VARIATIONAL CONVERGENCE OF BIVARIATE FUNCTIONS: LOPSIDED CONVERGENCE *

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Abstract. We explore convergence notions for bivariate functions that yield convergence and stability results for their maxinf (or minsup) points. This lays the foundations for the study of the stability of solutions to variational inequalities, the solutions of inclusions, of Nash equilibrium points of non-cooperative games and Walras economic equilibrium points, of fixed points, the primal and dual solutions of convex optimization problems and of zero-sum games. These applications will be dealt with in a couple of accompanying papers.

Keywords: lopsided convergence, maxinf-points, minsup-points,
Ky Fan Functions, variational inequalities, epi-convergence

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1 Variational convergence of bifunctions

A fundamental component of Variational Analysis is the analysis of the properties of bifunctions, or equivalently *bifunctions*. For example: the analysis of the Lagrangians associated with an optimization problem, of the Hamiltonians associated with Calculus of Variations and Optimal Control problems, the reward functions associated with cooperative or non-cooperative games, and so on. In a series of articles, we deal with the stability of the solutions of a wide collection of problems that can be re-cast as finding the maxinf-points of certain bifunctions.

So, more explicitly: given a bifunction $F : C \times D \to I\!R$ we are interested in finding a point, say $\bar{x} \in C$, that maximizes with respect to the first variable x, the infimum of F, $\inf_{y \in D} F(\cdot, y)$, with respect to the second variable y. One refers to such a point $\bar{x}$ as a *maxinf-point* and one writes

$$\bar{x} \in \operatorname{argmaxinf}_{C \times D} F \quad \text{or simply} \quad \bar{x} \in \operatorname{argmaxinf} F.$$

In some particular situations, for example when the bifunction is concave-convex, such a point can be a saddle point, but in many other situation it's just a maxinf-point, or a minsup-point when minimizing with respect to the first variable the supremum of F with respect to the second variable. To study the stability, and the existence, of such points, and the sensitivity of their associated values, one is lead to introduce and analyze convergence notion(s) for bifunctions that in turn will guarantee the convergence either of their saddle points or of just their maxinf-points.

This paper is devoted to the foundations. Two accompanying papers deal with the motivating examples [11, 10]: variational inequalities, fixed points, Nash equilibrium points of non-cooperative games, equilibrium points of zero-sum games, etc. We make a distinction between the situations when the bifunction is generated from a single-valued mapping [11] or when the mapping can also be set-valued [10].

The major tool is the notion of *lopsided convergence*, that was introduced in [2], but is modified here so that a wider class of applications can be handled. The major adjustment is that bifunctions aren't as in [2] no longer defined on all of $I\!R^n \times I\!R^m$ with values in the extended reals, but are now only finite-valued on a specific product $C \times D$ with C, D subsets of $I\!R^n$ and $I\!R^m$. Dealing with 'general' bifunctions defined on the full product space was in keeping with the elegant work of Rockafellar [13] on duality relations for convex-concave bifunctions and the subsequent work [3] on the epi/hypo-convergence of saddle functions. However, our present analysis actually shows that notwithstanding its aesthetic allurement one should not cast bifunctions, even in the convex-concave case, in the general extended-real valued framework. In some way, this is in contradiction with the univariate case where the extension, by allowing for the values $\pm\infty$, of functions defined on a (constrained) set to all of $I\!R^n$ has been so effectively exploited to derive a "unified" convergence and differentiation theory [5, 14]. We shall show that some of this can be recovered, but one must first make a clear distinction between max-inf problems and min-sup ones, and only then one can generate the appropriate extensions; after all, also in the univariate case one makes a clear distinction when extending a function in a minimization setting or a maximization setting.

In order to be consistent in our presentation, and to set up the results required later on, we begin by a presentation of the theory of epi-convergence for real-valued univariate functions that are *only* defined on a subset of $\mathbb{R}^n$. No new results are actually derived although a revised formulation is required. We make the connection with the standard approach, i.e., when these (univariate) functions are extended real-valued. We then turn to lopsided convergence and point out the shortcomings of an extended real-valued approach. Finally, we exploit our convergence result to obtain a extension of Ky Fan inequality [7] to situations when the domain of definition of the bifunction is not necessarily compact.

2 Epi-convergence

One can always represent an optimization problem, involving constraints or not, as one of minimizing an extended real-valued function. In the case of a constrained-minimization problem, simply redefine the objective as taking on the value ∞ outside the feasible region, the set determined by the constraints. In this framework, the canonical problem can be formulated as one of minimizing on all of $\mathbb{R}^n$ an extended real-valued function $f\colon \mathbb{R}^n \to \overline{\mathbb{R}}$. Approximation issues can consequently be studied in terms of the convergence of such functions. This has lead to the notion of *epi-convergence*[2] that plays a key role in "Variational Analysis" [1, 5, 14]; when dealing with a maximization problem, it is hypo-convergence, the convergence of the hypographs, that is the appropriate convergence notion.

Henceforth, we restrict our development to the minimization setting but, at the end of this section, we translate results and observations to the maximization case.

As already indicated, in Variational Analysis, one usually deals with

$$\mathrm{fcn}(\mathbb{R}^n) = \big\{ f\colon \mathbb{R}^n \to \overline{\mathbb{R}} \big\}$$

the space of extended real-valued functions that are defined on *all* of $\mathbb{R}^n$, even allowing for the possibility that they are nowhere finite-valued. Definitions, properties, limits, etc., generally do not refer to the domain on which they are finite. For reasons that will become clearer when we deal with the convergence of bifunctions, we need to depart from this simple, and very convenient, paradigm. Our focus will be on

$$fv\text{-}\mathrm{fcn}(\mathbb{R}^n) = \big\{ f\colon C \to \mathbb{R} \,\big|\, \text{ for some } \emptyset \neq C \subset \mathbb{R}^n \big\},$$

the class of all *finite-valued functions with non-empty domain* $C \subset \mathbb{R}^n$. It must be understood that in this notation, $\mathbb{R}^n$ doesn't refer to the domain of definition, but to the underlying space that contains the domains on which the functions are defined.

The *epigraph* of a function f is *always* the set of all points in $\mathbb{R}^{n+1}$ that lie on or above the graph of f, irrespective of f belonging to fv-fcn$(\mathbb{R}^n)$ or fcn$(\mathbb{R}^n)$. If $f\colon C \to \mathbb{R}$ belongs to fv-fcn$(\mathbb{R}^n)$, then

$$\mathrm{epi}\, f = \big\{ (x,\alpha) \in C \times \mathbb{R} \,\big|\, \alpha \geq f(x) \big\} \subset \mathbb{R}^{n+1},$$

[2]for extensive references and a survey of the field one can consult [1, 5], and, in particular, the Commentary section that concludes [14, Chapter 7]

and if f belongs to $\mathrm{fcn}(I\!\!R^n)$ then

$$\mathrm{epi}\, f = \big\{(x,\alpha) \in I\!\!R^{n+1} \,\big|\, \alpha \geq f(x)\big\}.$$

A function f is *lsc* (= *lower semicontinuous*) if its epigraph is closed as a subset of $I\!\!R^{n+1}$, i.e., $\mathrm{epi}\, f = \mathrm{cl}(\mathrm{epi}\, f)$ with cl denoting closure [14, Theorem 1.6][3].
So, when $f \in fv\text{-fcn}(I\!\!R^n)$, lsc implies[4] that for all $x^\nu \in C \to x$:

- if $x \in C$: $\liminf_\nu f(x^\nu) \geq f(x)$, and
- if $x \in \mathrm{cl}\, C \setminus C$: $f(x^\nu) \to \infty$.

In our minimization framework: $\mathrm{cl}\, f$ denotes the function whose epigraph is the closure relative to $I\!\!R^{n+1}$ of the epigraph of f, i.e., the lsc-regularization of f. It's possible that when $f \in fv\text{-fcn}$, $\mathrm{cl}\, f$ might be defined on a set that's strictly larger than C but always contained in $\mathrm{cl}\, C$.

Let's turn to convergence: *set-convergence*, in the Painlev´ e-Kuratowski sense [14, §4.B], is defined as follows: $C^\nu \to C \subset I\!\!R^n$ if

- (a_{set}) all cluster points of a sequence $\big\{x^\nu \in C^\nu\big\}_{\nu \in N}$ belong to C,
- (b_{set}) for each $x \in C$, one can find a sequence $x^\nu \in C^\nu \to x$.

When just condition (a_{set}) holds, then C is then the *outer limit* of the sequence $\big\{C^\nu\big\}_{\nu \in N}$, and when it's just ($\mathrm{b}_{set}$) that holds, then C is the *inner limit* [14, Chapter 4, §2]. Note, that whenever C is the limit, the outer- or the inner-limit, it's *closed* [14, Proposition 4.4] and that $C = \emptyset$ if and only if the sequence C^ν eventually escapes from any bounded set [14, Corollary 4.11]. Moreover, if the sequence $\big\{C^\nu\big\}_{\nu \in N}$ consists of convex sets, its inner limit, and its limit if it exists, are also convex [14, Proposition 4.15].

2.1 Definition (epi-convergence). *A sequence of functions* $\big\{f^\nu,\, \nu \in I\!\!N\big\}$, *whose domains lie in* $I\!\!R^n$, epi-converges *to a function* f *when* $\mathrm{epi}\, f^\nu \to \mathrm{epi}\, f$ *as subsets of* $I\!\!R^{n+1}$*; again irrespective of* f *belonging to* $fv\text{-fcn}(I\!\!R^n)$ *or* $\mathrm{fcn}(I\!\!R^n)$. *One then writes* $f^\nu \xrightarrow{e} f$.

Figure 1 provides an example of two functions f and f^ν that are close to each other in terms of the distance between their epigraphs —i.e., the distance between the location of the two jumps— but are pretty far from each other pointwise or with respect to the ℓ^∞-norm — i.e., the size of the jumps.

Let $\{f^\nu\}_{\nu \in N}$ be a sequence of functions with domains in $I\!\!R^n$. When,

- f is the *lower epi-limit* of the functions f^ν when $\mathrm{epi}\, f$ is the outer limit of $\{\mathrm{epi}\, f^\nu\}_{\nu \in N}$,
- f is the *upper epi-limit* of the functions f^ν when $\mathrm{epi}\, f$ is the inner limit of $\mathrm{epi}\, f^\nu$.

Of course, f is the epi-limit of the sequence if it's both the lower and upper epi-limit.

2.2 Proposition (properties of epi-limits) *Let* $\{f^\nu\}_{\nu \in N}$ *be a sequence of functions with domains in* $I\!\!R^n$. *Then, the lower and upper epi-limits and the epi-limit, if it exists, are all lsc. Moreover, if the functions* f^ν *are convex, so is the upper epi-limit, and the epi-limit, if it exists. In particular, this implies that the* family of lsc functions is closed under epi-convergence.

[3]throughout it's implicitly assumed that $I\!\!R^n$ is equipped with it's usual Euclidean topology

[4]Indeed, if $\liminf_\nu f(x^\nu) < \infty$, then for some subsequence $\{\nu_k\}$, $f(x^{\nu_k}) \to \alpha \in I\!\!R$ and because $\mathrm{epi}\, f$ is closed, it implies that $(x,\alpha) \in \mathrm{epi}\, f$ which would place x in the domain of f, contradicting $x \notin C$.

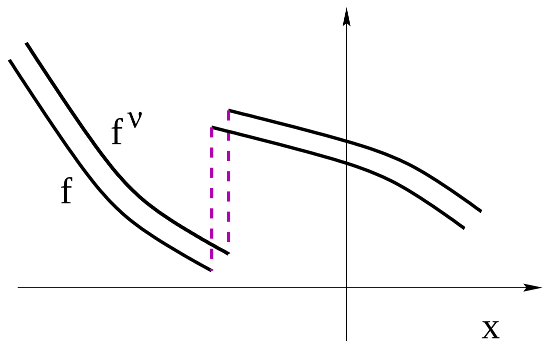

Figure 1: f and f^ν epigraphically close to each other

Proof. Follows immediately from the properties of set-limits. □

The definition of epi-convergence for families of functions in $\text{fcn}(I\!R^n)$ is the usual one [14, Chapter 7, §B] with all the implications concerning the convergence of the minimizers and infimal values [14, Chapter 7, §E]. But, in a certain sense, the definition is "new" when the focus is on epi-convergent families in $fv\text{-fcn}(I\!R^n)$, and it's for this class of functions that we need to know the conditions under which one can claim convergence of the minimizers and infimums. *We chose to make the presentation self-contained*, although as will be shown later, one could also embed $fv\text{-fcn}(I\!R^n)$ in a subclass of $\text{fcn}(I\!R^n)$ and then appeal to the standard results which unfortunately requires that one plows a substantial amount of material.

When f is *an epi-limit it's necessarily a lsc function* since its epigraph is the set-limit of a collection of sets in $I\!R^{n+1}$. It's epigraph is closed but its domain C is not necessarily closed. Simply think of the collection of functions $f^\nu = f$ for all ν with $C = (0, \infty)$ and $f(x) = 1/x$ on C. This collection clearly epi-converges to the lsc function f on C with closed epigraph but not with closed domain.

2.3 Lemma (epi-limit value at boundary points). *Suppose $f : C \to I\!R$ is the epi-limit of a sequence $\{f^\nu : C^\nu \to I\!R\}_{\nu \in N}$ with all functions in $fv\text{-fcn}(I\!R^n)$. Then, for any sequence $x^\nu \in C^\nu \to x$: $\liminf_\nu f^\nu(x^\nu) > -\infty$.*

Proof. We proceed by contradiction. Suppose that $x^\nu \in C^\nu \to x$ and $\liminf_\nu f^\nu(x^\nu) = -\infty$. By assumption $f > -\infty$ on C, thus the x^ν can't converge to a point in C, i.e., necessarily $x \notin C$. If that's the case and since $\text{epi}\, f^\nu \to \text{epi}\, f$, the line $\{x\} \times I\!R$ would have to lie in $\text{epi}\, f$ contradicting the assumption that f, the epi-limit of the f^ν, belongs to $fv\text{-fcn}(I\!R^n)$. □

2.4 Example (an epi-limit that's not in $fv\text{-fcn}(I\!R^n)$). *Consider the sequence of functions $\{f^\nu : [0, \infty) \to I\!R\}_{\nu \in N}$ with*

$$f^\nu(x) = \begin{cases} -\nu^2 x & \text{if } 0 \le x \le \nu^{-1}, \\ \nu^2 x - 2\nu & \text{if } \nu^{-1} \le x \le 2\nu^{-1}, \\ 0 & \text{for } x \ge 2\nu^{-1}. \end{cases}$$

Detail. The functions $f^\nu \in fv\text{-fcn}(I\!R)$ and for the sequence $x^\nu = \nu^{-1}$, $f^\nu(x^\nu) \to -\infty$ and $f^\nu \xrightarrow{e} f$ where $f\colon [0,\infty) \to I\!R$ with $f \equiv 0$ on $(0,\infty)$ and $f(0) = -\infty$. Thus, the functions f^ν epi-converge to f as functions in $\text{fcn}(I\!R)$, provided they are appropriately extended, i.e., taking on the value ∞ on $(-\infty, 0)$. But they don't epi-converge to a function in $fv\text{-fcn}(I\!R)$. ◻

In addition to our geometric definition, the next proposition provides an analytic characterization of epi-converging sequences in $fv\text{-fcn}(I\!R^n)$.

2.5 Proposition (epi-convergence in $fv\text{-fcn}(I\!R^n)$). *Let* $\{f : C \to I\!R, f^\nu : C^\nu \to I\!R, \nu \in I\!N\}$ *be a collection of functions in* $fv\text{-fcn}(I\!R^n)$*. Then,* $f^\nu \xrightarrow{e} f$ *if and only the following conditions are satisfied:*

(a) *for* $x^\nu \in C^\nu \to x$, $\liminf_\nu f^\nu(x^\nu) \geq f(x)$ *when* $x \in C$ *and* $f^\nu(x^\nu) \nearrow \infty$ *when* $x \notin C$, [5]

(b) $\forall x \in C$, $\exists\, x^\nu \in C^\nu \to x$ *such that* $\limsup_\nu f^\nu(x^\nu) \leq f(x)$.

Proof. If $\text{epi}\, f^\nu \to \text{epi}\, f$ and $x^\nu \in C^\nu \to x$ either $\liminf_\nu f^\nu(x^\nu) = \alpha < \infty$ or not; Lemma 2.3 reminds us that $\alpha = -\infty$ is not a possibility. In the first instance, (x, α) is a cluster point of $\{(x^\nu, f^\nu(x^\nu)) \in \text{epi}\, f^\nu\}_{\nu \in N}$ and thus belongs to $\text{epi}\, f$, i.e., $f(x) \leq \alpha$ and hence the first assertion in (a) holds; $\alpha > -\infty$ since otherwise f would not be finite valued on C. If $\alpha = \infty$ that means that $f^\nu(x^\nu) \nearrow \infty$ and x can't belong to C, and thus the second assertion in (a) holds. On the other hand, if $x \in C$ and thus $f(x)$ is finite, there is a $\{(x^\nu, \alpha^\nu) \in \text{epi}\, f^\nu\}_{\nu \in N}$ such that $x^\nu \in C^\nu \to x \in C$ and $\alpha^\nu \to f(x)$ with $\alpha^\nu \geq f^\nu(x^\nu)$, i.e., $\limsup_\nu f^\nu(x^\nu) \leq f(x)$, i.e., (b) is also satisfied.

Conversely, if (a) holds and $(x^\nu, \alpha^\nu) \in \text{epi}\, f^\nu \to (x, \alpha)$ then either $x \in C$ or not; recall also, that in view of Lemma 2.3, α can't be $-\infty$ since we are dealing with epi-convergence in $fv\text{-fcn}(I\!R^n)$. In the latter instance, by $f^\nu(x^\nu) \to \alpha = \infty$, so we aren't dealing with a converging sequence of points (in R^{n+1}) and there is no need to consider this situation any further. When $x \in C$, since then $\liminf_\nu f^\nu(x^\nu) \geq f(x)$ and $\alpha^\nu \geq f^\nu(x^\nu)$, one has $\alpha \geq f(x)$ and consequently (x, α) belongs to $\text{epi}\, f$; this means that condition (a_{set}) is satisfied. If $(x, \alpha) \in \text{epi}\, f$, from (b) follows the existence of a sequence $x^\nu \in C^\nu \to x$ such that $\limsup_\nu f^\nu(x^\nu) \leq f(x) \leq \alpha$. We can then choose the $\alpha^\nu \geq f^\nu(x^\nu)$ so that $\alpha^\nu \to \alpha$ that yields (b_{set}). ◻

2.6 Theorem (epi-convergence: basic properties). *Consider a sequence* $\{f^\nu : C^\nu \to I\!R, \nu \in I\!N\} \subset fv\text{-fcn}(I\!R^n)$ *epi-converging to* $f\colon C \to I\!R$*, also in* $fv\text{-fcn}(I\!R^n)$*. Then*

$$\limsup_{\nu \to \infty} (\inf f^\nu) \;\leq\; \inf f.$$

Moreover, if $x^k \in \text{argmin}_{C^{\nu_k}} f^{\nu_k}$ *for some subsequence* $\{\nu_k\}$ *and* $x^k \to \bar{x}$*, then* $\bar{x} \in \text{argmin}_C f$ *and* $\min_{C^{\nu_k}} f^{\nu_k} \to \min_C f$*.*

If $\text{argmin}_C f$ *is a singleton, every convergent subsequence of minimizers converges to* $\text{argmin}_C f$*.*

Proof. Let $\{x^l\}_{l=1}^\infty$ be a sequence in C such that $f(x^l) \to \inf f$. By 2.5(b), for each l one can find a sequence $x^{\nu,l} \in C^\nu \to x^l$ such that $\limsup_\nu f^\nu(x^{\nu,l}) \leq f(x^l)$. Since for all ν, $\inf f^\nu \leq f^\nu(x^{\nu,l})$, it

[5] $\nearrow$ means converging and non-decreasing but not necessarily monotonically

follows that for all l,

$$\limsup_\nu(\inf f^\nu) \le \limsup_\nu f^\nu(x^{\nu,l}) \le f(x^l),$$

and one has, $\limsup_\nu(\inf f^\nu) \le \inf f$ since $f(x^l) \to \inf f$.

For the sequence $x^k \in C^{\nu_k} \to \bar{x}$, from the above and 2.5(a),

$$\inf f \ge \limsup_k f^{\nu_k}(x^k) \ge \liminf_k f^{\nu_k}(x^k) \ge f(\bar{x}),$$

i.e., $\bar{x}$ minimizes f on C and $f^{\nu_k}(x^k) = \min_{C^{\nu_k}} f^{\nu_k} \to \min_C f$.

Finally, since every convergent subsequence of minimizers of the functions f^ν converges to a minimizer of f, it must converge to the unique minimizer when $\mathrm{argmin}_C f$ is a singleton. □

In most of the applications, we shall rely on a somewhat more restrictive notion than plain epi-convergence to guarantee the convergence of the infimums.

2.7 Definition (tight epi-convergence). *The sequence* $\{f^\nu : C^\nu \to I\!R\}_{\nu\in I\!N} \subset fv\text{-fcn}(I\!R^n)$ epi-converges tightly *to* $f : C \to I\!R \in fv\text{-fcn}(I\!R^n)$, *if* $f^\nu \xrightarrow{e} f$ *and for all* $\varepsilon > 0$, *there exist a compact set* B_ε *and an index* ν_ε *such that*

$$\forall\, \nu \ge \nu_\varepsilon : \quad \inf_{B_\varepsilon \cap C^\nu} f^\nu \le \inf_{C^\nu} f^\nu + \varepsilon.$$

2.8 Theorem (convergence of the infimums). *Let* $\{f^\nu : C^\nu \to I\!R\}_{\nu\in I\!N} \subset fv\text{-fcn}(I\!R^n)$ *be a sequence of functions that epi-converges to the function* $f : C \to I\!R$ *also in* $fv\text{-fcn}(I\!R^n)$, *with* $\inf_C f$ *finite. Then, they epi-converge tightly*

(a) *if and only if* $\inf_{C^\nu} f^\nu \to \inf_C f$.

(b) *if and only if there exists a sequence* $\varepsilon^\nu \searrow 0$ *such that* $\varepsilon^\nu\text{-}\mathrm{argmin}\, f^\nu \to \mathrm{argmin}\, f$.

Proof. Let's start with necessity in (a). For given $\varepsilon > 0$, the assumptions and Theorem 2.6 imply

$$\liminf_\nu(\inf_{C^\nu \cap B_\varepsilon} f^\nu) \le \liminf_\nu(\inf_{C^\nu} f^\nu) + \varepsilon \le \limsup_\nu(\inf_{C^\nu} f^\nu) + \varepsilon \le \inf_C f + \varepsilon < \infty.$$

If there is a subsequence $\{\nu_k\}$ such that $f(x^k) < \kappa$ for some $x^k \in C^{\nu_k} \cap B_\varepsilon$, it would follow that $\inf_C f < \kappa$. Indeed, since B_ε is compact, the sequence $\{x^k\}$ has a cluster point, say $\bar{x}$, and then condition (a) of Proposition 2.5 guarantee $f(\bar{x}) < \kappa$ with $\bar{x} \in C$, and consequently, also $\inf_C f < \kappa$. Since it's assumed that $\inf_C f$ is finite, it follows that there is no such sequences with κ arbitrarily negative. In other words, excluding possibly a finite number of indexes, $\inf_{C^\nu \cap B_\varepsilon} f^\nu$ stays bounded away from $-\infty$ and one can find $x^\nu \in \varepsilon\text{-}\,\mathrm{argmin}_{C^\nu \cap B_\varepsilon} f^\nu$. The sequence $\{x^\nu\}_{\nu\in I\!N}$ admits a cluster point, say $\bar{x}$, that lies in B_ε and again by 2.5(a), $f(\bar{x}) \le \mathrm{liminf}_\nu f^\nu(x^\nu)$. Hence,

$$\inf_C f - \varepsilon \le f(\bar{x}) - \varepsilon \le \mathrm{liminf}_\nu f^\nu(x^\nu) - \varepsilon \le \mathrm{liminf}_\nu(\inf_{C^\nu} f^\nu).$$

In combination with our first string of inequalities and the fact that $\varepsilon > 0$ can be chosen arbitrarily small, it follows that indeed $\inf_{C^\nu} f^\nu \to \inf_C f$.

Next, we turn to sufficiency in (a). Since $\inf f^\nu \to \inf f \in I\!R$ by assumption, it's enough, given any $\delta > 0$, to exhibit a compact set B such that $\limsup_\nu \big(\inf_{B\cap C^\nu} f^\nu\big) \le \inf_C f + \delta$. Choose any point x such that $f(x) \le \inf_C f + \delta$. Because $f^\nu \xrightarrow{e} f$ in fv-fcn($I\!R^n$), there exists a sequence, 2.5(a), $x^\nu \to x$ such that $\limsup_\nu f^\nu(x^\nu) \le f(x)$. Let B be any compact set large enough to contain all the points x^ν. Then $\inf_B f^\nu \le f^\nu(x^\nu)$ for all ν, so B has the desired property.

We derive (b) from (a). Let $\bar\alpha^\nu = \inf f^\nu \to \inf f = \bar\alpha$ that is finite by assumption, and consequently for ν large enough, also $\bar\alpha^\nu$ is finite. Since convergence of the epigraphs implies the convergence of the level sets [14, Proposition 7.7], one can find a sequence of $\alpha^\nu \searrow \bar\alpha$ such that $\mathrm{lev}_{\alpha^\nu} f^\nu \to \mathrm{lev}_{\bar\alpha} f = \mathrm{argmin}\, f$. Simply set $\varepsilon^\nu := \alpha^\nu - \bar\alpha^\nu$.

For the converse, suppose there's a sequence $\varepsilon^\nu \searrow 0$ with ε^ν-argmin $f^\nu \to \mathrm{argmin}\, f \ne \emptyset$. For any $x \in \mathrm{argmin}\, f$ one can select $x^\nu \in \varepsilon^\nu$-argmin f^ν with $x^\nu \to x$. Then because $f^\nu \xrightarrow{e} f$, one obtains

$$\begin{aligned}\inf f = f(x) &\le \liminf_\nu f^\nu(x^\nu) \le \liminf_\nu(\inf f^\nu + \varepsilon^\nu)\\ &\le \liminf_\nu(\inf f^\nu) \le \limsup_\nu(\inf f^\nu) \le \inf f,\end{aligned}$$

where the last inequality comes from Theorem 2.6. □

2.9 Remark (convergence of domains). Although, epi-convergence essentially implies convergence of the level sets [14, Proposition 7.7], it does not follow that it implies the convergence of their (effective) domains. Indeed, consider the following sequence $f^\nu\colon I\!R \to I\!R$ with $f^\nu \equiv \nu$ except for $f^\nu(0) = 0$ that epi-converges to $\delta_{\{0\}}$ the indicator function of $\{0\}$. We definitely don't have $\mathrm{dom}\, f^\nu = I\!R$ converging to $\mathrm{dom}\, \delta_{\{0\}} = \{0\}$. *This vigorously argues against the temptation of involving the convergence of their domains in the definition of epi-convergence,* even *for functions in* fv-fcn($I\!R^n$).

2.1 Epi-convergence for extended real-valued functions

This concluded the presentation of the results that will be used in the sequel. As indicated earlier, it's possible to relate certain results to those for extended real-valued functions. To do so, one identifies fv-fcn($I\!R^n$) with

$$pr\text{-fcn}(I\!R^n) := \big\{ f \in \mathrm{fcn}(I\!R^n) \,\big|\, -\infty < f \not\equiv \infty \big\},$$

the subset of proper functions in fcn($I\!R^n$); in a minimization context, a function f is said to be *proper* if $f > -\infty$ and $f \not\equiv \infty$, in which case, it's finite on its *(effective) domain*

$$\mathrm{dom}\, f = \big\{ x \in I\!R^n \,\big|\, f(x) < \infty \big\}.$$

There is an one-to-one correspondence, a *bijection*[6] denoted here by η, between the elements of fv-fcn($I\!R^n$) and those of pr-fcn($I\!R^n$): If $f \in fv$-fcn($I\!R^n$), its extension to all of $I\!R^n$ by setting $\eta f = f$ on its domain and $\eta f \equiv \infty$ on the complement of its domain, uniquely identifies a function in

[6] In fact, this bijection is a homeomorphism when we restrict our attention to lsc functions. The continuity of this correspondence is immediate if both of these function-spaces are equipped with the topology induced by the convergence of the epigraphs, see below.

pr-fcn($\mathbb{R}^n$). And, if $f \in pr$-fcn($\mathbb{R}^n$), the restriction of f to dom f, uniquely identifies a function $\eta^{-1}f$ in fv-fcn($\mathbb{R}^n$). It's important to observe that under this bijection, any function, either in pr-fcn($\mathbb{R}^n$) or fv-fcn($\mathbb{R}^n$), and the corresponding one in fv-fcn($\mathbb{R}^n$) or pr-fcn($\mathbb{R}^n$), have the *same* epigraphs!

Since, epi-convergence for sequences in fv-fcn($\mathbb{R}^n$) or in fcn($\mathbb{R}^n$) is always defined in terms of the convergence of the epigraphs, there is really no need to verify that the analytic versions (Proposition 2.5 and [14, Proposition 7.2]) also coincide. However, for completeness sake and to highlight the connections, we go through the details of an argument.

2.10 Proposition (epi-convergence in fv-fcn($\mathbb{R}^n$) and fcn($\mathbb{R}^n$)). *Let* $\{f : C \to \mathbb{R}, f^\nu : C^\nu \to \mathbb{R}, \nu \in \mathbb{N}\}$ *be a collection of functions in* fv-fcn($\mathbb{R}^n$). *Then,* $f^\nu \xrightarrow{e} f$ *if and only* $\eta f^\nu \xrightarrow{e} \eta f$ *where* η *is the bijection defined above.*

Proof. Now, $\eta f^\nu \xrightarrow{e} \eta f$ ([14, Proposition 7.2]) if and only if for all $x \in \mathbb{R}^n$:

(a$_{xt}$) $\liminf_\nu \eta f^\nu(x^\nu) \geq \eta f(x)$ for every sequence $x^\nu \to x$,

(b$_{xt}$) $\limsup_\nu \eta f^\nu(x^\nu) \leq \eta f(x)$ for some sequence $x^\nu \to x$.

Since for $x \notin C$, $\eta f(x) = \infty$, (a$_{xt}$) clearly implies (a). Conversely, if (a) holds, $x \in C$ and $x^\nu \to x$, when computing the $\liminf_\nu \eta f^\nu(x^\nu)$ one can ignore elements $x^\nu \notin C^\nu$ since then $\eta f^\nu(x^\nu) = \infty$. Hence, for $x \in C$, actually (a) implies (a$_{xt}$). If $x \notin C$ and $x^\nu \to x$, 2.5(a) and again the fact that $\eta f^\nu(x^\nu) = \infty$ when $x \notin C^\nu$, yield (a$_{xt}$).

If (b$_{xt}$) holds and $x \in C$, then the sequence $x^\nu \to x$ must, at least eventually, have $x^\nu \in C^\nu$ since otherwise the $\limsup_\nu \eta f^\nu(x^\nu)$ would be ∞ whereas $f(x) = \eta f(x)$ is finite. Thus, (b$_{xt}$) implies (b). Conversely, (b) certainly yields (b$_{xt}$) if $x \in C$. If $x \notin C$, $\eta f(x) = \infty$ and so the inequality in (b$_{xt}$) is also trivially satisfied in that case. □

As long as we restrict our attention to pr-fcn($\mathbb{R}^n$), in view of the preceding observations, *all the basic results*, cf. [14, Chapter 7, §E] of the theory of epi-convergence related to the convergence of infimums and minimizers apply equally well to functions in fv-fcn($\mathbb{R}^n$) and *not just those featured here*. In particular, if one takes into account the bijection between fv-fcn($\mathbb{R}^n$) and pr-fcn($\mathbb{R}^n$), then Theorem 2.6 is simply an adaptation of the standard results for epi-converging sequences in fcn($\mathbb{R}^n$), cf. [14, Proposition 7.30 & Theorem 7.31]. Similarly, again by relying on the bijection η to translate the statement of Theorem 2.8 into an equivalent one for functions $\eta f^\nu, \eta f$ that belong to fcn($\mathbb{R}^n$), one comes up with [14, Theorem 7.31] about the convergence of the infimal values.

Finally, in a maximization setting, one can simply pass from f to $-f$, or one can repeat the previous arguments with the following changes in the terminology: min to max (inf to sup), ∞ to $-\infty$, epi to hypo, $\leq$ to $\geq$ (and vice-versa), lim inf to lim sup (and vice-versa), and lsc to usc. The *hypograph* of f is the set of all points in $\mathbb{R}^{n+1}$ that lie on or below the graph of f, f is *usc* (= *upper semicontinuous*) if its hypograph is closed, and it's *proper*, in the maximization framework, if $-\infty \not\equiv f < \infty$; in the maximization setting cl f denotes the function whose hypograph is the closure, relative to $\mathbb{R}^{n+1}$ of hypo f, it's also called its *usc regularization.*

A sequence is said to *hypo-converge*, written $f^\nu \xrightarrow{h} f$, when $-f^\nu \xrightarrow{e} -f$, or equivalently if hypo $f^\nu \to$ hypo f, and it *hypo-converge tightly* if $-f^\nu$ epi-converge tightly to $-f$. And consequently, if the sequence hypo-converges tightly to f with $\sup_C f$ finite, then $\sup_{C^\nu} f^\nu \to \sup_C f$.

When hypo f is the inner set-limit of the hypo f^ν, then f is the *lower hypo-limit* of the functions f^ν and if it's the outer set-limit then it's their *upper hypo-limit*. It then follows from Proposition 2.2 that the lower and upper hypo-limits, and the hypo-limit, if it exists, are all usc. Moreover, if the functions f^ν are concave, so is the lower hypo-limit, and the hypo-limit, if it exists. Hence, also the *family of usc functions is closed under hypo-convergence*.

3 Lopsided convergence

Lopsided convergence for bifunctions was introduced in [2]; we already relied on this notion to formalize the convergence of pure exchange economies and to study the stability of their Walras equilibrium points [9]. It's aimed at the convergence of maxinf-points, or minsup-points but not at both; therefore the name *lopsided*, or *lop-convergence*. However, our present, more comprehensive, analysis has lead us to adjust the original definition since otherwise some apparently natural classes of extended real-valued bifunctions with domain and values like those depicted in Figure 2 would essentially be excluded, form the well-behaved lop-convergent families. Moreover, like in §2, in this section our

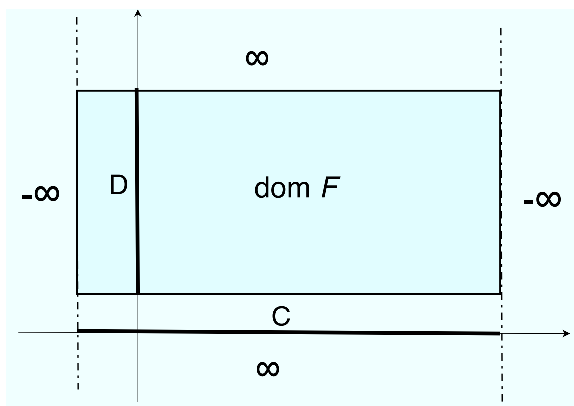

Figure 2: Partition of the domain of a *proper* bifunction: maxinf framework

focus will not be on extended real-valued functions but on *finite-valued bifunctions* that are only defined on a product of non-empty sets rather than on extended real-valued functions defined on the full product space. The motivation for proceeding in this manner, again, coming from the applications. But this time, it's not just one possible approach, it's in fact *mandated* by the underlying structure of the class of bifunctions that are of interest in the applications. In §4, we shall however, like in the previous section, provide a partial the bridge with the extended real-valued framework that was used in [2].

The definition of lop-convergence is necessarily one-sided. One is either interested in the convergence of maxinf-points or minsup-points but not in both. In general, the maxinf-points are not minsup-points and vice-versa. When, they identify the same points, such points are *saddle-points*. In this article, our concern is with the lopsided'-situation and will deal with the saddle-point'-situation in a different note.

Definitions and results can be stated either in terms of the convergence of maxinf-points or minsup-points with some obvious adjustments for signs and terminology. However, *it's important to know if we are working in a maxinf or a minsup framework* and this is in keeping with the (plain) univariate case where one has to focus on either minimization or maximization. Because most of the applications of interest are more naturally formulated in terms of maxinf-problems, that's the version that will be dealt with in this section. At the end of the section, one finds the necessary translations required to deal with minsup-problems.

Here, the term *bifunction* always refers to functions defined on the product of two non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively[7] one writes

$$\mathrm{biv}(\mathbb{R}^{n+m}) = \big\{ F\colon \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}} \big\}$$

for the class of bifunctions that are extended real-valued and defined on all of $\mathbb{R}^n \times \mathbb{R}^m$ and

$$fv\text{-}\mathrm{biv}(\mathbb{R}^{n+m}) = \big\{ F\colon C \times D \to \mathbb{R} \,\big|\, \emptyset \neq C \subset \mathbb{R}^n,\ \emptyset \neq D \subset \mathbb{R}^m \big\}$$

for the class of bifunctions that are real-valued and defined on the product $C \times D$ of non-empty subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$; here, it's understood that $\mathbb{R}^{n+m}$ doesn't refer to the domain of definition but to the (operational) product space that includes $C \times D$.

For a bifunction $F \in fv\text{-}\mathrm{biv}(\mathbb{R}^{n+m})$, one refers to $\bar{x}$ as a *maxinf-point* if

$$\bar{x} \in \underset{C\times D}{\mathrm{argmaxinf}}\, F = \underset{x\in C}{\mathrm{argmax}} \big[\inf_{y\in D} F(x,y) \big],$$

and a *minsup-point* if

$$\bar{x} \in \underset{C\times D}{\mathrm{argminsup}}\, F = \underset{x\in C}{\mathrm{argmin}} \big[\sup_{y\in D} F(x,y) \big].$$

When $F \in \mathrm{biv}(\mathbb{R}^{n+m})$, the sets C and D are then respectively $\mathbb{R}^n$ and $\mathbb{R}^m$.

For now, let's focus on $fv\text{-}\mathrm{biv}(\mathbb{R}^{n+m})$ always keeping in mind that we position ourselves in the maxinf framework.

3.1 Definition (lop-convergence, fv-biv). *The bifunctions* $\big\{F^\nu : C^\nu \times D^\nu \to \mathbb{R}\big\}_{\nu\in N}$ lop-converges (= converges lopsided) *to* $F : C \times D \to \mathbb{R}$*, also written* $F^\nu \xrightarrow{lop} F$*, if*

(a) $\forall\, y \in D$ & $x^\nu \in C^\nu \to x$, $\exists\, y^\nu \in D^\nu \to y$ *such that* $\mathrm{limsup}_\nu\, F^\nu(x^\nu, y^\nu) \leq F(x,y)$ *when* $x \in C$ *and* $F^\nu(x^\nu, y^\nu) \searrow -\infty$ *when* $x \notin C$,

(b) $\forall\, x \in C$, $\exists\, x^\nu \in C^\nu \to x$ *such that* $\forall\, y^\nu \in D^\nu \to y$, $\mathrm{liminf}_\nu\, F^\nu(x^\nu, y^\nu) \geq F(x,y)$ *when* $y \in D$ *and* $F^\nu(x^\nu, y^\nu) \nearrow \infty$ *when* $y \notin D$.

Epi-convergence of "slices" is an immediate consequence of this definition:

[7] A follow-up paper will deal with bifunctions defined on the product of non-empty subsets of two topological spaces potentially equipped with different topologies.

3.2 Proposition (epi-convergence of slices). *Suppose* $F^\nu \xrightarrow{lop} F$ *in* fv-biv$(I\!R^{n+m})$. *Then, for all* $x \in C$, *there exists* $x^\nu \in C^\nu \to x$ *such that the (univariate) functions* $F^\nu(x^\nu, \cdot) \xrightarrow{e} F(x, \cdot)$.

Proof. From 3.1(b) there exists $x^\nu \in C^\nu \to x$ such that the functions $\{F^\nu(x^\nu, \cdot), \nu \in I\!N\}$ and $F(x, \cdot)$ satisfy 2.5(a). On the other hand, from 3.1(a), for any $y \in D$ and $x^\nu \in C^\nu \to x$, one can find $y^\nu \in D^\nu \to y$ such that condition 2.5(b) is also satisfied. □

Although a number of properties can be immediately derived from the definition, cf. Theorem 5.2 for example, to obtain the convergence of the maxinf-points, however, we need to require "partial tightness"; refer to Theorem 3.4(b) when dealing with epi-convergence for univariate functions. The *ancillary-tightness* condition is new; it's inspired from A. Bagh's work [6] on approximation for optimal control problems. A more conventional condition, that implies ancillary-tightness, could be: *(b) holds and there is a compact set* $B \subset I\!R^m$ *such that*

$$\forall x \in I\!R^n : \quad B \supset \{y \mid F^\nu(x, y) < \infty\}.$$

This last condition, suggested in [2], is unfortunately too restrictive in many applications. In particular, ancillary-tightness allows for a generalization of Ky Fan's inequality, cf. §5, that can be exploited in situations when the domain of definition of the bifunctions is not compact.

3.3 Definition (ancillary-tight lop-convergence). *When* $F^\nu \xrightarrow{lop} F$, *all in* $f\,$biv$(I\!R^{n+m})$, *the convergence is said to be* ancillary tight *if 3.1(b) is strengthened as follows:*

(b-t) *not only,* $\forall\, x \in C$, $\exists\, x^\nu \in C^\nu \to x$ *such that* $\forall\, y^\nu \in D^\nu \to y$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(x, y)$ *or* $F(x^\nu, y^\nu) \to \infty$ *depending on* y *belonging or not to* D, *but also, for any* $\varepsilon > 0$ *one can find a compact set* B_ε, *depending possibly on the sequence* $\{x^\nu \to x\}$, *such that*

$$\inf_{D^\nu \cap B_\varepsilon} F^\nu(x^\nu, \cdot) \leq \inf_{D^\nu} F^\nu(x^\nu, \cdot) + \varepsilon \qquad \textit{for all } \nu \textit{ sufficient large} .$$

The convergence of the *inf-projections* (or marginal functions)

$$\{h^\nu = \inf_{y \in D^\nu} F^\nu(\cdot, y),\ \nu \in I\!N\} \qquad \text{and} \qquad h = \inf_{y \in D} F(\cdot, y)$$

plays a pivotal role in the argumentation and brought us to introduce ancillary tightness.

3.4 Theorem (hypo-convergence of the inf-projections). *For* $\{h^\nu\}_{\nu \in I\!N}$ *and* h, *the inf-projections of the bifunctions* $F^\nu \xrightarrow{lop} F$ *in* fv-biv$(I\!R^{n+m})$ *such that* dom h *non-empty, one has:*

- *(a)* h *is the upper hypo-limit of the functions* $\{h^\nu\}_{\nu \in I\!N}$,
- *(b)* h *is actually their hypo-limit if the lop-convergence is ancillary tight*[8].

Proof. Recall that all these bifunctions are finite-valued on their domain, in particular F is finite-valued on the "rectangle" $C \times D$ and also that $h \equiv -\infty$ has been excluded. Let $x^\nu \in C^\nu \to x$. When

[8] related results can be found in the literature when the bifunctions are extended real-valued, cf. in particular [12].

$x \in C$, given any $\varepsilon > 0$ arbitrarily small, pick $y^\varepsilon \in D$ such that $F(x, y^\varepsilon) - \varepsilon \leq h(x)$. Then 3.1(a) yields $y^\nu \in D^\nu \to y^\varepsilon$ such that

$$\mathrm{limsup}_\nu\, h^\nu(x^\nu) \leq \mathrm{limsup}_\nu\, F^\nu(x^\nu, y^\nu) \leq F(x, y^\varepsilon) \leq h(x) + \varepsilon,$$

implying $\mathrm{limsup}_\nu\, h^\nu(x^\nu) \leq h(x)$. Next, suppose that $x \notin C$. Pick any $y \in D$, again 3.1(a) guarantees a sequence $y^\nu \in D^\nu \to y$ such that

$$\mathrm{limsup}_\nu\, h^\nu(x^\nu) \leq \mathrm{limsup}_\nu\, F^\nu(x^\nu, y^\nu) \searrow -\infty.$$

So, in both situations, condition 2.5(a) is satisfied. In other words, h is their upper hypo-limit since $\mathrm{hypo}\, h$ is the outer limit hypographs of the functions $\mathrm{hypo}\, h^\nu$. This takes care of the first assertion.

So, let's turn to the second one. Let $x \in \mathrm{dom}\, h \neq \emptyset$, i.e., $h(x) \in I\!R$. Now, choose $x^\nu \in C^\nu \to x$ such that $F^\nu(x^\nu, \cdot) \xrightarrow{e} F(x, \cdot)$, cf. Proposition 3.2. In fact, they epi-converge tightly as an immediate consequence of ancillary-tightness, i.e.,

$$h^\nu(x^\nu) = \inf_{D^\nu} F^\nu(x^\nu, \cdot) \to \inf_D F(x, \cdot) = h(x),$$

via Theorem 2.8(a). ◻

3.5 Theorem (convergence of maxinf-points, fv-biv). *Suppose $F^\nu \xrightarrow{lop} F$, all in fv-biv$(I\!R^{n+m})$, ancillary-tight with $\{h^\nu\}_{\nu \in N}$ and h, their inf-projections such that $\mathrm{dom}\, h \neq \emptyset$. Let $x^\nu \in \mathrm{argmaxinf}\, F^\nu$ and $\bar{x}$ any cluster point of the sequence $\{x^\nu,\, \nu \in I\!N\}$, then $\bar{x}$ is a maxinf-point of the limit function F. Moreover, with $\{x^\nu, \nu \in N \subset I\!N\}$, the (sub)sequence converging to $\bar{x}$,*

$$\lim_{\nu \xrightarrow[N]{} \infty} \Big[\inf_{y \in D^\nu} F^\nu(x^\nu, y) \Big] = \inf_{y \in D} F(\bar{x}, y),$$

i.e., the maxinf-values converge for this (sub)sequence.

Proof. Theorem 3.4 tells us that the inf-projections h^ν hypo-converge to h. Maxinf-points of F^ν and F are then maximizers of the corresponding inf-projections h^ν and h. The assertions now follow immediately from the convergence of the argmax of hypo-converging sequences, cf. Theorem 2.6 translated to the maximization framework. ◻

Further approximation results, however, require "full tightness," not just ancillary-tightness.

3.6 Definition (tight lopsided convergence, fv-biv). *The bifunctions $\{F^\nu : C^\nu \times D^\nu \to I\!R\}_{\nu \in N}$* lop-converges tightly *to $F : C \times D \to I\!R$, all in fv-biv$(I\!R^{n+m})$, if the convergence is ancillary tight and, in addition,*

(a-t) *not only $\forall\, y \in D$ & $x^\nu \in C^\nu \to x$, $\exists\, y^\nu \in D^\nu \to y$ such that $\mathrm{limsup}_\nu\, F^\nu(x^\nu, y^\nu) \leq F(x, y)$ or $F^\nu(x^\nu, y^\nu) \searrow -\infty$ depending on x belonging or not to C, but also given any $\varepsilon > 0$ one can find $A_\varepsilon \subset I\!R^n$ such that for all ν large enough,*

$$\sup_{x \in C^\nu \cap A_\varepsilon} \inf_{y \in D^\nu} F^\nu(x, y) \geq \sup_{x \in C^\nu} \inf_{y \in D^\nu} F^\nu(x, y) - \varepsilon,$$

3.7 Theorem (approximating maxinf-points). *Suppose $F^\nu \xrightarrow{lop} F$ tightly in fv-biv$(\mathbb{R}^{n+m})$ such that* $\sup h = \sup_x \inf_y F(x,y)$ *is finite. Then,*

$$\sup_x \inf_y F^\nu(x,y) \to \sup_x \inf_y F(x,y).$$

If $\bar{x} \in$ argmaxinf F, one can always find sequences $\{\varepsilon^\nu \searrow 0,\ x^\nu \in \varepsilon^\nu\text{-argmax}_x(\inf_y F^\nu)\}_{\nu \in N \subset \mathcal{N}}$ such that $x^\nu \underset{N}{\to} \bar{x}$. Conversely, if such sequences exist, then $\sup_x(\inf_y F^\nu) \underset{N}{\to} \inf_y F(\bar{x}, \cdot)$.

Proof. The hypo-convergence of the inf-projections h^ν to $h = \inf_{y \in D} F(\cdot, y)$ follows from Theorem 3.4. From (a-t), it then follows that they also hypo-converge tightly. The assertions then proceed directly from Theorem 2.8 ☐

4 Extended real-valued bifunctions

Let's make a parenthesis to deal with bifunctions that are extended real-valued and defined on all of $\mathbb{R}^n \times \mathbb{R}^m$, keeping in mind that we remain in the maxinf setting. To define convergence, we can't proceed as in §2, where we tied the convergence of functions with that of their epigraphs. Here, there is no easily identifiable (unique) geometric object that can be associated with a bifunction.

Recall that biv$(\mathbb{R}^{n+m})$ is the family of all extended-real valued functions defined on $\mathbb{R}^n \times \mathbb{R}^m$. In our maxinf case, similar to [13], the *effective domain* $\operatorname{dom} F$ of a bifunction $F : \mathbb{R}^{n+m} \to \overline{\mathbb{R}}$ is by definition

$$\operatorname{dom} F = \operatorname{dom}_x F \times \operatorname{dom}_y F,$$

where

$$\operatorname{dom}_x F = \{x \mid F(x,y) > -\infty,\ \forall\, y \in \mathbb{R}^m\},$$
$$\operatorname{dom}_y F = \{y \mid F(x,y) < \infty,\ \forall\, x \in \mathbb{R}^n\}.$$

Thus, F is finite-valued on $\operatorname{dom} F$; it doesn't exclude the possibility that our "original" F might have been finite-valued at some points that don't belong to $\operatorname{dom} F$. In our maxinf framework, the term *proper* will however be reserved for bifunctions with non-empty domain and such that

$$F(x,y) = -\infty \text{ when } x \notin \operatorname{dom}_x F$$
$$F(x,y) = \infty \text{ when } x \in \operatorname{dom}_x F \text{ but } y \notin \operatorname{dom}_y F,$$

see Figure 2. Let's denote this subcollection of *proper* bivariate functions by *pr*-biv$(\mathbb{R}^{n+m})$.

4.1 Definition (biv, lop-convergence). *The sequence of bifunctions $\{F^\nu\}_{\nu \in \mathcal{N}} \subset$ biv$(\mathbb{R}^{n+m})$ lop-converges to $F : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ if*

(a$_{xt}$) $\forall\, y \in \operatorname{dom}_y F$ & $x^\nu \to x \in \mathbb{R}^n$, $\exists\, y^\nu \to y$ *such that* $\operatorname{limsup}_\nu F^\nu(x^\nu, y^\nu) \le F(x,y)$,

(b$_{xt}$) $\forall x \in \operatorname{dom}_x F$, $\exists\, x^\nu \to x$ *such that* $\forall\, y^\nu \to y \in \mathbb{R}^m$, $\operatorname{liminf}_\nu F^\nu(x^\nu, y^\nu) \ge F(x,y)$.

Observe that when the functions F^ν and F don't depend on x they lop-converge if and only if they epi-converge and if they don't depend on y they converge lopsided if and only if they hypo-converge. This later assertion follows from Proposition 2.10. Moreover, if for all (x, y), the functions $F^\nu(x, \cdot) \xrightarrow{e} F(x, \cdot)$ *and* $F^\nu(\cdot, y) \xrightarrow{h} F(\cdot, y)$, then the functions F^ν lop-converge to F; however, one should keep in mind that this is a sufficient condition but by <u>no means</u> a necessary one.

4.2 Remark ('83 versus new definition). *For example, the definition of lop-convergence in* [2] *required condition 4.1*(b_{xt}) *to hold not just for all* $x \in \mathrm{dom}_x F$ *but for all* $x \in \mathbb{R}^n$. *The implication is that then lop-convergent families must be restricted to those converging to a function* F *whose* $\mathrm{dom}_x F = \mathbb{R}^n$.

Detail. Indeed, consider the following simple example: For all $\nu \in \mathbb{N}$,

$$F^\nu(x, y) = F(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1] \times [0, 1], \\ -\infty & \text{if } y \in (0, 1), x \notin [0, 1], \\ \infty & \text{elsewhere.} \end{cases}$$

Then, in terms of Definition 4.1, the F^ν lop-converge to F, but they do not if one insists that condition 4.1(b_{xt}) holds for all $x \in \mathbb{R}^n$. Indeed, there is no way to find a sequence $x^\nu \to -1$, for example, such that for all $y^\nu \to 0$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(-1, 0) = \infty$; simply consider $y^\nu = 1/\nu \to 0$. □

As in §2, we set up a bijection, similarly denoted η, between the elements of fv-biv($\mathbb{R}^{n+m}$) and the (max-inf) proper bifunctions, *pr*-biv($\mathbb{R}^{n+m}$). For $F \in fv$-biv($\mathbb{R}^{n+m}$), set

$$\eta F(x, y) = \begin{cases} F(x, y) & \text{when } (x, y) \in C \times D, \\ -\infty & \text{when } x \notin C, \\ \infty & \text{when } x \in C \text{ but } y \notin D, \end{cases}$$

i.e., ηF extends F to all of $\mathbb{R}^n \times \mathbb{R}^m$. Then, for $F \in pr$-biv, $\eta^{-1} F$ will be the restriction of F to its domain of finiteness, namely $\mathrm{dom}_x F \times \mathrm{dom}_y F$.

4.3 Proposition (lop-convergence in fv-biv and biv). *For* F *and* $\{F^\nu\}_{\nu \in \mathbb{N}}$ *in* fv-biv($\mathbb{R}^{n+m}$), $F^\nu \xrightarrow{lop} F$ *if and only* $\eta F^\nu \xrightarrow{lop} \eta F$ *in pr*-biv($\mathbb{R}^{n+m}$), *cf. Definition 4.1, where* η *is the bijection between* fv-biv($\mathbb{R}^{n+m}$) *and pr*-biv($\mathbb{R}^{n+m}$) *described earlier.*

Proof. Suppose (a_{xt}) holds, $y \in D$ and $x^\nu \in C^\nu \to x$, then there exists $y^\nu \to y$ such that $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) \leq \eta F(x, y) = F(x, y)$. Referring to 3.1(a), only $y \in D$ needs to be considered. When $x \in C$, $\eta F(x, y) = F(x, y) \in R$ and consequently, for ν sufficiently large, all y^ν must belong to D^ν, since otherwise $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = \infty$ contradicting (a_{xt}). On the other hand, if $x \notin C$, $\eta F(x, y) = -\infty$ and if a (sub)sequence of $\{y^\nu\}_{\nu \in \mathbb{N}}$ was such that its elements didn't belong to the corresponding D^ν then $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = \infty$, again contradicting ($\mathrm{a}_{xt}$). Hence, in this situation, $\limsup_\nu F^\nu(x^\nu, y^\nu) = \limsup_\nu \eta F^\nu(x^\nu, y^\nu) \to -\infty$. This confirms that condition 3.1(a) holds.

When (b_{xt}) holds and $x \in C = \mathrm{dom}_x\, \eta F$, which in particular means $\eta F(x,\cdot) > -\infty$, there exists $x^\nu \to x$ such that $\liminf_\nu \eta F^\nu(x^\nu, y^\nu) \geq \eta F(x,y)$ for all $y^\nu \to y \in I\!R^m$. To recover 3.1(b), again we split the proof between the cases when y belongs to, or not, to $D = \mathrm{dom}_y\, \eta F$. In both instances, if there is a (sub)sequence $\{x^\nu \notin C^\nu\}$ $\liminf_\nu \eta F^\nu(x^\nu, y^\nu) = -\infty$ contradicting (b_{xt}). So, necessarily, at least for ν sufficiently large $x^\nu \in C^\nu$. If $y \in D$ and any $y^\nu \in D^\nu \to y$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(x,y)$. When $y \notin D$, $\eta F(x,y) = \infty$, then for any $y^\nu \in D^\nu \to y$, $\eta F^\nu(x^\nu, y^\nu) = F(x^\nu, y^\nu) \to \infty$. This means that 3.1(b) also holds and this completes the proof that $\eta F^\nu \xrightarrow{lop} \eta F$ implies $F^\nu \xrightarrow{lop} F$.

For the converse, taking into account the definition of the bijection η, let's start by showing that 3.1(a) yields (a_{xt}). First suppose $x \in C$. [Note, if $y \notin D$, then $\eta F(x,y) = \infty \geq \limsup_\nu \eta F^\nu(x^\nu, y^\nu)$ for any sequences $x^\nu \to x$, $y^\nu \to y$.] With $y \in D$, if the sequence $x^\nu \to x$ doesn't include, at least, a subsequence $\{x^\nu \in C^\nu,\, \nu \in N \subset I\!N\}$ then $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = -\infty$ and the inequality in (a_{xt}) is certainly satisfied, otherwise 3.1(a) guarantees $y^\nu \underset{N}{\to} y$ such that $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) \leq \eta F(x,y) = F(x,y)$. Next, when $x \notin C$, $\eta F(x,\cdot) \equiv -\infty$. When $y \in \mathrm{dom}_Y\, F + D$, either (i) the sequence $x^\nu \to x$ doesn't include a subsequence $\{x^\nu \in C^\nu,\, \nu \in N \subset I\!N\}$ in which case $F(x^\nu,\cdot) \equiv -\infty$ for ν sufficiently large, whence $\limsup_\nu \eta F^\nu(x^\nu, y^\nu) = -\infty = \eta F(x,y)$, or (ii) the sequence $x^\nu \to x$ includes a (sub)sequence $\{x^\nu \in C^\nu,\, \nu \in N \subset I\!N\}$ in which case 3.1(a) guarantees a sequence $y^\nu \underset{N}{\to} y$ such that $\eta F^\nu(x^\nu, y^\nu) = F^\nu(x^\nu, y^\nu) \underset{-}{\to} \infty$. This takes care of ($\mathrm{a}_{xt}$).

There remains to show that 3.1(b) yields (b_{xt}). Only $x \in \mathrm{dom}_x\, \eta F = C$ needs to be considered in which case $\eta F(x,\cdot) > -\infty$ and let $x^\nu \in C^\nu \to x$ the sequence predicated in 3.1(b). When $y^\nu \to y \in D = \mathrm{dom}_y\, \eta F$ either (i) there is a (sub)sequence $y^\nu \in D^\nu \underset{N}{\to} y$ in which case the desired inequality follows immediately from the one in 3.1(b) or (ii) when there is no such (sub)sequence $\eta F^\nu(x^\nu, y^\nu) = F^\nu(x^\nu, y^\nu) = \infty$ and $\liminf_\nu \eta F(x^\nu, y^\nu) = \infty \geq \eta F(x,y)$. When $y^\nu \to y \notin D = \mathrm{dom}_y\, \eta F$, making the same distinction as previously, one either gets (i) $\eta F^\nu(x^\nu, y^\nu) \underset{N}{\to} \infty$ from 3.1(b) or (ii) as above. This, then, concludes the proof of the equivalence. ☐

Ancillary-tight lop-convergence is also the key to the convergence of the maxinf-points of extended real-valued bifunctions.

4.4 Definition (ancillary-tight lop-convergence , biv). *The lop-convergence of a sequence of bifunctions in* $\mathrm{biv}(I\!R^{n+m})$ *is* ancillary-tight *if it converges lopsided and for all* $x \in C$*, the following augmented condition of 4.1(b) holds:*

(b_{xt}-t) *not only* $\exists\, x^\nu \to x \in \mathrm{dom}_x\, F$ *such that* $\forall\, y^\nu \to y$, $\liminf_\nu F^\nu(x^\nu, y^\nu) \geq F(x,y)$, *but also, for any* $\varepsilon > 0$ *one can find a compact set* B_ε*, possibly depending on the sequence* $\{x^\nu \to x\}$*, such that for all* ν *larger than some* ν_ε*,*

$$\inf_{B_\varepsilon} F^\nu(x^\nu,\cdot) \leq \inf F^\nu(x^\nu,\cdot) + \varepsilon.$$

4.5 Proposition (ancillary-tight lop-convergence: fv-biv & biv). $F^\nu \xrightarrow{lop} F$ *in* $fv\text{-}\mathrm{biv}(I\!R^n \times I\!R^m)$ *ancillary tight if and only if* $\eta F^\nu \xrightarrow{lop} \eta F$ *ancillary-tight in* $pr\text{-}\mathrm{biv}(I\!R^{n+m})$ *where* η *is the bijection from defined earlier.*

Proof. Proposition 4.3 already set up the lop-convergence equivalence, so one only has to verify the ancillary-tight condition. But that's immediate because in both cases it only involves points that belong to $C \times I\!R^m = \mathrm{dom}_x\, \eta F \times I\!R^m$ and sequences converging to such points. □

4.6 Proposition (hypo-convergence of the inf-projections, biv). *Suppose* $F^\nu \xrightarrow{lop} F$, *in* $\mathrm{biv}(I\!R^{n+m})$, *ancillary-tight. Then, the inf-projections* $h^\nu = \inf_{y\in D^\nu} F^\nu(\cdot, y)$ *hypo-converge to* $h = \inf_{y\in D} F(\cdot, y)$ *as functions defined on* $I\!R^n$.

Proof. The proof is the same as that of Theorem 3.4 with the obvious adjustments when the sequences don't belong to $\mathrm{dom}\, F^\nu$ and the limit point doesn't lie in $\mathrm{dom}\, F$. □

4.7 Theorem (convergence of maxinf-points, biv). *Suppose* $F^\nu \xrightarrow{lop} F$, *in* $\mathrm{biv}(I\!R^{n+m})$, *ancillary-tight,* $x^\nu \in \mathrm{argmaxinf}\, F^\nu$, *at least for all* ν *large enough and* $\bar{x}$ *is a cluster point of this sequence. Then,* $\bar{x}$ *is a maxinf-point of the limit function* F. *Moreover, with* $\{x^\nu, \nu \in N \subset I\!N\}$ *a (sub)sequence converging to* $\bar{x}$,

$$\lim_{\nu \xrightarrow[N]{} \infty} \inf F^\nu(x^\nu, \cdot) = \inf F(\bar{x}, \cdot)\,],$$

i.e., the maxinf-values also converge for this (sub)sequence.

Proof. The previous proposition yields the hypo-convergence of the inf-projections h^ν to h. Maxinf-points for F^ν and F are then maximizers of the corresponding functions h^ν and g. The assertions now follow immediately from the convergence of the argmax of hypo-converging sequences, cf. Theorem [14, Theorem 7.31] translated to the maximization framework. □

To deal with a minsup situations one can either repeat all the arguments changing inf to sup, liminf to limsup and vice-versa, or simply re-integrate the questions issues to the maxinf framework by changing signs of the approximating and limit bifunctions.

5 Ky Fan's Inequality extended

The class of usc functions is closed under hypo-converge [14, Theorem 7.4], and so is the class of concave usc functions [14, Theorem 7.17]. The class of *Ky Fan bifunctions* is closed under lopsided convergence, Theorem 5.2. We exploit this result to obtain a generalization of Ky Fan Inequality that allows us to claim existence of maxinf-points in situations when the domain of definition of the Ky Fan function is not necessarily compact.

5.1 Definition *A bifunction* $F : C \times D \to I\!R$, *in* $fv\text{-}\mathrm{biv}(I\!R^{n+m})$, *with* C*and* D *non-empty convex sets, is called* a Ky Fan function *if*

(a) $\forall\, y \in D$: $x \mapsto F(x, y)$ *is usc on* C,

(b) $\forall\, x \in C$: $y \mapsto F(x, y)$ *is convex on* D;

note that the sets C *or* D *are not required to be compact.*

5.2 Theorem (lop-limits of Ky Fan functions). *The lopsided limit $F : C \times D \to I\!R$ of a sequence $\{F^\nu : C^\nu \times D^\nu \to I\!R\}_{\nu \in N}$ of Ky Fan functions in fv-biv$(I\!R^{n+m})$ is also a Ky Fan function.*

Proof. For the convexity of $y \mapsto F(x, y)$, let $x^\nu \in C^\nu \to x \in C$ be the sequence set forth by 3.1(b) and $y^0, y^\lambda, y^1 \in D$ with $y^\lambda = (1-\lambda)y^0 + \lambda y^1$ for $\lambda \in [0, 1]$. In view of 3.1(a), we can choose sequences $\{y^{0,\nu} \in D^\nu \to y^0\}$, $\{y^{1,\nu} \in D^\nu \to y^1\}$ such that $F^\nu(x^\nu, y^{0,\nu}) \to F(x, y^0)$ and $F^\nu(x^\nu, y^{1,\nu}) \to F(x, y^1)$. Let $y^{\lambda,\nu} = (1-\lambda)y^{0,\nu} + \lambda y^{1,\nu}$; $y^{\lambda,\nu} \in D^\nu$ since the functions $F^\nu(x, \cdot)$ are convex and the sequence $\{y^{\lambda,\nu}\}_{\nu \in N}$ certainly converges to y^λ. For all ν, one has

$$F^\nu(x^\nu, y^{\lambda,\nu}) \le (1-\lambda)F^\nu(x^\nu, y^{0,\nu}) + \lambda F^\nu(x^\nu, y^{1,\nu}),$$

Taking liminf on both sides yields

$$F(x, y^\lambda) \le \liminf_\nu F^\nu(x^\nu, y^{\lambda,\nu}) \le (1-\lambda)F(x, y^0) + \lambda F(x, y^1),$$

that establishes the convexity of $F(x, \cdot)$.

To prove the upper semicontinuity of F with respect to x-variable, we show that for $y \in D$,

$$\text{hypo}\, F(\cdot, y) \quad \text{is the inner-limit of} \quad \text{hypo}\, F^\nu(\cdot, y^\nu),$$

where the limit is with respect to all sequences $\{y^\nu \in D^\nu\}_{\nu \in N}$ converging to y and $\nu \to \infty$. This yields the upper semicontinuity since the inner set-limit is always closed and a function is usc if and only if its hypograph is closed. One needs to show that if $(x, \alpha) \in \text{hypo}\, F(\cdot, y)$, then whenever $y^\nu \in D^\nu \to y$, one can find $(x^\nu, \alpha^\nu) \in \text{hypo}\, F^\nu(\cdot, y^\nu)$ such that $(x^\nu, \alpha^\nu) \to (x, \alpha)$. But that follows immediately from 3.1(a) since we can adjust the $\alpha^\nu \le F^\nu(x^\nu, y^\nu)$ so that they converge to $\alpha \le F(x, y)$. ◻

For a Ky Fan function with compact domain and non-negative on its diagonal, one has the following important existence result:

5.3 Lemma (Ky Fan's Inequality; [7], [4, Theorem 6.3.5]). *Suppose $F : C \times C \to I\!R$ is a Ky Fan function with C compact. Then, the set of maxinf-points of F is a nonempty subset of C. Moreover, if $F(x, x) \ge 0$ (on $C \times C$), then for every maxinf-point $\bar{x}$ of C one has that $F(\bar{x}, \cdot) \ge 0$ on C.*

One of the consequences of the lopsided convergence is an extension of the Ky Fan's Inequality to the case when it is not possible to apply it directly because $\text{dom}\, F = C \times C$ is not compact. However, one might be able to approach F by a sequence $\{F^\nu\}_{\nu \in N}$ defined on compact sets C^ν. This procedure could be useful in many situation where the original maxinf-problem is unbounded, and then the problem is approximated by a family of truncated maxinf-problems. Such is the case, for example, when we consider Lagrangian bifunctions where the multipliers associated to inequality constraints are not bounded, or when the original problem is a Walras equilibrium with a positive orthant as consumption set; in [8] one is precisely confronted with such situations. Another simple, illustrative example follows the statement of the theorem.

5.4 Theorem (Extension of Ky Fan's Inequality). *Let F be a Ky Fan function defined on $C \times C$. Suppose one can find sequences of compact convex sets $\{C^\nu \subset I\!R^n\}$ and (finite-valued) Ky Fan functions $\{F^\nu : C^\nu \times C^\nu \to I\!R\}_{\nu \in N}$ whose lop-convergence to F is ancillary-tight, then every cluster point $\bar{x}$ of any sequence $\{x^\nu,\, \nu \in I\!N\}$ of maxinf-points of the F^ν is a maxinf-point of F*

Proof. Ky Fan's Inequality 5.3 implies that for all ν, the set of maxinf-points of F^ν is non-empty. On the other hand, in view of Theorems 5.2 and 3.5 any cluster point of such maxinf-points will be a maxinf-point of F. □

5.5 Example (Extended Ky Fan's Inequality applied). *We consider a Ky Fan function $F(x, y) = \sin x + (y+1)^{-1}$ defined on the set $[0,\infty)^2$. Although,*

$$\inf_{y\in[0,\infty)} F(x,y) = \sin x,$$

and the set maxinf-points is not empty, we can't apply Ky Fan Inequality because the domain of F is not compact; the function $F(\cdot, y)$ is not even sup-compact.

Detail. If we consider the functions $F^\nu(x, y) = \sin x + (y + 1){-}1$ on the compact domains $[0, \nu]^2$, one can apply Ky Fan's Inequality. Indeed, in this case we have,

$$\inf_{y\in[0,\nu)} F(x,y) = \sin x + (\nu+1)^{-1},$$

that hypo- and pointwise-converges to $\sin x$, and

$$\operatorname*{argmax}_{x\in[0,\nu]} \inf_{y\in[0,\nu]} F(x,y) = \{\pi/2 + 2k\pi \mid k \in I\!N\}.$$

Thus, $x^\nu = \pi/2$ and $\tilde{x}^\nu = \pi/2 + 2\nu\pi$ are maxinf-points of the F^ν. The sequence $\{x^\nu\}_{\nu\in N}$ converges to a maxinf-point of F, the second sequence $\{\tilde{x}^\nu\}_{\nu\in N}$ does not. □

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