Permutations with short monotone subsequences

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Abstract

We consider permutations of $1, 2, ..., n^2$ whose longest monotone subsequence is of length n and are therefore extremal for the Erdős-Szekeres theorem. Such permutations correspond via the Robinson-Schensted correspondence to pairs of square $n \times n$ Young tableaux. We show that all the bumping sequences are constant and therefore these permutations have a simple description in terms of the pair of square tableaux. We deduce a limit shape result for the plot of values of the typical such permutation, and other properties of these extremal permutations.

1 Introduction

In this paper, we consider a class of permutations which have a certain extremality property with respect to the length of their monotone subsequences. The well-known Erdős-Szekeres theorem states that a permutation $\pi = (\pi(1), \pi(2), ..., \pi(n^2))$ of the numbers $1, 2, ..., n^2$ must contain a monotone (either increasing or decreasing) subsequence $\pi(i_1), \pi(i_2), ..., \pi(i_n), \quad i_1 < i_2 < ... < i_n$. Our main object of study will be those permutations which do not have any longer monotone subsequences than those guaranteed to exist by this theorem.

Definition 1. *A permutation $\pi \in \mathcal{S}_{n^2}$ is called an* **Extremal Erdős-Szekeres** *(EES) permutation if π does not have a monotone subsequence of length $n+1$. Denote by EES_n the EES permutations in $\mathcal{S}_{n^2}$.*

The famous example showing sharpness of the Erdős-Szekeres theorem is the permutation

$$n, n-1, ..., 1, \quad 2n, 2n-1, ..., n+1, \quad 3n, 3n-1, ..., 2n+1,$$

$$..., n^2, n^2-1, ..., n^2-n+1.$$

However, there are many more examples. Here are the 4 EES permutations in $\mathcal{S}_4$:

$$2\ 1\ 4\ 3, \qquad 2\ 4\ 1\ 3, \qquad 3\ 1\ 4\ 2, \qquad 3\ 4\ 1\ 2,$$

and here are a few of the 1764 EES permutations in $\mathcal{S}_9$:

$$5\ 6\ 3\ 1\ 4\ 9\ 2\ 8\ 7, \qquad 7\ 3\ 8\ 9\ 1\ 4\ 2\ 6\ 5, \qquad 4\ 7\ 3\ 5\ 9\ 8\ 1\ 2\ 6,$$

$$6\ 2\ 8\ 5\ 9\ 1\ 7\ 3\ 4, \qquad 3\ 2\ 8\ 9\ 1\ 6\ 4\ 7\ 5, \qquad 4\ 8\ 9\ 3\ 6\ 1\ 2\ 7\ 5.$$

Here is an EES permutation in $\mathcal{S}_{25}$:

$$13\ 10\ 20\ 15\ 3\ 22\ 23\ 2\ 9\ 25\ 17\ 21\ 14\ 7\ 8\ 1\ 4\ 5\ 16\ 11\ 24\ 19\ 18\ 6\ 12 \qquad (1)$$

It was observed by Knuth [2, Exer. 5.1.4.9] that the EES permutations in EES_n are in bijection with pairs of (standard) Young tableaux of square shape $(n, n, ..., n)$ via the Robinson-Schensted correspondence, and that, since the number of square Young tableaux can be computed using the hook formula of Frame-Robinson-Thrall ([2, Th. 5.1.4.H]), this gives a formula for the number of EES permutations:

Theorem 1.

$$|EES_n| = \left(\frac{(n^2)!}{1 \cdot 2^2 \cdot 3^3 \cdot ... \cdot n^n \cdot (n+1)^{n-1} \cdot (n+2)^{n-2} \cdot ... \cdot (2n-1)^1} \right)^2.$$

Apart from this surprising but elementary observation, no one has yet undertaken a systematic study of these permutations. In particular, it seems interesting to study the behavior of the *typical* EES permutations - what different properties do they have from ordinary random permutations? An initial step in this direction was taken in [3].

Here is another elementary observation on EES permutations, which is an immediate corollary of the fact that taking the inverse of a permutation does not change the maximal lengths of increasing and decrasing subsequences.

Theorem 2. *If $\pi \in EES_n$ then $\pi^{-1} \in EES_n$.*

We prove two main results about EES permutations. Our first result concerns the structure of the deterministic EES permutation. The Robinson-Schensted correspondence gives a description of EES permutations in terms of pairs of square Young tableaux. This description may not seem like a useful one, since in general the Robinson-Schensted correspondence is an algorithmic procedure which can be difficult to analyze. However, we show that when the inverse correspondence is applied to square Young tableaux, it in fact degenerates to a simple mapping which can be described explicitly. First, we introduce some useful notation. If $\mathbf{a}$ is a sequence of distinct numbers and u is one of the numbers, denote

$$\begin{aligned}
\text{lis}(\mathbf{a}) &= \text{the maximal length of an increasing subsequence in } \mathbf{a}, \\
\text{lds}(\mathbf{a}) &= \text{the maximal length of a decreasing subsequence in } \mathbf{a}, \\
\text{lis}_u(\mathbf{a}) &= \text{the maximal length of an increasing subsequence in } \mathbf{a} \\
&\quad\ \text{containing } u, \\
\text{lds}_u(\mathbf{a}) &= \text{the maximal length of a decreasing subsequence in } \mathbf{a} \\
&\quad\ \text{containing } u.
\end{aligned}$$

Then we have:

Theorem 3. *Let $\mathcal{T}_n$ be the set of square $n \times n$ standard Young tableaux. There is a bijection from $\mathcal{T}_n \times \mathcal{T}_n$ to EES_n, defined as follows: to each pair of tableaux $P = (p_{i,j})_{i,j=1}^n$, $Q = (q_{i,j})_{i,j=1}^n$ corresponds the permutation $\pi \in EES_n$ given by*

$$\pi(q_{i,j}) = p_{n+1-i,j}, \quad (1 \le i, j \le n). \tag{2}$$

In the inverse direction, P and Q can be constructed from π as follows:

$$\begin{aligned} q_{i,j} &= \text{ the unique } 1 \le k \le n^2 \text{ such that} \\ &\quad lds_{\pi(k)}(\pi(1), \pi(2), ..., \pi(k)) = i \quad \text{and} \\ &\quad lis_{\pi(k)}(\pi(1), \pi(2), ..., \pi(k)) = j. \end{aligned} \tag{3}$$

$$\begin{aligned} p_{i,j} &= \text{ the unique } 1 \le k \le n^2 \text{ such that} \\ &\quad lds_{\pi^{-1}(k)}(\pi^{-1}(1), ..., \pi^{-1}(k)) = i \quad \text{and} \\ &\quad lis_{\pi^{-1}(k)}(\pi^{-1}(1), ..., \pi^{-1}(k)) = j. \end{aligned} \tag{4}$$

Next, we explore the properties of *random* EES permutations. For each n, let $\mathbb{P}_n$ be the uniform probability measure on EES_n. One result concerning these permutations was proved in [3], and is a corollary of the connection between EES permutations and square Young tableaux and the main result of [3] on the limit shape of random square Young tableaux:

Theorem 4. *[3] Let $0 < \alpha < 1$, let $n \to \infty$ and $k = k(n) \to \infty$ in such a way that $k/n^2 \to \alpha$. Then for all $\epsilon > 0$,*

$$\mathbb{P}_n \left[\pi \in EES_n \ : \ \left| \frac{1}{n} lis\Big(\pi(1), \pi(2), ..., \pi(k(n))\Big) - 2\sqrt{\alpha(1-\alpha)} \right| > \epsilon \right] \xrightarrow[n\to\infty]{} 0.$$

(See [3] for a stronger statement including some rate of convergence estimates.)

If $\pi \in EES_n$, define the **plot** of π to be the set A_π given by

$$A_\pi = (i, \pi(i))_{1 \le i \le n^2}.$$

What does this set look like for a typical $\pi \in \mathcal{S}_n$? Figure 1(a) shows A_π for a randomly chosen $\pi \in EES_{100}$. For comparison, figure 1(b) shows A_π for a permutation π chosen at random from *all* the permutations in $\mathcal{S}_{10000}$. Clearly the points in A_π for a random EES permutation cluster inside a certain subset of the square $[1, 10000] \times [1, 10000]$. The phenomenon is explained by the following limit shape theorem, and is illustrated in Figure 2.

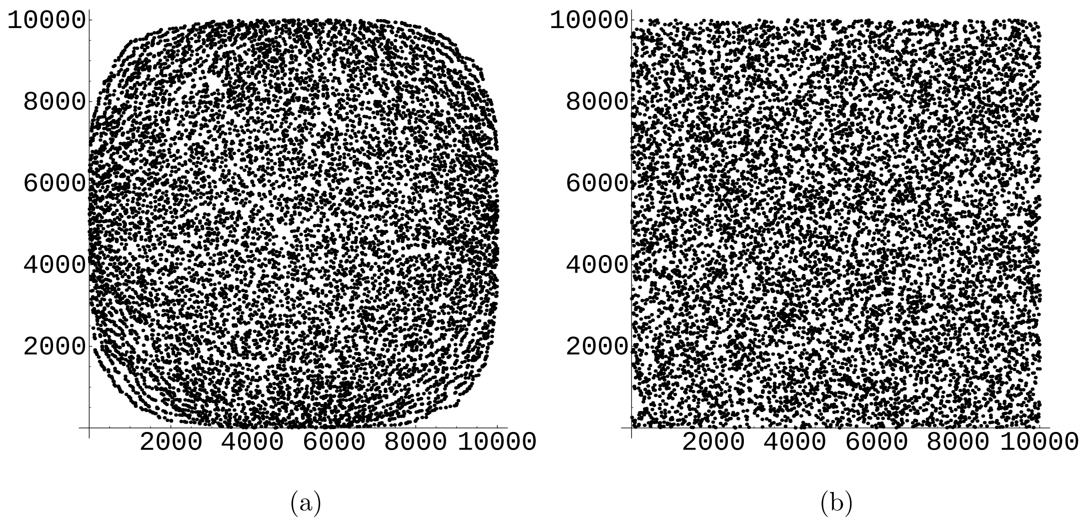


Figure 1: A uniform random EES permutation and a uniform random permutation of $1, 2, ..., 10000$.

Theorem 5. *Define the set*

$$\mathcal{Z} = \left\{ (x,y) \in [-1,1] \times [-1,1] \ : \ (x^2-y^2)^2 + 2(x^2+y^2) \leq 3 \right\}.$$

Then: (i) For any open set U containing $\mathcal{Z}$,

$$\mathbb{P}_n \left[\pi \in EES_n \ : \ \left(\frac{2}{n^2} A_\pi - (1,1) \right) \subset U \right] \xrightarrow[n\to\infty]{} 1.$$

(ii) For any open set $U \subset \mathcal{Z}$,

$$\mathbb{P}_n \left[\pi \in EES_n \ : \ \left(\frac{2}{n^2} A_\pi - (1,1) \right) \cap U \neq \emptyset \right] \xrightarrow[n\to\infty]{} 1.$$

In section 4 we state and prove a stronger version of Theorem 5(ii), which describes the density of points of (the correctly scaled) A_π in any small region in $\mathcal{Z}$, and mention additional results.

2 EES permutations and square Young tableaux

In this section, we prove Theorem 3. Our proof uses the Robinson-Schensted correspondence. Although the bijection between EES permutations and pairs of

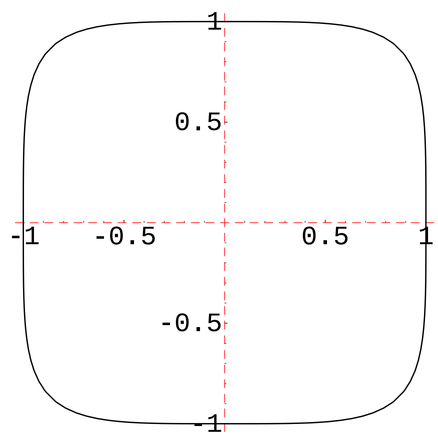


Figure 2: The limiting shape of the plot of a random EES permutation. The boundary is the quartic curve $(x^2 - y^2)^2 + 2(x^2 + y^2) = 3$.

square Young tableaux is a special case of the Robinson-Schensted correspondence, this special case is much simpler than the general case. For instance, the worst-case computational complexity of (2) is $O(n^2)$, and the worst-case complexity of (3) and (4) is $O(n^2 \log n)$; compare this with the *average-case* complexity of $\theta(m^{3/2} \log m)$ of the Robinson-Schensted correspondence applied to a general permutation of m elements (note that in our case $m = n^2$), see [4].

We assume that the reader is familiar with the definition and basic properties of the Robinson-Schensted correspondence; for background consult [2, section 5.1.4]. Recall that the Robinson-Schensted correspondence attaches to each permutation $\pi \in \mathcal{S}_m$ two standard Young tableaux P and Q whose shape is the same Young diagram λ of size m. The length of the first row λ_1 of λ is equal to $\mathrm{lis}(\pi)$, and the length of the first column λ'_1 of λ is equal to $\mathrm{lds}(\pi)$. In particular, if $\pi \in \mathit{EES}_n$, then λ is a Young diagram of size n^2 whose first row and column are both of length n; the only such diagram is the square diagram of shape $(n, n, ..., n)$, and this proves Knuth's observation mentioned in the introduction. Theorem 1 follows using the hook formula.

Our proof of (2) now relies on the following lemma.

Lemma 1. *When the Robinson-Schensted correspondence is applied to an EES permutation $\pi \in \mathit{EES}_n$ to compute the tableaux P, Q, all the* **bumping sequences** *are constant.*

We encourage the reader to try applying the Robinson-Schensted correspondence to the permutation given in (1) before reading on, to get a feeling for what is happening.

Proof. We prove the obviously equivalent statement that in the application of the *inverse* Robinson-Schensted correspondence to two square $n \times n$ Young tableaux P and Q, all the bumping sequences are constant.

Recall that the inverse Robinson-Schensted correspondence consists of n^2 deletion operations, where at each step a corner element is deleted from the shape of P and Q corresponding to where the maximal entry in Q is located, and P is modified by bumping the entry of P that was in the deleted corner up to the next higher row, then repeatedly bumping up an element from each row until reaching the top row.

The proof will be by induction on k, the number of deletion operations performed. For a given $k \geq 1$, let λ be the shape of the tableaux P and Q after $k-1$ deletion operations (so λ is the shape of the subtableau of the original Q consisting of all entries $\leq n^2 - k + 1$). Denote by $P = (p_{i.j})_{i,j=1}^n$ the entries of the original tableau P, and denote by $\hat{P} = (\hat{p}_{i,j})_{i,j}$ the entries of the tableau P after $k-1$ deletion operations. Assume that the k-th corner element to be deleted is at location (i_0, j_0). A little reflection will convince the reader that the k-th bumping sequence will be constant if and only if for all $2 \leq i \leq i_0$ we will have that $\hat{p}_{i,j_0} < \hat{p}_{i-1,j_0+1}$ (where we take $\hat{p}_{i-1,j_0+1} = \infty$ if location $(i-1, j_0+1)$ lies outside λ).

By the induction hypothesis, all the bumping sequences before time k were constant; another way to express this is via the equation

$$\hat{p}_{i,j} = p_{i+n-\lambda'(j),j}, \qquad (1 \leq j \leq n,\ 1 \leq i \leq \lambda'(j)),$$

where $\lambda'(j)$ is the length of the j-th column of λ, which simply says that the j-th row of $\hat{P}$ contains the $\lambda'(j)$ bottom elements of the j-th row of P, in the same order. So we have

$$\begin{aligned} \hat{p}_{i,j_0} &= p_{i+n-\lambda'(j_0),j_0}, \\ \hat{p}_{i-1,j_0+1} &= p_{i-1+n-\lambda'(j_0+1),j_0+1}. \end{aligned}$$

But (i_0, j_0) is a corner element of λ, so $\lambda'(j_0) = i_0 > \lambda'(j_0+1)$. This implies that $i + n - \lambda'(j_0) \leq i - 1 + n - \lambda'(j_0+1)$, and therefore $\hat{p}_{i,j_0} < \hat{p}_{i-1,j_0+1}$, since P is a Young tableau. ∎

Lemma 1 easily implies (2). At the k-th deletion step, if the corner cell being deleted is at location (i_0, j_0) (so $\lambda'(j_0) = i_0$), then $q_{i_0,j_0} = n^2 - k + 1$, and the element bumped out of the first row will be $\hat{p}_{1,j_0} = p_{n+1-\lambda'(j_0),j_0}$. As a consequence we get $\pi(n^2 - k + 1) = \pi(q_{i_0,j_0}) = p_{n+1-i_0,j_0}$.

To conclude the proof of Theorem 3, we now prove (3) and (4). Clearly it is enough to prove (3), since replacing π by π^{-1} has the effect of switching P and Q in the output of the Robinson-Schensted correspondence. Note that $q_{i,j} = k$ if and only if (i,j) was the corner cell that was added to the tableau P at the k-th insertion step. Because of the properties of the Robinson-Schensted correspondence (specifically, [2, Th. 5.1.4.D(b)] and [2, Exer. 5.1.4.2]), this implies in particular that

$$\mathrm{lds}_{\pi(k)}(\pi(1), \pi(2), ..., \pi(k)) \geq i, \qquad (5)$$

$$\text{lis}_{\pi(k)}(\pi(1), \pi(2), ..., \pi(k)) \geq j. \tag{6}$$

Now, it is easy to see that

$$\left\{ \Big(\text{lds}_{\pi(k)}\big(\pi(1), \pi(2), ..., \pi(k)\big), \text{lis}_{\pi(k)}\big(\pi(1), \pi(2), ..., \pi(k)\big) \Big) \ : \ 1 \leq k \leq n^2 \right\}$$

is a set of distinct points in $\mathbb{Z}^2$ - this is the fact used in the well-known proof of the Erdős-Szekeres theorem using the pigeon-hole principle (and this fact also validates the use of the word "unique" in (3) and (4)). However, since π is an EES, all these n^2 points lie in $[1, n] \times [1, n]$. So in fact the inequality in (5) and (6) must be an equality, and (3) holds. ■

3 Proof of the limit shape result

We now prove Theorem 5. First, we recall the limit shape result for random square Young tableaux proved in [3]. For each $n \in \mathbb{N}$, let μ_n denote the uniform probability measure on $\mathcal{T}_n$, the set of $n \times n$ square Young tableaux. Pittel and Romik [3] proved that there is an (explicitly describable) function $L : [0,1] \times [0,1] \to [0,1]$ that describes the limiting surface of the typical square Young tableau (see Figure 3). More precisely:

Theorem 6. *[3] For all $\epsilon > 0$,*

$$\mu_n \left[T = (t_{i,j})_{i,j=1}^n \in \mathcal{T}_n \ : \ \max_{1 \leq i,j \leq n} \left| \frac{1}{n^2} t_{i,j} - L(i/n, j/n) \right| > \epsilon \right] \xrightarrow[n\to\infty]{} 0.$$

For a stronger result with explicit rates of convergence, see [3]. The only properties of the limit surface L that we will need are that it is an increasing function of either coordinate, and that its values on the boundary of the square are given by

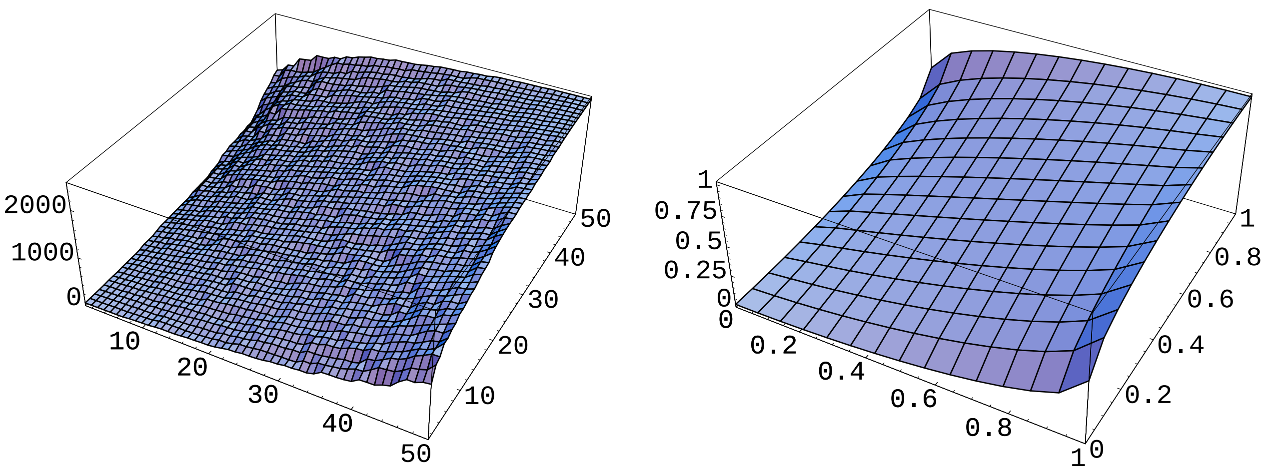
$$L(t, 0) \ = \ L(0, t) \ = \ \frac{1 - \sqrt{1 - t^2}}{2}, \tag{7}$$

$$L(t, 1) \ = \ L(1, t) \ = \ \frac{1 + \sqrt{2t - t^2}}{2}. \tag{8}$$

Let π be a uniform random permutation in EES_n. By Theorem 3, its plot can be described in terms of the tableaux P and Q (which are uniform random $n \times n$ square tableaux) by

$$A_\pi = \left\{ \big(q_{i,j}, p_{n+1-i,j}\big) \ : \ 1 \leq i, j \leq n \right\}.$$

By Theorem 6, each point $n^{-2}(q_{i,j}, p_{n+1-i,j})$ is with high probability (as $n \to \infty$) uniformly close to the point $(L(u, v), L(1 - u, v))$, where $u = i/n$, $v = j/n$.



(a) 3D plot of a random tableau

(b) The limit surface $L(x,y)$

Figure 3: A random 50×50 square tableau and the limit surface

It follows that Theorem 5 is true with the limit shape set

$$\mathcal{Z}' = \Big\{ \big(2L(u,v) - 1, 2L(1-u,v) - 1\big) \ : \ 0 \le u, v \le 1 \Big\}.$$

By (7) and (8), it follows that the mapping $(u,v) \to (2L(u,v)-1, 2L(1-u,v)-1)$ maps the boundary of the square $[0,1] \times [0,1]$ into the four curves described parametrically by

$$\Big(-\sqrt{1-t^2}, -\sqrt{2t-t^2}\Big)_{0\le t\le 1}, \qquad \Big(-\sqrt{1-t^2}, \sqrt{2t-t^2}\Big)_{0\le t\le 1},$$

$$\Big(\sqrt{2t-t^2}, \sqrt{1-t^2}\Big)_{0\le t\le 1}, \qquad \Big(\sqrt{2t-t^2}, -\sqrt{1-t^2}\Big)_{0\le t\le 1}.$$

Setting $x = \pm\sqrt{2t-t^2}$, $y = \pm\sqrt{1-t^2}$, it is easy to verify that

$$(x^2-y^2)^2 + 2(x^2+y^2) = 3,$$

so these curves are the parametrizations of the boundary of the set $\mathcal{Z}$. It is also easy to check that the interior of the square is mapped to the interior of $\mathcal{Z}$, so $\mathcal{Z}' = \mathcal{Z}$. ∎

4 Concluding remarks

Theorem 3 shows that square Young tableaux behave in a simpler and more rigid way than tableaux of arbitrary shapes in relation to the Robinson-Schensted algorithm. This turns out to be true for other tableau algorithms as well. We show as a consequence of Theorem 3 that the Schützenberger evacuation involution also takes on an especially simple form for square tableaux, a result originally due to Schützenberger [**?**]:

Theorem 7. *[**?**] For a standard Young tableau P, let $evac(P)$ denote the* **evacuation tableau** *of P, as defined in [5, p. 425-426]. Then for a square tableau $P = (p_{i,j})_{i,j=1}^n \in \mathcal{T}_n$, we have*

$$(evac(P))_{i,j} = n^2 + 1 - p_{n+1-i,n+1-j}.$$

Proof. This is an immediate corollary of (2) and [5, Th. 7.A1.2.10]. ■

The Edelman-Greene bijection [1] between Young tableaux and balanced tableaux also degenerates to a very simple mapping when the tableau shape is a square. See also [1, Cor. 7.23] for a result analogous to Theorem 7 for tableaux of staircase shape.

We mention some additional results on random EES permutations. A special case of Theorem 5 which seems particularly noteworthy is the following:

Theorem 8. *For all $\epsilon > 0$,*

$$\mathbb{P}_n\Big[\pi \in EES_n \ : \ |\pi(1) - n^2/2| > \epsilon \cdot n^2\Big] \xrightarrow[n\to\infty]{} 0.$$

We can also strengthen Theorem 5(ii) somewhat, by counting approximately how many points of the plot of a typical EES permutation fall in any small region in $\mathcal{Z}$:

Theorem 9. *Let $\varphi : [0,1] \times [0,1] \to \mathcal{Z}$ be the 1-1 and onto mapping defined by*

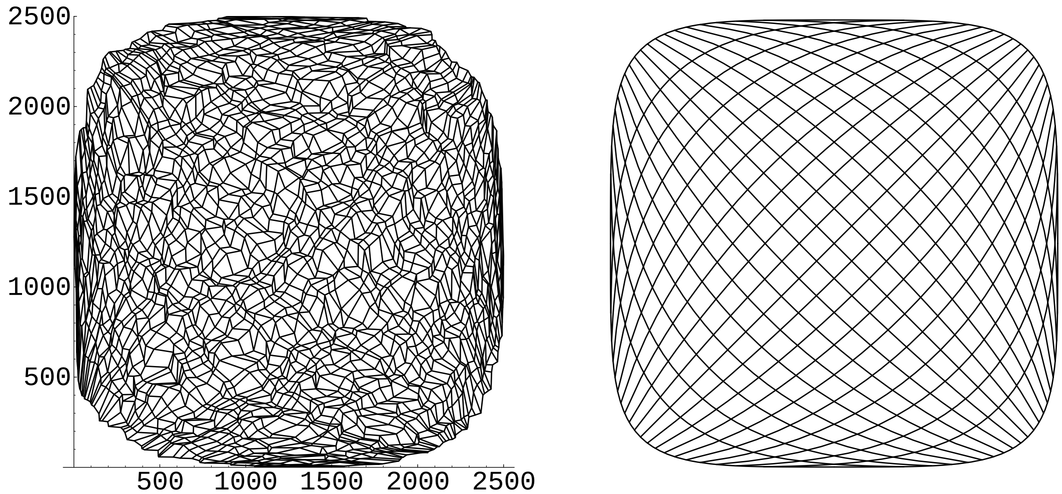
$$\varphi(u,v) = (2L(u,v) - 1, 2L(1-u,v) - 1).$$

For any open set $U \subset \mathcal{Z}$ and for any $\epsilon > 0$, we have

$$\mathbb{P}_n\left[\pi \in \mathcal{S} \ : \ \left|\frac{1}{n^2} card\bigg(\Big(\frac{2}{n^2}A_\pi - (1,1)\Big) \cap U\bigg) - \int_U |J_{\varphi^{-1}}(x,y)| dx\, dy\right| > \epsilon\right]$$

$$\xrightarrow[n\to\infty]{} 0,$$

where $J_{\varphi^{-1}}$ is the Jacobian of the mapping φ^{-1} and $card(\cdot)$ is the cardinality of a set.



(a) (b)

Figure 4: The grid decomposition of an EES permutation

The proof is an obvious extension of the proof of Theorem 5, and is omitted. The function $J_{\varphi^{-1}}$ does not seem to have a simple explicit formula. See [3] for the explicit description of the limit surface function L.

Finally, we mention an interesting way of looking at Fig. 1(a) suggested by Omer Angel. The grid structure inherited from the square tableau can be imposed on the picture by connecting two points $(k, \pi(k)), (m, \pi(m)$ with $k < m$ in the plot of π if $k = q_{i,j}, m = q_{i,j+1}$ or $k = q_{i,j}, m = q_{i+1,j}$. This is related to the observation that an EES permutation can be decomposed in a unique way as a union of n disjoint increasing subsequences of length n and simultaneously a union of n disjoint decreasing subsequences of length n such that the intersection of any of the increasing subsequences with any of the decreasing subsequences contains exactly one element. The grid picture is a way of representing this decomposition graphically.

Fig. 3(a) shows the deformed grid that is obtained as a result. Fig. 3(b) shows the ideal grid that is typically obtained in the limit as $n \to \infty$. This is also a simple corollary of Theorem 6 and Theorem 3. It seems interesting to study the small-scale behavior of this grid near a fixed point in $\mathcal{Z}$ for a random EES permutation as n grows large.

Acknowledgements. I thank Omer Angel for useful discussions and for suggesting the grid decomposition of EES permutations described above.

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