On the Relations between Two Types of Convergence for Convex Functions

GABRIELLA SALINETTI* AND ROGER J.-B. WETS†

Università di Roma, Roma, Italy; University of Kentucky, Lexington, Kentucky and Stanford University, Stanford, California

Submitted by G. L. Lions

Theory and applications have shown that there are two important types of convergence for convex functions: pointwise convergence and convergence in a topology induced by the convergence of their epigraphs. We show that these two types of convergence are equivalent on the class of convex functions which are equi-lower semicontinuous. This turns out to be maximal classes of convex functions for which this equivalence can be obtained. We also indicate a number of implications of these results to the convergence of convex sets and the corresponding support functions and to the convergence of the infima of sequences of convex minimization problems.

1. INTRODUCTION

The study of the convergence of convex sets and convex functions has usually been undertaken to the foundations for approximation results in statistics [1, 2], in convex optimization [3–5], in the theory of variational inequalities [6, 7], and for control problems [8, 9]. The key question always boils down to: Given a sequence of minimization problems \( \{P_n\} \) whose constraints and objective converge in some sense, usually pointwise, to the constraints and objective of a minimization problem \( P \), does it follow that the sequence of optimal values of \( P_n \) converge to the optimal value of \( P \)? (There are a number of related questions such as convergence of the optimality sets to the optimality set of \( P \), convergence of the associated price systems, ...). To each minimization problem \( P_n \) we can associate an extended real-valued function \( f_n \) with \( f_n(x) = +\infty \) if \( x \) fails to satisfy the constraints of \( P_n \), and otherwise \( f_n(x) \) is the value of the objective of \( P_n \) evaluated at \( x \). In this framework, we have that

\[
\inf P_n = \inf_x f_n(x) = -f_n^*(0),
\]

* Supported in part by C.N.R. (Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni).
† Supported in part by the National Science Foundation under Grant MPS 75-07028 at the University of Kentucky.
where $f^*$ is the conjugate of $f$. The convergence of the infimum values of the $P_v$ to the infimum of $P$ is then equivalent to the (pointwise) convergence of the conjugate functions $f_v^*$ to $f^*$ at $0$. Of more general interest is the (pointwise) convergence of $f_v^*$ to $f^*$; each point $x^*$ corresponding to a perturbed version of the original problem, since

$$f_v^*(x^*) = -\inf_x [f_v(x) - \langle x, x^* \rangle].$$  \hspace{1cm} (1.2)

The study of these and related convergence questions leads to the definition of a topology $\tau$ on the class of lower semicontinuous convex functions. Convergence of $f_v$ to $f$ in this topology is equivalent to the convergence of $\text{epi } f_v$, the epigraphs of $f_v$, to $\text{epi } f$. This topology $\tau$ was introduced by Mosco [7, Sect. 1.7]; various extensions and refinements were obtained by Joly [10, 11] and in [12], Robert works with a variant of this topology. It plays an important role in the study of convergence of convex sets and also convex functions because it turns out that conjugation is bicontinuous with respect to this topology; in particular, we have that

$$f_v \overset{\tau}{\longrightarrow} f \quad \text{if and only if} \quad f_v^* \overset{\tau^*}{\longrightarrow} f^*$$

for $\tau^*$ defined in a way similar to $\tau$. This property, first established by Wakup and Wets [13] (in the framework of closed convex cones) and later, independently by Mosco [14] and Joly [11], plays a key role in the study of the convergence of convex sets and functions. In particular, it allows us to relate the convergence of convex sets to the pointwise convergence of their support functions. In the compact case (for compact convex sets) the relations between these two types of convergences have been investigated by Wigsman [15] and Van Cutsem [16].

The main objective of this note is to delineate the relations between pointwise convergence and $\tau$-convergence for sequences of convex functions. In particular, we identify the largest class of functions for which these two types of convergences coincide. We also show the implications of these results to the convergence of convex sets and their support functions and to the convergence of the infima of sequences of convex programs.

The terminology and notation is the standard one of convex analysis. For $f$ a convex function, we write $\text{epi } f$ for the epigraph of $f$, i.e.,

$$\text{epi } f = \{(\eta, x) \mid \eta \geq f(x)\};$$

dom $f$ is the effective domain of $f$, i.e.,

$$\text{dom } f = \{x \mid f(x) < +\infty\};$$

the indicator function of a set $D$

$$\psi_D(x) = 0 \quad \text{if} \quad x \in D,$$

$$= +\infty \quad \text{if} \quad x \notin D;$$
the support function of $D$ is $\psi_D^*$, the conjugate of the indicator function of $D$. We deviate from the standard terminology only in the use of the term closed; here a closed convex function is automatically lower semicontinuous and proper, i.e., $f > -\infty$ and $f \neq +\infty$. Accordingly, closed convex sets are always nonempty, their indicator functions being closed, in this sense, only if they are closed and nonempty.

Finally, to avoid fruitless repetitions, we always say that a sequence (of points, of sets, of functions) converges for all indices in the index set when, actually, we only need or can only assert that convergence occurs for all indices excluding a finite number.

2. Convergence of Convex Sets and $\tau$-Convergence of Convex Functions

Let $\mathbb{N}$ be the positive integers. By $M$ we always denote an infinite subset of $\mathbb{N}$. Let $E$ be a reflexive Banach space and $\{K_v, v \in \mathbb{N}\}$ a sequence of subsets of $E$. Following Mosco [7, Sect. 1.1] we say that the sequence of sets $\{K_v\}$ converges to a set $K$ in $E$ if

$$w\text{-}\lim \sup K_v \subseteq K \subseteq s\text{-}\lim \inf K_v,$$

where

$$w\text{-}\lim \sup K_v = \{x = w\text{-}\lim x_\mu \mid x_\mu \in K_\mu, \mu \in M \text{ and } M \subseteq \mathbb{N}\} \tag{2.1}$$

and

$$s\text{-}\lim \inf K_v = \{x = s\text{-}\lim x_v \mid x_v \in K_v, v \in \mathbb{N}\}. \tag{2.2}$$

Here $x = w\text{-}\lim x_\mu$ means that $x$ is the limit point of the sequence $\{x_\mu\}$ with respect to the weak topology on $E$ and $x = s\text{-}\lim x_v$ means that $x$ is the limit point of the sequence $\{x_v\}$ with respect to the norm (or strong) topology on $E$. Since for any sequence of sets $\{K_v\}$, we always have that

$$w\text{-}\lim \sup K_v \supseteq s\text{-}\lim \inf K_v,$$

then $\{K_v, v \in \mathbb{N}\}$ converges to $K$ if and only if

$$w\text{-}\lim \sup K_v = K = s\text{-}\lim \inf K_v,$$

and we write simply $K_v \rightarrow K$. Note that if the strong and weak topologies on $E$ coincide, as would be the case if $E = \mathbb{R}^n$, then the above notion of convergence for sets is the classical one as defined by Kuratowski [17, p. 83].

$\tau$-Convergence of a sequence $\{f_v, v \in \mathbb{N}\}$ of closed (proper and lower semicontinuous) convex functions on $E$ to a closed convex function $f$, is defined in terms of convergence of the epigraphs of the $f_v$ to the epigraph of $f$. We say that
the sequence \( \{f_v\} \) \( \tau \)–converges to \( f \), which we write \( f_v \to f \), if the sets \( \{\text{epi } f_v \} \) converge to \( \text{epi } f \), i.e.,

\[
f_v \xrightarrow{\tau} f \quad \text{if and only if} \quad \text{epi } f_v \to \text{epi } f.
\]

(2.6)

\( \tau \)-Convergence is defined in a similar way, but now for closed convex functions defined on \( E^* \), the conjugate space of \( E \). In particular, we have that

\[
f_v^* \xrightarrow{\tau^*} f^* \quad \text{if and only if} \quad \text{epi } f_v^* \to \text{epi } f^*.
\]

Mosco obtained the following important characterization of \( \tau \)-convergence:

**Lemma 1** [7, Lemma I.10]. Suppose that \( f \) and \( \{f_v, v \in \mathbb{N}\} \) are closed convex functions defined on \( E \). Then \( f_v \to f \) if and only if

(i) every \( x \in E \) is the s–limit of a sequence \( \{x_v, v \in \mathbb{N}\} \) such that

\[
\limsup_v f_v(x_v) \leq f(x); \quad \text{and}
\]

(ii) given any sequence \( \{x_\mu, \mu \in \mathcal{M}\} \) with \( x = \omega-lim x_\mu \) we have that

\[
\liminf_\mu f_\mu(x_\mu) \geq f(x).
\]

**Proof.** It is straightforward to verify that (i) is equivalent to

\[
\text{s–lim inf epi } f_v \supset \text{epi } f
\]

(2.7)

and that (ii) is equivalent to

\[
\text{w–lim sup epi } f_v \subset \text{epi } f,
\]

(2.8)

which, in view of (2.1) and the definition of \( \tau \)-convergence, yields the desired result. A detailed proof can be found in [7].

3. **Pointwise Convergence and \( \tau \)-Convergence**

**Pointwise convergence** is defined in the usual way. The sequence of functions \( \{f_v, v \in \mathbb{N}\} \) is said to converge pointwise to the function \( f \), written as \( f_v \to f \), if for all \( x \in E \), \( f(x) = \lim f_v(x) \) or equivalently if

\[
\limsup_v f_v(x) \leq f(x) \leq \liminf_v f_v(x).
\]

(3.1)

Equivalence of pointwise convergence and \( \tau \)-convergence for closed convex functions has been proved in some special instances. Using bicontinuity of conjugation, one can easily deduce from Van Cutsem’s results [16, I.10], reproduced in [4], that when \( E \) is finite-dimensional and the sets \( K_v \) and \( K \) are convex compact, we have that

\[
\psi_{K_v}^* \to \psi_K^* \quad \text{if and only if} \quad \psi_{K_v} \xrightarrow{\tau^*} \psi_K^*.
\]
It is also easy to see that if the \( \{K_n\} \) is a decreasing sequence of closed convex sets; then

\[
\psi_{K_n} \to \psi_K \quad \text{if and only if} \quad \psi_{K_n}^\tau \to \psi_K.
\]

Finally, in [12], Robert shows that if in \( \mathbb{R}^n \) pointwise convergence implies \( \tau \)-convergence for a restricted class of closed convex functions having \( \inf \text{ dom } f \) and \( \text{int dom } f^* \) nonempty. (For closed convex functions defined on \( \mathbb{R}^n \) and satisfying these additional conditions, convergence in Robert’s topology is equivalent to \( \tau \)-convergence.)

Simple examples show that equivalence of pointwise convergence and \( \tau \)-convergence cannot be expected unless one restricts oneself to a certain subclass of closed convex functions. The theorems below show that a maximal subclass can actually be identified.

A sequence of closed convex functions \( \{f_\nu, \nu \in \mathbb{N}\} \) is said to be equi-lower semicontinuous at \( x \) (relative to the weak topology) if for every \( \varepsilon > 0 \), there exists \( \mathcal{W} \) a \( w \)-neighborhood of \( x \) such that for all \( \nu \in \mathbb{N} \),

\[
f_\nu(x) - \varepsilon \leq f_\nu(y),
\]

whenever \( y \in \mathcal{W} \). A sequence of functions \( \{f; f_\nu, \nu \in \mathbb{N}\} \) is equi-lower semicontinuous if the following three conditions are satisfied:

(a) \( \{f_\nu, \nu \in \mathbb{N}\} \) is equi-lower semicontinuous at every point in \( \text{dom } f \);

(\( \beta \)) for all \( x \in \text{dom } f \), there exists \( \nu_x \) such that for all \( \nu \geq \nu_x \), \( x \in \text{dom } f_\nu \);

(\( \gamma \)) \( \{f_\nu, \nu \in \mathbb{N}\} \) goes uniformly to \( +\infty \) on every \( w \)-compact subset of \( E \setminus \text{cl dom } f \), the complement of the closure of dom \( f \).

This last condition essentially requires, in a local sense, a uniform divergence of \( f_\nu \) to \( +\infty \) on \( E \setminus \text{cl dom } f \). The two following lemmas yield some of the immediate consequences of pointwise and \( \tau \)-convergence, respectively. In particular, they show that condition (\( \beta \)) is automatically satisfied if \( f_\nu \to f \) and that condition (\( \gamma \)) is automatically satisfied if \( f_\nu \to^\tau f \). Later, we show that together pointwise and \( \tau \)-convergence imply condition (a).

**LEMMA 2.** Suppose that \( f \) and \( \{f_\nu, \nu \in \mathbb{N}\} \) are closed convex functions such that \( f_\nu \to f \). Then

(i) \( \varepsilon \text{-lim inf epi } f_\nu \supseteq \text{epi } f \).

(ii) Condition (\( \beta \)) for the equi-lower semicontinuity of the sequence \( \{f; f_\nu, \nu \in \mathbb{N}\} \) is satisfied. Moreover, if \( E \) is finite-dimensional, then

(iii) Condition (\( \gamma \)) for the equi-lower semicontinuity of the sequence \( \{f; f_\nu, \nu \in \mathbb{N}\} \) is satisfied, when \( E \) is finite dimensional.

**Proof.** Statements (i) and (ii) follow directly from the definitions and Lemma 1(i). Statement (iii) is a direct consequence of the definition of pointwise con-
vergence and the fact that in finite dimension simplical neighborhoods of a point have a finite number of extreme points.

**Lemma 3.** Suppose that \( f \) and \( \{ f_v, v \in \mathbb{N} \} \) are closed convex functions such that \( f_v \to^* f \). Then

(i) \( \liminf f_v(x) \geq f(x) \);

(ii) condition (\( \gamma \)) for the equi-lower semicontinuity of the sequence \( \{ f; f_v, v \in \mathbb{N} \} \) is satisfied.

**Proof.** By condition (ii) of Lemma 1 for every \( M \subset \mathbb{N} \), \( \{ x_u, \mu \in M \} \) and \( x = w\lim x_u \), we have that \( \lim f_u(x_u) \geq f(x) \). Thus, in particular for the sequence \( \{ x_v = x, v \in \mathbb{N} \} \) we have that \( \lim f_v(x) \geq f(x) \). This proves part (i) of the lemma.

To prove (ii) we proceed by contradiction. Now suppose that \( C \) is a \( w \)-compact set in \( E \setminus \text{cl dom} f \) and \( \{ f_v \} \) does not go uniformly to \( +\infty \) on \( C \). Then there exists a sequence \( \{ x_u, \mu \in M \} \) in \( C \) such that for some \( a \in \mathbb{R}, f_u(x_u) < a \) for all \( \mu \in M \). The sequence \( \{ x_u \} \) has at least one \( w \)-cluster point in \( C \), say \( x \). Since \( (x, a) \in \text{epi} f_u \) and a subsequence of \( \{ (x_u, a), \mu \in M \} \) \( w \)-converges to \( (x, a) \) it follows that \( (x, a) \in w\limsup \text{epi} f_v \). Hence \( (x, a) \in \text{epi} f \), since \( w\limsup \text{epi} f_v = \text{epi} f \) by \( \tau \)-convergence; cf. (2.5). This is in contradiction to \( x \notin \text{cl dom} f \).

The two following lemmas yield some of the implications of pointwise convergence when combined with equi-lower semicontinuity.

**Lemma 4.** Suppose that \( \{ f_v, v \in \mathbb{N} \} \) and \( f \) are closed convex functions such that the collection \( \{ f_v \} \) is equi-lower semicontinuous at \( x \in \text{dom} f \) and \( f_v \to f \). Then, for any sequence \( \{ x_u, \mu \in M \subset \mathbb{N} \} \) with \( x = w\lim x_u \), we have that

\[
\liminf f_u(x_u) \geq f(x).
\]

**Proof.** By equi-lower semicontinuity of \( \{ f_u \} \) at \( x, \) for all \( \epsilon > 0 \) there exists a \( w \)-neighborhood \( \mathcal{U} \) of \( x \) such that (3.2) is satisfied for all \( y \in \mathcal{U} \). Thus in particular there exists \( \mu_0 \) such that

\[
f_u(x) - \epsilon \leq f_u(x_u)
\]

for all \( \mu \in M, \mu \geq \mu_0 \). Taking \( \liminf \) on both sides of the inequality and using pointwise convergence, we get that

\[
f(x) - \epsilon \leq \liminf f_u(x_u), \tag{3.3}
\]

from which the lemma follows directly since (3.3) holds for all \( \epsilon > 0 \).

**Lemma 5.** Suppose that \( \{ f_v, v \in \mathbb{N} \} \) and \( f \) are closed convex functions, \( f_v \to f \) and the sequence \( \{ f; f_v, v \in \mathbb{N} \} \) is equi-lower semicontinuous. Then \( (x, a) \in w\limsup \text{epi} f_v \) implies that \( x \in \text{dom} f \).
Proof. First we show that if \((x, a) \in \omega \text{-} \lim \sup epi f_v\), then \(x \in \cl \text{dom}\ f\).

It suffices to prove that if \(x \notin \cl \text{dom}\ f\), there is no \(a \in R\) such that \((x, a) \in \omega \text{-} \lim \sup epi f_v\). Take \(\bar{x} \notin \cl \text{dom}\ f\) and any real number. Then there exists a \(\omega\)-compact convex neighborhood \(\mathcal{N}\) of \(\bar{x}\) such that \(\mathcal{N}\) is contained in \(E \setminus \cl \text{dom}\ f\).

By equi-lower semicontinuity of \(\{f; f_v, \nu \in N\}\), more precisely by (y), the uniform convergence of \(\{f_v\}\) to \(+\infty\) on \(\mathcal{N}\), there exists a \(\omega\)-compact convex neighborhood of \((\bar{x}, a)\), say \(\mathcal{M}\), such that the closed convex sets \(epi f_v\) and \(\mathcal{M}\) are strictly disjoint for all \(\nu\) larger than some \(\bar{\nu}\). Hence, there exists a sequence of hyperplanes \(\{H_v, \nu \geq \bar{\nu}\}\) separating strictly \(\mathcal{M}\) from \(epi f_v\), i.e., for all \(\nu \geq \bar{\nu}\), we have that

\[
H_v^{-} \supset \mathcal{M}
\]

and

\[
H_v^{+} \supset epi f_v,
\]

where \(H_v^{-}\) and \(H_v^{+}\) denote the opposing half-spaces determined by \(H_v\). This implies that

\[
\bigcup_{\nu \geq \bar{\nu}} epi f_v \subset \bigcup_{\nu \geq \bar{\nu}} H_v^{+}
\]

and also that

\[
\omega \text{-cl}(\bigcup_{\nu \geq \bar{\nu}} epi f_v) \subset D = \omega \text{-cl}(\bigcup_{\nu \geq \bar{\nu}} H_v^{+}).
\]

Now \(D \cap \omega \text{-int} \mathcal{M} = \emptyset\) and \((\bar{x}, a) \notin \omega \text{-cl}(\bigcup_{\nu \geq \bar{\nu}} epi f_v)\) since \((\bar{x}, a) \notin D\). Consequently, \((\bar{x}, a) \notin \omega \text{-} \lim \sup epi f_v\), since

\[
\omega \text{-lim} \sup epi f_v = \bigcap_{\nu \in N} \omega \text{-cl} \left( \bigcup_{\nu \geq \bar{\nu}} epi f_v \right).
\]

The above when combined with (2.4) and Lemma 2(ii) implies that

\[
\text{dom}\ f \subset \{x \mid (x, a) \in \omega \text{-} \lim \sup epi f_v\} \subset \cl \text{dom}\ f.
\]

Now, suppose that \((\bar{x}, a) \in \omega \text{-} \lim \sup epi f_v\) and \(\bar{x} \notin \text{dom}\ f\). In view of the theorem above, this implies \(\bar{x} \in \cl \text{dom}\ f\) and \(f(\bar{x}) = +\infty\). It is also easy to see that

\[
C = \{(x, a) = (1 - \lambda) (\bar{x}, a) + \lambda (y, b) \mid \lambda \in [0, 1], (y, b) \in epi f\}
\]

is contained in \(\omega \text{-} \lim \sup epi f_v\), by (2.4) and Lemma 2(ii). Let \(\mathcal{N}'\) again denote a convex \(\omega\)-neighborhood of \(\bar{x}\). \(\mathcal{N}' \cap \text{dom}\ f\) is nonempty because \(\bar{x} \in \text{dom}\ f\) and \(\text{dom}\ f\) is nonempty by properness of \(f\). On the other hand, \(f\) closed and \(f(\bar{x}) = +\infty\) implies that \(f(y)\) tend to \(+\infty\) as \(y\) \(\omega\)-converges to \(\bar{x}\). Take \((x_1, a_1) \in C\) such that \(x_1 \in \mathcal{N}' \cap \text{dom}\ f\) and \(f(x_1) > a_1\). The existence of such a point is guaranteed by the construction. The pointwise convergence of the \(f_v\) to \(f\) implies that for some \(\bar{\nu} \in N\) and an \(\epsilon > 0\) (sufficiently small)

\[
a_1 + 2\epsilon < f_v(x_1) \quad \text{for all } \nu \geq \bar{\nu}.
\]
Since $x_1 \in \text{dom} f$, the collection $\{f_\nu\}$ is equi-lower semicontinuous at $x_1$, i.e., there exists a $w$-neighborhood $\mathcal{N}$ of $x_1$ such that
\[ f_\nu(x_1) - \varepsilon \leq f_\nu(y) \quad \text{for all } y \in \mathcal{N}. \] (3.11)

Combining (3.10) and (3.11) we have that
\[ a_1 + \varepsilon < f_\nu(x_1) - \varepsilon \leq f_\nu(y) \quad \text{for } y \in \mathcal{N} \quad \text{and } \nu \geq \tilde{\nu}. \] (3.12)

The point $(x_1, a_1)$ being in $C \subseteq w\text{-lim sup } \text{epi } f_\nu$ implies the existence of a sequence $\{(x_\mu, a_\mu), \mu \in \mathbb{M} | (x_\mu, a_\mu) \in \text{epi } f_\mu\}$ $w$-converging to $(x_1, a_1)$. Every such sequence must have all $x_\mu \in \mathcal{N}$ for $\mu$ larger than some $\tilde{\mu}$. Thus the elements of the tail of the sequence must satisfy (3.12), i.e.,
\[ a_1 + \varepsilon < f_\mu(x_\mu) - \varepsilon \leq f_\mu(x_\mu) \leq a_\mu \quad \text{for } \mu \geq \tilde{\mu}. \] (3.13)

Taking limits in (3.13), we get a contradiction, since (3.13) would imply that $a_1 + \varepsilon \leq a_1$. Thus $(x_1, a_1) \notin w\text{-lim sup } \text{epi } f_\nu$. This, in turn, invalidates the working assumption, that $(x, a) \in w\text{-lim sup } \text{epi } f$.

Note that the first part of the proof of the lemma and statement (iii) of Lemma 2 also yield the following:

**Corollary.** Suppose that $\{f_\nu, \nu \in \mathbb{N}\}$ and $f$ are closed convex functions defined on $\mathbb{R}^n$ and $f_\nu \to f$. Then $(x, a) \in \text{lim sup } \text{epi } f_\nu$ implies that $x \in \text{cl dom } f$.

**Theorem 1.** Suppose that $\{f_\nu, \nu \in \mathbb{N}\}$ and $f$ are closed convex functions. Then $f_\nu \to f$ and $f_\nu \to f'$ imply that $\{f; f_\nu, \nu \in \mathbb{N}\}$ is an equi-lower semicontinuous collection.

**Proof.** We proceed by contradiction. Suppose that there exists $x \in \text{dom } f$ such that $\{f_\nu\}$ is not equi-lower semicontinuous at $x$. Then by definition of equi-lower semicontinuity and Lemma 2(ii) there exists $\varepsilon > 0$ such that in every $w$-neighborhood $\mathcal{W}$ of $x$, there exists $y_\alpha$ and a corresponding index $\alpha$ such that
\[ f_{\alpha}(x) - \varepsilon > f_{\alpha}(y_\alpha). \] (3.14)

Take $\{\mathcal{W}_\alpha, \alpha \in A\}$ a nested sequence of neighborhoods of $x$ such that $\bigcap \mathcal{W}_\alpha = \{x\}$. Then, from pointwise convergence, it follows that
\[ f(x) - \varepsilon = \liminf \alpha f_{\alpha}(x) - \varepsilon \geq \liminf \alpha f_{\alpha}(y_\alpha), \]
contradicting criterion (ii) in Lemma 1 for $\tau$-convergence. This proves condition (a) for equi-lower semicontinuity. The remainder now follows directly from statements (ii) in Lemmas 2 and 3.

**Theorem 2.** Suppose that $\{f_\nu, \nu \in \mathbb{N}\}$ and $f$ are closed convex functions such
that $f_v \rightarrow f$. Then $f_v \rightarrow^w f$ if and only if the collection $\{f; f_v\}$ is equi-lower semicontinuous.

Proof. In view of Lemma 2(ii) if suffices to show that
\[ \text{epi } f \supset \text{w-lim sup } \text{epi } f_v. \] (3.15)

Take $(x, a) \in \text{w-lim sup } \text{epi } f_v$, i.e., there exists a sequence $\{(x_{\mu}, a_{\mu}), \mu \in M\}$ such that $(x, a) = \text{w-lim}(x_{\mu}, a_{\mu})$ with
\[ f_{\mu}(x_{\mu}) \leq a_{\mu} \quad \text{for all } \mu \in M. \]

Moreover, by equi-lower semicontinuity and Lemma 5, $x \in \text{dom } f$. Then again by equi-lower semicontinuity and Lemma 4, we have that
\[ f(x) \leq \lim \inf f_{\mu}(x_{\mu}) \leq \lim \inf a_{\mu} = a, \]
which implies that $(x, a) \in \text{epi } f$ and thus proves (3.15). The remainder now follows from Theorem 1.

Some special cases of particular interest are given in the corollaries below. We give a separate proof of Corollary 2A.

Corollary 2A. Suppose that $\{f_v, v \in N\}$ and $f$ are closed convex functions and $f_v \rightarrow f$. Suppose, moreover, that $\text{w-int dom } f \neq \emptyset$ and that to each $x \in \text{w-int dom } f$ there corresponds a $\text{w}$-neighborhood $\mathcal{U}$ of $x$ and $\nu_x \in N$ such that $\mathcal{U} \subset \text{w-int dom } f_v$ for all $\nu \geq \nu_x$ and $f_v \rightarrow f$ uniformly on $\mathcal{U}$. Finally, suppose that the functions $f_v$ go uniformly to $+\infty$ on every $\text{w}$-compact subset of $E \setminus \text{cl dom } f$. Then $f_v \rightarrow^w f$.

Proof. Take $x \in \text{w-int dom } f$ and take $\mathcal{U}$, the postulate $\text{w}$-neighborhood of $x$, to be $\text{w}$-compact with $\mathcal{U} \subset \text{w-int dom } f$. The function $f$ is continuous on $\mathcal{U}$ and for $\nu \geq \nu_x$ and so are the functions $f_v$. Now the $f_v$ converge uniformly on $\mathcal{U}$. Take any sequence $\{x_{\mu}, \mu \in M\}$ with $x = \text{w-lim } x_{\mu}$. Without loss of generality, we may assume that $x_{\mu} \in \mathcal{U}$ for all $\mu \in M$. Given any $\epsilon > 0$, continuity of $f$ at $x$ and uniform convergence of $\{f_v\}$ yield the existence of $\mu_x \in M$ such that for all $\mu \geq \mu_x$,
\[ |f(x) - f_{\mu}(x_{\mu})| \leq |f(x) - f(x_{\mu})| + |f(x_{\mu}) - f_{\mu}(x_{\mu})| < \epsilon. \]

In particular, we have that $\lim \inf f_{\mu}(x_{\mu}) \geq f(x)$ for every subsequence $\{x_{\mu}, \mu \in M\}$ $\text{w}$-converging to $x$. Hence for $x \in \text{w-int dom } f$, and $(x, a) \in \text{w-lim sup } \text{epi } f_v$ we also have that $(x, a) \in \text{epi } f$. This follows from Lemma 2(i) and the above; since $(x, a) = \text{w-lim}(x_{\mu}, a_{\mu})$, $a_{\mu} \geq f_{\mu}(x_{\mu})$ implies, in this case, that $a \geq f(x)$.

The first part of the proof of Lemma 5, which only relies on the uniform "divergence" of $\{f_v\}$ to $+\infty$ on $\text{w}$-compact subsets of $E \setminus \text{cl dom } f$, shows that if $(x, a) \in \text{w-lim sup } f_v$, then $x \in \text{cl dom } f$. In view of this and of the above, if $\text{w-lim sup } \text{epi } f_v$ and $\text{epi } f$ differ in any way, there must be a point $(x, a) \in$...
$w$-$\lim$ sup $\text{epi } f$, $x$ on the boundary of $\text{dom } f$ such that $(x, a) \notin \text{epi } f$. It is easy to see that $\text{con}[(x, a), \text{epi } f]$, the convex hull of $(x, a)$, and $\text{epi } f$, must then be contained in $w$-$\lim$ sup $\text{epi } f$. Since $(x, a) \notin \text{epi } f$ and $\text{epi } f$ is closed it follows that there exists a $w$-open neighborhood $\mathcal{N}$ of $(x, a)$ such that $\mathcal{N}$ is disjoint of $\text{epi } f$. Clearly $\mathcal{N}$ intersects the $w$-interior of $\text{con}[(x, a), \text{epi } f]$. This implies the existence of a point $(x_1, a_1) \in w$-$\lim$ sup $\text{epi } f$, with $x_1 \in w$-int $\text{dom } f$ and $a_1 < f(x_1)$. This is in contradiction to the fact, established above, that $y \in w$-int $\text{dom } f$, $(y, b) \in w$-$\lim$ sup $\text{epi } f$ implies that $f(y) < b$. Hence,

$$w$-$\lim$ sup $\text{epi } f \subset \text{epi } f.$$ 

This with Lemma 2(i) yields $\tau$-convergence of $\{f_v\}$ to $f$.

**Corollary 2B.** Suppose that $\{f_v, v \in \mathbb{N}\}$ and $f$ are closed convex functions, $w$-int $\text{dom } f \neq \emptyset$, $\text{dom } f_v \supset \text{dom } f$, and $f_v \to f$ uniformly. Suppose also that the functions $f_v$ go uniformly to $+\infty$ on every $w$-compact subset of $E \setminus \text{cl } \text{dom } f$. Then $f_v \to^\tau f$.

Corollary 2B is just a restatement of Corollary 2A in the case when "convergence" to $\text{dom } f$ occurs by a sequence of sets $\{\text{dom } f_v\}$ containing $\text{dom } f$. The next corollary is a significant strengthening of a result of Robert [12, Theorem 4.7]. It is a consequence of Corollary 2A, Lemma 2(iii), and that in finite dimension the existence of a neighborhood $\mathcal{U}$, as postulated in Corollary 2A, follows directly from [22, Theorem 10.6], Lemma 2(ii) and the existence of a simplicial neighborhood (with a finite number of extreme points) for every point $x$ in int $\text{dom } f$.

**Corollary 2C.** Suppose that $\{f_v, v \in \mathbb{N}\}$ and $f$ are closed convex functions defined on $\mathbb{R}^n$, $f_v \to f$ and int $\text{dom } f \neq \emptyset$. Then $f_v \to^\tau f$.

**Corollary 2D.** Suppose that $f$ and $\{f_v, v \in \mathbb{N}\}$ are closed convex functions defined on $\mathbb{R}^n$ such that $f_v \to f$ and

$$\text{aff } \text{dom } f_v \subset \text{aff } \text{dom } f,$$

where $\text{aff } C$ is the affine hull of $C$. Then $f_v \to^\tau f$.

**Proof.** Lemma 2(ii) shows that $\text{aff } \text{dom } f_v \supset \text{aff } \text{dom } f$ when $f_v \to f$. We can thus apply Corollary 2C, replacing $\mathbb{R}^n$ by the affine space $\text{aff } \text{dom } f$.

**Corollary 2E.** Suppose that $\{f_v, v \in \mathbb{N}\}$ and $f$ are closed convex functions $\text{dom } f_v = \text{dom } f$ and $f_v \to f$. Then $f_v \to^\tau f$.

**Proof.** This follows directly from Corollary 2A, if we view the underlying space as being the smallest closed affine subspace containing $\text{aff } \text{dom } f$ and recast the proof so that everything is relative to this affine subspace of $E$.

We now turn to the converse of Theorem 2. (Remember that by Lemma 3(ii),
condition (γ) of equi-lower semicontinuity is repetitious when the sequence \{f_v\} is \(\tau\)-converging to \(f\).

**Theorem 3.** Suppose that \(f\) and \(\{f_v, \nu \in \mathbb{N}\}\) are closed convex functions such that \(f_v \to^\tau f\). Then \(f_v \to f\) if and only if the collection \(\{f; f_v, \nu \in \mathbb{N}\}\) is equi-lower semicontinuous.

**Proof.** Suppose that \(f_v \to^\tau f\) and \(\{f; f_v\}\) is equi-lower semicontinuous. In view of (3.1) and Lemma 3(i), it suffices to show that \(\limsup f_v(x) \leqslant f(x)\). First suppose that \(x \notin \text{dom } f\); then the preceding inequality is trivially satisfied. Now take \(x \in \text{dom } f\); then by equi-lower semicontinuity (β), \(x \in \text{dom } f_v\) for \(\nu\) sufficiently large. Also, by equi-lower semicontinuity (α) at every \(x \in \text{dom } f\), we have that for all \(\epsilon > 0\)

\[
f_v(x) - \epsilon \leqslant f_v(y)
\]

for all \(\nu \in \mathbb{N}\), provided that \(y \in \mathcal{U}\), for \(\mathcal{U}\) a \(w\)-neighborhood of \(x\). \(\tau\)-Convergence of the \(f_v\) to \(f\) implies that there exists a sequence \(\{x_v, \nu \in \mathbb{N}\}\) such that \(x = s\lim x_v\) and such that \(\limsup f_v(x_v) \leqslant f(x)\); cf. Lemma 1(i). Convergence of \(x_v\) to \(x\) implies that for \(\nu\) sufficiently large, \(x_v \in \mathcal{U}\) and then \(f_v(x) - \epsilon \leqslant f_v(x_v)\). Hence

\[
\limsup f_v(x) - \epsilon \leqslant \limsup f_v(x_v) \leqslant f(x).
\]

The above holding for all \(\epsilon > 0\) shows that for \(x \in \text{dom } f\)

\[
\limsup f_v(x) \leqslant f(x).
\]

The necessity of equi-lower semicontinuity follows directly from Theorem 1.

**Corollary 3A.** Suppose that \(\{f_v, \nu \in \mathbb{N}\}\) and \(f\) are closed convex functions such that \(\text{dom } f_v \supset \text{dom } f\) for all \(\nu \in \mathbb{N}\) and \(f_v \to^\tau f\). Then \(f_v \to f\) if and only if the collection \(\{f_v\}\) is equi-lower semicontinuous at every point of \(\text{dom } f\).

**Corollary 3B** [12, Proposition 4.101]. Suppose that \(f\) and \(\{f_v, \nu \in \mathbb{N}\}\) are closed convex functions defined on \(\mathbb{R}^n\) with \(f_v \to^\tau f\). Then \(f_v \to f\) on \(\text{int } \text{dom } f\).

**Proof.** It suffices to observe that if \(x \in \text{int } \text{dom } f \subset \mathbb{R}^n\) and \(f_v \to^\tau f\) then \(x \in \text{int } \text{dom } f_v\) for \(\nu\) sufficiently large and that consequently the sequence \(\{f_v\}\) convergence uniformly on a closed simplicial neighborhood of \(x\); see [22, Theorem 10.8].

4. Pointwise Convergence of Closed Convex Functions and Their Conjugates

\(\tau\)-Convergence of a sequence of closed convex functions \(\{f_v, \nu \in \mathbb{N}\}\) to a closed convex function \(f\) is equivalent to \(\tau^*\)-convergence of \(\{f^*, \nu \in \mathbb{N}\}\). This result proved in [14] and [11] can also be viewed as a consequence of a result of [13].
for closed convex cones. To see this simply observe that for $f$ closed and convex we have that
\[
\text{pol cl pos}((-1, \eta, x) | (\nu, x) \in \text{epi } f) = \text{cl pos}((\eta^*, -1, x^*) | (\eta^*, x^*) \in \text{epi } f^*),
\]
(4.1)
where $\text{pol}$ associates to a closed convex cone $C$ its polar cone, $\text{pol } C = \{x^* \in E^* | \langle x, x^* \rangle \leq 0\}$, and where $\text{pos}$ denotes the positive hull, $\text{pos } S$ is the intersection of all convex cones containing $S$. Combining (4.1) with the fact that $\text{pol}$ is an isometry [13, Theorem 1] and observing that a collection of closed convex sets $\{K_\nu, \nu \in \mathbb{N}\}$ converges to $K$ if and only if $\{\text{cl pos}((-1) \times K_\nu), \nu \in \mathbb{N}\}$ converges to $\text{cl pos}((-1) \times K)$, shows directly that
\[
f_{\nu} \rightarrow f\quad \text{if and only if}\quad f_{\nu}^* \rightarrow f^*,
\]
(4.2)
provided, naturally, that $\{f_{\nu}, \nu \in \mathbb{N}\}$ and $f$ are closed convex functions.

It is now easy to combine Theorems 2 and 3 and their corollaries with (4.2) to obtain various relations between pointwise convergence of a class of closed convex functions and pointwise convergence of their conjugate functions. In particular, we get the following:

**Theorem 4.** Suppose that $f$ and $\{f_\nu, \nu \in \mathbb{N}\}$ are closed convex functions such that $\{f; f_\nu, \nu \in \mathbb{N}\}$ and $\{f^*; f_\nu^*, \nu \in \mathbb{N}\}$ are equi-lower semicontinuous sequences. Then $f_\nu \rightarrow f$ if and only if $f_{\nu}^* \rightarrow f^*$.

As an example, we give below one of the many implications of this theorem.

**Corollary 4A.** Suppose that $f$ and $\{f_\nu, \nu \in \mathbb{N}\}$ are closed convex functions defined on $\mathbb{R}^n$ such that $\{f_\nu\}$ and $\{f_\nu^*\}$ are equi-lower semicontinuous at every point of $\text{dom } f$ and $\text{dom } f^*$, respectively. Then $f_\nu \rightarrow f_\nu$ and $\text{dom } f_\nu^* \supset \text{dom } f^*$ implies $f_\nu \rightarrow f^*$. Dually $f_\nu^* \rightarrow f^*$ and $\text{dom } f_\nu \supset \text{dom } f$ implies $f_\nu \rightarrow f$.

There does not seem to be a simple condition which can be imposed on the collection $\{f_\nu\}$ which is equivalent to equi-lower semicontinuity of the collection $\{f^*; f_\nu^*, \nu \in \mathbb{N}\}$. There are, however, some special cases, as we see in Section 5, where this property is immediate. In other instances we might be satisfied with equi-lower semicontinuity at a given point as exemplified in the following theorem.

**Theorem 5.** Suppose that $\{f_\nu, \nu \in \mathbb{N}\}$ and $f$ are inf-compact convex functions defined on $\mathbb{R}^n$ such that $\{f; f_\nu\}$ is an equi-lower semicontinuous collection. Then $f_\nu \rightarrow f$ implies that
\[
\text{Min } f_\nu \rightarrow \text{Min } f.
\]
(4.3)
Proof. Inf-compactness of the \( f_v \) and of \( f \) implies that the \( f_v^* \) and \( f^* \) are continuous at \( 0 \); cf. [18] or [19]. The result now follows from Theorem 2, (4.2), Corollary 3B, and the fact that \( \inf f(x) = -f^*(0) \).

Note that in the previous theorem, it would be sufficient to assume that \( f \) is inf-compact, since pointwise convergence \( f_v \to f \) implies that the \( f_v \) are inf-compact from some \( \nu \) on. Taking a somewhat different viewpoint, we extract from (4.2) and the proof of Theorem 3 the following important result for approximation theory.

**Theorem 6.** Suppose that \( \{f_v, \nu \in \mathbb{N}\} \) and \( f \) are closed convex functions such that \( \{f, f_v\} \) is an equi-lower semicontinuous sequence and \( f_v \to f \). Suppose, moreover, that \( \inf f \) is finite. Then there exists \( \{x_{\nu}^*, \nu \in \mathbb{N}\} \) converging to \( 0 \) such that

\[
\lim \{\inf [f_v - \langle \cdot, x_{\nu}^* \rangle]\} = \inf f. \tag{4.4}
\]

**Proof.** Equi-lower semicontinuity of the \( \{f_v\} \) and pointwise convergence of \( \{f_v\} \) to \( f \) implies that \( f_v^* \to f^* \); cf. Theorem 2 and (4.2). Now Lemma 1(i) implies that there exists a sequence \( \{x_{\nu}^*\} \) that s-converges to \( 0 \) such that \( \limsup f_v^*(x_{\nu}^*) \leq f^*(0) = -\inf f \). Also by Lemma 1(ii) we are assured that \( \lim \inf f_v^*(x_{\nu}^*) \geq f^*(0) \). The theorem now follows from the simple observation that \( f_v^*(x_{\nu}^*) = -\inf [f_v - \langle \cdot, x_{\nu}^* \rangle] \).

### 5. Convergence of Convex Sets and Their Support Functions

We can now apply Theorem 4, or variants thereof, to the theory of convergence of convex sets. It is easy to see that convergence of convex sets \( \{K_{\nu}, \nu \in \mathbb{N}\} \) to a (closed) convex set \( K \) is equivalent to the \( \tau \)-convergence of the indicator functions \( \{\psi_{K_{\nu}}, \nu \in \mathbb{N}\} \) to the function \( \psi_K \). Moreover, by (4.2) we have that

\[
\psi_{K_{\nu}} \xrightarrow{\tau} \psi_K \quad \text{if and only if} \quad \psi_{K_{\nu}}^* \xrightarrow{\tau^*} \psi_K^*. \tag{5.1}
\]

However, to obtain pointwise convergence of the indicator functions \( \psi_{K_{\nu}} \) to \( \psi_K \) we must demand (cf. condition \( \beta \) of equi-lower semicontinuity) that for all \( x \in K \) there exists \( \nu_x \) such that \( x \in K_{\nu}, \) for all \( \nu \in \mathbb{N}, \nu \geq \nu_x \). This is akin to requiring that \( \{K_{\nu}, \nu \in \mathbb{N}\} \) be a decreasing sequence. The pointwise convergence of the support functions is characterized in the following theorem. For \( C \) a convex subset of \( E \), we denote by \( 0^+C \) the recession cone of \( C \), i.e., \( 0^+C \) is the maximal convex cone such that

\[
C \supset C + 0^+C. \tag{5.2}
\]

**Theorem 7.** Suppose that \( \{K_{\nu}, \nu \in \mathbb{N}\} \) and \( K \) are closed convex subsets of \( E \) such that for all \( \nu \in \mathbb{N} \) and \( \{\psi_K^*; \psi_{K_{\nu}}^*\} \) is an equi-lower semicontinuous collection with closed effective domain. Then \( K_{\nu} \to K \) implies that \( \psi_{K_{\nu}}^* \to \psi_K^* \)
Proof. In view of (5.1) and Corollary 3A, it suffices to show that for all \( n \in \mathbb{N} \), \( \text{dom } \psi_{K_n}^* \supseteq \text{dom } \psi_K^* \). But this follows directly from the facts that 
\[ \text{dom } \psi_{K_n}^* = \text{pol } 0^+K_n \text{ and dom } \psi_K^* = \text{pol } 0^+K, \] 
and also that 
\[ 0^+K \supseteq 0^+K_n \] 
implies 
\[ \text{pol } 0^+K \supseteq \text{pol } 0^+K_n. \] (5.3)

**Corollary 7A** [16, I.10; 14, Theorem 3.1]. Suppose that \( \{K_n, n \in \mathbb{N}\} \) is a collection of compact convex sets converging to a compact convex set \( K \). Then \( \psi_{K_n}^* \rightarrow \psi_K^* \).

Proof. In this case \( 0^+K_n = 0^+K = \{0\} \) and thus \( \text{dom } \psi_{K_n}^* = \text{dom } \psi_K^* = E^* \) from which follows trivially the equi-lower semicontinuity of \( \{\psi_K^*\} \).

**Corollary 7B.** Suppose that \( \{K_n = C_n + 0^+K_n, n \in \mathbb{N}\} \) is a sequence of closed convex sets converging to a closed convex set \( K = C + 0^+K \), with \( C_n \) converging to \( C \) such that \( C \) and \( \{C_n, n \in \mathbb{N}\} \) are compact and \( 0^+K \supseteq 0^+K_n \). Then \( \psi_{K_n}^* \rightarrow \psi_K^* \).

Proof. Observe that \( \psi_K = \psi_{C_n + 0^+K_n} \) implies that 
\[ \psi_{K_n}^* = \psi_{C_n}^* + \psi_0^*. \] Pointwise convergence now follows from Corollary 7A, \( \psi_{0^+K_n}^* = \psi_{\text{pol } 0^+K_n} \) and (5.3).

Let us also note that if convergence of closed convex sets is defined in terms of convergence of the Hausdorff distance, convergence in this sense can only occur if the sets \( \{K_n, n \in \mathbb{N}\} \) and \( K \) satisfy the hypotheses of Corollary 7B. This corollary thus implies that convergence, in the sense of Hausdorff distance, always implies pointwise convergence of the support functions. This, combined with Theorem 7 can be used to extend the results of [16, Chap. I, Sect. 1] to the noncompact case. We can also use this observation to generalize a result of Hörmander [20] and Ghoula-Houri [21] on the relation between the Hausdorff distance of two compact convex sets and the distance, in a certain norm, of the corresponding support functions, cf. [23].

Of particular interest is a version of Theorem 6 applied to the special case \( \{K_n = C_n + 0^+K_n, n \in \mathbb{N}, C_n \text{ compact}\} \) and the functions \( f_n \) have the special form 
\[ f_n(x) = \langle x, y_n \rangle + \psi_{K_n}, \] (5.4)
for \( K_n \) closed convex sets. Then 
\[ f_n^*(x^*) = \psi_{K_n}^*(x^* - y_n). \] (5.5)
Clearly, if the \( y_n \) are converging to some \( y \), and \( f = \langle \cdot, y \rangle + \psi_K, \{f; f_n\} \) is an equi-lower semicontinuous sequence and 
\[ \text{dom } f_n^* = y_n + \text{dom } \psi_{K_n}^* = y_n + \text{pol}(0^+K_n). \] (5.6)
Corollary 7C. Suppose that \( \{K_v = C_v + 0^+K, v \in \mathbb{N}\} \) and \( K = C + 0^+K \) are closed convex sets in \( E \) with \( \{C_v, v \in \mathbb{N}\} \) and \( C \) compact, and \( 0^+K, \subset 0^+K \). Let \( \{y_v, v \in \mathbb{N}\} \) be any collection in \( E^* \) such that \( y = w\)-lim \( y_v \) with

\[
y_v \in (\text{pol } 0^+K) + y. \tag{5.7}
\]

Suppose, moreover, that the linear functional \( \langle \cdot, y \rangle \) is bounded below on \( K \). Then \( K_v \to K \) implies that

\[
(\inf \langle \cdot, y_v \rangle \text{ on } K_v) \to (\inf \langle \cdot, y \rangle \text{ on } K). \tag{5.8}
\]

Proof. From (5.7), \( 0^+K_v \subset 0^+K \), and (5.3) it follows that for all \( v \in \mathbb{N} \),
\[
dom f_v^* \supset \dom f^* \text{ for } f_v = \langle \cdot, y_v \rangle + \psi_{K_v} \text{ and } f = \langle \cdot, y \rangle + \psi_K \text{ since}
\]
\[
f_v^* = \psi_{K_v}^*(\cdot - y_v) = \psi_v^*(\cdot - y_v) + \psi_{0^+K}^*(\cdot - y_v).
\]

Consequently, to show (5.8), it suffices to establish that

\[
\psi_v^*(-y_v) \to \psi_C^*(-y).
\]

This in turn follows from the fact that since \( \dom \psi_v^* = \dom \psi_C^* = E^* \), we have uniform (pointwise) convergence of \( \psi_v^* \) to \( \psi_C^* \) on every \( w \)-compact subset of \( E^* \).

Corollary 7D [16, I.13]. Suppose that \( \{K_v, v \in \mathbb{N}\} \) is a collection of compact convex sets in \( E \) converging to a compact set \( K \) and \( \{y_v, v \in \mathbb{N}\} \) is a collection of points in \( E^* \) such that \( y = w\)-lim \( y_v \). Then

\[
(\inf \langle \cdot, y_v \rangle \text{ on } K_v) \to (\inf \langle \cdot, y \rangle \text{ on } K). \tag{5.9}
\]

References

8. A. BENSOUSSAN, A. BOSSAVET, AND J. NEDELEC, Approximation de problème de
$L^p$, Ms., Univ. de Bordeaux, France, 1974.
10. J.-L. JOLY, "Une famille de topologies et de convergences sur l'ensemble des fonc-
11. J.-L. JOLY, Une famille de topologies sur l'ensemble des fonctions convexes pour
lesquelles la polarité est bicontinue, Technical Report, Univ. de Bordeaux, France,
533–555.
15. R. WIJSMAN, Convergence of sequences of convex sets, cones and functions, II,
16. B. VAN CUTSEM, "Eléments aléatoires à valeurs convexes compactes," Thèse,
18. R. T. ROCKAFELLAR, Level sets and continuity of conjugate convex functions, Trans.
19. J.-J. MOREAU, Sur la fonction polaire d'une fonction semi-continue superieurement,
20. L. HÖRMANDER, Sur la fonction d'appui des ensembles convexes dans un espace
21. A. GHOULA-HOURI, Cônes de Banach-Application aux problèmes de contrôle, Ms.,
Univ. de Caen, France, 1966.
23. G. SALINETTI AND R. WETS, On the convergence of sequences of convex sets in finite