Some Comments on Euler's Series for $\pi^2/6$

Dan Romik

1. Introduction

Let $R_N = \sum_{n=N}^{\infty} \frac{1}{n^2}$ be the remainder term in Euler's series $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. In [2], it is shown that R_N has an asymptotic expansion in powers of $\frac{1}{N}$:

$$R_N = \frac{1}{N} + \frac{1}{2N^2} + \frac{1}{6N^3} - \frac{1}{30N^5} + \ldots + \frac{B_{2K}}{N^{2K+1}} + O\left(\frac{(2K+1)!}{(2\pi N)^{2K+1}}\right) =$$

$$= \frac{1}{2N^2} + \sum_{m=0}^{K} \frac{B_{2m}}{N^{2m+1}} + O\left(\frac{(2K+1)!}{(2\pi N)^{2K+1}}\right),$$

where B_{2m} are the Bernoulli numbers. An analogous expansion is shown for the remainder term of the alternating harmonic series, which sums to $\log 2$, involving the so-called tangent numbers. These expansions, coupled with some of the known methods for calculating the Bernoulli numbers and tangent numbers, lead to a fairly simple and efficient method of evaluating Euler's series and the alternating harmonic series.
In this note, we show *exact formulae* for the remainder term of these series. These formulae are derived by repeated application of a well-known trick for accelerating the convergence of a series. We present the formulae as Theorem 1:

Theorem 1.

$$\text{(1)} \qquad \frac{\pi^2}{6} = \sum_{n=1}^{N-1} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(N-1)!}{n^2(n+1)(n+2)\ldots(n+N-1)}$$

$$\text{(2)} \quad \log 2 = \sum_{n=1}^{2N} \frac{(-1)^{n-1}}{n} +$$

$$+ \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2N-1)}{(2n) \cdot (2n-1)(2n+1)(2n+3)(2n+5) \cdots (2n+2N-1)}$$

School of Mathematical Sciences, Tel-Aviv University, Tel-Aviv 69978, Israel
romik@math.tau.ac.il

Remark. Formula (1) has appeared in [4] (chapter 6, example 9). Formula (2) we believe is new.

Using these formulae, one can sum the original series fairly rapidly using only simple rational operations, by summing first the original series up to $N-1$, then summing the series for the remainder term up to $N-1^*$. In the next section, we show how to derive (and prove) the formulae in Theorem 1, concentrating mainly on formula (1). In section 3 we derive error estimates for these formulae, thus enabling their use for effective computations of the values of these series.

2. The trick

Start with Euler's series. The remainder term R_N is of the order of $\frac{1}{N}$. It is well-known (see e.g. [1], chapter 3) that the convergence of such a series can be accelerated in the following manner: replace the summand $\frac{1}{n^2}$ by a summand which is close to it, but for which the corresponding series is telescopic, then add the difference between them - which is of smaller order of magnitude - separately. We choose $\frac{1}{n(n+1)}$ as the approximant. Thus:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{n(n+1)}\right) =$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)} = 1 + \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}$$

And so we have represented the original series as a telescopic series which sums to 1, plus another series which converges more rapidly - the remainder term for it is of the order of $\frac{1}{n^2}$.

We now iterate this process, i.e. try to represent the new series as a telescopic series plus the difference which is of a smaller order than magnitude. We "guess" $\frac{1}{n(n+1)(n+2)}$ to be a suitable approximant:

*Actually, there seems to be no special reason, other than the want of simplicity, to sum the same number of terms of the original series and the remainder term series. Computationally speaking, taking the number of terms summed from the remainder term series to be some other linear function of the number of terms from the original series (or possibly some *non*linear function) would probably yield slightly better results.

$$1+\sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}=1+\sum_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}+\sum_{n=1}^{\infty}\frac{2}{n^2(n+1)(n+2)}=$$

$$=1+\sum_{n=1}^{\infty}\left(\frac{1}{2n(n+1)}-\frac{1}{2(n+1)(n+2)}\right)+\sum_{n=1}^{\infty}\frac{2}{n^2(n+1)(n+2)}=$$

$$=1+\frac{1}{4}+\sum_{n=1}^{\infty}\frac{2}{n^2(n+1)(n+2)}$$

Again, we have represented the series as a telescopic series which we can calculate plus another series which converges more rapidly. A curious suspicion now arises: the sums of the first two telescopic series are exactly the first two terms of the original series $\sum_{n=1}^{\infty}\frac{1}{n^2}$! We would like now to iterate the convergence-acceleration trick, and suspect that at each stage we would get a new telescopic series, which sums to exactly the corresponding element of the original series, plus another series, which would be the remainder term. To verify this, we execute one more step:

$$1+\frac{1}{4}+\sum_{n=1}^{\infty}\frac{2}{n^2(n+1)(n+2)}=$$

$$=1+\frac{1}{4}+\sum_{n=1}^{\infty}\frac{2}{n(n+1)(n+2)(n+3)}+\sum_{n=1}^{\infty}\frac{6}{n^2(n+1)(n+2)(n+3)}=$$

$$=1+\frac{1}{4}+\sum_{n=1}^{\infty}\left(\frac{2}{3n(n+1)(n+2)}-\frac{2}{3(n+1)(n+2)(n+3)}\right)+$$

$$+\sum_{n=1}^{\infty}\frac{6}{n^2(n+1)(n+2)(n+3)}=$$

$$=1+\frac{1}{4}+\frac{1}{9}+\sum_{n=1}^{\infty}\frac{6}{n^2(n+1)(n+2)(n+3)}$$

We can see that a pattern emerges, and from here, the passage to the general form of the formula which appears in Theorem 1 is a simple induction step, which we omit. The formula for $\log 2$ is obtained in a similar fashion, after first representing the alternating harmonic series as a series of positive signs $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n)}$.

3. An error estimate

We now cast Theorem 1 into a form more readily suitable for actual computation. The basic method would be to sum the first $N-1$ terms of the original series, then the first $N-1$ terms of the series for the remainder term. We estimate the remaining error term easily by comparing it with a close telescopic series; the error term is (in the case of Euler's series):

$$\sum_{n=N}^{\infty} \frac{(N-1)!}{n^2(n+1)(n+2)\dots(n+N-1)} <$$

$$< \sum_{n=N}^{\infty} \frac{(N-1)!}{(n-1)n(n+1)(n+2)\dots(n+N-1)} =$$

$$= \frac{(N-1)!}{N} \sum_{n=N}^{\infty} \left(\frac{1}{(n-1)n\dots(n+N-2)} - \frac{1}{n(n+1)\dots(n+N-1)} \right) =$$

$$= \frac{(N-1)!}{N} \cdot \frac{1}{(N-1)N(N+1)\dots(2N-2)} = \frac{(N-1)!^2}{(N-1)N(2N-2)!}$$

And by the well-known asymptotic relation $\binom{2m}{m} \sim \frac{4^m}{\sqrt{\pi m}}$ (a version of Stirling's formula) we can now write the following version of Theorem 1 (the estimate for the error term in the formula for $\log 2$ comes from a similar analysis):

Theorem 1'.

$$\frac{\pi^2}{6} = \sum_{n=1}^{N-1} \frac{1}{n^2} + \sum_{n=1}^{N-1} \frac{(N-1)!}{n^2(n+1)(n+2)\dots(n+N-1)} + O\left(\frac{1}{N^{3/2} \cdot 4^N} \right)$$

$$\log 2 = \sum_{n=1}^{2N} \frac{(-1)^{n-1}}{n} +$$

$$+ \sum_{n=1}^{N} \frac{1 \cdot 3 \cdot 5 \cdots (2N-1)}{(2n) \cdot (2n-1)(2n+1)(2n+3) \cdots (2n+2N-1)} + O\left(\frac{N}{4^N} \right)$$

The constant concealed by the O-sign can in both cases be taken as 10.

4. A generating function

By comparing the formula in Theorem 1' with the asymptotic expansion in [2], we can now arrive at a generating function for the Bernoulli numbers - that is, a function $f(x)$ that has an asymptotic expansion in powers of x near $x = 0$, where the coefficients are exactly the Bernoulli numbers. Note that the function $x/(e^x - 1)$ is not such a function - it is more properly called an *exponential* generating function, because of the additional factorial factor in the coefficients.
First, rewrite the terms of the series for R_N as

$$
\begin{aligned}
R_N = \sum_{n=N}^{\infty} \frac{1}{n^2} &= \sum_{n=1}^{\infty} \frac{(N-1)!}{n^2(n+1)(n+2)\dots(n+N-1)} = \\
&= \sum_{n=1}^{\infty} \frac{(n-1)!}{nN(N+1)(N+2)\dots(N+n-1)} = \\
&\qquad = \frac{1}{N} + \frac{1}{2N(N+1)} + \frac{2}{3N(N+1)(N+2)} + \dots
\end{aligned}
$$

Each of the summands can now be developed as a power series in 1/N, and comparison of the error terms will show that the coefficients are exactly the Bernoulli numbers:

Theorem 2. For $x > 0$, define

$$
\begin{aligned}
f(x) = x + \frac{x^2}{2(1+x)} &+ \frac{2x^3}{3(1+x)(1+2x)} + \frac{6x^4}{4(1+x)(1+2x)(1+3x)} = \dots \\
&= \sum_{k=1}^{\infty} \frac{(k-1)!}{k} \cdot \frac{x^k}{(1+x)(1+2x)(1+3x)\dots(1+(k-1)x)}.
\end{aligned}
$$

Then $f(x)$ has the following asymptotic expansion as $x \searrow 0$:

$$
\begin{aligned}
f(x) = x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{30}x^5 + \dots + B_{2K}x^{2K+1} + O(x^{2K+2}) = \\
\frac{1}{2}x^2 + \sum_{m=0}^{K} B_{2m}x^{2m+1} + O(x^{2K+2})
\end{aligned}
$$

5. Additional comments

1. The method of repeatedly applying the convergence-acceleration trick can be applied to other series (try doing it for Gregory's series $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$!). In fact, for a given series there could be more than one way to perform this. However, it is a remarkable and mysterious fact that the two series treated here seem to be the only ones for which the resulting telescopic series sum exactly to the terms of the original series, thereby leading to a formula for the remainder term of the series.

2. It should be noted that as an algorithm for computing π, the method presented here does not compare with some of the known existing methods in terms of speed (for a description of some of the known methods, see [3]). However, its main advantage seems to be in its simplicity and elegance, being based essentially on Euler's famous series. Furthermore, unlike some of the known methods, it requires only rational operations (except one square-root operation at the end to convert π^2 to π).

References

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