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CONTINUITY OF SOME CONVEX-CONE-VALUED MAPPINGS

DAVID W. WALKUP AND ROGER J.-B. WETS

1. **Introduction.** We consider the class $\mathcal{C}(\mathcal{B})$ of all closed convex cones in a reflexive Banach space $\mathcal{B}$ as a topological space and investigate the resulting topological properties of certain mappings into $\mathcal{C}(\mathcal{B})$. In §2 we show that, with the proper choice of Hausdorff metrics for $\mathcal{C}(\mathcal{B})$ and $\mathcal{C}(\mathcal{B}^*)$, the operation of taking the polar cone is an isometry between these spaces. In §3 we consider the operation of forming the positive hull of a finite set of points as a mapping into $\mathcal{C}(R^m)$ and obtain some sufficient conditions for the continuity of this mapping. The results of §3 have application in the theory of stochastic programming [6].

2. **Properties of the polar map.** Let $\mathcal{B}$ denote a reflexive Banach space with norm $\|\cdot\|$ and unit ball B and let $\mathcal{B}^*$ be the conjugate space of $\mathcal{B}$ with norm $\|y^*\| = \sup_{x\in B} |\langle x, y^*\rangle|$. Let $\mathcal{C}(\mathcal{B})$ denote the class of all closed convex cones in $\mathcal{B}$ (with vertex at the origin). We define a metric distance d between $P, Q \in \mathcal{C}(\mathcal{B})$ as follows:

$$\delta(P, Q) = \sup_{p\in P\cap B} \inf_{q\in Q} \|p - q\|, \qquad d(P, Q) = \max\{\delta(P, Q), \delta(Q, P)\}.$$

Thus d is essentially the Hausdorff metric on B. Distances δ^* and d^* are defined similarly on $\mathcal{C}(\mathcal{B}^*)$ in terms of the conjugate norm.

To each subset D of $\mathcal{B}$ is associated its polar, pol $D \subset \mathcal{B}^*$, defined by

$$\text{pol } D = \{y^* \in \mathcal{B}^* \mid \langle x, y^*\rangle \leqq 0 \quad \text{for all } x \in D\}.$$

It is well known that the restriction of pol to $\mathcal{C}(\mathcal{B})$ is a one-to-one map from $\mathcal{C}(\mathcal{B})$ onto $\mathcal{C}(\mathcal{B}^*)$ and that pol(pol C) $= C$ for each $C \in \mathcal{C}(\mathcal{B})$.

THEOREM 1. *The polar map is an isometry from* $\mathcal{C}(\mathcal{B})$ *onto* $\mathcal{C}(\mathcal{B}^*)$.

PROOF. Clearly the theorem will follow if we show that

$$\delta^*(\text{pol } Q, \text{pol } P) = \delta(P, Q) \quad \text{for all } P, Q \text{ in } \mathcal{C}(\mathcal{B}). \tag{1}$$

In fact, since the case $P = Q$ is trivial, (1) will follow from the identity pol(pol C) $= C$ if we show that

$$\delta^*(\text{pol } Q, \text{pol } P) \geqq \delta(P, Q) \tag{2}$$

whenever P, Q are members of $\mathcal{C}(\mathcal{B})$ such that $\delta(P, Q) = \Delta > 0$. We

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begin by proving that (2) holds when P is a ray, Q is a closed half-space, and $P \cap Q = \{0\}$. Let p be the element of P with norm 1. Then

$$\delta(P, Q) = \delta(p, Q) = \inf_{q \in Q} \|p - q\| = \langle p, q^* \rangle = \Delta > 0,$$

where q^* is the element of pol Q with norm 1. From the definition of δ^* and the conjugate norm, it follows

$$\begin{aligned} \delta^*(\text{pol } Q, \text{pol } P) = \delta^*(q^*, \text{pol } P) &= \inf_{p^* \in \text{pol} P} \|q^* - p^*\| \\ &= \inf_{p^* \in \text{pol} P} \sup_{x \in B} |\langle x, q^* \rangle - \langle x, p^* \rangle|. \end{aligned}$$

Since the supremum is taken over a set containing p, we obtain

$$\delta^*(\text{pol } Q, \text{pol } P) \geqq |\langle p, q^* \rangle - \langle p, p^* \rangle| \geqq \Delta.$$

This establishes (2) for the special case. In general, if $\delta(P, Q) = \Delta > 0$, then for any ϵ, $0 < \epsilon < \Delta$, there exists $\bar{p} \in P \cap B$ such that $\delta(\bar{p}, Q) = \Delta' \geqq \Delta - \epsilon$. Let $N(\bar{p}) = \{x \mid x = \lambda z, \|\bar{p} - z\| \leqq \Delta', \lambda \geqq 0\}$. Then by a well-known separation theorem there exists a half-space $\overline{Q}$ containing Q and missing the interior of $N(\bar{p})$. Thus $\delta(\bar{p}, \overline{Q}) = \Delta'$ and hence, by the special case considered above, $\delta^*(\text{pol } \overline{Q}, \text{ pol } \bar{p}) \geqq \Delta'$. Since $\bar{p} \in P$, $\overline{Q} \supset Q$ it follows pol $\bar{p} \supset$ pol P, pol $\overline{Q} \in$ pol Q and thus

$$\delta^*(\text{pol } Q, \text{ pol } P) \geqq \delta^*(\text{pol } \overline{Q}, \text{ pol } \bar{p}) \geqq \Delta' \geqq \Delta - \epsilon,$$

from which (2) and hence the theorem follows.

The above theorem bears an interesting relationship with a result of V. Klee on convex bodies in a normed linear space [4, p. 253]. Let H be a reflexive Banach space, and let $\mathcal{K}(H)$ be the class of closed bounded convex bodies in H with the origin in their interior. The space H can be imbedded in a reflexive Banach space $\mathfrak{B}$ in such a way that H is a closed hyperplane at distance 1 from the origin 0 of $\mathfrak{B}$ and the origin of H is a point p of $\mathfrak{B}$ which is also the closest point of H to 0. Let P^* be the set of points of $\mathfrak{B}^*$ whose products with p is -1, and let h^* be the point of $\mathfrak{B}^*$ whose products with points of H are all equal to -1. It is easily checked that $P^* - h^*$ is (isometric to) H^*, the conjugate of H. In addition, if K is a member of $\mathcal{K}(H)$, then $(P^* \cap \text{pol } K) - h^*$ is the polar body $K^\circ = \{t^* \in H^* \mid \sup \langle (s - p), t^* \rangle \leqq 1, s \in H\}$, which is a member of $\mathcal{K}(H^*)$. Moreover, the map ζ, which associates with each point s of H the ray $\{\lambda s \mid \lambda \geqq 0\}$, and its inverse ζ^{-1} can be seen to be Lipschitzian homeomorphisms of bounded portions of H and their images in $\mathcal{C}(\mathfrak{B})$. (It suffices to consider 2-dimensional subspaces of $\mathfrak{B}$ spanned by two arbitrary points of H and establish a Lipschitz constant depending only on the norms of these

points.) If $\mathcal{K}(H)$ and $\mathcal{K}(H^*)$ are given the Hausdorff metrics derived from H and H^*, then by Theorem 1 the polar map $^\circ$ is a locally Lipschitzian homeomorphism of $\mathcal{K}(H)$ onto $\mathcal{K}(H^*)$. This is essentially the content of Klee's result.

Let ψ be a mapping from a topological space $\mathfrak{X}$ into $\mathcal{C}(\mathfrak{B})$. Then ψ may be interpreted as a set-valued mapping into $\mathfrak{B}$, in connection with which we employ the concepts of closed and lower semicontinuous mappings as defined by Berge in [1, Chapter VI]. It is easily seen that ψ is closed or lower semicontinuous with respect to either the strong or weak topology of $\mathfrak{B}$ if and only if the same holds for ψ_B, where $\psi_B(x) = \psi(x) \cap B$.

PROPOSITION 1. *Let ψ be a set-valued mapping of the topological space $\mathfrak{X}$ into $\mathfrak{B}$.*

(i) *If ψ is lower semicontinuous, then* pol ψ *is weakly closed.*

(ii) *If ψ is weakly closed and $\psi(x)$ is a convex cone for each $x \in \mathfrak{X}$, then* pol ψ *is lower semicontinuous.*

PROOF. Part (i) follows in a straightforward fashion from the elementary definitions of lower semicontinuity, the map pol, and weakly closed maps. For part (ii) assume pol ψ is not lower semicontinuous. Then there exists $\hat{x} \in \mathfrak{X}$, a spherical neighborhood N_ϵ about $\hat{z}^* \in$ pol $\psi(x)$, and for each neighborhood M about $\hat{x}$ a point $x = x(M) \in M$ such that pol $\psi(x(M)) \cap N_\epsilon = \varnothing$. For each M the two convex sets N_ϵ and pol $\psi(x(M))$ may be separated by a hyperplane, that is, there exists $y(M)$ such that

$$\|y(M)\| = 1,$$

$$\langle y(M), \hat{z}^* \rangle \geqq \epsilon, \tag{3}$$

$$\langle y(M), z^* \rangle \leqq 0 \quad \text{for all } z^* \in \text{pol } \psi(x(M)),$$

i.e.

$$y(M) \in \psi(x(M)). \tag{4}$$

Since each $y(M)$ belongs to the weakly compact ball B, there exists $\hat{y} \in B$ such that $\hat{y}$ is a weak limit point of $y(M_\alpha)$ for every system of neighborhoods M_α of x converging on $\hat{x}$. But then (3) implies $\langle y, z^* \rangle \geqq \epsilon$ i.e. $\hat{y} \notin \psi(\hat{x})$, which contradicts (4) and the assumption ψ is weakly closed.

When $\mathfrak{B}$ is of finite dimension the following conditions are equivalent:

(a) ψ is continuous with respect to the topology of $\mathcal{C}(\mathfrak{B})$.

(b) For all $x \in \mathfrak{X}$, see [3, p. 295]

$$\lim_{x\to\hat{x}} \delta(\psi(\hat{x}), \psi(x)) = 0, \qquad \lim_{x\to\hat{x}} \delta(\psi(x), \psi(\hat{x})) = 0.$$

(c) ψ_B is upper semicontinuous (or closed) and lower semicontinuous.

(d) ψ is closed and lower semicontinuous.

These remarks yield a direct proof of Proposition 1 for finite-dimensional spaces.

3. **Properties of the map** pos. In this section we concentrate on the continuity properties of the operator pos, which we interpret as associating with each $m\times n$ real matrix A the closed convex polyhedral cone spanned positively by the points in R^m determined by the columns of A. We identify the $m\times n$ matrices with the points of R^{mn}. We shall also write lin A and con A respectively, for the subspace of R^m spanned linearly by the columns of A and the convex hull of the columns of A.

In order to show that the restriction of pos to a subset Z of R^{mn} is continuous in the sense of continuity of the associated set-valued map into the unit ball B^m in R^m or the associated map into the space $\mathfrak{C}$ of closed convex cones in R^m, it suffices to show that the restriction of pos to Z is closed with respect to the relative topology of Z, since:

PROPOSITION 2. *Considered as mappings from R^{mn} into R^m,* pos *and* lin *are lower semicontinuous,* polpos *and* pollin *are closed, and* con *is continuous.*

PROOF. The continuity of con is intuitively obvious and easily proven. To prove pos is lower semicontinuous we must show that if the sequence A_i converges to $\hat{A}$ and $\hat{p}$ is any point of pos $\hat{A}$, then there exists points $p_i\in$pos A_i such that lim $p_i=\hat{p}$. But if $\hat{p}\in$pos $\hat{A}$, then $\hat{p}=\hat{A}y$ for some $y\in R^n_\oplus$ and the sequence of points $p_i=A_iy$ has the required properties. The rest of the proposition follows from Proposition 1 and the remark that lin $A=$pos$[A, -A]$.

We may now state the principal result of this section. We use the usual notations of dim C for the dimension of a cone C and $\mathfrak{L}C$ for the lineality space of C, i.e. the maximal linear subspace contained in C.

THEOREM 2. *Suppose Z is a subset of R^{mn}, k is an integer, and for every matrix $\hat{A}\in Z$,*

(a) dim $\mathfrak{L}$ pos $\hat{A}=k$,

(b) *there exists a neighborhood N about $\hat{A}$ such that if any column $\hat{A}^i$ of $\hat{A}$ lies in* $\mathfrak{L}$ pos $\hat{A}$, *then the corresponding column A^i of any matrix A in $N\cap Z$ lies in* $\mathfrak{L}$ pos A.

Then the restriction of pos *to Z is continuous.*

Two corollaries to this theorem, representing extreme cases of the hypotheses, are stated below. The result of Corollary 1 will be used in the proof of the theorem. Corollary 2 will follow trivially from the theorem.

COROLLARY 1. *Suppose* pos $\hat{A}$ *is a pointed cone, i.e.* dim £ pos $\hat{A}=0$, *and none of the columns of $\hat{A}$ is the zero vector. Then* pos *is continuous in a neighborhood of $\hat{A}$.*

COROLLARY 2. *Suppose Z is a subset of R^{mn}, k is an integer, and for every matrix $A\in Z$ the cone* pos A *is a subspace of R^m of dimension k. Then the restriction of* pos *to Z is continuous.*

PROOF OF COROLLARY 1. Suppose pos $\hat{A}$ is pointed and none of the columns of $\hat{A}$ is identically zero. Then there exists a hyperplane H in R^m missing the origin and intersecting pos $\hat{A}$ in a polytope $P(\hat{A})$. On a sufficiently small neighborhood N of R^{mn} about $\hat{A}$, $P(A)=H\cap\text{pos } A$ is the convex hull of uniformly bounded points $p_i(A)$ each of which is the intersection with H of the ray in R^m generated by a column A^i of A. Clearly the points $p_i(A)$ are continuous functions of A on N and, by the continuity of con, so is $P(A)$. But since $P(A)$ is a closed function, so is pos A. From Proposition 2 it follows pos is continuous on N.

PROOF OF THEOREM 2. For the proof we restrict our attention to matrices A in a sufficiently small neighborhood N of the relative topology for Z containing an arbitrary point $\hat{A}$ of Z. It will suffice to show that pos is a closed map of N into the column space of A. Without loss of generality we may assume the rows and columns of A are so arranged that A may be partitioned in the form

$$A = [L \quad T] = \begin{bmatrix} L_{11} & L_{12} & T_1 \\ L_{21} & L_{22} & T_2 \end{bmatrix},$$

where £ pos $A=$ pos $L=$ lin L, each column of T lies outside £ pos A, and $\hat{L}_{11}$ is a maximal nonsingular square submatrix of $\hat{L}$. It is clear from hypothesis (a) that $\hat{L}_{11}$ is a $k\times k$ matrix. If N is sufficiently small, then L_{11}, B, and B^{-1} are nonsingular (throughout N), where B is the $m\times m$ matrix

$$B = \begin{bmatrix} L_{11} & 0 \\ L_{21} & I \end{bmatrix}.$$

Now consider

$$D = B^{-1}A = \begin{bmatrix} I & L'_{12} & T'_1 \\ 0 & L'_{22} & T'_2 \end{bmatrix}$$

and note that D is a continuous function of A. Since B^{-1} is nonsingular, the combinatorial structures of the cones pos D and pos A are identical. This fact, together with (a), (b), and the assumption that N is sufficiently small, yields the following sequence of results: The columns of $B^{-1}L$ lie in $\mathfrak{L}$ pos D,

$$\mathfrak{L} \text{ pos } D = \text{lin} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

the matrix L'_{22} is identically zero, no column of $B^{-1}T$ lies in $\mathfrak{L}$ pos D, and, therefore,

$$\text{pos } D = \text{lin} \begin{bmatrix} I \\ 0 \end{bmatrix} + \text{pos} \begin{bmatrix} 0 \\ T'_2 \end{bmatrix},$$

where

$$\text{pos} \begin{bmatrix} 0 \\ T'_2 \end{bmatrix}$$

is a pointed cone and none of the columns of

$$\begin{bmatrix} 0 \\ T'_2 \end{bmatrix}$$

are zero. It follows readily from Corollary 1 and the fact

$$\text{pos} \begin{bmatrix} 0 \\ T'_2 \end{bmatrix}$$

is orthogonal to the fixed flat

$$\text{lin} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

that the map $D \to \text{pos } D$ is closed. Thus the map $A \to \text{pos } D$ is closed since D is a continuous function of A. Finally, the map $A \to \text{pos } A = B\{\text{pos } D\}$ is closed since it is the composition of the closed map $A \to \text{pos } D$ and the continuous map $A \to B$ with the continuous map, $(A, B) \to BA$. This completes the proof.

Curiously, although Theorem 2 is obviously related to the theory of positive linear dependence [2, 5], the concepts of combinatorial type and frame (minimal independent spanning set) fundamental in

that theory do not appear in the hypotheses of Theorem 2. There are simple counterexamples in three dimensions which show that fixed combinatorial type of pos A on Z or the existence of a continuous frame for pos A on Z are not sufficient to make pos A continuous on Z.

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