

Dan Romik

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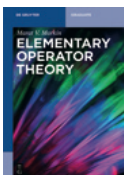


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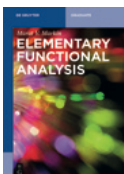


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# Topics in Complex Analysis

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# Contents

## Preface — XI

### 0 Prerequisites and notation — 1

- 0.1 Prerequisites — 1
- 0.2 Notation — 1
- Exercises for Chapter 0 — 3

### 1 Basic theory — 4

- 1.1 Motivation: why study complex analysis? — 4
- 1.2 The fundamental theorem of algebra — 9
- 1.3 Holomorphicity, conformality, and the Cauchy–Riemann equations — 12
- 1.4 Additional consequences of the Cauchy–Riemann equations — 18
- 1.5 Power series — 20
- 1.6 Contour integrals — 22
- 1.7 The Cauchy, Goursat, and Morera theorems — 28
- 1.8 Simply connected regions and the general version of Cauchy’s theorem — 32
- 1.9 Consequences of Cauchy’s theorem — 36
- 1.10 Zeros, poles, and the residue theorem — 46
- 1.11 Meromorphic functions, holomorphicity at  $\infty$ , and the Riemann sphere — 50
- 1.12 Classification of singularities and the Casorati–Weierstrass theorem — 52
- 1.13 The argument principle and Rouché’s theorem — 53
- 1.14 The open mapping theorem and maximum modulus principle — 57
- 1.15 The logarithm function — 58
- 1.16 The local behavior of holomorphic functions — 60
- 1.17 Infinite products and the product representation of the sine function — 63
- 1.18 Laurent series — 68
- Exercises for Chapter 1 — 71

### 2 The prime number theorem — 82

- 2.1 Motivation: analytic number theory and the distribution of prime numbers — 82
- 2.2 The Euler gamma function — 83
- 2.3 The Riemann zeta function: definition and basic properties — 89
- 2.4 A theorem on the zeros of the Riemann zeta function — 97
- 2.5 Proof of the prime number theorem — 99
- Exercises for Chapter 2 — 110

**3 Conformal mapping — 118**

- 3.1 Motivation: classifying complex regions up to conformal equivalence — **118**
- 3.2 First singleton conformal equivalence class: the complex plane — **121**
- 3.3 Second singleton conformal equivalence class: the Riemann sphere — **123**
- 3.4 The Riemann mapping theorem — **124**
- 3.5 The unit disc and its automorphisms — **126**
- 3.6 The upper half-plane and its automorphisms — **129**
- 3.7 The Riemann mapping theorem: a more precise formulation — **131**
- 3.8 Proof of the Riemann mapping theorem, part I: technical background — **132**
- 3.9 Proof of the Riemann mapping theorem, part II: the main construction — **137**
- 3.10 Annuli and doubly connected regions — **140**
- Exercises for Chapter 3 — **145**

**4 Elliptic functions — 146**

- 4.1 Motivation: elliptic curves — **146**
- 4.2 Doubly periodic functions — **149**
- 4.3 Poles and zeros; the order of a doubly periodic function — **151**
- 4.4 Construction of the Weierstrass  $\wp$ -function — **154**
- 4.5 Eisenstein series and the Laurent expansion of  $\wp(z)$  — **158**
- 4.6 The differential equation satisfied by  $\wp(z)$  — **159**
- 4.7 A recurrence relation for the Eisenstein series — **160**
- 4.8 Half-periods; factorization of the associated cubic — **161**
- 4.9  $\wp(z)$  and  $\wp'(z)$  generate all doubly periodic functions — **163**
- 4.10  $\wp(z)$  as a conformal map for rectangles — **165**
- 4.11 The discriminant of a cubic polynomial — **168**
- 4.12 The discriminant of a lattice — **170**
- 4.13 The  $J$ -invariant of a lattice — **170**
- 4.14 The modular variable  $\tau$ : from elliptic functions to elliptic modular functions — **171**
- 4.15 The classification problem for complex tori — **172**
- 4.16 Equivalence between complex tori and elliptic curves — **177**
- Exercises for Chapter 4 — **179**

**5 Modular forms — 182**

- 5.1 Motivation: functions of lattices — **182**
- 5.2 The modular group  $\Gamma = \text{PSL}(2, \mathbb{Z})$  — **184**
- 5.3 The modular group as a group of Möbius transformations — **185**
- 5.4 The fundamental domain and the modular surface  $\mathbb{H}/\Gamma$  — **186**
- 5.5 The classification problem for complex tori, part II — **190**
- 5.6 The point at  $i\infty$ , premodular forms, and their Fourier expansions — **191**
- 5.7 Fourier expansions and number-theoretic identities — **194**

5.8	Modular functions —	<b>199</b>
5.9	Klein's $J$ -invariant —	<b>205</b>
5.10	The $J$ -invariant as a conformal map —	<b>208</b>
5.11	The classification problem for complex tori, part III —	<b>209</b>
5.12	Modular forms —	<b>209</b>
5.13	Examples of modular forms —	<b>214</b>
5.14	Infinite products for modular forms —	<b>218</b>
	Exercises for Chapter 5 —	<b>228</b>
<b>6</b>	<b>Sphere packing in 8 dimensions —</b>	<b>233</b>
6.1	Motivation: the sphere packing problem in $d$ dimensions —	<b>233</b>
6.2	A high-level overview of the proof —	<b>236</b>
6.3	Preparation: some remarks on Fourier eigenfunctions —	<b>237</b>
6.4	The (+1)-Fourier eigenfunction —	<b>239</b>
6.5	The (-1)-Fourier eigenfunction —	<b>250</b>
6.6	A modular form inequality —	<b>256</b>
6.7	Proof of Theorem 6.1 —	<b>263</b>
	Exercises for Chapter 6 —	<b>265</b>
<b>A</b>	<b>Appendix: Background on sphere packings —</b>	<b>267</b>
A.1	Sphere packings and their densities —	<b>267</b>
A.2	Lattices and lattice packings —	<b>268</b>
A.3	Periodic sphere packings —	<b>268</b>
A.4	Lattice covolume —	<b>269</b>
A.5	Dual lattices —	<b>269</b>
A.6	The Poisson summation formula for lattices —	<b>270</b>
A.7	Construction of the lattice $E_8$ —	<b>271</b>
A.8	The Cohn–Elkies sphere packing bounds —	<b>276</b>
A.9	Magic functions —	<b>278</b>
A.10	Radial functions and their Fourier transforms —	<b>279</b>
A.11	Structural properties of $E_8$ magic functions —	<b>281</b>
A.12	Summary —	<b>284</b>
	Exercises for Appendix A —	<b>286</b>

**Bibliography — 289**

**Web Bibliography — 291**

**Index — 293**