collection of sets, $\{C^\nu \subset Y, \nu \in N\}$ to a set C is defined in terms of the pointwise convergence of the distance functions:

$$C^\nu \xrightarrow{\text{cw}} C \quad \text{if} \quad d_{C^\nu} \xrightarrow{p} d_C.$$

The next proposition records an important (and well-known) relationship between set convergence and the pointwise convergence of the associated distance functions. The *distance function* d_C or $d(\cdot, C)$ associated to a set $C \subset Y$ is defined by

$$d_C(y) = d(y, C) := \begin{cases} \inf\{\, d(y, z) \,|\, z \in C \,\} & \text{if } C \neq \emptyset; \\ \infty & \text{if } C = \emptyset. \end{cases}$$

The continuity of the function $y \mapsto d(y, C)$ is immediate from the following observations: Let y^ν converge to y, i.e., such that $d(y^\nu, y) \to 0$, then the (triangle) inequalities

$$d(y^\nu, C) \leq d(y, C) + d(y^\nu, y), \qquad d(y, C) \leq d(y^\nu, C) + d(y^\nu, y)$$

yield $\limsup_\nu d(y^\nu, C) \le d(y, C) \le \liminf_\nu d(y^\nu, C)$ if $C \neq \emptyset$; if $C = \emptyset$ then $d_C \equiv \infty$. Actually, distance functions are equi-continuous (with modulus of continuity 1) since for any set C one has:

$$d_C(y_1) \le d_C(y_2) + d(y_1, y_2).$$

Proposition 4.1 [**8, 11**, propositions 2.1 and 2.2]**.** *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets, and $C \subset Y$, a closed set. Then, for all $y \in Y$,*

(a) $\liminf_{\mathcal{H}} d(y, C^\nu) \ge d(y, C)$ *if and only if for any pair $0 < \varepsilon < \eta$, there exists $H \in \mathcal{H}$ such that for all $\nu \in H$,*

$$C \cap \mathbb{B}(y, \eta) = \emptyset \quad \textit{implies} \quad C^\nu \cap \mathbb{B}(y, \varepsilon) = \emptyset;$$

(b) $\limsup_{\mathcal{H}} d(y, C^\nu) \le d(y, C)$ *if and only if there exists $H \in \mathcal{H}$ such that for all $\nu \in H$,*

$$C \cap \mathbb{B}^o(y, \varepsilon) \neq \emptyset \quad \textit{implies} \quad C^\nu \cap \mathbb{B}^o(y, \varepsilon) \neq \emptyset.$$

The relationship between Choquet-Wijsman convergence and Painlevé-Kuratowski (plain) convergence is clarified by the next proposition.

Proposition 4.2 [**11**, proposition 2.3, theorem 2.6]**.** *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets, and $C \subset Y$, a closed set. If $d(y, C^\nu) \to d(y, C)$ for every $y \in Y$ then $C^\nu \to C$, more precisely*

(a) $C \supset \mathrm{ls}\, C^\nu$ *if* $\liminf_{\mathcal{H}} d(y, C^\nu) \ge d(y, C)$ *for all $y \in Y$;*

(b) $C \subset \mathrm{li}\, C^\nu$ *if and only if* $\limsup_{\mathcal{H}} d(y, C^\nu) \le d(y, C)$ *for all $y \in Y$.*

Thus, cw-convergence implies (plain) convergence. These two notions are identical if the closed balls of Y are compact, i.e., (a) becomes also an "if and only if" condition.

Moreover, if the distance functions d_{C^ν} converge pointwise to d_C then they also converge uniformly on every compact subset of Y.

The difference between (plain) convergence and cw-convergence is illustrated by the following simple example. Again consider the sequence $\{\, C^\nu = \{e^\nu\} \subset \ell^1, \nu \in \mathbb{N} \,\}$ with e^ν the unit vector in ℓ^1 with a 1 in the ν-th position. One has $\mathrm{lm}\, C^\nu = \emptyset$, but there is *no* "cw-limit" set; a sequence converging to the empty set must eventually escape from every compact set, whereas a sequence cw-converging to the empty set must eventually escape from every *bounded* set.

Cw-convergence does imply the (cw-)convergence of ε-fattenings.

Proposition 4.3. *Let (Y, d_Y) be a metric space, $\{C^\nu \subset Y,\, \nu \in N\}$ a filtered collection of sets, and $C \subset Y$, a closed set. Then, for any $y \in Y$ and $\varepsilon \ge 0$,*

$$d(y, C^\nu) \to d(y, C) \qquad \Longrightarrow \qquad d(y, \varepsilon C^\nu) \to d(y, \varepsilon C).$$

Proof. Propositions 1.4 and 4.2, together, already yield $\limsup_{\mathcal{H}} d(y, \varepsilon C^\nu) \le d(y, \varepsilon C)$ for all $y \in Y$, it thus suffices to show that $\liminf_{\mathcal{H}} d(y, \varepsilon C^\nu) \ge d(y, \varepsilon C)$ for all $y \in Y$, or equivalently (proposition 4.1), for any pair $0 < \theta < \eta$, there exists $H \in \mathcal{H}$ such that for all $\nu \in H$,

$$d(y, \varepsilon C) > \eta \;\Rightarrow\; d(y, \varepsilon C^\nu) > \theta,$$

given that for any pair $0 < \theta' < \eta'$, there exists $H' \in \mathcal{H}$ such that for all $\nu \in H'$,

$$d(y, C) > \eta' \;\Rightarrow\; d(y, C^\nu) > \theta'.$$

If $d(y, \varepsilon C) > \theta$, then $d(y, C) > \varepsilon + \theta$, and this implies that there exists $H \in \mathcal{H}$ such that $d(y, C^\nu) > \theta'$ for all $\theta' \in (0, \varepsilon + \theta)$ which in turn implies that $d(y, \varepsilon C^\nu) > \theta$ for all $\theta \in (0, \theta)$ for all $\nu \in H$. ◻

Remark 4.4. A mapping $S : X \rightrightarrows Y$ is *cw-osc (cw-isc, cw-continuous) at* $\bar{x} \in X$ if for all $y \in Y$, $\liminf_{\mathcal{N}(\bar{x})} d(y, S(x)) \geq d(y, S(\bar{x}))$ ($\limsup_{\mathcal{N}(\bar{x})} d(y, S(x)) \leq d(y, S(\bar{x}))$, $\lim_{\mathcal{N}(\bar{x})} d(y, S(x)) = d(y, S(\bar{x}))$). Observe that as consequence of proposition 4.2, one has that cw-osc conincides with osc, and that a mapping is isc (continuous) at $\bar{x}$ whenever it's cw-isc (cw-continuous) at $\bar{x}$.

5. Continuous and uniform convergence

Two more classical concepts of convergence are often useful in identifying graph convergence and equi-outer semicontinuity properties.

Again, (X, τ) is a topological space, (Y, d) a metric space and N and index space with filter $\mathcal{H}$. A filtered collection of mappings $S^\nu : X \rightrightarrows Y$ *converges continuously* to a mapping S at $\bar{x}$ if $S^\nu(x^\nu) \to S(\bar{x})$ for all $x^\nu \to \bar{x}$. It does so *relative to a set* $D \subset X$ if this holds at all $\bar{x} \in D$ when $x^\nu \in D$.

The mappings S^ν *converge uniformly* to S on a set $D \subset X$ if for every $\varepsilon > 0$ and compact set $B \subset Y$ there exists $H \in \mathcal{H}$ such that

$$\left\{ \begin{array}{l} S^\nu(x) \cap B \subset \varepsilon S(x) \\ S(x) \cap B \subset \varepsilon S^\nu(x) \end{array} \right\} \text{ for all } x \in D \text{ when } \nu \in H. \tag{5.1}$$

The inclusion property for uniform convergence should be compared to the one automatically present by virtue of proposition 1.1. when the mappings S^ν converge pointwise, i.e., $S = \text{p-lm}\, S^\nu$ at $\bar{x}$: for every $\varepsilon > 0$ and compact set $B \subset Y$, there exists $H \in \mathcal{H}$ such that

$$\left. \begin{array}{l} S^\nu(\bar{x}) \cap B \subset \varepsilon S(\bar{x}) \\ S(\bar{x}) \cap B \subset \varepsilon S^\nu(\bar{x}) \end{array} \right\} \text{ when } \nu \in H. \tag{5.2}$$

By the same token, continuous convergence of S^ν to S at $\bar{x}$ can be identified with the condition that for every $\varepsilon > 0$ and compact set $B \subset Y$, there exists $H \in \mathcal{H}$ along with a neighborhood $V \in \mathcal{N}(\bar{x})$ such that

$$\left. \begin{array}{l} S^\nu(x) \cap B \subset \varepsilon S(\bar{x}) \\ S(\bar{x}) \cap B \subset \varepsilon S^\nu(x) \end{array} \right\} \text{ for all } x \in V \text{ when } \nu \in H. \tag{5.3}$$

For continuous convergence relative to D, x must of course be restricted to D. Continuous convergence can be viewed as a "localized" version of uniform convergence.

It is also possible to define these notions of convergence for mappings in terms of the cw-convergence of sets (as well as in terms of any other notions of convergence for sets). The collection of mappings $\{S^\nu : X \rightrightarrows Y,\, \nu \in N\}$ *cw-converge pointwise at* $\bar{x} \in X$ if

$$d(y, S^\nu(\bar{x})) \to d(y, S(\bar{x})), \quad \forall\, y \in Y; \tag{5.4}$$

if this holds at all $\bar{x} \in X$, the mappings S^ν are said to cw-converge pointwise. *Continuous cw-convergence at* $\bar{x} \in X$ means that for all $x^\nu \to \bar{x}$, one has

$$d(y, S^\nu(x^\nu)) \to d(y, S(\bar{x})), \quad \forall\, y \in Y; \tag{5.5}$$

and the mappings S^ν are said to continuously cw-converge to S if (5.5) holds for all $\bar{x} \in X$. Finally, *uniform cw-convergence on a set* D means that for all $y \in Y$ and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that

$$|d(y, S^\nu(x)) - d(y, S(x))| \le \varepsilon, \quad \forall\, x \in D, \forall\, \nu \in H. \tag{5.6}$$

Remember that in view of proposition 4.2, the notions based on the Choquet-Wijsman convergence of sets are more restrictive than those based on (Painlevé-Kuratowski) convergence; they coincide if Y has compact balls.

Proposition 5.1. *Let (X, τ) be a topological space, (Y, d) a metric space, and let $\{\, S^\nu : X \rightrightarrows Y,\, \nu \in N \,\}$ a filtered collection of closed-valued mappings. If the mappings S^ν converge continuously to S relative to a set D, then S is continuous relative to D.*

Proof. Let $\bar{x} \in D$. From (5.3) it follows that given any compact set $B \subset Y$ and $\varepsilon > 0$, there exist $V \in \mathcal{N}(\bar{x})$ and $H \in \mathcal{H}$ such that for all $x \in V, \nu \in H$: $S(x) \cap B \subset \varepsilon S^\nu(\bar{x})$, and $S^\nu(\bar{x}) \cap B \subset \varepsilon S(\bar{x})$. These two inclusions imply $S(x) \cap B \subset (2\varepsilon S)(\bar{x})$. Since this holds for every compact B and $\varepsilon > 0$, it follows from proposition 1.1 that $S(\bar{x}) \supset \mathrm{ls}_{\mathcal{N}(\bar{x})}\, S(x)$.

Again from (5.3), continuous convergence implies that for every compact set B and $\varepsilon > 0$, there exist $V \in \mathcal{N}(\bar{x})$ and $H \in \mathcal{H}$ such that $S(\bar{x}) \cap B \subset \varepsilon S^\nu(x)$ for all $x \in V \cap D, \nu \in H$. Moreover, since for every $x \in V \cap D$, $S(x) = \mathrm{lm}\, S^\nu(x)$ as a consequence of continuous convergence on D, there exists $H_x \in \mathcal{H}$ such that for all $\nu \in H \cap H_x$, $S^\nu(x) \cap B \subset \varepsilon S(x)$, cf. proposition 1.1. Hence $S(\bar{x}) \cap B \subset (2\varepsilon S)(x)$ for all $x \in V$, and from proposition 1.1 it then follows that $S(\bar{x}) \subset \mathrm{li}_{\mathcal{N}(\bar{x})}\, S(x)$. This with the inclusion obtained in the previous paragraph, yields the continuity of S at $\bar{x}$ for all $\bar{x} \in D$. $\square$

Theorem 5.2. *Let (X, τ) be a topological space, (Y, d) a metric space and $\{\, S^\nu : X \rightrightarrows Y,\, \nu \in N \,\}$ a filtered collection of closed-valued mappings.*

(a) *Continuous convergence of the mappings S^ν to S relative to D implies uniform convergence on any compact subset of D.*

(b) *Assuming that S is cw-continuous relative to $D \subset X$, then uniform cw-convergence on all compact subsets of D implies continuous cw-convergence relative to D.*

Proof. We begin with (a). Suppose to the contrary that the first inclusion in (5.1) fails, i.e., that there exists a compact set $C \subset X$, $\varepsilon > 0$, a compact set $B \subset Y$, $H \in \mathcal{H}^\#$ and $x^\nu \in C, \nu \in H$ such that $S^\nu(x^\nu) \cap B \not\subset \varepsilon S(x^\nu)$ for all $\nu \in H$. For $\nu \in H$, let $y^\nu \in S^\nu(x^\nu) \cap B \setminus \varepsilon S(x^\nu)$ and $(\bar{x}, \bar{y})$ a cluster point of the collection $\{(x^\nu, y^\nu),\, \nu \in H\}$, say $(x^\nu, y^\nu) \underset{H_0}{\rightrightarrows} (\bar{x}, \bar{y})$ for $H_0 \in \mathcal{H}^\#$.

Continuous convergence of S^ν to S relative to D implies in particular that $S(\bar{x}) \supset \mathrm{ls}_{H_0} S^\nu(x^\nu)$, implying that $\bar{y} \in S(\bar{x})$. On the other hand, $d(y^\nu, S(x^\nu)) > \varepsilon$ for all $\nu \in H_0$. Since $d(\bar{y}, S(x^\nu)) + d(\bar{y}, y^\nu) \geq d(y^\nu, S(x^\nu) > \varepsilon$ and $y^\nu \underset{H_0}{\rightrightarrows} \bar{y}$, one has

$$\limsup\nolimits_{\mathcal{H}} d(\bar{y}, S(x^\nu)) \geq \limsup\nolimits_{H_0} d(\bar{y}, S(x^\nu)) \geq \varepsilon.$$

Since S is continuous (proposition 5.1) at $\bar{x}$ relative to D, $d(\bar{y}, S(\bar{x})) \geq \varepsilon > 0$ (proposition 4.2(b)), it also follows that $\bar{y} \notin S(\bar{x})$ and this is in contradiction with the earlier assertion that $\bar{y} \in S(\bar{x})$.

The same argument applies if the second inclusion in (5.1) fails, after interchanging the roles played by the sets $S^\nu(x^\nu)$ and $S(x^\nu)$.

We now proceed with (b). Let $x^\nu \to \bar{x}$, all in D. We have to show that given any $y \in Y$ for all $\varepsilon > 0$ there exists $H_\varepsilon \in \mathcal{H}$ such that $|d(y, S^\nu(x^\nu)) - d(y, S(\bar{x}))| \leq \varepsilon$, but that's an immediate consequence of $|d(y, S^\nu(x^\nu)) - d(y, S(x^\nu))| \leq \varepsilon/2$ for all $\nu \in H_0$ for some $H_0 \in \mathcal{H}$ as follows from uniform cw-convergence (5.6) on the compact set $\{x^\nu, \nu \in H\}$, and $|d(y, S(x^\nu)) - d(y, S(\bar{x}))| \leq \varepsilon/2$ for all $\nu \in H \subset H_0$ with $H \in \mathcal{H}$ and this follows from the cw-continuity of S relative to D. $\square$

Theorem 5.3. *Let (X, τ) be a topological space, (Y, d) a metric space and $\{\, S^\nu : X \rightrightarrows Y,\ \nu \in N \,\}$ a filtered collection of closed-valued mappings. For mappings S, $S^\nu : X \rightrightarrows Y$ and a set $D \subset X$, the following properties at a point $\bar{x} \in X$ are equivalent:*

(a) *S^ν converges continuously to S at $\bar{x}$ relative to D;*

(b) *S^ν converges graphically to S at $\bar{x}$ relative to D, and the collection is (asymptotically) equi-continuous at $\bar{x}$ relative to D.*

Proof. Suppose first that (a) holds. Obviously this condition implies that S^ν converges both pointwise and graphically to S at $\bar{x}$ relative to X, and therefore by theorem 3.3 that the collection is equi-osc relative to D. We must show that the collection is also equi-isc relative to D. If not, there would exist $\varepsilon > 0$, $B \subset Y$ compact, $H \in \mathcal{H}^\#$ and $x^\nu \underset{H}{\rightrightarrows} \bar{x}$ in X such that

$$S^\nu(\bar{x}) \cap B \not\subset \varepsilon S^\nu(x^\nu) \quad \text{when } \nu \in N.$$

It would be possible then to choose for each $\nu \in H$ an element $y^\nu \in S^\nu(\bar{x}) \cap B \setminus \varepsilon S^\nu(x^\nu)$. Let $\bar{y}$ be a cluster point of the y^ν, i.e., such that $y^\nu \underset{H'}{\rightrightarrows} \bar{y}$ for $H' \subset H$, $H' \in \mathcal{H}^\#$. This point $\bar{y}$ necessarily belongs to $S(\bar{x}) \cap B$ because $S^\nu(\bar{x}) \to S(\bar{x})$. On the other hand, since $d(y^\nu, S^\nu(x^\nu)) > \varepsilon$ and $y^\nu \underset{H'}{\rightrightarrows} \bar{y}$, $d(\bar{y}, S^\nu(x^\nu)) > \varepsilon/2$ for all $\nu \in H'_0 \subset H'$, $H'_0 \in \mathcal{H}^\#$. Hence,

$$\varepsilon/2 \le \limsup_{H'_0} d(\bar{y}, S^\nu(x^\nu) \le \limsup_{\mathcal{H}} d(\bar{y}, S^\nu(x^\nu) \le d(\bar{y}, S(\bar{x})),$$

with the last inequality coming from proposition 4.2(b) since $S^\nu(x^\nu) \to S(\bar{x})$. And one would have that $\bar{y}$ can't belong to $S(\bar{x})$ in contradiction with our earlier statement, consequently, the collection of mappings S^ν must also be equi-isc.

Now suppose that (b) holds. Consider any $x^\nu \to \bar{x}$ in X. Graphical convergence yields that $\operatorname{ls} S^\nu(x^\nu) \subset S(\bar{x})$, so there remains only to show for arbitrary $\bar{y} \in S(\bar{x})$ that $\bar{y} \in \operatorname{li} S^\nu(x^\nu)$. Because the mappings S^ν are equi-osc and $S^\nu \xrightarrow{g} S$ at $\bar{x}$ relative to D, one has $S^\nu(\bar{x}) \to S(\bar{x})$ by theorem 3.3, so for indices ν in some set $H_0 \in \mathcal{H}$ one can find $H_1 \subset H_0, H_1 \in \mathcal{H}$, $y^\nu \in S^\nu(\bar{x})$ with $y^\nu \underset{H_1}{\rightrightarrows} \bar{y}$. Now, since the mappings S^ν are also equi-isc at $\bar{x}$, with $B = \{y^\nu,\, \nu \in H\}$, for all $\varepsilon > 0$ there exists $V_\varepsilon \in \mathcal{N}(\bar{x})$ and $H_2 \subset H_1, H_2 \in \mathcal{H}$ such that

$$S^\nu(\bar{x}) \cap B \;\subset\; \varepsilon S^\nu(x), \quad \forall\, x \in V, \forall\, \nu \in H_1.$$

Then for some $H_\varepsilon \in \mathcal{H}$ with $H_\varepsilon \subset H_2$ one has $x^\nu \in V_\varepsilon$, $y^\nu \in \varepsilon S^\nu(x^\nu)$ for all $\nu \in H_\varepsilon$. Since this holds for all $\varepsilon > 0$, by letting $\varepsilon \searrow 0$, one can generate points $\hat{y}^\nu \in S^\nu(x^\nu)$ that converge to $\bar{y}$, and this guarantees $\bar{y} \in \operatorname{li} S^\nu(x^\nu)$. ☐

Corollary 5.4. *Let (X, τ) be a topological space, (Y, d) a metric space and $\{\, F^\nu : X \to Y,\, \nu \in N\,\}$ a filtered collection of single-valued mappings. The following hold:*

(a) *F^ν converges graphically to F at $\bar{x}$ implies F^ν converges continuously to F at $\bar{x}$ provided the collection $\{F^\nu,\, \nu \in N\}$ is eventually locally compact, i.e., there exist $V \in \mathcal{N}(\bar{x})$, $H \in \mathcal{H}$ and a compact set $B \subset Y$ such that $F^\nu(x) \in B$ for all $x \in V$ when $\nu \in N$;*

(b) *F^ν converges continuously to F at $\bar{x}$ implies F^ν converges graphically to F at $\bar{x}$.*

Proof. From theorem 5.3 and the definition of continuous convergence, one has that (b) implies (a). On the other hand, the local compactness in (a) implies that every collection of points $\{F^\nu(x^\nu)\}$ with $x^\nu \to \bar{x}$ is eventually in B, while the graphical convergence in (a) along with the single-valuedness of F ensures that the only possible cluster point of such a collection is $F(\bar{x})$. Hence, $F^\nu(x^\nu) \to F(\bar{x})$ whenever $x^\nu \to \bar{x}$. ☐

This section conclude with a statement of a result that becomes "classical" when collections of equi-continuous single-valued mappings (equivalently, functions from X to Y) are involved. To do so, we are in need of the remarkable compactness property of the space of mappings equipped with the topology induced by graph-convergence:

Lemma 5.5. *Let (X, d_X), (Y, d_Y) be metric spaces and $\{S^\nu : X \rightrightarrows Y,\, \nu \in N\}$ a filtered collection of set-valued mappings. Then, either this collection escapes to the horizon, i.e., if for every pair of compact sets $D \subset X$, $B \subset Y$, there exists $H \in \mathcal{H}$ such that $S^\nu(x) \cap B = \emptyset$*

for all $x \in D$, $\nu \in H$, *or there exists a subcollection of the mappings that converges graphically to a mapping* $S : X \rightrightarrows Y$ *with* $\operatorname{dom} S = \{\, x \in X \mid S(x) \neq \emptyset \,\}$ *nonempty.*

Proof. The key to the proof is to view graph convergence as the set convergence of the graphs and appeal to Mrówka's theorem [**6**, p.149, **12**] which asserts that every filtered family of subsets of $X \times Y$ has a subfamily that converges, in the Painlevé-Kuratowski sense, to some set. This could be the empty set which would correspond to the graphs of the S^ν "escaping to the horizon". This eventuality is characterized by Corollary 1.2 which leads to the criterion adopted here; note that if the graphs $\operatorname{gph} S^\nu$ of the mappings S^ν eventually miss every compact set of type $D \times B$ with $D \subset X$ and $B \subset Y$, then they also eventually miss every compact subset of $X \times Y$. □

With the help of this lemma, we obtain a generalization of the classical Arzelà-Ascolì as a corollary of theorem 5.3.

Corollary 5.6 (Arzelà-Ascolì Theorem for mappings)**.** *Let* (X, d_X), (Y, d_Y) *be metric spaces and* $\{S^\nu : X \rightrightarrows Y,\, \nu \in N\}$ *a filtered collection of mappings, equi-continuous relative to a set* $D \subset X$*. Then, there exists a subcollection converging uniformly on all compact subsets of* D *to a mapping* $S : X \rightrightarrows Y$ *that is continuous relative to* D*.*

Moreover, if at some point $\bar{x}$ *in* D *this collection of mappings is eventually locally compact, then* $D = \operatorname{dom} S = \{x | S(x) \neq \emptyset\}$*; a collection* $\{S^\nu, \nu \in I\!N\}$ *is eventually locally compact at* $\bar{x}$ *if there exists* $V \in \mathcal{N}(\bar{x})$, $H \in \mathcal{H}$ *and a compact set* $B \subset Y$ *such that* $S^\nu(x) \cap B \neq \emptyset$ *for all* $x \in V$, $\nu \in H$*.*

Proof. The preceding lemma guarantees the existence of a subcollection, say for $\nu \in H, H \in \mathcal{H}^\#$, graph converging to a mapping $S : X \rightrightarrows Y$, possibly empty-valued. One now appeals to theorem 5.3 to claim that in fact these mappings $\{S^\nu, \nu \in H\}$ actually continuously converge to S, since the S^ν are equi-continuous, and S must then be continuous. Uniform convergence on compact subsets of D then follows from theorem 5.2.

If the $\{S^\nu\}$ are locally compact a some point $\bar{x} \in D$ then $S(\bar{x})$ cannot be empty, and since S is continuous on D it implies that S in nonempty valued on D, i.e., $\operatorname{dom} S = D$. □

6. Equi-lower semicontinuity

A filtered collection of lower semicontinuous (lsc) functions $\{f^\nu : X \to \overline{I\!R}, \nu \in (N, \mathcal{H})\}$ is (eventually) *equi-lower semicontinuous (equi-lsc)* at $\bar{x}$ if to every $\varepsilon > 0$ and $\rho > 0$ one can associate some $V \in \mathcal{N}(\bar{x})$ and $H \in \mathcal{H}$ such that

$$\forall\, \nu \in H : \quad \inf_{x \in V} f^\nu(x) \;\geq\; \min\,[\, f^\nu(\bar{x}) - \varepsilon,\, \rho\,].$$

The f^ν are *equi-lsc* if this holds at every $\bar{x} \in X$.

This notion was introduced in [**14**] for sequences of convex lsc functions defined on a Banach space, and for filtered families of (arbitrary) functions defined on a topological space in [**10**]. In particular, it was shown that the equi-lsc condition is both necessary and sufficient for pointwise and epi-convergence to coincide. The only purpose of this section is to record the fact that equi-lower semicontinuity for functions corresponds to the equi-outer semi-continuity of the (upper) profile mappings associated with these functions.

For $f : X \to \overline{I\!R}$, let's call

$$U_f : X \to I\!R \text{ with } U_f(x) = \begin{cases} I\!R & \text{if } f(x) = -\infty, \\ [\, f(x), \infty) & \text{if } f(x) \in I\!R, \\ \emptyset & \text{if } f(x) = \infty, \end{cases}$$

the *(upper) profile mapping* associated with the function f. One has,

$$\operatorname{gph} U_f = \operatorname{epi} f = \{\, (x, \alpha) \,|\, \alpha \geq f(x) \,\} \subset X \times I\!R. \tag{6.1}$$

Recall that a filtered collection of lsc functions $\{f^\nu : X \to \overline{I\!R}, \nu \in (N, \mathcal{H})\}$ *epi-converges to f at $\bar{x}$* if

$$\text{e-ls}_{\mathcal{H}}\, f^\nu(\bar{x}) \leq f(\bar{x}) \leq \text{e-li}_{\mathcal{H}}\, f^\nu(\bar{x}),$$

where

$$\begin{aligned} \text{e-li}_{\mathcal{H}}\, f^\nu(\bar{x}) &= \sup_{V \in \mathcal{N}(\bar{x})} \inf_{H \in \mathcal{H}} \sup_{\nu \in H} \inf_{x \in V} f^\nu(x), \\ \text{e-ls}_{\mathcal{H}}\, f^\nu(\bar{x}) &= \sup_{V \in \mathcal{N}(\bar{x})} \sup_{H \in \mathcal{H}} \inf_{\nu \in H} \inf_{x \in V} f^\nu(x). \end{aligned}$$

Proposition 6.1. *Let (X, τ) be a topological space, $\{f^\nu : X \to \overline{I\!R}, \nu \in (N, \mathcal{H})\}$ a filtered collection of lsc functions, and $\{U_{f^\nu} : X \to \overline{I\!R}, \nu \in N\}$ the associated (upper) profile mappings. Then, the f^ν epi-converge to f at $\bar{x}$ if and only if the profile mappings U_{f^ν} graph-converge to U_f at $\bar{x}$.*

Proof. This is immediate from the definitions and (6.1). □

Proposition 6.2. *Let (X, τ) be a topological space and $\{f^\nu : X \to \overline{I\!R}, \nu \in N\}$ a filtered collection of extended real-valued lsc functions. This collection is equi-lsc at $\bar{x}$ if and only if the (upper) profile mappings $\{U_{f^\nu} : X \to \overline{I\!R}, \nu \in N\}$ are equi-osc at $\bar{x}$.*

Proof. For $B \subset I\!R$, compact, let $\rho := \max\,[\, \alpha \,|\, \alpha \in B\,]$. Then, the condition:

$$U_{f^\nu}(x) \cap B \subset U_f^\nu(\bar{x}) + \varepsilon[-1, 1], \quad \forall\, x \in V \in \mathcal{N}(\bar{x}),\ \forall\, \nu \in H \in \mathcal{H}$$

will be satisfied if and only if for all $x \in V$, $\nu \in H$,

$$\begin{cases} f^\nu(x) > \rho & \text{when } f^\nu(\bar{x}) - \varepsilon > \rho, \\ f^\nu(x) \geq f^\nu(\bar{x}) - \varepsilon & \text{when } f^\nu(\bar{x}) - \varepsilon \leq \rho, \end{cases}$$

which is just another way to formulate the equi-lsc condition. □

When theorem 3.3, relating pointwise and graph-convergence of mappings, is applied to the profile mappings, one is able to relate the pointwise and epi-convergence of lsc functions.

6.3 Theorem [10]. *Let (X,τ) be a topological space and $\{f^\nu : X \to \overline{I\!R}, \nu \in (N,\mathcal{H})\}$ a filtered collection of lsc functions and a point $\bar{x} \in X$. If the collections is equi-lsc at $\bar{x}$, then*

$$\begin{aligned}\text{e-li}\, f^\nu(\bar{x}) = \text{p-li}\, f^\nu(\bar{x}) &= \liminf_{\nu \in N} f^\nu(\bar{x}),\\ \text{e-ls}\, f^\nu(\bar{x}) = \text{p-ls}\, f^\nu(\bar{x}) &= \limsup_{\nu \to \infty} f^\nu(\bar{x}).\end{aligned}$$

Thus, when the f^ν are equi-lsc, $f^\nu \xrightarrow{e} f$ if and only if $f^\nu \xrightarrow{p} f$.

More generally, relative to an arbitrary set $D \subset X$ containing $\bar{x}$, any two of the following conditions implies the third:

(a) *the collection is (eventually) equi-lsc at $\bar{x}$ relative to D;*
(b) *$f^\nu \xrightarrow{e} f$ at $\bar{x}$ relative to D;*
(c) *$f^\nu \xrightarrow{p} f$ at $\bar{x}$ relative to D.*

Proof. This is immediate from the corresponding result for mappings in theorem 3.3 by virtue of the equivalences in propositions 6.1 and 6.2. □

Remark 6.4. The following is a sufficient, but not necessary condition for equi-lower semicontinuity at $\bar{x} \in X$: Given a filtered collection of lsc functions $\{f^\nu : X \to \overline{I\!R}, \nu \in N\}$, suppose that to every $\varepsilon > 0$ and $\rho > 0$, one can associate $V \in \mathcal{N}(\bar{x})$ and $H \in \mathcal{H}$ such that

$$\inf_{x \in V} f^\nu(x) \geq \begin{cases} f^\nu(\bar{x}) - \varepsilon & \text{for all } \nu \in H \text{ such that } f^\nu(\bar{x}) < \infty,\\ \rho & \text{for all } \nu \in H \text{ such that } f^\nu(\bar{x}) = \infty.\end{cases}$$

To see that this condition isn't necessary, consider the sequence of lsc functions

$$\{\, f^\nu(x) = \delta_{(-\infty,1]}(x) + \max[\,0,\, \nu x\,], \quad \nu \in I\!N\,\},$$

which is equi-lsc at $\bar{x} = 1$, but doesn't satisfy the preceding condition at $\bar{x} = 1$.

7. Mosco-graph-convergence

Let Y be a normed linear space, and let's denote by σ and ω the strong and weak topologies on Y. A filtered collection of sets $\{\, C^\nu \subset Y, \nu \in (N,\mathcal{H})\,\}$ is said to *Mosco-converge* to a set $C \subset Y$ if

$$C = \text{lm}_\sigma\, C^\nu = \text{lm}_\omega\, C^\nu,$$

with lm_σ and lm_ω indicating that the limits are calculated with respect to the strong and weak topologies. Since $\sigma \supset \omega$, this is equivalent to requiring that

$$\text{ls}_\omega\, C^\nu \subset C \subset \text{li}_\sigma\, C^\nu.$$

One then writes,

$$C^\nu \xrightarrow{M} C \quad \text{or} \quad C = \text{M-}\,\text{lm}\, C^\nu.$$

Let's begin with an observation that complements proposition 1.1.

Proposition 7.1. *Let (Y,σ) be a normed linear space, $\{C^\nu \subset Y,\ \nu \in N\}$ a filtered collection of sets and $C \subset Y$ a closed set. If for all $B \subset Y$, ω-compact and $\varepsilon > 0$ there exists $H \in \mathcal{H}$ such that for all $\nu \in H$, $C^\nu \cap B \subset \varepsilon C$, then $C \supset \operatorname{ls}_\omega C^\nu$. Consequently, C^ν Mosco-converges to C whenever this condition and that in 1.1(b) are satisfied.*

Proof. One can use the same argument as for "sufficiency in (a)" of proposition 1.1, except that convergence of the $y^\nu \underset{H}{\rightarrow} \bar{y}$ will be with respect to the weak topology and one observes that $B = \{\, \bar{y}, y^\nu,\ \nu \in H \,\}$ is weakly compact. $\square$

Remark 7.2. The converse doesn't hold in general, and thus Mosco-convergence doesn't imply the condition of proposition 1.1. To see this, let X be a Hilbert space and consider the sequence of sets $\{\, C^\nu = \{0, e^\nu\},\ \nu \in \mathbb{N} \,\}$ where the e^ν are the unit vectors of an orthonormal base. Clearly, C^ν Mosco-converges to $C = \{0\}$, but the condition of proposition 7.1 doesn't hold .

A filtered collection of mappings $\{\, S^\nu : X \rightrightarrows Y,\ \nu \in N \,\}$ with (X, σ_X), (Y, σ_Y) normed linear spaces, *Mosco-pointwise converges* to $S: X \rightrightarrows Y$ at $\bar{x}$ if $S^\nu(\bar{x}) \xrightarrow{\mathrm{M}} S(\bar{x})$, i.e.,

$$\operatorname{ls}_\omega S^\nu(\bar{x}) \subset S(\bar{x}) \subset \operatorname{li}_\sigma S^\nu(\bar{x})$$

where the subscripts of ls and li refer to the weak and strong topologies on Y. One writes $S^\nu \overset{\text{M-p}}{\rightrightarrows} S$ when this holds for all $\bar{x} \in X$. The collection *Mosco-graph-converges* to S if $\operatorname{gph} S^\nu \xrightarrow{\mathrm{M}} \operatorname{gph} S$, and one then writes $S^\nu \overset{\text{M-g}}{\rightrightarrows} S$. Graph-convergence at a point $\bar{x}$ means that

$$\big(\text{g-ls}_{\times\omega} S^\nu\big)(\bar{x}) \subset S(\bar{x}) \subset \big(\text{g-li}_{\times\sigma} S^\nu\big)(\bar{x})$$

where the subscripts of ls and li now refer to the weak and strong product topologies on $X \times Y$, i.e.,

$$\begin{aligned} \big(\text{g-li}_{\times\sigma} S^\nu\big)(\bar{x}) &= \bigcup_{\{x^\nu \to \bar{x}\}} \operatorname{li}_\sigma S^\nu(x^\nu), \\ \big(\text{g-ls}_{\times\omega} S^\nu\big)(\bar{x}) &= \bigcup_{\{x^\nu \rightharpoonup \bar{x}\}} \operatorname{ls}_\omega S^\nu(x^\nu), \end{aligned}$$

where the unions are taken over all $x^\nu \to \bar{x}$ for the inner limit, and all $x^\nu \rightharpoonup \bar{x}$ for the outer limit, $\rightharpoonup$ denoting weak convergence.

To relate Mosco-pointwise convergence to Mosco-graph-convergence, one relies on strengthened version of equi-outer semicontinuity.

Definition 7.3. *Let (X, σ_X), (Y, σ_Y) be normed linear spaces, and $(N, \mathcal{H})$ a filtered index set. A collection of set-valued mappings $\{\, S^\nu : X \rightrightarrows Y,\ \nu \in N \,\}$ is (asymptotically) M-equi-osc at $\bar{x} \in X$, if for every ω_Y-compact set $B \subset Y$ and $\varepsilon > 0$, one can find $H \in \mathcal{H}$, and $V \in \mathcal{N}_{\omega_X}(\bar{x})$, i.e., a neighborhood of $\bar{x}$ with respect to the weak topology, such that*

$$\forall\, x \in V, \forall\, \nu \in H : \qquad S^\nu(x) \cap B \subset \varepsilon S^\nu(\bar{x}).$$

The mappings are said to be M-equi-osc if this property hold at every $\bar{x} \in X$.

When $Y = I\!R^n$ is finite dimensional, $\omega_Y = \sigma_Y$, and M-equi-outer semicontinuity takes on a simplified form to which one could refer as *"weak equi-outer semicontinuity"*.

Definition 7.4. *The definition of ω-equi-osc at a point $\bar{x}$ is the same as that of equi-osc at $\bar{x}$ (definition 3.1) except that it guarantees the existence of a neighborhood $V \in \mathcal{N}_\omega(\bar{x})$ (with respect to the weak topology) and $H \in \mathcal{H}$ so that*

$$\forall\, x \in V,\, \forall\, \nu \in H: \qquad S^\nu(x) \cap B \subset \varepsilon S^\nu(\bar{x}).$$

Theorem 7.5. *Let (X, σ_X), (Y, σ_Y) be normed linear spaces, and consider $\{\, S^\nu\colon X \rightrightarrows Y,\ \nu \in N\,\}$ a filtered collection of closed-valued mappings. If this collection is M-equi-osc at $\bar{x}$, then*

$$(\text{g-ls}_{\times\omega}\, S^\nu)(\bar{x}) = (\text{p-ls}_\omega\, S^\nu)(\bar{x}),$$
$$(\text{g-li}_{\times\sigma}\, S^\nu)(\bar{x}) = (\text{p-li}_\sigma\, S^\nu)(\bar{x}).$$

Thus in particular, if the collection is M-equi-osc , one has

$$S^\nu \xrightarrow{\text{M-g}} S \text{ if and only if } S^\nu \xrightarrow{\text{M-p}} S.$$

Moreover, when Y is finite dimensional, any two of the following conditions implies the third:

(a) *the collection is ω-equi-osc at $\bar{x}$;*
(b) $S^\nu \xrightarrow{\text{M-g}} S$ *at $\bar{x}$;*
(c) $S^\nu \to S$ *at $\bar{x}$.*

Proof. Since $\sigma \supset \omega$, M-equi-osc implies equi-osc, and thus from theorem 3.3 it follows that

$$\forall\, x \in X: \qquad \text{p-li}_\sigma\, S^\nu(x) = \text{g-li}_{\times\sigma}\, S^\nu(x).$$

From the definitions of pointwise and graphical outer limits (2.3), one has,

$$\forall\, x \in X: \qquad \text{p-ls}_\omega\, S^\nu(x) \subset \text{g-ls}_{\times\omega}\, S^\nu(x).$$

Therefore, there remains only to show that

$$\forall\, x \in X: \qquad \text{g-ls}_{\times\omega}\, S^\nu(x) \subset \text{p-ls}_\omega\, S^\nu(x).$$

Let $\bar{y} \in \text{g-ls}_{\times\omega}\, S^\nu(\bar{x})$. i.e., there exist $H \in \mathcal{H}^\#$, $x^\nu \underset{H}{\rightharpoonup} \bar{x}$, $y^\nu \underset{H}{\rightharpoonup} \bar{y}$ such that $y^\nu \in S^\nu(x^\nu)$. Since the set $B = \{\bar{y}, y^\nu,\, \nu \in H\}$ is ω_Y-compact, M-equi-osc implies that for all $\varepsilon > 0$, there exists $H' \in \mathcal{H}$ such that $S^\nu(x^\nu) \cap B \subset \varepsilon S^\nu(\bar{x})$ for all $\nu \in H \cap H'$. This means that for all $\varepsilon > 0$, there exists $H_\varepsilon \in \mathcal{H}^\#$ such that $\bar{y} \in \varepsilon S^\nu(\bar{x})$ for all $\nu \in H_\varepsilon$, from which it follows that $\bar{y} \in \text{p-ls}_\omega S^\nu(\bar{x})$.

Next, let's show that (b) and (c) $\Longrightarrow$ (a) when Y is finite dimensional. Then

$$\begin{aligned} \text{p-ls}_\omega S^\nu(\bar{x}) &= \text{p-ls}\, S^\nu(\bar{x}), \\ \big(\text{g-ls}_{\times\omega} S^\nu\big)(\bar{x}) &= \bigcup_{\{x^\nu \rightharpoonup \bar{x}\}} \text{ls}\, S^\nu(x^\nu). \end{aligned}$$

Now suppose, in particular, that $S(\bar{x}) = \text{g-ls}_{\times\omega} S^\nu(\bar{x}) = \text{p-ls}\, S^\nu(\bar{x})$ but the collection $\{\, S^\nu: X \rightrightarrows Y,\ \nu \in N \,\}$ is not ω-equi-osc at $\bar{x}$, i.e., there exist a compact set B, $\varepsilon > 0$, $H \in \mathcal{H}^\#$, $x^\nu \underset{H}{\rightharpoonup} \bar{x}$, and $y^\nu \in \big(S^\nu(x^\nu) \cap B\big) \setminus \varepsilon S^\nu(\bar{x})$. Since B is compact, $\{y^\nu,\ \nu \in H\}$ must have a cluster point, say $\bar{y}$. Mosco-graph-convergence implies

$$\bar{y} \in \text{g-ls}_{\times\omega} S^\nu(\bar{x}) = S(\bar{x}).$$

On the other hand, $\bar{y} \notin \text{p-ls}\, S^\nu(\bar{x}) = S(\bar{x})$ since by assumption $S^\nu(\bar{x}) \to S(\bar{x})$; using the same argument as in the proof of theorem 3.3. Thus $\text{g-ls}_{\times\omega} S^\nu(\bar{x})$ and $\text{p-ls}\, S^\nu(\bar{x})$ couldn't be equal as assumed. □

Remark 7.6. When dealing with the profile mappings associated with a collection of functions, one has $Y = I\!R$. Following the same analysis as that in §6, one shows that a collection of functions $\{f^\nu: X \to \overline{I\!R},\ \nu \in N\}$ is ω-equi-lsc (at a point $\bar{x}$) if and only if the (upper) profile mappings defined by $\{U_{f^\nu},\ \nu \in N\}$ are ω-equi-osc (at $\bar{x}$). The notion of an ω-equi-lsc collection of functions was introduced in [**10**].

8. Pointwise convergence of subgradients mappings

This first example shows that the subgradient mappings of a convergent family of convex functions are equi-osc at every point where the limit function is differentiable. Somewhat weaker versions of theorem 8.3 below were already obtained by Rockafellar [**13**, theorem 24.5] and Birge and Qi [**7**] by (much) different techniques.

The following fact about the convergence of connected sets will be used; for a proof one could consult [**15**]. The Fréchet filter on $I\!N$ will be denoted by $\mathcal{N}_\infty$, and $\mathcal{N}(\bar{x})$ will designate the neighborhood system of $\bar{x}$.

Lemma 8.1. *Let $\{\, C^\nu \subset I\!R^n, \nu \in \mathcal{H} \,\}$ be connected, and assume that $\text{ls}_{\mathcal{H}} C^\nu$ is bounded and nonempty. Then, for all $\varepsilon > 0$ there exists an index set $H \in \mathcal{H}$ such that $C^\nu \subset \text{ls}_{\mathcal{H}} C^\nu + \varepsilon I\!B$ for all $\nu \in \mathcal{H}$, where $I\!B$ is the unit ball in $I\!R^n$.*

Proposition 8.2. *Let $\{\, S^\nu : I\!R^n \rightrightarrows I\!R^d,\ \nu \in I\!N \,\}$ be closed- and connected-valued. If $\text{g-ls}\, S^\nu(\bar{x})$ is bounded, then for all $\varepsilon > 0$ there exist $N \in \mathcal{N}_\infty$ and $V \in \mathcal{N}(\bar{x})$ such that for all $\nu \in N$, $x \in V$, $S^\nu(x) \subset \text{g-ls}\, S^\nu(\bar{x}) + \varepsilon I\!B$.*

Proof. The assertion follows from the preceding lemma and formula (2.3) for g-ls $S^\nu(\bar{x})$ with $\mathcal{H} = \mathcal{N}_\infty \otimes \mathcal{N}(\bar{x})$. ◻

Theorem 8.3. *Let* $\{ f, f^\nu : I\!R^n \to \overline{I\!R},\, \nu \in I\!N \}$ *be lsc, convex functions such that* $f^\nu \xrightarrow{e} f$, *and let* $\bar{x} \in \operatorname{int}\operatorname{dom} f$. *Then, for all* $\varepsilon > 0$, *there exist* $H \in \mathcal{N}_\infty$, $V \in \mathcal{N}(\bar{x})$ *such that*

$$\partial f^\nu(x) \subset \partial f(\bar{x}) + \varepsilon I\!B, \quad \forall\, x \in V, \nu \in N.$$

If f *is differentiable at* $\bar{x}$, *then actually*

$$\operatorname{lm} \partial f^\nu(\bar{x}) = \nabla f(\bar{x}),$$

and the subgradient mappings $\{ \partial f^\nu : I\!R^n \rightrightarrows I\!R,\, \nu \in I\!N \}$ *are equi-osc at* $\bar{x}$.

Proof. Since the function f^ν epi-converge to f, it follows from Attouch's Theorem [**1**, **5**, Theorem 7.6.4], that the subgradient mappings ∂f^ν graph-converge to ∂f. These mappings are closed-, convex-valued and if $\bar{x} \in \operatorname{int}\operatorname{dom} f$, $\partial f(\bar{x})$ is bounded, from which the first assertion follows via proposition 8.2.

If f is differentiable at $\bar{x}$, one then has that for all $\varepsilon > 0$ there exists N_ε such that for all $\nu \in N_\varepsilon$, $\partial f^\nu(\bar{x}) \subset I\!B(\nabla f(\bar{x}), \varepsilon)$. Since graph-convergence also implies that $\operatorname{ls} \partial f^\nu(\bar{x}) \subset \{\nabla f(\bar{x})\}$, it follows $\operatorname{lm} \partial f^\nu(\bar{x}) = \nabla f(\bar{x})$. Theorem 3.3, then guarantees the (eventual) equi-outer semicontinuity (at $\bar{x}$) of the subgradient mappings, since one has that p-lm $S^\nu(\bar{x}) =$ g-lm $S^\nu(\bar{x})$. ◻

9. Pointwise limits of maximal monotone operators

Let H be a Hilbert space. Recall that $A : H \rightrightarrows H$ is called a *monotone operator* if

$$\forall\, x, x' \in H, y \in A(x), y' \in A(x') : \quad \langle y - y', x - x' \rangle \geq 0,$$

where $\operatorname{dom} A = \{ x \in H \,|\, A(x) \neq \emptyset \}$. Furthermore, A is said to be *maximal monotone* if it is monotone and there is no other monotone operator whose graph includes gph A. The following results are well known:

Proposition 9.1 [**1**, **5**, proposition 7.1.7]**.** *For* $\{ A, A^\nu : H \rightrightarrows H,\, \nu \in I\!N \}$ *a countable collection of maximal monotone operators defined on a Hilbert space* H,

$$A \subset \text{g-li}\, A^\nu \quad \Longleftrightarrow \quad A^\nu \xrightarrow{g} A.$$

Proposition 9.2 [**3**, proposition 3.1]**.** *Let* $\{ A^\nu : I\!R^n \rightrightarrows I\!R^n,\, \nu \in I\!N \}$ *be a sequence of maximal monotone operators defined on* $I\!R^n$. *Then,*

$$A_n \xrightarrow{g} A \quad \Longrightarrow \quad A \text{ maximal monotone}.$$

However, the pointwise limit of a sequence of maximal monotone operators is not, in general, maximal monotone. For an example of a sequence of maximal monotone operators with a graphical limit different from its pointwise limit, cf. [**2**, proposition 3.56]. The following proposition will provide the exact condition under which the pointwise limit is maximal monotone:

Proposition 9.3. *Let* $\{ A^\nu : I\!R^n \rightrightarrows I\!R^n, \nu \in I\!N \}$ *be a sequence of maximal monotone operators defined on* $I\!R^n$*, and* $A := \text{p-lm}\, A^\nu$*. Then,*

$$A \text{ maximal monotone} \quad \Longleftrightarrow \quad \text{the collection } \{ A^\nu, \nu \in I\!N \} \text{ is equi-osc.}$$

Proof. Under equi-outer semicontinuity, pointwise convergence implies graph-convergence (theorem 3.3), and thus by proposition 9.1, A is maximal monotone. On the other hand, if $A = \text{p-lm}\, A^\nu$ is maximal monotone, since $\text{p-lm}\, A^\nu \subset \text{g-li}\, A^\nu$ (proposition 2.2), from proposition 9.2, it follows that $A^\nu \xrightarrow{g} A$. Equi-outer semicontinuity then follows from theorem 3.3 since the A^ν converge to A both pointwise and graphically. □

10. Differential inclusions

Let's consider the following *differential inclusion*:

$$\begin{aligned} \dot{x}(t) &\in_{a.e.} A(x(t)), \quad t \in [0, T], \\ x(t) &\in_{a.e.} K, \qquad\quad\; t \in [0, T], \\ x &\in W^{1,1}([0, T]; I\!R^n), \end{aligned}$$

where $A : I\!R^n \rightrightarrows I\!R^n$ is closed-valued, $K \subset I\!R^n$ is closed, and $W^{1,1}([0, T]; I\!R^n)$ is the Sobolev space of $L^1([0, T]; I\!R^n)$-functions with (distributional) derivatives also in L^1. The formulation includes as special cases systems of ordinary linear or nonlinear differential equations, as well as many control problems. Consider, for example, the following closed loop control problem:

$$\begin{aligned} \dot{x}(t) &=_{a.e.} f(t, x(t), u(t)), \quad t \in [0, T], \\ u(t) &\in U(t, x(t)), \qquad\quad t \in [0, T], \\ x(t) &\in_{a.e.} K, \qquad t \in [0, T], \end{aligned}$$

where u, the control function, must be measurable, and $U : [0, T] \times I\!R^n \rightrightarrows I\!R^m$ is a closed-valued mapping. For the corresponding differential inclusion, simply set $A(t, x(t)) := f(t, x(t), U(t, x(t)))$. For more about differential inclusion, consult [**4**] and the reference therein.

We shall begin with a result based on the existence of a $W^{1,1}$-cluster point to a sequence of solutions of the approximating problems $(\mathcal{DI}_\nu)$. This condition will be relaxed later, but this will require imposing some additional condition of the mappings A^ν.

We shall begin with a result based on the existence of a $W^{1,1}$-cluster point to a sequence of solutions of the approximating problems $(\mathcal{DI}_\nu)$. This condition will be relaxed later, but will in some sense be replaced by extra conditions on the mappings A^ν.

Proposition 10.1. *Consider the differential inclusions: for* $\nu \in I\!N$*,*

$$\begin{aligned} \dot{x}(t) &\in_{a.e.} A^\nu(x(t)), \quad t \in [0, T], \\ x(t) &\in_{a.e.} K^\nu, \qquad t \in [0, T], \\ x &\in W^{1,1}([0, T]; I\!R^n), \end{aligned} \tag{$\mathcal{DI}_\nu$}$$

and

$$\begin{aligned} &\dot{x}(t) \in_{a.e.} A(x(t)), \quad t \in [0,T], \\ &x(t) \in_{a.e.} K, \qquad t \in [0,T], \qquad\qquad (\mathcal{DI}) \\ &\quad x \in W^{1,1}([0,T]; I\!R^n), \end{aligned}$$

where $\{ K^\nu \subset I\!R^n,\ \nu \in I\!N \}$ *are closed sets,* $K = \operatorname{lm} K^\nu$, $\{ A^\nu : I\!R^n \rightrightarrows I\!R^n, \nu \in I\!N \}$ *are closed-valued mappings and* $A = \text{p-lm}\, A^\nu$.

If the mappings A^ν *are equi-osc, and* $\{x^\nu,\ \nu \in I\!N\}$ *is a sequence of solutions to* $(\mathcal{DI}_\nu)$ *and* x^0 *a* $W^{1,1}$*-cluster point of this sequence, then* x^0 *is a solution of* $(\mathcal{DI})$.

Proof. Let $\{x^{\nu_k}, k \in I\!N\}$ be the subsequence $W^{1,1}$-converging to x^0, i.e., there is a (sub)sequence $\{x^{\nu_k}, k \in I\!N\}$ such that $x^{\nu_k} \xrightarrow{p} x^0$ and $\dot{x}^{\nu_k} \xrightarrow{p} \dot{x}^0$. $A = \text{p-lm}\, A^\nu$ and equi-outer semicontinuity of the A^ν imply, via theorem 3.3, that $A = \text{g-lm}\, A^\nu$, for any fixed t, one has

$$\dot{x}^{\nu_k}(t) \in A^{\nu_k}(x^{\nu_k}(t)),\ k = 1, \ldots, \quad \Longrightarrow \quad \dot{x}^0(t) \in A(x^0(t)),$$

and since $K = \operatorname{lm} K^\nu$, $x^{\nu_k}(t) \in K^{\nu_k}$ implies $x^0(t) \in K$. □

A mapping A is *sublinear* (also called a “convex process” [**13, 5**]) if $\operatorname{gph} A$ is a closed convex cone. Differential inclusions with A a sublinear mapping include as special cases that of systems of linear ordinary differential equations as well as various applications involving nonlinear dynamics, cf. [**5**, chapter 2]

10.2 Corollary. *For* $\nu \in I\!N$, *let* $A^\nu : I\!R^n \rightrightarrows I\!R^m$ *be sublinear closed-valued mappings such that for all* x, $\sup_\nu d(0, A^\nu(x)) < \infty$. *Then the* A^ν *are equi-continuous.*

Proof. It follows from the Uniform Boundedness Principle [**5**, theorem 2.3.1] for sublinear mappings that there exist $\rho > 0$ and $\{y^\nu_x \in A^\nu(x), \nu \in I\!N\}$ such that $|y^\nu_x| \le \rho|x|$. Let x^1, x^2 be any point in $I\!R^n$, and $y^\nu \in A^\nu(x^2 - x^1)$ such that $|y^\nu| \le \rho|x^1 - x^2|$. For $\nu \in I\!N$, let $y^\nu_1 \in A^\nu(x^1)$, then

$$y^\nu_2 = y^\nu + y^\nu_1 \in A^\nu(x^2) \supset A^\nu(x^2 - x^1) + A^\nu(x^1), \quad |y^\nu_2 - y^\nu_1| \le \rho'|x^1 - x^2|.$$

Since the choice of the y^ν_1 was arbitrary, one has $A^\nu(x^1) \subset A^\nu(x^2) + \rho|x^1 - x^2|$, which certainly implies equi-continuity since this inclusion holds for all $\nu \in I\!N$. □

Note that a bit more has actually been proved, namely, that these sublinear mappings are equi-lipschitzian continuous with respect to the Pompeiu-Hausdorff distance. Finally, combining the two last propositions, yields the following:

Corollary 10.3. *For* $\nu \in I\!N$, *let* x^ν *be a solution to the differential inclusion:*

$$\begin{aligned} &\dot{x}(t) \in_{a.e.} A^\nu(x(t)), \quad t \in [0,T], \\ &x(t) \in_{a.e.} K^\nu, \qquad t \in [0,T], \\ &\quad x \in W^{1,1}([0,T]; I\!R^n). \end{aligned}$$

If for the closed-valued sublinear mappings $A^\nu : I\!R^n \rightrightarrows I\!R^m$, $\sup_\nu d(0, A^\nu(x)) < \infty$ *for all* x, $A = \text{p-lm}\, A^\nu$, *and* $K = \text{lm}\, K^\nu$, *then any* $W^{1,1}$*-cluster point of the* $\{x^\nu,\, \nu \in I\!N\}$ *is a solution of the differential inclusion:*

$$\begin{aligned} \dot{x}(t) &\in_{a.e.} A(x(t)), \quad t \in [0,T], \\ x(t) &\in_{a.e.} K, \qquad\quad t \in [0,T], \\ x &\in W^{1,1}([0,T]; I\!R^n). \end{aligned}$$

Rather than assuming that the solutions of the approximating problems $(\mathcal{DI}_\nu)$ have a $W^{1,1}$ cluster point, let's now relax this rather strong requirement to only inisting on weak convergence of the derivatives. To do this we rely on a convergence theorem for differential inclusions.

Theorem 10.4 [**4, 5**, theorem 7.2.1]**.** *Let* $\{A^\nu : I\!R^n \rightrightarrows I\!R^n, \nu \in I\!N\}$ *be locally bounded. Suppose* $\dot{x}^\nu(t) \in_{a.e.} A^\nu(x^\nu(t))$ *for all* ν, $x^\nu \xrightarrow{p} x^0$ *almost everywhere,* $\dot{x}^\nu \rightharpoonup z$ *(weakly in* L^1*) with* $z \in L^1$*. Then* $z(t) \in_{a.e.} \text{cl}\,\text{con}\, A(x(t))$ *when* $A^\nu \xrightarrow{g} A$*.*

Proposition 10.5. *Again consider the differential inclusions* $(\mathcal{DI}_\nu)$ *and* $(\mathcal{DI})$ *where* $\{\, K^\nu \subset I\!R^n,\, \nu \in I\!N \,\}$ *are closed sets,* $K = \text{lm}\, K^\nu$, $\{\, A^\nu : I\!R^n \rightrightarrows I\!R^n, \nu \in I\!N \,\}$ *are locally bounded, closed-, convex-valued mappings and* $A = \text{p-lm}\, A^\nu$*.*

If the mappings A^ν *are equi-osc, and* $\{x^\nu,\, \nu \in I\!N\}$ *is a sequence of solutions to* $(\mathcal{DI}_\nu)$ *such that for some subsequence with* $\nu \in N$, $x^\nu \underset{N}{\to} x^0$ *in* L^1 *and* $\dot{x}^\nu \underset{N}{\to} \dot{x}^0$ *weakly in* L^1*, then* x^0 *is a solution of* $(\mathcal{DI})$*.*

Proof. Passing to a subsequence, if necessary, one has that with $\nu \in N$, $x^\nu \xrightarrow{p} x^0$ for almost all $t \in [0,T]$. Since equi-outer semicontinuity and pointwise convergence imply graph-convergence (theorem 3.3), from the convergence theorem 10.4 for differential inclusions follows that $\dot{x}(t) \in_{a.e.} A(x^0(t))$; recall that by assumption the mapping A is convex-valued. Hence x^0 is a solution of $(\mathcal{DI})$. □

A particular case to which this proposition is applicable is when the mappings A^ν are maximal monotone. It then follows from proposition 9.3, that A is also maximal monotone, and then these mappings are closed-, convex-valued.

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