JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **60**, 211–226 (1977)

On the Relations between Two Types of Convergence for Convex Functions

GABRIELLA SALINETTI* AND ROGER J.-B. WETS†

Università di Roma, Roma, Italy; University of Kentucky, Lexington, Kentucky and Stanford University, Stanford, California

Submitted by G. L. Lions

Theory and applications have shown that there are two important types of convergence for convex functions: pointwise convergence and convergence in a topology induced by the convergence of their epigraphs. We show that these two types of convergence are equivalent on the class of convex functions which are equi-lower semicontinuous. This turns out to be maximal classes of convex functions for which this equivalence can be obtained. We also indicate a number of implications of these results to the convergence of convex sets and the corresponding support functions and to the convergence of the infima of sequences of convex minimization problems.

1. INTRODUCTION

The study of the convergence of convex sets and convex functions has usually been undertaken to the foundations for approximation results in statistics [1, 2], in convex optimization [3–5], in the theory of variational inequalities [6, 7], and for control problems [8, 9]. The key question always boils down to: Given a sequence of minimization problems $\{\mathbf{P}_\nu\}$ whose constraints and objective converge in some sense, usually pointwise, to the constraints and objective of a minimization problem $\mathbf{P}$, does it follow that the sequence of optimal values of $\mathbf{P}_\nu$ converge to the optimal value of $\mathbf{P}$? (There are a number of related questions such as convergence of the optimality sets to the optimality set of $\mathbf{P}$, convergence of the associated price systems,...). To each minimization problem $\mathbf{P}_\nu$ we can associate an extended real-valued function f_ν with $f_\nu(x) = +\infty$ if x fails to satisfy the constraints of $\mathbf{P}_\nu$ and otherwise $f_\nu(x)$ is the value of the objective of $\mathbf{P}_\nu$ evaluated at x. In this framework, we have that

$$\inf \mathbf{P}_\nu = \inf_x f_\nu(x) = -f_\nu{}^*(0), \tag{1.1}$$

* Supported in part by C.N.R. (Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni).

† Supported in part by the National Science Foundation under Grant MPS 75-07028 at the University of Kentucky.

ISSN 0022–247X

where f^* is the *conjugate* of f. The convergence of the infimum values of the $\mathbf{P}_\nu$ to the infimum of $\mathbf{P}$ is then equivalent to the (pointwise) convergence of the conjugate functions f_ν^* to f^* at 0. Of more general interest is the (pointwise) convergence of f_ν^* to f^*; each point x^* corresponding to a perturbed version of the original problem, since

$$f_\nu^*(x^*) = -\inf_x [f_\nu(x) - \langle x, x^* \rangle]. \tag{1.2}$$

The study of these and related convergence questions leads to the definition of a topology τ on the class of lower semicontinuous convex functions. Convergence of f_ν to f in this topology is equivalent to the convergence of epi f_ν, the epigraphs of f_ν, to epi f. This topology τ was introduced by Mosco [7, Sect. 1.7]; various extensions and refinements were obtained by Joly [10, 11] and in [12], Robert works with a variant of this topology. It plays an important role in the study of convergence of convex sets and also convex functions because it turns out that conjugation is bicontinuous with respect to this topology; in particular, we have that

$$f_\nu \xrightarrow{\tau} f \qquad \text{if and only if} \qquad f_\nu^* \xrightarrow{\tau^*} f^*$$

for τ^* defined in a way similar to τ. This property, first established by Wakup and Wets [13] (in the framework of closed convex cones) and later, independently by Mosco [14] and Joly [11], plays a key role in the study of the convergence of convex sets and functions. In particular, it allows us to relate the convergence of convex sets to the pointwise convergence of their support functions. In the compact case (for compact convex sets) the relations between these two types of convergences have been investigated by Wÿsman [15] and Van Cutsem [16].

The main objective of this note is to delineate the relations between pointwise convergence and τ-convergence for sequences of convex functions. In particular, we identify the largest class of functions for which these two types of convergences coincide. We also show the implications of these results to the convergence of convex sets and their support functions and to the convergence of the infima of sequences of convex programs.

The terminology and notation is the standard one of convex analysis. For f a convex function, we write epi f for the *epigraph* of f, i.e.,

$$\operatorname{epi} f = \{(\eta, x) \mid \eta \geqslant f(x)\};$$

dom f is the *effective domain* of f, i.e.,

$$\operatorname{dom} f = \{x \mid f(x) < +\infty\};$$

the *indicator function* of a set D

$$\begin{aligned} \psi_D(x) &= 0 && \text{if} \quad x \in D, \\ &= +\infty && \text{if} \quad x \notin D; \end{aligned}$$

the *support function* of D is $\psi_D{}^*$, the conjugate of the indicator function of D. We deviate from the standard terminology only in the use of the term *closed*; here a closed convex function is automatically lower semicontinuous and *proper*, i.e., $f > -\infty$ and $f \not\equiv +\infty$. Accordingly, *closed convex sets are always nonempty*, their indicator functions being closed, in this sense, only if they are closed *and* nonempty.

Finally, to avoid fruitless repetitions, we always say that a sequence (of points, of sets, of functions) converges *for all* indices in the index set when, actually, we only need or can only assert that convergence occurs for all indices excluding a finite number.

2. Convergence of Convex Sets and τ-Convergence of Convex Functions

Let $\mathbf{N}$ be the positive integers. By $\mathbf{M}$ we *always* denote an infinite subset of $\mathbf{N}$. Let E be a reflexive Banach space and $\{K_\nu, \nu \in \mathbf{N}\}$ a sequence of subsets of E. Following Mosco [7, Sect. 1.1] we say that *the sequence of sets* $\{K_\nu\}$ *converges to a set* K *in* E if

$$w\text{-}\lim\sup K_\nu \subset K \subset s\text{-}\lim\inf K_\nu, \tag{2.1}$$

where

$$w\text{–}\lim\sup K_\nu = \{x = w\text{–}\lim x_\mu \mid x_\mu \in K_\mu, \mu \in \mathbf{M} \text{ and } \mathbf{M} \subset \mathbf{N}\} \tag{2.2}$$

and

$$s\text{–}\lim\inf K_\nu = \{x = s\text{–}\lim x_\nu \mid x_\nu \in K_\nu, \nu \in \mathbf{N}\}. \tag{2.3}$$

Here $x = w\text{–}\lim x_\mu$ means that x is the limit point of the sequence $\{x_\mu\}$ with respect to the weak topology on E and $x = s\text{–}\lim x_\nu$ means that x is the limit point of the sequence $\{x_\nu\}$ with respect to the norm (or strong) topology on E. Since for any sequence of sets $\{K_\nu\}$, we always have that

$$w\text{–}\lim\sup K_\nu \supset s\text{–}\lim\inf K_\nu, \tag{2.4}$$

then $\{K_\nu, \nu \in \mathbf{N}\}$ converges to K if and only if

$$w\text{–}\lim\sup K_\nu = K = s\text{–}\lim\inf K_\nu, \tag{2.5}$$

and we write simply $K_\nu \to K$. Note that if the strong and weak topologies on E coincide, as would be the case if $E = R^n$, then the above notion of convergence for sets is the classical one as defined by Kuratowski [17, p. 83].

τ-Convergence of a sequence $\{f_\nu, \nu \in \mathbf{N}\}$ of *closed* (proper *and* lower semicontinuous) convex functions on E to a closed convex function f, is defined in terms of convergence of the epigraphs of the f_ν to the epigraph of f. We say that

the sequence $\{f_\nu\}$ τ-converges to f, which we write $f_\nu \to^\tau f$, if the sets $\{\text{epi} f_\nu, \nu \in \mathbf{N}\}$ converge to epi f, i.e.,

$$f_\nu \xrightarrow{\tau} f \qquad \text{if and only if} \qquad \text{epi} f_\nu \to \text{epi} f. \tag{2.6}$$

τ^*-Convergence is defined in a similar way, but now for closed convex functions defined on E^*, the conjugate space of E. In particular, we have that

$$f_\nu{}^* \xrightarrow{\tau^*} f^* \qquad \text{if and only if} \qquad \text{epi} f_\nu{}^* \to \text{epi} f^*.$$

Mosco obtained the following important characterization of τ-convergence:

LEMMA 1 [7, Lemma 1.10]. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions defined on E. Then $f_\nu \to^\tau f$ if and only if*

(i) *every $x \in E$ is the s–limit of a sequence $\{x_\nu, \nu \in \mathbf{N}\}$ such that $\limsup_\nu f_\nu(x_\nu) \leqslant f(x)$; and*

(ii) *given any sequence $\{x_\mu, \mu \in \mathbf{M}\}$ with $x = w\text{–}\lim x_\mu$ we have that $\liminf_\mu f_\mu(x_\mu) \geqslant f(x)$.*

Proof. It is straigthforward to verify that (i) is equivalent to

$$s\text{–}\liminf \text{epi} f_\nu \supset \text{epi} f \tag{2.7}$$

and that (ii) is equivalent to

$$w\text{–}\limsup \text{epi} f_\nu \subset \text{epi} f, \tag{2.8}$$

which, in view of (2.1) and the definition of τ-convergence, yields the desired result. A detailed proof can be found in [7].

3. Pointwise Convergence and τ-Convergence

Pointwise convergence is defined in the usual way. The sequence of functions $\{f_\nu, \nu \in \mathbf{N}\}$ is said to converge pointwise to the function f, written as $f_\nu \to f$, if for all $x \in E$, $f(x) = \lim f_\nu(x)$ or equivalently if

$$\limsup f_\nu(x) \leqslant f(x) \leqslant \liminf f_\nu(x). \tag{3.1}$$

Equivalence of pointwise convergence and τ-convergence for closed convex functions has been proved in some special instances. Using bicontinuity of conjugation, one can easily deduce from Van Cutsem's results [16, I.10], reproduced in [4], that when E is finite-dimensional and the sets K_ν and K are convex compact, we have that

$$\psi^*_{K_\nu} \to \psi_K{}^* \qquad \text{if and only if} \qquad \psi^*_{K_\nu} \xrightarrow{\tau^*} \psi_K{}^*.$$

It is also easy to see that if the $\{K_\nu\}$ is a decreasing sequence of closed convex sets; then

$$\psi_{K_\nu} \to \psi_K \qquad \text{if and only if} \qquad \psi_{K_\nu} \xrightarrow{\tau} \psi_K .$$

Finally, in [12], Robert shows that in R^n pointwise convergence implies τ-convergence for a restricted class of closed convex functions having inf dom f and int dom f^* nonempty. (For closed convex functions defined on R^n and satisfying these additional conditions, convergence in Robert's topology is equivalent to τ-convergence.)

Simple examples show that equivalence of pointwise convergence and τ-convergence cannot be expected unless one restricts oneself to a certain subclass of closed convex functions. The theorems below show that a maximal subclass can actually be identified.

A sequence of closed convex functions $\{f_\nu, \nu \in \mathbf{N}\}$ is said to be *equi-lower semicontinuous* at x (relative to the weak topology) if for every $\epsilon > 0$, there exists $\mathscr{U}$ a w-neighborhood of x such that for all $\nu \in N$,

$$f_\nu(x) - \epsilon \leqslant f_\nu(y), \tag{3.2}$$

whenever $y \in \mathscr{U}$. A sequence of functions $\{f; f_\nu, \nu \in \mathbf{N}\}$ is *equi-lower semicontinuous* if the following three conditions are satisfied:

(α) $\{f_\nu, \nu \in \mathbf{N}\}$ is equi-lower semicontinuous at every point in dom f;

(β) for all $x \in \operatorname{dom} f$, there exists ν_x such that for all $\nu \geqslant \nu_x$, $x \in \operatorname{dom} f_\nu$;

(γ) $\{f_\nu, \nu \in \mathbf{N}\}$ goes uniformly to $+\infty$ on every w-compact subset of $E\backslash$cl dom f, the complement of the closure of dom f.

This last condition essentially requires, in a local sense, a uniform divergence of f_ν to $+\infty$ on $E\backslash$cl dom f. The two following lemmas yield some of the immediate consequences of pointwise and τ-convergence, respectively. In particular, they show that condition (β) is automatically satisfied if $f_\nu \to f$ and that condition (γ) is automatically satisfied if $f_\nu \to^\tau f$. Later, we show that together pointwise and τ-convergence imply condition (α).

LEMMA 2. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions such that $f_\nu \to f$. Then*

(i) s–lim inf epi $f_\nu \supset$ epi f.

(ii) *Condition (β) for the equi-lower semicontinuity of the sequence $\{f; f_\nu, \nu \in \mathbf{N}\}$ is satisfied. Moreover, if E is finite-dimensional, then*

(iii) *Condition (γ) for the equi-lower semicontinuity of the sequence $\{f; f_\nu, \nu \in \mathbf{N}\}$ is satisfied, when E is finite dimensional.*

Proof. Statements (i) and (ii) follow directly from the definitions and Lemma 1(i). Statement (iii) is a direct consequence of the definition of pointwise convergence and the fact that in finite dimension simplical neighborhoods of a point have a finite number of extreme points.

LEMMA 3. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions such that $f_\nu \to^\tau f$. Then*

(i) $\liminf f_\nu(x) \geqslant f(x)$;

(ii) *condition* (γ) *for the equi-lower semicontinuity of the sequence* $\{f; f_\nu, \nu \in \mathbf{N}\}$ *is satisfied.*

Proof. By condition (ii) of Lemma 1 for every $\mathbf{M} \subset \mathbf{N}$, $\{x_\mu, \mu \in \mathbf{M}\}$ and $x = w\text{-}\lim x_\mu$, we have that $\liminf f_\mu(x_\mu) \geqslant f(x)$. Thus, in particular for the sequence $\{x_\nu = x, \nu \in \mathbf{N}\}$ we have that $\liminf f_\nu(x) \geqslant f(x)$. This proves part (i) of the lemma.

To prove (ii) we proceed by contradiction. Now suppose that C is a w-compact set in $E \backslash \mathrm{cl}\,\mathrm{dom}\, f$ and $\{f_\nu\}$ does not go uniformly to $+\infty$ on C. Then there exists a sequence $\{x_\mu, \mu \in \mathbf{M}\}$ in C such that for some $a \in R$, $f_\mu(x_\mu) < a$ for all $\mu \in \mathbf{M}$. The sequence $\{x_\mu\}$ has at least one w-cluster point in C, say x. Since $(x, a) \in \mathrm{epi}\, f_\mu$ and a subsequence of $\{(x_\mu, a), \mu \in \mathbf{M}\}$ w-converges to (x, a) it follows that $(x, a) \in w\text{-}\limsup \mathrm{epi}\, f_\nu$. Hence $(x, a) \in \mathrm{epi}\, f$, since $w\text{-}\limsup \mathrm{epi}\, f_\nu = \mathrm{epi}\, f$ by τ-convergence; cf. (2.5). This is in contradiction to $x \notin \mathrm{cl}\,\mathrm{dom}\, f$.

The two following lemmas yield some of the implications of pointwise convergence when combined with equi-lower semicontinuity.

LEMMA 4. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions such that the collection $\{f_\nu\}$ is equi-lower semicontinuous at $x \in \mathrm{dom}\, f$ and $f_\nu \to f$. Then, for any sequence $\{x_\mu, \mu \in \mathbf{M} \subset \mathbf{N}\}$ with $x = w\text{-}\lim x_\mu$, we have that*

$$\liminf f_\mu(x_\mu) \geqslant f(x).$$

Proof. By equi-lower semicontinuity of $\{f_\nu\}$ at x, for all $\epsilon > 0$ there exists a w-neighborhood $\mathscr{U}$ of x such that (3.2) is satisfied for all $y \in \mathscr{U}$. Thus in particular there exists μ_0 such that

$$f_\mu(x) - \epsilon \leqslant f_\mu(x_\mu)$$

for all $\mu \in \mathbf{M}$, $\mu \geqslant \mu_0$. Taking lim inf on both sides of the inequality and using pointwise convergence, we get that

$$f(x) - \epsilon \leqslant \liminf f_\mu(x_\mu), \tag{3.3}$$

from which the lemma follows directly since (3.3) holds for all $\epsilon > 0$.

LEMMA 5. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions, $f_\nu \to f$ and the sequence $\{f; f_\nu, \nu \in \mathbf{N}\}$ is equi-lower semicontinuous. Then $(x, a) \in w\text{-}\limsup$ $\mathrm{epi}\, f_\nu$ implies that $x \in \mathrm{dom}\, f$.*

Proof. First we show that if $(x, a) \in w\text{–}\limsup \operatorname{epi} f_\nu$, then $x \in \operatorname{cl} \operatorname{dom} f$. It suffices to prove that if $x \notin \operatorname{cl} \operatorname{dom} f$, there is no $a \in R$ such that $(x, a) \in w\text{–}\limsup \operatorname{epi} f_\nu$. Take $\bar{x} \notin \operatorname{cl} \operatorname{dom} f$ and a any real number. Then there exists a w-compact convex neighborhood $\mathscr{N}$ of $\bar{x}$ such that $\mathscr{N}$ is contained in $E \backslash \operatorname{cl} \operatorname{dom} f$. By equi-lower semicontinuity of $\{f; f_\nu, \nu \in \mathbf{N}\}$, more precisely by ($\gamma$), the uniform convergence of $\{f_\nu\}$ to $+\infty$ on $\mathscr{N}$, there exists a w-compact convex neighborhood of $(\bar{x}, a)$, say $\mathscr{M}$, such that the closed convex sets $\operatorname{epi} f_\nu$ and $\mathscr{M}$ are strictly disjoint for all ν larger than some $\bar{\nu}$. Hence, there exists a sequence of hyperplanes $\{H_\nu, \nu \geqslant \bar{\nu}\}$ separating strictly $\mathscr{M}$ from $\operatorname{epi} f_\nu$, i.e., for all $\nu \geqslant \bar{\nu}$, we have that

$$H_\nu^- \supset \mathscr{M} \tag{3.4}$$

and

$$H_\nu^+ \supset \operatorname{epi} f_\nu, \tag{3.5}$$

where H_ν^+ and H_ν^- denote the opposing half-spaces determined by H_ν. This implies that

$$\bigcup_{\nu \geqslant \bar{\nu}} \operatorname{epi} f_\nu \subset \bigcup_{\nu \geqslant \bar{\nu}} H_\nu^+ \tag{3.6}$$

and also that

$$w\text{–cl}\Big(\bigcup_{\nu \geqslant \bar{\nu}} \operatorname{epi} f_\nu\Big) \subset D = w\text{–cl}\Big(\bigcup_{\nu \geqslant \bar{\nu}} H_\nu^+\Big). \tag{3.7}$$

Now $D \cap w\text{–int}\, \mathscr{M} = \varnothing$ and $(\bar{x}, a) \notin w\text{–cl}(\bigcup_{\nu \geqslant \bar{\nu}} \operatorname{epi} f_\nu)$ since $(\bar{x}, a) \notin D$. Consequently, $(\bar{x}, a) \notin w\text{–}\limsup \operatorname{epi} f_\nu$, since

$$w\text{–}\limsup \operatorname{epi} f_\nu = \bigcap_{\nu \in \mathbf{N}} w\text{–cl}\left(\bigcup_{\mu \geqslant \nu} \operatorname{epi} f_\mu\right). \tag{3.8}$$

The above when combined with (2.4) and Lemma 2(ii) implies that

$$\operatorname{dom} f \subset \{x \mid (x, a) \in w\text{–}\limsup \operatorname{epi} f_\nu\} \subset \operatorname{cl} \operatorname{dom} f. \tag{3.9}$$

Now, suppose that $(\bar{x}, \bar{a}) \in w\text{–}\limsup \operatorname{epi} f_\nu$ and $\bar{x} \notin \operatorname{dom} f$. In view of the the above, this implies $\bar{x} \in \operatorname{cl} \operatorname{dom} f$ and $f(\bar{x}) = +\infty$. It is also easy to see that

$$C = \{(x, a) = (1 - \lambda)(\bar{x}, \bar{a}) + \lambda(y, b) \mid \lambda \in [0, 1], (y, b) \in \operatorname{epi} f\}$$

is contained in $w\text{–}\limsup \operatorname{epi} f_\nu$, by (2.4) and Lemma 2(ii). Let $\mathscr{N}$ again denote a convex w-neighborhood of $\bar{x}$. $\mathscr{N} \cap \operatorname{dom} f$ is nonempty because $\bar{x} \in \operatorname{dom} f$ and $\operatorname{dom} f$ is nonempty by properness of f. On the other hand, f closed and $f(\bar{x}) = +\infty$ implies that $f(y)$ tend to $+\infty$ as y w-converges to $\bar{x}$. Take $(x_1, a_1) \in C$ such that $x_1 \in \mathscr{N} \cap \operatorname{dom} f$ and $f(x_1) > a_1$. The existence of such a point is guaranteed by the construction. The pointwise convergence of the f_ν to f implies that for some $\bar{\nu} \in \mathbf{N}$ and an $\epsilon > 0$ (sufficiently small)

$$a_1 + 2\epsilon < f_\nu(x_1) \qquad \text{for all } v \geqslant \bar{\nu}. \tag{3.10}$$

Since $x_1 \in \operatorname{dom} f$, the collection $\{f_\nu\}$ is equi-lower semicontinuous at x_1, i.e., there exists a w-neighborhood $\mathcal{N}$ of x_1 such that

$$f_\nu(x_1) - \epsilon \leqslant f_\nu(y) \qquad \text{for all } y \in \mathcal{N}. \tag{3.11}$$

Combining (3.10) and (3.11) we have that

$$a_1 + \epsilon < f_\nu(x_1) - \epsilon \leqslant f_\nu(y) \qquad \text{for } y \in \mathcal{N} \quad \text{and} \quad \nu \geqslant \bar{\nu}. \tag{3.12}$$

The point (x_1, a_1) being in $C \subset w$–lim sup epi f_ν implies the existence of a sequence $\{(x_\mu, a_\mu), \mu \in \mathbf{M} \mid (x_\mu, a_\mu) \in \operatorname{epi} f_\mu\}$ w-converging to (x_1, a_1). Every such sequence must have all $x_\mu \in \mathcal{N}$ for μ larger than some $\bar{\mu}$. Thus the elements of the tail of the sequence must satisfy (3.12), i.e.,

$$a_1 + \epsilon < f_\mu(x_1) - \epsilon \leqslant f_\mu(x_\mu) \leqslant a_\mu \qquad \text{for } \mu \geqslant \bar{\mu}. \tag{3.13}$$

Taking limits in (3.13), we get a contradiction, since (3.13) would imply that $a_1 + \epsilon \leqslant a_1$. Thus $(x_1, a_1) \notin w$–lim sup epi f_ν. This, in turn, invalidates the working assumption, that $(\bar{x}, \bar{a}) \in w$–lim sup and $\bar{x} \notin \operatorname{dom} f$.

Note that the first part of the proof of the lemma and statement (iii) of Lemma 2 also yield the following:

COROLLARY. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions defined on R^n and $f_\nu \to f$. Then $(x, a) \in \limsup \operatorname{epi} f_\nu$ implies that $x \in \operatorname{cl} \operatorname{dom} f$.*

THEOREM 1. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions. Then $f_\nu \to f$ and $f_\nu \to^\tau f$ imply that $\{f; f_\nu, \nu \in \mathbf{N}\}$ is an equi-lower semicontinuous collection.*

Proof. We proceed by contradiction. Suppose that there exists $x \in \operatorname{dom} f$ such that $\{f_\nu\}$ is not equi-lower semicontinuous at x. Then by definition of equi-lower semicontinuity and Lemma 2(ii) there exists $\epsilon > 0$ such that in every w-neighborhood $\mathcal{U}_\alpha$ of x, there exists y_α and a corresponding index ν_α such that

$$f_{\nu_\alpha}(x) - \epsilon > f_{\nu_\alpha}(y_\alpha). \tag{3.14}$$

Take $\{\mathcal{U}_\alpha, \alpha \in \mathbf{A}\}$ a nested sequence of neighborhoods of x such that $\bigcap \mathcal{U}^\alpha = \{x\}$. Then, from pointwise convergence, it follows that

$$f(x) - \epsilon = \liminf_\alpha f_{\nu_\alpha}(x) - \epsilon \geqslant \liminf_\alpha f_{\nu_\alpha}(y_\alpha),$$

contradicting criterion (ii) in Lemma 1 for τ-convergence. This proves condition (α) for equi-lower semicontinuity. The remainder now follows directly from statements (ii) in Lemmas 2 and 3.

THEOREM 2. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions such* *that $f_\nu \to f$. Then $f_\nu \to^\tau f$ if and only if the collection $\{f; f_\nu\}$ is equi-lower semi-continuous.*

Proof. In view of Lemma 2(ii) if suffices to show that

$$\operatorname{epi} f \supset w\text{–}\lim\sup \operatorname{epi} f_\nu . \tag{3.15}$$

Take $(x, a) \in w\text{–}\lim\sup \operatorname{epi} f_\nu$, i.e., there exists a sequence $\{(x_\mu, a_\mu), \mu \in \mathbf{M}\}$ such that $(x, a) = w\text{–}\lim(x_\mu, a_\mu)$ with

$$f_\mu(x_\mu) \leqslant a_\mu \qquad \text{for all } \mu \in \mathbf{M}.$$

Moreover, by equi-lower semicontinuity and Lemma 5, $x \in \operatorname{dom} f$. Then again by equi-lower semicontinuity and Lemma 4, we have that

$$f(x) \leqslant \liminf f_\mu(x_\mu) \leqslant \liminf a_\mu = a,$$

which implies that $(x, a) \in \operatorname{epi} f$ and thus proves (3.15). The remainder now follows from Theorem 1.

Some special cases of particular interest are given in the corollaries below. We give a separate proof of Corollary 2A.

Corollary 2A. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions and $f_\nu \to f$. Suppose, moreover, that w–int dom $f \neq \varnothing$ and that to each $x \in w$–int dom f there corresponds a w-neighborhood $\mathscr{U}$ of x and $\nu_x \in \mathbf{N}$ such that $\mathscr{U} \subset w$–int dom f_ν for all $\nu \geqslant \nu_x$ and $f_\nu \to f$ uniformly on $\mathscr{U}$. Finally, suppose that the functions f_ν go uniformly to $+\infty$ on every w-compact subset of $E\backslash$cl dom f. Then $f_\nu \to^\tau f$.*

Proof. Take $x \in w$–int dom f and take $\mathscr{U}$, the postulate w-neighborhood of x, to be w-compact with $\mathscr{U} \subset w$–int dom f. The function f is continuous on $\mathscr{U}$ and for $\nu \geqslant \nu_x$ and so are the functions f_ν. Now the f_ν converge uniformly on $\mathscr{U}$. Take any sequence $\{x_\mu, \mu \in \mathbf{M}\}$ with $x = w\text{–}\lim x_\mu$. Without loss of generality, we may assume that $x_\mu \in \mathscr{U}$ for all $\mu \in \mathbf{M}$. Given any $\epsilon > 0$, continuity of f at x and uniform convergence of $\{f_\nu\}$ yield the existence of $\mu_\epsilon \in \mathbf{M}$ such that for all all $\mu \geqslant \mu_\epsilon$,

$$|f(x) - f_\mu(x_\mu)| \leqslant |f(x) - f(x_\mu)| + |f(x_\mu) - f_\mu(x_\mu)| < \epsilon.$$

In particular, we have that $\liminf f_\mu(x_\mu) \geqslant f(x)$ for every subsequence $\{x_\mu, \mu \in \mathbf{M}\}$ w–converging to x. Hence for $x \in w$–int dom f, and $(x, a) \in w\text{–}\lim\sup \operatorname{epi} f_\nu$ we also have that $(x, a) \in \operatorname{epi} f$. This follows from Lemma 2(i) and the above; since $(x, a) = w\text{–}\lim(x_\mu, a_\mu)$, $a_\mu \geqslant f_\mu(x_\mu)$ implies, in this case, that $a \geqslant f(x)$.

The first part of the proof of Lemma 5, which only relies on the uniform "divergence" of $\{f_\nu\}$ to $+\infty$ on w-compact subsets of $E\backslash$cl dom f, shows that if $(x, a) \in w\text{–}\lim\sup f_\mu$, then $x \in \text{cl}\backslash\text{dom} f$. In view of this and of the above, if $w\text{–}\lim\sup \operatorname{epi} f_\nu$ and $\operatorname{epi} f$ differ in any way, there must be a point $(x, a) \in$

w–lim sup epi f_ν, x on the boundary of dom f such that $(x, a) \notin$ epi f. It is easy to see that con$\{(x, a)$, epi $f\}$, the convex hull of (x, a), and epi f, must then be contained in w–lim sup epi f_ν. Since $(x, a) \notin$ epi f and epi f is closed it follows that there exists a w-open neighborhood $\mathcal{N}$ of (x, a) such that $\mathcal{N}$ is disjoint of epi f. Clearly $\mathcal{N}$ intersects the w-interior of con$\{(x, a)$, epi $f\}$. This implies the existence of a point $(x_1, a_1) \in w$–lim sup epi f_ν with $x_1 \in w$–int dom f and $a_1 < f(x_1)$. This is in contradiction to the fact, established above, that $y \in w$–int dom f, $(y, b) \in w$–lim sup epi f_ν implies that $f(y) \leqslant b$. Hence,

$$w\text{–lim sup epi } f_\nu \subset \text{epi } f.$$

This with Lemma 2(i) yields τ-convergence of $\{f_\nu\}$ to f.

COROLLARY 2B. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions, w–int dom $f \neq \varnothing$, dom $f_\nu \supset$ dom f, and $f_\nu \to f$ uniformly. Suppose also that the functions f_ν go uniformly to $+\infty$ on every w-compact subset of $E \backslash$cl dom f. Then $f_\nu \to^\tau f$.*

Corollary 2B is just a restatement of Corollary 2A in the case when "convergence" to dom f occurs by a sequence of sets $\{\text{dom } f_\nu\}$ containing dom f. The next corollary is a significant strenghtening of a result of Robert [12, Theorem 4.7]. It is a consequence of Corollary 2A, Lemma 2(iii), and that in finite dimension the existence of a neighborhood $\mathcal{U}$, as postulated in Corollary 2A, follows directly from [22, Theorem 10.6], Lemma 2(ii) and the existence of a simplicial neighborhood (with a finite number of extreme points) for every point x in int dom f.

COROLLARY 2C. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions defined on R^n, $f_\nu \to f$ and* int dom $f \neq \varnothing$. *Then $f_\nu \to^\tau f$.*

COROLLARY 2D. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions defined on R^n such that $f_\nu \to f$ and*

$$\text{aff dom } f_\nu \subset \text{aff dom } f,$$

where aff C *is the affine hull of C. Then $f_\nu \to^\tau f$.*

Proof. Lemma 2(ii) shows that aff dom $f_\nu \supset$ aff dom f when $f_\nu \to f$. We can thus apply Corollary 2C, replacing R^n by the affine space aff dom f.

COROLLARY 2E. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions* dom $f_\nu =$ dom f *and $f_\nu \to f$. Then $f_\nu \to^\tau f$.*

Proof. This follows directly from Corollary 2A, if we view the underlying space as being the smallest closed affine subspace containing aff dom f and recast the proof so that everything is relative to this affine subspace of E.

We now turn to the converse of Theorem 2. (Remember that by Lemma 3(ii), condition (γ) of equi-lower semicontinuity is repetitious when the sequence $\{f_\nu\}$ is τ-converging to f.)

THEOREM 3. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions such that $f_\nu \to^\tau f$. Then $f_\nu \to f$ if and only if the collection $\{f; f_\nu, \nu \in \mathbf{N}\}$ is equi-lower semicontinuous.*

Proof. Suppose that $f_\nu \to^\tau f$ and $\{f; f_\nu\}$ is equi-lower semicontinuous. In view of (3.1) and Lemma 3(i), it suffices to show that $\limsup f_\nu(x) \leqslant f(x)$. First suppose that $x \notin \operatorname{dom} f$; then the preceding inequality is trivially satisfied. Now take $x \in \operatorname{dom} f$; then by equi-lower semicontinuity (β), $x \in \operatorname{dom} f_\nu$ for ν sufficiently large. Also, by equi-lower semicontinuity (α) at every $x \in \operatorname{dom} f$, we have that for all $\epsilon > 0$

$$f_\nu(x) - \epsilon \leqslant f_\nu(y)$$

for all $\nu \in \mathbf{N}$, provided that $y \in \mathscr{U}$, for $\mathscr{U}$ a w-neighborhood of x. τ-Convergence of the f_ν to f implies that there exists a sequence $\{x_\nu, \nu \in \mathbf{N}\}$ such that $x = s\text{–}\lim x_\nu$ and such that $\limsup f_\nu(x_\nu) \leqslant f(x)$; cf. Lemma 1($i$). Convergence of x_ν to x implies that for ν sufficiently large, $x_\nu \in \mathscr{U}$ and then $f_\nu(x) - \epsilon \leqslant f_\nu(x_\nu)$. Hence

$$\limsup f_\nu(x) - \epsilon \leqslant \limsup f_\nu(x_\nu) \leqslant f(x).$$

The above holding for all $\epsilon > 0$ shows that for $x \in \operatorname{dom} f$

$$\limsup f_\nu(x) \leqslant f(x).$$

The necessity of equi-lower semicontinuity follows directly from Theorem 1.

COROLLARY 3A. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions such that* $\operatorname{dom} f_\nu \supset \operatorname{dom} f$ *for all $\nu \in \mathbf{N}$ and $f_\nu \to^\tau f$. Then $f_\nu \to f$ if and only if the collection $\{f_\nu\}$ is equi-lower semicontinuous at every point of* $\operatorname{dom} f$.

COROLLARY 3B [12, Proposition 4.10]. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions defined on R^n with $f_\nu \to^\tau f$. Then $f_\nu \to f$ on* $\operatorname{int} \operatorname{dom} f$.

Proof. It suffices to observe that if $x \in \operatorname{int} \operatorname{dom} f \subset R^n$ and $f_\nu \to^\tau f$ then $x \in \operatorname{int} \operatorname{dom} f_\nu$ for ν sufficiently large and that consequently the sequence $\{f_\nu\}$ convergence uniformly on a closed simplicial neighborhood of x; see [22, Theorem 10.8].

4. POINTWISE CONVERGENCE OF CLOSED CONVEX FUNCTIONS AND THEIR CONJUGATES

τ-Convergence of a sequence of closed convex functions $\{f_\nu, \nu \in \mathbf{N}\}$ to a closed convex function f is equivalent to τ^*-convergence of $\{f^*, \nu \in \mathbf{N}\}$. This result proved in [14] and [11] can also be viewed as a consequence of a result of [13] for closed convex cones. To see this simply observe that for f closed and convex we have that

$$\text{pol cl pos}\{(-1, \eta, x) \mid (\nu, x) \in \text{epi} f\} = \text{cl pos}\{(\eta^*, -1, x^*) \mid (\eta^*, x^*) \in \text{epi} f^*\}, \tag{4.1}$$

where pol associates to a closed convex cone C its polar cone, pol $C = \{x^* \in E^* \mid \langle x, x^* \rangle \leqslant 0\}$, and where pos denotes the positive hull, pos S is the intersection of all convex cones containing S. Combining (4.1) with the fact that pol is an isometry [13, Theorem 1] and observing that a collection of closed convex sets $\{K_\nu, \nu \in \mathbf{N}\}$ converges to K if and only if $\{\text{cl pos}(\{-1\} \times K_\nu), \nu \in \mathbf{N}\}$ converges to cl pos$(\{-1\} \times K)$, shows directly that

$$f_\nu \xrightarrow{\tau} f \qquad \text{if and only if} \qquad f_\nu^* \xrightarrow{\tau^*} f^*, \tag{4.2}$$

provided, naturally, that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are closed convex functions.

It is now easy to combine Theorems 2 and 3 and their corollaries with (4.2) to obtain various relations between pointwise convergence of a class of closed convex functions and pointwise convergence of their conjugate functions. In particular, we get the following:

THEOREM 4. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions such that $\{f; f_\nu, \nu \in \mathbf{N}\}$ and $\{f^*; f_\nu^*, \nu \in \mathbf{N}\}$ are equi-lower semicontinuous sequences. Then $f_\nu \to f$ if and only if $f_\nu^* \to f^*$.*

As an example, we give below one of the many implications of this theorem.

COROLLARY 4A. *Suppose that f and $\{f_\nu, \nu \in \mathbf{N}\}$ are closed convex functions defined on R^n such that $\{f_\nu\}$ and $\{f_\nu^*\}$ are equi-lower semicontinuous at every point of* dom f *and* dom f^*, *respectively. Then $f_\nu \to f_\nu$ and* dom $f_\nu^* \supset$ dom f^* *implies $f_\nu \to f^*$. Dually $f_\nu^* \to f^*$ and* dom $f_\nu \supset$ dom f *implies $f_\nu \to f$.*

There does not seem to be a simple condition which can be imposed on the collection $\{f_\nu\}$ which is equivalent to equi-lower semicontinuity of the collection $\{f^*; f_\nu^*, \nu \in \mathbf{N}\}$. There are, however, some special cases, as we see in Section 5, where this property is immediate. In other instances we might be satisfied with equi-lower semicontinuity at a given point as examplified in the following theorem.

THEOREM 5. *Suppose that $\{f_\nu, \nu \in \mathbf{N}\}$ and f are inf-compact convex functions defined on R^n such that $\{f; f_\nu\}$ is an equi-lower semicontinuous collection. Then $f_\nu \to f$ implies that*

$$\text{Min} f_\nu \to \text{Min} f. \tag{4.3}$$

Proof. Inf-compactness of the f_ν and of f implies that the f_ν^* and f^* are continuous at 0; cf. [18] or [19]. The result now follows from Theorem 2, (4.2), Corollary 3B, and the fact that $\inf f_\nu(x) = -f_\nu^*(0)$.

Note that in the previous theorem, it would be sufficient to assume that f is inf-compact, since pointwise convergence $f_\nu \to f$ implies that the f_ν are inf-compact from some ν on. Taking a somewhat different viewpoint, we extract from (4.2) and the proof of Theorem 3 the following important result for approximation theory.

THEOREM 6. *Suppose that* $\{f_\nu, \nu \in \mathbf{N}\}$ *and* f *are closed convex functions such that* $\{f; f_\nu\}$ *is an equi-lower semicontinuous sequence and* $f_\nu \to f$. *Suppose, moreover, that* $\inf f$ *is finite. Then there exists* $\{x_\nu^*, \nu \in \mathbf{N}\}$ *converging to* 0 *such that*

$$\lim\{\inf[f_\nu - \langle \cdot, x_\nu^* \rangle]\} = \inf f. \tag{4.4}$$

Proof. Equi-lower semicontinuity of the $\{f_\nu\}$ and pointwise convergence of $\{f_\nu\}$ to f implies that $f_\nu^* \to^{\tau^*} f^*$; cf. Theorem 2 and (4.2). Now Lemma 1(i) implies that there exists a sequence $\{x_\nu^*\}$ that s-converges to 0 such that $\limsup f_\nu^*(x_\nu^*) \leqslant f^*(0) = -\inf f$. Also by Lemma 1(ii) we are assured that $\liminf f_\nu^*(x_\nu^*) \geqslant f^*(0)$. The theorem now follows from the simple observation that $f_\nu^*(x_\nu^*) = -\inf[f_\nu - \langle \cdot, x_\nu^* \rangle]$.

5. Convergence of Convex Sets and Their Support Functions

We can now apply Theorem 4, or variants thereof, to the theory of convergence of convex sets. It is easy to see that convergence of convex sets $\{K_\nu, \nu \in \mathbf{N}\}$ to a (closed) convex set K is equivalent to the τ-convergence of the indicator functions $\{\psi_{K_\nu}, \nu \in \mathbf{N}\}$ to the function ψ_K. Moreover, by (4.2) we have that

$$\psi_{K_\nu} \xrightarrow{\tau} \psi_K \qquad \text{if and only if} \qquad \psi_{K_\nu}^* \xrightarrow{\tau^*} \psi_K^*. \tag{5.1}$$

However, to obtain pointwise convergence of the indicator functions ψ_{K_ν} to ψ_K we must demand (cf. condition (β) of equi-lower semicontinuity) that for all $x \in K$ there exists ν_x such that $x \in K_\nu$ for all $\nu \in \mathbf{N}$, $\nu \geqslant \nu_x$. This is akin to requiring that $\{K_\nu, \nu \in \mathbf{N}\}$ be a decreasing sequence. The pointwise convergence of the support functions is characterized in the following theorem. For C a convex subset of E, we denote by 0^+C the *recession cone* of C, i.e., 0^+C is the maximal convex cone such that

$$C \supset C + 0^+C. \tag{5.2}$$

THEOREM 7. *Suppose that* $\{K_\nu, \nu \in \mathbf{N}\}$ *and* K *are closed convex subsets of* E *such that for all* $\nu \in \mathbf{N}$ *and* $\{\psi_K^*; \psi_{K_\nu}^*\}$ *is an equi-lower semicontinuous collection with closed effective domain. Then* $K_\nu \to K$ *implies that* $\psi_{K_\nu}^* \to \psi_K^*$.

Proof. In view of (5.1) and Corollary 3A, it suffices to show that for all $\nu \in \mathbf{N}$, $\operatorname{dom} \psi_{K_\nu}^* \supset \operatorname{dom} \psi_K^*$. But this follows directly from the facts that $\operatorname{dom} \psi_{K_\nu}^* = \operatorname{pol} 0^+K_\nu$ and $\operatorname{dom} \psi_K^* = \operatorname{pol} 0^+K$, and also that

$$0^+K \supset 0^+K_\nu \qquad \text{implies} \qquad \operatorname{pol} 0^+K \,\mathbf{P}\, \operatorname{pol} 0^+K_\nu . \tag{5.3}$$

COROLLARY 7A [16, I.10; 14, Theorem 3.1]. *Suppose that $\{K_\nu, \nu \in \mathbf{N}\}$ is a collection of compact convex sets converging to a compact convex set K. Then $\psi_{K_\nu}^* \to \psi_K^*$.*

Proof. In this case $0^+K_\nu = 0^+K = \{0\}$ and thus $\operatorname{dom} \psi_{K_\nu}^* = \operatorname{dom} \psi_K^* = E^*$ from which follows trivially the equi-lower semicontinuity of $\{\psi_K^*\}$.

COROLLARY 7B. *Suppose that $\{K_\nu = C_\nu + 0^+K_\nu, \nu \in \mathbf{N}\}$ is a sequence of closed convex sets converging to a closed convex set $K = C + 0^+K$, with C_ν converging to C such that C and $\{C_\nu, \nu \in \mathbf{N}\}$ are compact and $0^+K \supset 0^+K_\nu$. Then $\psi_{K_\nu}^* \to \psi_K^*$.*

Proof. Observe that $\psi_{K_\nu} = \psi_{C_\nu + 0^+K_\nu}$ implies that

$$\psi_{K_\nu}^* = \psi_{C_\nu}^* + \psi_{0^+K_\nu}^* .$$

Pointwise convergence now follows from Corollary 7A, $\psi_{0^+K_\nu}^* = \psi_{\operatorname{pol} 0^+K_\nu}$ and (5.3).

Let us also note that if convergence of closed convex sets is defined in terms convergence of the Hausdorff distance, convergence in this sense can only occur if the sets $\{K_\nu, \nu \in \mathbf{N}\}$ and K satisfy the hypotheses of Corollary 7B. This corollary thus implies that convergence, in the sense of Hausdorff distance, always implies pointwise convergence of the support functions. This, combined with Theorem 7 can be used to extend the results of [16, Chap. I, Sect. 1] to the noncompact case. We can also use this observation to generalize a result of Hörmander [20] and Ghoula–Houri [21] on the relation between the Hausdorff distance of two compact convex sets and the distance, in a certain norm, of the corresponding support functions, cf. [23].

Of particular interest is a version of Theorem 6 applied to the special case $\{K_\nu = C_\nu + 0^+K_\nu, \nu \in \mathbf{N}, C_\nu \text{ compact}\}$ and the functions f_ν have the special form

$$f_\nu(x) = \langle x, y_\nu \rangle + \psi_{K_\nu} \tag{5.4}$$

for K_ν closed convex sets. Then

$$f_\nu^*(x^*) = \psi_{K_\nu}^*(x^* - y_\nu). \tag{5.5}$$

Clearly, if the y_ν are converging to some y, and $f = \langle \cdot, y \rangle + \psi_K$, $\{f; f_\nu\}$ is an equi-lower semicontinuous sequence and

$$\operatorname{dom} f_\nu^* = y_\nu + \operatorname{dom} \psi_{K_\nu}^* = y_\nu + \operatorname{pol}(0^+K_\nu). \tag{5.6}$$

COROLLARY 7C. *Suppose that* $\{K_\nu = C_\nu + 0^+K_\nu, \nu \in \mathbf{N}\}$ *and* $K = C + 0^+K$ *are closed convex sets in* E *with* $\{C_\nu, \nu \in \mathbf{N}\}$ *and* C *compact, and* $0^+K_\nu \subset 0^+K$. *Let* $\{y_\nu, \nu \in \mathbf{N}\}$ *be any collection in* E^* *such that* $y = w\text{-}\lim y_\nu$ *with*

$$y_\nu \in (\operatorname{pol} 0^+K) + y. \tag{5.7}$$

Suppose, moreover, that the linear functional $\langle \cdot, y\rangle$ *is bounded below on* K. *Then* $K_\nu \to K$ *implies that*

$$(\inf\langle \cdot, y_\nu\rangle \text{ on } K_\nu) \to (\inf\langle \cdot, y\rangle \text{ on } K). \tag{5.8}$$

Proof. From (5.7), $0^+K_\nu \subset 0^+K$, and (5.3) it follows that for all $\nu \in \mathbf{N}$, $\operatorname{dom} f_\nu^* \supset \operatorname{dom} f^*$ for $f_\nu = \langle \cdot, y_\nu\rangle + \mathscr{U}_{K_\nu}$ and $f = \langle \cdot, y\rangle + \psi_K$ since

$$f_\nu^* = \psi_{K_\nu}^*(\cdot - y_\nu) = \psi_{C_\nu}^*(\cdot - y_\nu) + \psi_{0^+K_\nu}^*(\cdot - y_\nu).$$

Consequently, to show (5.8), it suffices to establish that

$$\psi_{C_\nu}^*(-y_\nu) \to \psi_C^*(-y).$$

This in turn follows from the fact that since $\operatorname{dom} \psi_{C_\nu}^* = \operatorname{dom} \psi_C^* = E^*$, we have uniform (pointwise) convergence of $\psi_{C_\nu}^*$ to ψ_C^* on every w-compact subset of E^*.

COROLLARY 7D [16, I.13]. *Suppose that* $\{K_\nu, \nu \in \mathbf{N}\}$ *is a collection of compact convex sets in* E *converging to a compact set* K *and* $\{y_\nu, \nu \in \mathbf{N}\}$ *is a collection of points in* E^* *such that* $y = w\text{-}\lim y_\nu$. *Then*

$$(\inf\langle \cdot, y_\nu\rangle \text{ on } K_\nu) \to (\inf\langle \cdot, y\rangle \text{ on } K). \tag{5.9}$$

REFERENCES

1. D. L. BURKHOLDER AND R. WIJSMAN, Optimum properties and admissibility of sequential tests, *Ann. Math. Statist.* **34** (1963), 1–17.
2. V. VAN CUTSEM, Un théorème de convergence des suites d'estimateurs ensemblistes du maximum de vraisemblance, Technical Report, Grenoble, France, 1973.
3. G. DANTZIG, J. FOLKMAN, AND N. SHAPIRO, On the continuity of the minimum set of a continuous function, *J. Math. Anal. Appl.* **17** (1967), 519–548.
4. E. IDÉE, "Minimization d'une fonction quasi-convexe aléatoires: Applications," Thèse, Grenoble, France, 1973.
5. B. VAN CUTSEM, Problems of convergence in stochastic linear programming, *in* "Techniques of Optimization" (A. Balakrishnan, Ed.), pp. 445–454, Academic Press, New York, 1972.
6. U. MOSCO, Approximation of the solutions of some variational inequalities, *Ann. Scuola Normale Sup. Pisa* **21** (1967), 373–394, 765.
7. U. MOSCO, Convergence of convex set and of solutions of variational inequalities, *Advances Math.* **3** (1969), 510–585.

8. A. Bensoussan, A. Bossavet, and J. Nédelec, Approximation de problème de contrôle optimal, Technical Report I.R.I.A., Rocquencourt, France, 1967.
9. J.-L. Joly and F. de Thelin, Convergence d'intégrales convexes dans les espaces L^p, Ms., Univ. de Bordeaux, France, 1974.
10. J.-L. Joly, "Une famille de topologies et de convergences sur l'ensemble des fonctionelles convexes," Thèse, Grenoble, France, 1970.
11. J.-L. Joly, Une famille de topologies sur l'ensemble des fonctions convexes pour lesquelles la polarité est bicontinue, Technical Report, Univ. de Bordeaux, France, 1973.
12. R. Robert, Convergence de fonctionelles convexes, *J. Math. Anal. Appl.* **45** (1974), 533–555.
13. D. Wakup and R. Wets, Continuity of some convex-cone valued mappings, *Proc. Amer. Math. Soc.* **18** (1967), 229–235.
14. U. Mosco, On the continuity of the Young–Fenchel transform, *J. Math. Anal. Appl.* **35** (1971), 518–535.
15. R. Wijsman, Convergence of sequences of convex sets, cones and functions, II, *Trans. Amer. Math. Soc.* **123** (1966), 32–45.
16. B. Van Cutsem, "Éléments aléatoires à valeurs convexes compactes," Thèse, Grenoble, France, 1971.
17. C. Kuratowski, "Topologie," Vol. I, Pańistowowe Wydawnictioo Naukowe, Warszawa, 1958.
18. R. T. Rockafellar, Level sets and continuity of conjugate convex functions, *Trans. Amer. Math. Soc.* **123** (1966), 46–63.
19. J.-J. Moreau, Sur la fonction polaire d'une fonction semi-continue superieurement, *C. R. Acad. Sci. Paris* **258** (1964), 1128–1131.
20. L. Hörmander, Sur la fonction d'appui des ensembles convexes dans un espace localement convexe, *Arkiv Math.* **3** (1954), 181–186.
21. A. Ghoula-Houri, Cônes de Banach-Application aux problèmes de contrôle, Ms., Univ. de Caen, France, 1966.
22. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, N.J., 1970.
23. G. Salinetti and R. Wets, On the convergence of sequences of convex sets in finite dimension, *SIAM Review*, to appear.