Explicit formulas for hook walks on continual Young diagrams *†

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Abstract

We consider, following the work of S. Kerov, random walks which are continuous-space generalizations of the Hook Walks defined by Greene-Nijenhuis-Wilf, performed under the graph of a continual Young diagram. The limiting point of these walks is a point on the graph of the diagram. We present several explicit formulas giving the probability densities of these limiting points in terms of the shape of the diagram. This partially resolves a conjecture of Kerov concerning an explicit formula for the so-called Markov transform. We also present two inverse formulas, reconstructing the shape of the diagram in terms of the densities of the limiting point of the walks. One of the two formulas can be interpreted as an inverse formula for the Markov transform. As a corollary, some new integration identities are derived.

1. Introduction

A Young diagram is a graphic representation of a partition $\lambda : \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$ of an integer $n = \sum \lambda_i$. A *continual* Young diagram is the continous analogue of this, namely a positive increasing function f defined on some interval $[a, b]$.

Greene, Nijenhuis and Wilf (1979, 1984) introduced two random walks on Young diagrams called the *Hook walks*. These random walks continue until reaching some terminal point on the boundary of the diagram. By analyzing the probability distributions of these terminal points, they reproved two important formulas in the combinatorics of Young diagrams. Kerov (1993) generalized one of the walks to continual Young diagrams. This random walk converges to a limiting point on the boundary of the diagram, and Kerov conjectured a formula

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for the density of this limiting point, in the case where this random variable is in fact absolutely continuous.

In this paper, we give a unified treatment of both random walks. We prove Kerov's formula under fairly mild assumptions on the smoothness of the diagram, and present a new explicit formula for the density of the limiting point of the second hook walk. The fact that these expressions are probability densities, and thus integrate to 1, leads to some surprising integration relations; two examples are:

$$\int_0^1 f(x)dx =$$

$$\int_0^1 \left\{ \frac{1}{\pi}(1+f'(x)) \cdot \sin\left(\frac{\pi}{1+f'(x)}\right) \cdot \right.$$

$$\cdot (x+f(x))^{\frac{f'(x)}{1+f'(x)}} \cdot (1-x+f(1)-f(x))^{\frac{1}{1+f'(x)}} \cdot$$

$$\left. \cdot \exp\left[-\int_0^1 \frac{1}{u-x+f(u)-f(x)} \cdot \frac{f'(u)-f'(x)}{1+f'(x)} du \right] \right\} dx$$

which holds for any positive, increasing, smooth function f on $[0,1]$ that satisfies $f(0) = 0$, and having first derivative bounded away from 0 and infinity and second derivative bounded; and

$$\pi = \int_0^1 \left[\cos(\pi g(x)/2) \cdot x^{-(1+g(x))/2} \cdot (1-x)^{-(1-g(x))/2} \cdot \right.$$

$$\left. \exp\left(\frac{1}{2} \int_0^1 \frac{g(u)-g(x)}{u-x} du \right) \right] dx,$$

which holds for any smooth function g on $[0,1]$ which is bounded between $-1+\epsilon$ and $1-\epsilon$ for some $\epsilon > 0$.

We also solve the inverse problem: that of finding the shape of the diagram, when given the probability density of the limiting point of the random walk. Two inverse formulas are given, one for each of the two walks. For the walk that was treated by Kerov, the correspondence between the shape of the diagram and the probability density of the limiting point of the walk is closely related to the so-called Markov transform (see Kerov (1998)). In this case, our explicit formulas enable the direct calculation of the Markov transform and its inverse. The Markov transform has found several applications, notably to Dirichlet priors in statistics, so the inverse formula may well be applicable to that problem, a possibility that Diaconis and Kemperman (1996) seem to hint at in their very readable review of the subject.

We remark that the importance of the continual hook walk is best understood in connection with the asymptotic theory of Plancherel measure on the symmetric group. Kerov (1999) showed that the probability density of the limiting point of the walk (the so-called transition measure - see Section 2 below) governs the dynamical system of the evolution of a random (Plancherel-distributed) Young diagram, and used this to illuminate the beautiful Vershik-Kerov/Logan-Shepp limit shape theorem for irreducible representations of the symmetric group. The transition measure has also appeared in recent work of Ivanov and Olshanski (2001), where it was shown that the transition measure of a random Plancherel-distributed Young diagram converges to the semicircle distribution, and that the deviation from the semicircle distribution satisfies a central limit theorem. The form of the limiting Gaussian process in this central limit theorem exhibits a surprising resemblance to empirical eigenvalue distribution deviations appearing in the GUE random matrix model, a phenomenon that is not yet fully understood.

In section 2 we give the required definitions and terminology of continual Young diagrams and the hook walks. We concentrate on "rotated" Young diagrams, so that instead of increasing functions we shall be dealing with 1-Lipschitz functions. However, because of the esthetic appeal in working with increasing functions, we translate some of the formulas for those functions. In section 3 we present the main results, namely the formulas for the densities of the terminating point of the hook walks, together with the associated integration relations, and the inversion formulas. We also include a simple asymptotic result on the location of the roots of the polynomial $\frac{d}{dt}(t(t-1)(t-2)...(t-n))$, which in a sense inspires the computation for the inverse formulas.

In section 4 we review some of the elementary properties of the hook walks. The approach using moments is emphasized and some of the results there may be of independent interest, although the main goal is to prepare for the proofs of the main results, which are given in sections 5 and 6. Section 7 contains the formulas for increasing functions and some more curious formulas related to the hook walks.

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2. Definitions

Continual diagrams. While a continual Young diagram is most easily described as an increasing function on an interval, it turns out that for computational purposes, it is vastly preferable to use a coordinate system whereby the diagram is rotated clockwise by an angle of $\pi/4$. We thus define a *diagram*, following Kerov (1993, 1999) as a 1-Lipschitz function ω on an interval $[a, b]$, such that $a + \omega(a) = b - \omega(b)$. We denote $z = z(\omega) = a + \omega(a)$, the *center* of the diagram. This latter condition makes sure that the graph of ω hinges on the graph of the function $x \to |x - z|$, to make for a true rotated diagram (see Figure 1 below). Note that equivalently, one may think of a diagram as a 1-Lipschitz function defined on $\mathbb{R}$, such that outside of some interval $[a, b]$ and for some z, the graph of ω identifies with $x \to |x-z|$. The *domain* D_ω is the set $\{(x,y) : a \le x \le b, |x - z(\omega)| \le y \le \omega(x)\}$. The *dual domain* D'_ω is the set $\{(x,y) : a \le x \le b, \omega(x) \le y \le \min(\omega(a) + x - a, \omega(b) + b - x)\}$ (see Figure 1). The *area* of ω is $A(\omega) = \int_a^b (\omega(x) - |x-z|)dx$. (Note: although this is the true area, it is *twice* the area as defined by Kerov (1993, 1999))

Denote by $\mathcal{D}[a,b]$ the set of diagrams on the interval $[a,b]$.

Smooth diagrams. We denote by $\mathcal{S}[a,b]$ the set of diagrams $\omega \in \mathcal{D}[a,b]$ satisfying the following smoothness conditions: ω is piecewise twice-continuously-differentiable, ω'' is bounded, and for some two constants $-1 < c_1 < c_2 < 1$, the derivative satisfies $c_1 < \omega'(x) < c_2$ whereever it is defined.

Hooks and hook walks. For a point $(x,y) \in D_\omega$ ($\omega \in \mathcal{D}[a,b]$), the (interior) *hook* of (x,y) is the set

$$\{(x',y') \in D_\omega : (x' \le x \text{ and } y' - y = x - x') \text{ or } (x' > x \text{ and } y' - y = x' - x)\}.$$

(*In words:* The union of the two rays starting at (x,y) and going diagonally up-left and up-right, respectively, until they intersect the graph of ω. The intersection with the graph can be a segment, in which case all the segment is included.)

For a point $(x,y) \in D'_\omega$, the (exterior) hook of (x,y) is the set

$$\{(x',y') \in D'_\omega : (x' \le x \text{ and } y - y' = x - x') \text{ or } (x' > x \text{ and } y - y' = x' - x)\}.$$

(*In words:* The union of the two rays starting at (x,y) and going diagonally down-left and down-right, respectively, until they intersect the graph of ω.)

The two *hook walks*, the main subjects of this paper, are random walks on the domain (dual domain, respectively) of the diagram, that, from a given point, change at each step to a point which is chosen at random (uniformly, by arc length) from the hook of the last point.

The *exterior corner* walk (or simply: exterior walk) starts at the exterior corner point $(b - \omega(a), \omega(a) + \omega(b))$ and moves at each step to a uniformly chosen point in the (exterior) hook.

The *interior uniform* walk (or: interior walk) starts at a uniformly chosen point (by surface area) in D_ω, and moves at each step to a uniformly chosen point in the (interior) hook.

The transition measures. It is clear that the consecutive steps of either hook walk must converge almost surely to a limit point which is on the graph of the diagram. We call the distribution of the x-coordinate of this limiting point the *transition measure* of the diagram (relative to the given walk - thus we may talk about the interior transition measure or exterior transition measure). The

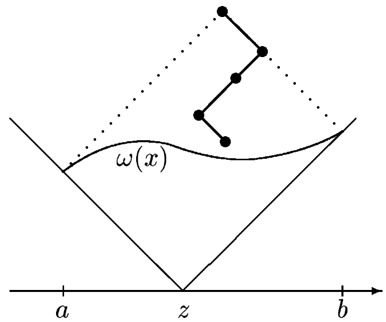

Figure 1: A continual Young diagram and some steps of an exterior hook walk

origin of this terminology is in the theory of discrete Young diagrams, where the transition measures for the exterior-corner- and interior-uniform- walks are in fact the transition measures of some Markov chains, which describe, respectively, the Plancherel growth of a random Young diagram, and a random Young tableau of given shape.

Remark. Note that the starting point of the exterior hook walk depends on the interval $[a, b]$ where the diagram is defined. This may cause some confusion in the formulation whereby the diagram is thought of as a function on $\mathbb{R}$, with the interval $[a, b]$ left unspecified (the purpose of this formulation was precisely to have a common ground to discuss diagrams on different intervals, which will be necessary in section 4). However, we remark that the *transition measure* is in fact *independent* of the choice of interval (as long as the diagram has its essential support inside the interval, that is, as long as the diagram identifies with $x \to |x - z|$ outside of the interval). This was proven for the discrete version of the hook walk in Greene-Nijenhuis-Wilf (1979, 1984) - the so-called "constant zone effect" - and since, in a sense to be specified in section 4, the discrete hook walk approximates the general one, the general case follows. So the choice of interval is in fact immaterial.

3. The main results

We now present the main results:

Theorem 1. Densities of the transition measures. Let $\omega \in \mathcal{S}[a,b]$. Then:

(1a) The density of the exterior transition measure for ω is equal to

$$\frac{1}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-a)^{-(1+\omega'(x))/2}\cdot(b-x)^{-(1-\omega'(x))/2}\cdot$$
$$\exp\left(\frac{1}{2}\int_a^b\frac{\omega'(u)-\omega'(x)}{u-x}du\right)$$

(1b) The density of the interior transition measure for ω is equal to

$$\frac{2}{\pi\cdot A(\omega)}\cos(\pi\omega'(x)/2)\cdot(x-a)^{(1+\omega'(x))/2}\cdot(b-x)^{(1-\omega'(x))/2}\cdot$$
$$\exp\left(-\frac{1}{2}\int_a^b\frac{\omega'(u)-\omega'(x)}{u-x}du\right)$$

Theorem 2. The continuous "hook" integration formulas. Let $\omega \in \mathcal{S}[a,b]$. Then:

(2a)

$$\pi=\int_a^b\left[\cos(\pi\omega'(x)/2)\cdot(x-a)^{-(1+\omega'(x))/2}\cdot(b-x)^{-(1-\omega'(x))/2}\cdot\right.$$
$$\left.\exp\left(\frac{1}{2}\int_a^b\frac{\omega'(u)-\omega'(x)}{u-x}du\right)\right]dx$$

(2b)

$$\int_a^b(\omega(x)-|x-z(\omega)|)dx=$$
$$=\int_a^b\left[\frac{2}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-a)^{(1+\omega'(x))/2}\cdot(b-x)^{(1-\omega'(x))/2}\cdot\right.$$
$$\left.\exp\left(-\frac{1}{2}\int_a^b\frac{\omega'(u)-\omega'(x)}{u-x}du\right)\right]dx$$

Theorem 3. The inversion formulas. Let $\omega \in \mathcal{D}[a,b]$. Then:

(3a) If the exterior transition measure of ω is absolutely continuous, and its density $g(x)$ is piecewise-continuously-differentiable, has a bounded derivative, and is bounded away from 0 (that is, $\forall x\in[a,b]\ g(x)>c$ for some $c>0$), then for almost all $x\in[a,b]$

$$\omega'(x)=-1+\frac{2}{\pi}\operatorname{arccot}\left[\frac{1}{\pi}\left(\log\left(\frac{b-x}{x-a}\right)+\frac{1}{g(x)}\int_a^b\frac{g(u)-g(x)}{u-x}du\right)\right]$$

(here, and below, arccot is the branch of the inverse cotangent function which returns values between 0 and π.)

(3b) If the interior transition measure of ω is absolutely continuous, and its density $h(x)$ is piecewise-continuously-differentiable, has a bounded derivative, and is bounded away from 0, then for almost all $x \in [a, b]$

$$\omega'(x) = 1 - \frac{2}{\pi}\operatorname{arccot}\left[\frac{1}{\pi}\left(\log\left(\frac{b-x}{x-a}\right) + \frac{1}{g(x)}\left(\int_a^b \frac{g(u)-g(x)}{u-x}du + \frac{2(x-z(\omega))}{A(\omega)}\right)\right)\right]$$

As will be seen in section 6, where the proof of Theorem 3 is given, at the heart of the proof is a limiting calculation involving approximation of the transition measure by atomic measures. In the special case where $g(x)$ is the uniform density on $[0, 1]$, this calculation is particularly simple and may be thought of as a result on the location of the roots of a certain polynomial. This seems worthy of mention both for its own sake and as an aid in following the proof of the general case:

Theorem 4. Let $p_n(t) = t(t-1)(t-2)(t-3)...(t-n)$. The derivative p'_n has a root between each two roots of p_n, so write $p'_n(t) = n \cdot \prod_{k=0}^{n-1}(t-(k+\lambda_{n,k}))$, where $0 < \lambda_{n,k} < 1$ are the fractional parts of the roots of p'_n. Then we have for all $0 < x < 1$,

$$\lim_{n\to\infty} \lambda_{n,\lfloor x\cdot n\rfloor} = \frac{1}{\pi}\operatorname{arccot}\left[\frac{1}{\pi}\log\left(\frac{1-x}{x}\right)\right]$$

In other words, a plot of the fractional parts of the roots of p'_n, in order of appearance, converges to a continuous curve. Figure 2 below shows a sample plot of the roots (in this example, $n = 30$) shown against the limiting curve.

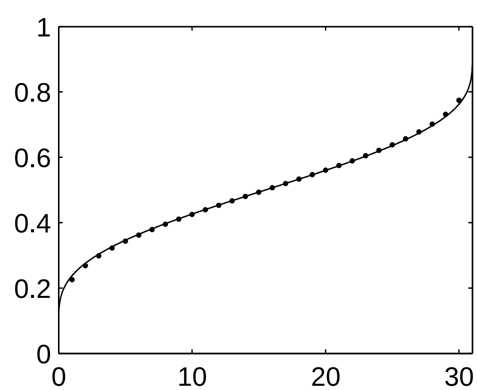

Figure 2: Fractional parts of the roots of p'_{30} and the limiting curve

Kerov's conjecture. S. Kerov (1993, 1998) conjectured a formula equivalent to formula (1a), as the correct expression not just for the density of the exterior transition measure in the case when this measure is absolutely continuous, but more generally for the *absolutely continuous part* of the exterior transition measure, for any diagram in $\mathcal{D}[a,b]$. Cifarelli and Regazzini (1990) proved this for *convex* diagrams. Our Theorem (1a) verifies the conjecture for the restricted class of diagrams $\mathcal{S}[a,b]$. However, we remark that it is quite easy, using the techniques presented in this paper, to further extend the domain of validity of the formula to a more general class of diagrams, covering partially the case where the exterior transition measure is a mixture of an absolutely continuous part and a discrete part (with no singular component): This is the class of all the positive, continuous functions $\omega : [a,b] \to \mathbb{R}$ that are piecewise twice-continuously-differentiable, that satisfy $a + \omega(a) = b - \omega(b)$, and such that on any segment of smoothness of ω, either the derivative of ω is bounded between two constants in $(-1,1)$, or it is identically equal to either -1 or 1. (It is these linear segments which add the atomic parts to the transition measure.)

4. Uniqueness, continuity, and moments

We now review some of the properties of the hook walks on general diagrams. A special class of diagrams, the *rectangular* diagrams, will play an important role. These are the diagrams for which the transition measures are atomic measures with finite support, so in a sense they are at the opposite end of the spectrum from the smooth diagrams, and it is using approximation by these diagrams that the theorems of section 3 will be proven.

A diagram $\omega \in \mathcal{D}[a,b]$ is called rectangular if it is piecewise linear and its derivative is equal to ± 1, whereever it exists (Figure 3). Rectangular diagrams have a particularly simple description using their sets of local minima and maxima: Let $x_1 < x_2 < x_3 < \ldots < x_n$ be the set of minima of ω, and $y_1 < y_2 < \ldots < y_{n-1}$ be its set of maxima. (x_k) and (y_k) are interlacing sequences, that is, we can write $x_1 < y_1 < x_2 < y_2 < \ldots < y_{n-1} < x_n$. The interlacing sequence pair $(x_k < y_k < x_{k+1})_{k=1}^{n-1}$ determines ω uniquely. (In this definition, we think of ω as a function on $\mathbb{R}$, identifying outside $[a,b]$ with the function $x \to |x-z|$; in other words, a and b may not be considered as local maxima, and may be considered as local minima only if $\omega'(a+) = 1$ and $\omega'(b-) = -1$, respectively.)

The center and the area of a rectangular diagram may be expressed using the minima and maxima:

$$z(\omega) = \sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} y_k, \qquad A(\omega) = 2 \sum_{1 \le j \le k \le n-1} (y_j - x_j)(x_{k+1} - y_k)$$

Denote by $\mathcal{D}_0[a,b]$ the set of all rectangular diagrams on $[a,b]$.

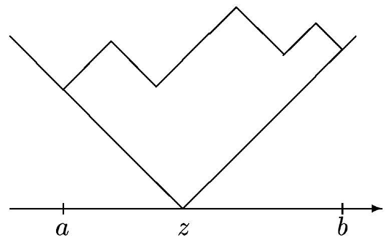

Figure 3: A rectangular diagram

4.1. The exterior walk

The results of this subsection have appeared in Kerov (1993), we include them for completeness and to motivate the analogous results of the following subsection, which discusses the interior walk.

Our starting point is the formula for the exterior transition measure of a rectangular diagram. The transition measure μ in this case is clearly atomic, and concentrated on the set of minimum points (x_k). We have for $k = 1, 2, ..., n$,

$$(1)\quad \mu(x_k) = \frac{\prod_i (x_k - y_i)}{\prod_{i \neq k}(x_k - x_i)} = \prod_{i<k}\left(1 - \frac{y_i - x_i}{x_k - x_i}\right) \cdot \prod_{i>k}\left(1 - \frac{x_i - y_{i-1}}{x_i - x_k}\right)$$

For the proof see Kerov (1993). Equivalently, one may define $\mu(x_k)$ using the partial fraction decomposition

$$(2)\qquad \sum_{k=1}^{n} \frac{\mu(x_k)}{x - x_k} = \frac{\prod_{i=1}^{n-1}(x - y_i)}{\prod_{i=1}^{n}(x - x_i)}$$

(2) can be rewritten as

$$(3)\qquad \int_{\mathbb{R}} \frac{d\mu(t)}{x - t} = \frac{1}{x} \exp\left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t - x}\right)$$

where σ is the *charge* of the diagram ω, defined as the function $\sigma(x) = (\omega(x) - |x|)/2$. This holds for real $x \notin [a, b]$. We now show that (3) can in fact be taken as an alternative defining equation for the exterior transition measure of *any* diagram (i.e. not just a rectangular one):

Lemma 1. For any diagram $\omega \in \mathcal{D}[a, b]$ with charge σ and exterior transition measure μ, (3) holds.

Proof. Equip $\mathcal{D}[a, b]$ with the topology of uniform convergence, and the set of measures on $[a, b]$ with the weak topology. Clearly, $\mathcal{D}_0[a, b]$ is dense in $\mathcal{D}[a, b]$.

Let ω_n be a sequence of rectangular diagrams converging to ω. It is easy to see that the distribution of the entire exterior hook walk on ω_n converges weakly to the distribution of the exterior hook walk on ω, and in particular, the transition measures μ_n of ω_n converge to μ. Also, the signed measures $d\sigma_n$ corresponding to the charges of ω_n, converge weakly to $d\sigma$. Therefore, for $x \notin [a,b]$ we have

$$\int_{\mathbb{R}} \frac{d\mu_n(t)}{x-t} \xrightarrow[n\to\infty]{} \int_{\mathbb{R}} \frac{d\mu(t)}{x-t},$$

$$\frac{1}{x}\exp\left(\int_{\mathbb{R}} \frac{d\sigma_n(t)}{t-x}\right) \xrightarrow[n\to\infty]{} \frac{1}{x}\exp\left(\int_{\mathbb{R}} \frac{d\sigma(t)}{t-x}\right)$$

Since (3) holds for each of the ω_n, it holds for ω. ■

We now rephrase equation (3) using moments. For a diagram $\omega \in \mathcal{D}[a,b]$ with charge σ and exterior transition measure μ, define $p_n = -n\int_{\mathbb{R}} u^{n-1} d\sigma(u)$ $(n = 1,2,3,...)$, and $h_n = \int_{\mathbb{R}} u^n d\mu(u)$ $(n = 0,1,2,...)$. By expanding into power series the integrands on both sides of (3), we can rewrite it as an identity of generating functions

$$\sum_{n=0}^{\infty} h_n x^{-n} = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} x^{-n}\right) \tag{4}$$

The two series converge when $|x|$ is large enough. By equating the coefficients on both sides one obtains the relation

$$h_n = \sum_{\rho\vdash n} \prod_{k\geq 1} \left(\frac{p_k}{k}\right)^{\rho_k} / \rho_k!, \tag{5}$$

where $\rho = (\rho_1, \rho_2, \rho_3, ...)$ runs over all partitions of n. (ρ_k indicates the number of times k appears in the partition, so $n = \sum k\rho_k$.) From this it is easy to see by induction that h_n also determines p_n uniquely (in fact each p_n is a polynomial in $h_1, ..., h_n$).

We are now in a position to prove:

Theorem 5. The correspondence $\omega \to \mu$ which assigns to a diagram ω its exterior transition measure μ establishes a homeomorphism between the set $\mathcal{D}[a,b]$ and the set $\mathcal{M}[a,b]$ of probability measures on the interval $[a,b]$ (with the topologies defined above).

Proof. Proofs may be found in Kerov (1993), Krein-Nudelman (1977). We give a proof which is a variation on Kerov's proof: If $\omega_n \to \omega$ in $\mathcal{D}[a,b]$, then, using the same argument as in the proof of Lemma 1 above, because of weak convergence of the distribution of the entire hook walk, we also have convergence $\mu_n \to \mu$ of the transition measures. So the correspondence $\omega \to \mu$ is continuous. We now prove that it is invertible and its inverse is continuous: if $\mu \in \mathcal{M}[a,b]$, take a sequence of atomic probability measures with finite support $\mu_n \in \mathcal{M}[a,b]$ converging weakly to μ. For each such μ_n there exists a (unique) rectangular diagram ω_n whose transition measure is μ_n (this follows directly from (2) - the (x_k) are the atoms of μ_n and the (y_k) are the roots of the equation $\sum \mu(x_k)/(x - x_k) = 0$). Since the ω_n are 1-Lipschitz and satisfy $0 \le \omega(a) \le b - a$, they are equicontinuous and uniformly bounded, therefore by the Arzela-Ascoli theorem have a uniformly convergent subsequence $\omega_{n_k} \to \omega$. By the continuity proven above, $\mu_{n_k} \to \mu'$ where μ' is the transition measure of ω. But $\mu_{n_k} \to \mu$, so $\mu = \mu'$ and ω is the desired inverse image of μ. The uniqueness of the inverse image of μ follows from the fact stated above that the moments h_n of μ determine uniquely the moments p_n of σ, which determine σ (and therefore ω) uniquely. Finally, if $\mu_n \to \mu$ and ω, ω_n are the inverse images of μ, μ_n, respectively, then any subsequence ω_{n_k} contains (by Arzela-Ascoli) a convergent subsequence $\omega_{n_{k_j}}$, which, by uniqueness and continuity, must converge to ω. Therefore ω_n itself must converge to ω. This establishes continuity of the inverse correspondence and finishes the proof. ∎

4.2. The interior walk

The interior walk exhibits an interesting duality with the exterior walk, so the ideas of the previous subsection copy over, with some minor changes, to the case of the interior walk. One notable complication is that the correspondence assigning to a diagram its interior transition measure is not one-to-one, so we do not have uniqueness of an inverse image. However, uniqueness can be restored if we fix two parameters, the center and the area of the diagram.

We start, as before, with the formula for the interior transition measure of a rectangular diagram. The transition measure ν will in this case be concentrated on the maximum points (y_k) of the diagram, with the sizes of the atoms being

$$(6) \qquad \nu(y_k) = -\frac{2}{A(\omega)} \cdot \frac{\prod_i (y_k - x_i)}{\prod_{i \neq k}(y_k - y_i)} =$$

$$= \frac{2}{A(\omega)} \cdot (x_{k+1} - y_k)(y_k - x_k) \cdot \prod_{i<k} \left(1 + \frac{y_i - x_i}{y_k - y_i}\right) \cdot \prod_{j>k} \left(1 + \frac{x_{j+1} - y_j}{y_j - y_k}\right)$$

for $k = 1, 2, ..., n-1$. While formula (5) appears explicitly in Kerov (1993, 1999), (6) appears in a somewhat different form in Greene-Nijenhuis-Wilf's (1979) treatment of the discrete hook walk. To formally deduce it from their result, one needs to consider first a walk on rectangular diagrams having their center at 0 and all of whose local extrema lie on integer points of the plane. For those diagrams, their formula translates to (6) upon conversion to rotated coordinates. Next, it can be seen by scaling reasons that the formula is valid for diagrams whose local extrema lie on rational points of the plane. And then, by approximation the result follows. Alternatively, one may form a Markov chain analogous to the one in proposition 4.1 of Kerov (1993) and use arguments similar to the ones in the original paper of Greene-Nijenhuis-Wilf (1979) (see also Pittel (1986)) to give a direct proof.

As in the exterior walk case, (6) is equivalent to the partial fraction decomposition

$$(7) \qquad -\frac{A(\omega)}{2}\sum_{k=1}^{n-1}\frac{\nu(y_k)}{x-y_k}+x-z(\omega)=\frac{\prod_{i=1}^{n}(x-x_i)}{\prod_{i=1}^{n-1}(x-y_i)}$$

(this is one way of verifying that the $\nu(y_k)$ sum to 1) which can be rewritten as

$$(8) \qquad -\frac{A(\omega)}{2}\int_{\mathbb{R}}\frac{d\nu(t)}{x-t}+x-z(\omega)=x\exp\left(-\int_{\mathbb{R}}\frac{d\sigma(t)}{t-x}\right)$$

We state for the record:

Lemma 2. (8) holds for any diagram $\omega\in\mathcal{D}[a,b]$ with charge σ and interior transition measure ν.

Proof. Take a sequence $\omega_n\in\mathcal{D}[a,b]$ of diagrams approximating ω and having the same area and center as ω, and continue as in the proof of Lemma 1. ■

As before, we translate (8) into the language of moments. With $p_n=-n\int_{\mathbb{R}}u^{n-1}d\sigma(u)\ \ (n=1,2,...)$ as before, and $g_n=\int_{\mathbb{R}}u^n d\nu(u)\ \ (n=0,1,2,...)$, we have the equation

$$(9) \qquad 1-\frac{z}{x}-\frac{A}{2}\sum_{n=0}^{\infty}g_n x^{-(n+2)}=\exp\left(-\sum_{n=1}\frac{p_n}{n}x^{-n}\right),$$

valid for large enough $|x|$. Upon equation of the coefficients one obtains

$$(10) \quad g_{n-2}=-\frac{2}{A}\sum_{\rho\vdash n}\prod_{k\geq 1}\left(\frac{-p_k}{k}\right)^{\rho_k}/\rho_k!=\frac{2}{A}\sum_{\rho\vdash n}(-1)^{\sum\rho_k+1}\prod_{k\geq 1}\left(\frac{p_k}{k}\right)^{\rho_k}/\rho_k!$$

(We also get $z=p_1$ and, since $g_0=1$, $A=p_2-p_1^2$.) We note as before that this implies that ν determines ω uniquely, *provided the area and center are fixed.* Another small complication relative to the case of the exterior walk, is that the support of ν is generally smaller than the support of $\omega(x)-|x-z|$. Also note that the trivial rectangular diagram $x\to|x-z|$ does *not* have an interior transition measure. This leads us to the following analogue of Theorem 5:

Theorem 6. Fix $z\in\mathbb{R}, A>0$. Let $\mathcal{D}_{A,z}[a,b]$ be the set of all diagrams on $[a,b]$ having area A and center z, equipped with the topology of uniform convergence. The correspondence $\omega\to\nu$ assigning to a diagram $\omega\in\mathcal{D}_{A,z}[a,b]$ its interior transition measure ν is a homeomorphism between $\mathcal{D}_{A,z}[a,b]$ and some closed subset of $\mathcal{M}[a,b]$. In the inverse direction, for each probability measure $\nu \in \mathcal{M}[a,b]$, there exists a unique diagram $\omega \in \mathcal{D}_{A,z}[c,d]$ for some interval $[c,d] \supset [a,b]$ such that ν is its interior transition measure. c and d may be taken to depend only on a and b, and not on ν. The correspondence $\nu \to \omega$ is a homeomorphism between $\mathcal{M}[a,b]$ and some closed subset of $\mathcal{D}_{A,z}[c,d]$.

Proof. This is basically a repetition of the arguments used in the proof of Theorem 5. The only new fact left to prove, in order to enable the use of the Arzela-Ascoli theorem and to prove the last claim in Theorem 6, is the following: For a given area A, center z, and interval $[a,b]$, there exists an interval $[c,d] \supset [a,b]$ such that for every atomic measure ν with finite support, the unique rectangular diagram ω having ν as its interior transition measure (whose existence is guaranteed by (7)) has its support (or more precisely the support of $\omega(x) - |x-z|$) contained in $[c,d]$.

We proceed to prove this fact: Let $x_1 < y_1 < x_2 < ... < y_{n-1} < x_n$ be the interlacing sequences of minima and maxima of ω. $y_1, y_2, ..., y_{n-1}$ are the atoms of ν, so it suffices to find an interval $[c,d]$ guaranteed to contain x_1, x_n. x_n is the root of the equation

$$-\frac{A}{2}\sum_{k=1}^{n-1}\frac{\nu(y_k)}{x-y_k} + x - z = 0$$

which lies to the right of y_{n-1}. But since

$$-\frac{A}{2}\sum_{k=1}^{n-1}\frac{\nu(y_k)}{x-y_k} + x - z > -\frac{A}{2}\cdot\frac{1}{x-y_{n-1}} + x - z,$$

and both of these functions are increasing on (y_{n-1},∞), x_n can be no greater than the root of the equation

$$-\frac{A}{2}\cdot\frac{1}{x-y_{n-1}} + x - z = \frac{x^2-(y_{n-1}+z)x+(zy_{n-1}-A/2)}{x-y_{n-1}} = 0$$

Remembering that $y_{n-1} \le b$, we have $x_n \le (b+z+\sqrt{(b+z)^2+2A})/2$, which gives us the upper bound d for the support of ω. d depends only on a and b since we have trivially $a \le z \le b$ and $0 < A \le (b-a)^2/2$, otherwise $\mathcal{D}_{A,z}[a,b] = \emptyset$. The lower bound is obtained in a similar way. ■

Remark. Theorem 6 implies that the correspondence $\omega \to (A(\omega), z(\omega), \nu)$ gives a bijection between $\mathcal{D}_{\mathbb{R}} := \cup_{a<b}\mathcal{D}[a,b] \setminus \{\text{trivial diagrams } x \to |x-z|\}$ and $(0,\infty)\times\mathbb{R}\times\mathcal{M}_{\mathbb{R}}$, where $\mathcal{M}_{\mathbb{R}} := \cup_{a<b}\mathcal{M}[a,b]$. However, it can be seen that this bijection is *not* a homeomorphism when $\mathcal{D}_{\mathbb{R}}$ and $\mathcal{M}_{\mathbb{R}}$ are equipped with the topologies of uniform and weak convergence, respectively. It is interesting to ask what is the precise topological nature of this correspondence. For our purposes, however, the results of Theorem 6 suffice.

5. Calculation of the densities

In this section, we calculate the densities of the exterior and interior transition measures for smooth diagrams. The basic tool is to approximate smooth diagrams by rectangular ones and use formulas (1) and (6). We write a detailed analysis of the exterior case, and go rapidly through the calculation in the interior case.

5.1. The exterior transition measure

We shall prove Theorem (1a) in two approximation steps. First, we prove it for diagrams $\omega \in \mathcal{S}[a,b]$ which are piecewise linear. Then, we shall approximate arbitrary smooth diagrams by piecewise linear ones. We shall use the two following well-known relations involving the gamma function:

$$(11) \quad \prod_{k=1}^{n}\left(1+\frac{t}{k}\right) \sim \frac{n^t}{\Gamma(t+1)}, \qquad (12) \quad \Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(\pi t)}$$

5.1.1. Piecewise linear diagrams

Let $\omega \in \mathcal{S}[a,b]$ be piecewise linear. Let μ be the measure on $[a,b]$ whose density $g(x)$ is given by the right-hand side of (1a). We define a sequence ω_n of rectangular diagrams approximating ω, as follows: First note that a rectangular diagram is determined uniquely by giving its local minima x_i and the values there. Now define ω_n by having its local minima be the points dividing each of the segments of linearity of ω into n equal parts, together with the requirement that ω_n interpolate ω at these points.

Let μ_n be the exterior transition measure of ω_n. Our claim is that $\mu_n \to \mu$ weakly as $n \to \infty$. We will work within each segment of linearity of ω, and finally "glue" the results together.

Let $[A,B] \subset [a,b]$ be a segment of linearity of ω. Thinking of n as fixed for the moment, let $A = x_N < x_{N+1} < x_{N+2} < \ldots < x_{N+n} = B$ be the minima of ω_n within the segment $[A,B]$, that is, $x_{N+k} = A + (B-A)k/n, k = 0,1,2,\ldots,n$. It is easy to calculate that $y_{N+k} = A + (B-A)(k+1/2)/n + (B-A)d/2n$, where $d = (\omega(B)-\omega(A))/(B-A)$ is the slope of ω on $[A,B]$ (y_{N+k} is calculated by reasoning that it lies on the intersection of the two lines whose equations are $t \to \omega(x_{N+k}) + t - x_{N+k}$, $t \to \omega(x_{N+k+1}) - t + x_{N+k+1}$).

We now calculate the asymptotics of the probabilities $\mu_n(x_{N+k})$, hoping to get approximately $(B-A)/n$ ("Δx") times the density of μ at x_{N+k}. To make the argument rigorous, first replace μ_n by an absolutely continuous version of it, μ'_n, by dispersing the measure of each x_{N+k} uniformly over the interval $[x_{N+k}, x_{N+k+1})$, and eliminating the measure of the last point x_{N+n}. Since μ_n and μ'_n clearly converge or diverge weakly together (it will be easy to see from the calculation below that the measure of the eliminated point x_{N+n} is negligible), the claim that $\mu_n \to \mu$ thus reduces to checking that the sequence of densities g_n of μ'_n converges to $g(x)$ and is majorized by an integrable function, so that the dominated convergence theorem can be applied.

Fix $x \in (A, B)$, and let k such that $x_{N+k} \le x < x_{N+k+1}$ (note that there is an implicit dependence of n, of the partition points x_{N+j} as well as of k; as n grows to infinity, k behaves like $n \cdot (x - A)/(B - A)$). Then:

$$g_n(x) = \left(\frac{B-A}{n}\right)^{-1} \cdot \mu_n(x_{N+k}) =$$

$$= \left(\frac{B-A}{n}\right)^{-1} \cdot \prod_{i<N+k} \left(1 - \frac{y_i - x_i}{x_{N+k} - x_i}\right) \cdot \prod_{i>N+k} \left(1 - \frac{x_i - y_{i-1}}{x_i - x_{N+k}}\right) =$$

$$= \left[\prod_{i<N} \left(1 - \frac{y_i - x_i}{x_{N+k} - x_i}\right) \cdot \prod_{i>N+n} \left(1 - \frac{x_i - y_{i-1}}{x_i - x_{N+k}}\right)\right] \cdot$$

$$\cdot \left[\left(\frac{B-A}{n}\right)^{-1} \cdot \prod_{i=N}^{N+k-1} \left(1 - \frac{y_i - x_i}{x_{N+k} - x_i}\right) \cdot \prod_{i=N+k+1}^{N+n} \left(1 - \frac{x_i - y_{i-1}}{x_i - x_{N+k}}\right)\right]$$

We treat the two parenthesized expressions in the last equation separately: The second one is equal to

$$\left(\frac{B-A}{n}\right)^{-1} \cdot \prod_{i=0}^{k-1} \left(1 - \frac{y_{N+i} - x_{N+i}}{x_{N+k} - x_{N+i}}\right) \cdot \prod_{i=k+1}^{n} \left(1 - \frac{x_{N+i} - y_{N+i-1}}{x_{N+i} - x_{N+k}}\right) =$$

$$\left(\frac{B-A}{n}\right)^{-1} \cdot \prod_{i=0}^{k-1} \left(1 - \frac{(1+d)/2}{k-i}\right) \cdot \prod_{i=k+1}^{n} \left(1 - \frac{(1-d)/2}{i-k}\right) =$$

$$\left(\frac{B-A}{n}\right)^{-1} \cdot \prod_{i=1}^{k} \left(1 - \frac{(1+d)/2}{i}\right) \cdot \prod_{i=1}^{n-k} \left(1 - \frac{(1-d)/2}{i}\right) \sim_{n\to\infty}$$

$$\left(\frac{B-A}{n}\right)^{-1} \cdot \frac{k^{-(1+d)/2}}{\Gamma((1-d)/2)} \cdot \frac{(n-k)^{-(1-d)/2}}{\Gamma((1+d)/2)} =$$

$$= \frac{1}{\pi} \sin(\pi(1+d)/2) \cdot \left(\frac{k}{n}(B-A)\right)^{-(1+d)/2} \cdot \left(\frac{n-k}{n}(B-A)\right)^{-(1-d)/2} =$$

$$\frac{1}{\pi} \sin(\pi(1+d)/2) \cdot (x_{N+k} - A)^{-(1+d)/2} \cdot (B - x_{N+k})^{-(1-d)/2} \sim$$

$$\frac{1}{\pi} \sin(\pi(1+d)/2) \cdot (x - A)^{-(1+d)/2} \cdot (B - x)^{-(1-d)/2}$$

It remains to evaluate the asymptotics of the products in the first parentheses; this is in fact simpler, since in these products the individual terms tend to zero. In general, a product of the form $\prod(1 + \Delta x_i h(x_i))$ converges to $\exp(\int h(u)du)$. In our case, the limit of the two products is then easily seen to be

$$\exp\left(\int_a^A \frac{-(1+\omega'(u))/2}{x-u} du + \int_B^b \frac{-(1-\omega'(u))/2}{u-x} du\right)$$

Putting the pieces together, we have the formula

$$\lim_{n\to\infty} g_n(x) = \frac{1}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-A)^{-(1+\omega'(x))/2}\cdot(B-x)^{-(1-\omega'(x))/2}\cdot$$
$$\exp\left(\int_a^A \frac{-(1+\omega'(u))/2}{x-u}du + \int_B^b \frac{-(1-\omega'(u))/2}{u-x}du\right)$$

We now rearrange the terms slightly, noting that

$$(x-A)^{-(1+\omega'(x))/2} = (x-a)^{-(1+\omega'(x))/2}\cdot\exp\left(\int_a^A \frac{(1+\omega'(x))/2}{x-u}du\right)$$

and

$$(B-x)^{-(1-\omega'(x)/2)} = (b-x)^{-(1-\omega'(x))/2}\cdot\exp\left(\int_B^b \frac{(1-\omega'(x))/2}{u-x}du\right)$$

to finally arrive at

$$\lim_{n\to\infty} g_n(x) = \frac{1}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-a)^{-\frac{1+\omega'(x)}{2}}\cdot(b-x)^{-\frac{1-\omega'(x)}{2}}\cdot$$
$$\exp\left(\frac{1}{2}\int_{[a,A]\cup[B,b]}\frac{\omega'(u)-\omega'(x)}{u-x}du\right)$$

One more cosmetic change of the formula is to write

$$\lim_{n\to\infty} g_n(x) = \frac{1}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-a)^{-\frac{1+\omega'(x)}{2}}\cdot(b-x)^{-\frac{1-\omega'(x)}{2}}\cdot$$
$$\exp\left(\frac{1}{2}\int_a^b\frac{\omega'(u)-\omega'(x)}{u-x}du\right) = g(x),$$

since within the segment $[A,B]$ there is no contribution to the integral inside the exponent.

To complete this part of the proof, we need to show that the densities $g_n(x)$ are uniformly bounded by some integrable function. Looking back at the two parenthesized expressions, we see that the first is bounded by 1, and the second is

$$\left(\frac{B-A}{n}\right)^{-1}\cdot\prod_{i=1}^{k}\left(1-\frac{(1+d)/2}{i}\right)\cdot\prod_{i=1}^{n-k}\left(1-\frac{(1-d)/2}{i}\right)\le$$
$$\le c_1\cdot\left(\frac{B-A}{n}\right)^{-1}\cdot\prod_{i=1}^{k+1}\left(1-\frac{(1+d)/2}{i}\right)\cdot\prod_{i=1}^{n-k}\left(1-\frac{(1-d)/2}{i}\right)\le$$
$$\le c_1\cdot\left(\frac{B-A}{n}\right)^{-1}\cdot\exp\left(-\sum_{i=1}^{k+1}\frac{(1+d)/2}{i}-\sum_{i=1}^{n-k}\frac{(1-d)/2}{i}\right)\le$$

$$\leq c_2 \cdot \left(\frac{B-A}{n}\right)^{-1} \cdot (k+1)^{-(1+d)/2} \cdot (n-k)^{-(1-d)/2} =$$

$$= c_2 \cdot \left(\frac{(k+1)(B-A)}{n}\right)^{-(1+d)/2} \cdot \left(\frac{(n-k)(B-A)}{n}\right)^{-(1-d)/2} \leq$$

$$\leq c_2 \cdot (x-A)^{-(1+d)/2} \cdot (B-x)^{-(1-d)/2}$$

for some constants c_1, c_2 (depending on ω). Thus we have shown that

$$g_n(x) \leq c_2 \cdot (x-A)^{-(1+\omega'(x))/2} \cdot (B-x)^{-(1-\omega'(x))/2}$$

inside any maximal segment of linearity $[A,B]$, and since $\sup_{x\in[a,b]} |\omega'(x)| < 1$, this is an integrable function. Together with the fact that $g_n(x) \to g(x)$ for all x in the interior of a segment of linearity of ω, this finishes the proof that $\mu_n \to \mu$ weakly. This proves that μ is indeed the transition measure of ω, as was claimed, and therefore (1a) is true for piecewise linear diagrams.

5.1.2. Smooth diagrams

We now turn to the final approximation step, that of going from piecewise linear diagrams to piecewise C^2 ones. Let $\omega \in \mathcal{S}[a,b]$, and define a sequence of approximating piecewise-linear diagrams ω_n, as follows: for each n, partition each of the segments of smoothness of ω into 2^n equal parts. Then ω_n is the diagram that is linear on each of the partition intervals and interpolates ω at their endpoints. We denote by $\mathcal{P}$ the set of all these endpoints (a countable set), and denote $L = \sup_{x\in[a,b]} |\omega''(x)| < \infty$, $M = \sup_{x\in[a,b]} |\omega'(x)| < 1$.

Let μ_n be the exterior transition measure, with density $g_n(x)$, of ω_n. Let μ be the measure whose density $g(x)$ is given by (1a) (for the diagram ω). For the same reasons as in the previous subsection, it will suffice to prove that μ_n converges weakly to μ as $n \to \infty$, to imply that μ is indeed the transition measure of ω. We shall show this in two steps: first, we show that $g_n(x) \to g(x)$ for *almost all* $x \in [a,b]$ (somewhat surprisingly, this fails on a large set of x's, though a set of measure zero). Finally, a suitable boundedness argument will assure the weak convergence.

The first step. Define

$$p(x) = \frac{1}{\pi}\cos(\pi\omega'(x)/2) \cdot (x-a)^{-(1+\omega'(x))/2}(b-x)^{-(1-\omega'(x))/2}$$

$$q(x) = \exp\left(\frac{1}{2}\int_a^b \frac{\omega'(u)-\omega'(x)}{u-x}du\right)$$

$$p_n(x) = \frac{1}{\pi}\cos(\pi\omega_n'(x)/2) \cdot (x-a)^{-(1+\omega_n'(x))/2}(b-x)^{-(1-\omega_n'(x))/2}$$

$$q_n(x) = \exp\left(\frac{1}{2}\int_a^b \frac{\omega_n'(u) - \omega_n'(x)}{u-x}du\right),$$

so that $g_n(x) = p_n(x)q_n(x), g(x) = p(x)q(x)$. Clearly $p_n(x) \to p(x)$ for all $x \in [a,b] \setminus \mathcal{P}$, we now try to show $q_n(x) \to q(x)$ (this will fail for some x's, but succeed for most): For a given $x \in [a,b] \setminus \mathcal{P}$,

$$\frac{\omega_n'(u) - \omega_n'(x)}{u-x} \xrightarrow[n\to\infty]{} \frac{\omega'(u) - \omega'(x)}{u-x}$$

for all $u \in [a,b] \setminus \mathcal{P} \setminus \{x\}$. To deduce that $q_n(x) \to q(x)$, some kind of boundedness argument is now required. Let $[A,B]$ be the maximal segment of smoothness of ω containing x. Then for all $u \in [a,A] \setminus \mathcal{P}$ we have

$$\left|\frac{\omega_n'(u) - \omega_n'(x)}{u-x}\right| \le \frac{2M}{x-A},$$

and for all $u \in [B,b] \setminus \mathcal{P}$

$$\left|\frac{\omega_n'(u) - \omega_n'(x)}{u-x}\right| \le \frac{2M}{B-x},$$

which implies by the dominated convergence theorem that

$$\int_{[a,A]\cup[B,b]} \frac{\omega_n'(u) - \omega_n'(x)}{u-x}du \xrightarrow[n\to\infty]{} \int_{[a,A]\cup[B,b]} \frac{\omega'(u) - \omega'(x)}{u-x}du$$

Now let $0 \le k = k(n) \le 2^n - 1$ be such that x is in the kth partition interval of the segment $[A,B]$, i.e. $A + (B-A)k/2^n < x < A + (B-A)(k+1)/2^n$. We bound $(\omega_n'(u) - \omega_n'(x))/(u-x)$ (as a function of u) separately on the different partition intervals within $[A,B]$. On the interval $(A+(B-A)k/2^n, A+(B-A)(k+1)/2^n)$ this expression is zero. On the other intervals: We can write $\omega_n'(u) = \omega'(u')$, $\omega_n'(x) = \omega'(x')$, where u' is in the same partition interval as u and x' is in the same partition interval as x (since ω_n is, within each partition interval, a linear interpolation of ω). Thus if u is in the jth interval, then

$$\left|\frac{\omega_n'(u) - \omega_n'(x)}{u-x}\right| = \left|\frac{\omega'(u') - \omega'(x')}{u'-x'}\right| \cdot \left|\frac{u'-x'}{u-x}\right| =$$

$$= |\omega''(u'')| \cdot \left|1 + \frac{(u'-u) + (x'-x)}{u-x}\right| \le L \cdot \left(1 + \frac{2(B-A)}{2^n \cdot |u-x|}\right) \le$$

$$\le L \cdot \left(1 + \frac{2}{|j-k| - 1}\right)$$

For $j < k-1$ or $j > k+1$ this gives an effective bound of $3L$. For the $(k-1)$th and $(k+1)$th interval we are left with the bound of $(1 + 2(B-A)/(2^n|u-x|))$, which is not effective at all, since when x and u are in adjacent intervals they can be arbitrarily close! One may describe exactly how close they may be using the binary expansion of $(x-A)/(B-A)$: if we denote by $s_n(x)$ the length of the sequence of zeroes in this binary expansion starting at the nth place, and by $t_n(x)$ the length of the sequence of ones starting at the nth place, then for u in the $(k \pm 1)$th interval we have $|u-x| \geq (B-A)2^{-(n+s_n(x)\vee t_n(x))}$, and so

$$\int_{A+(B-A)(k-1)/2^n}^{A+(B-A)k/2^n} \left| \frac{\omega_n'(u)-\omega_n'(x)}{u-x} \right| du \leq$$

$$\leq \frac{L(B-A)}{2^n} + \frac{2L(B-A)}{2^n} \int_{A+(B-A)(k-1)/2^n}^{A+(B-A)k/2^n} \frac{du}{x-u} =$$

$$= \frac{L(B-A)}{2^n} + \frac{2L(B-A)}{2^n} \log\left(\frac{x-(A+(B-A)(k-1)/2^n)}{x-(A+(B-A)k/2^n)} \right) \leq$$

$$\leq \frac{L(B-A)}{2^n} + \frac{2L(B-A)}{2^n} \log\left(\frac{2(B-A)/2^n}{(B-A)/2^{n+s_n(x)\vee t_n(x)}} \right) \leq$$

$$\leq \frac{L(B-A)}{2^n} + \frac{4L(B-A)}{2^n} \cdot (s_n(x) \vee t_n(x)).$$

In a similar manner, integrating on the $(k+1)$th interval gives the same bound

$$\int_{A+(B-A)(k+1)/2^n}^{A+(B-A)(k+2)/2^n} \left| \frac{\omega_n'(u)-\omega_n'(x)}{u-x} \right| du \leq \frac{L(B-A)}{2^n} + \frac{4L(B-A)}{2^n} \cdot (s_n(x) \vee t_n(x)).$$

So, our attempt to prove boundedness of the sequence of integrands failed - but we are rescued by the fact that it failed on a set of small measure, namely the two intervals adjacent to the kth, and where the values of the integrands are not too big. In other words, we claim that, under some further restrictions on x, the sequence $(\omega_n'(u)-\omega_n'(x))/(u-x)$ shall be uniformly integrable in u. Indeed, we have proved uniform boundedness on all but the two intervals adjacent to the kth, and on them we have the estimate

$$\int_{I_{k-1} \cup I_{k+1}} \left| \frac{\omega_n'(u)-\omega_n'(x)}{u-x} \right| du \leq \frac{2L(B-A)}{2^n} + \frac{8L(B-A)}{2^n} \cdot (s_n(x) \vee t_n(x)),$$

where $I_j = [A+(B-A)j/2^n, A+(B-A)(j+1)/2^n]$ is the jth interval. This bound tends to 0 (which is what we need to prove uniform integrability) for those $x \in [A,B] \setminus \mathcal{P}$ for which $s_n(x) \vee t_n(x)$ grows asymptotically at a rate smaller than, say, $2^{n/2}$. But in fact it is a well-known fact in number theory that almost every x has this property (one may prove this directly, or appeal to the stronger theorem from Feller (1957), p. 197, which says that for almost all $z \in [0,1]$, the length $t_n(z)$ of the sequence of zeros in the binary expansion starting at place n, satisfies $\limsup t_n(z)/\log_2(n) = 1$). Thus, for almost every x the sequence $(\omega_n'(u)-\omega_n'(x))/(u-x)$ is uniformly integrable, and therefore $q_n(x) \to q(x)$, as was claimed.

We note this as a lemma, to be used in section 6:

Lemma 3. If $f : [a,b] \to \mathbb{R}$ is piecewise-continuously-differentiable with bounded derivative, and f_n is a sequence of piecewise-constant functions obtained by dividing each interval of differentiability of f into 2^n equal parts and defining f_n on each subinterval as the average value of f on that subinterval. Then for almost all $x \in [a,b]$

$$\int_a^b \frac{f_n(u)-f_n(x)}{u-x}du \xrightarrow[n\to\infty]{} \int_a^b \frac{f(u)-f(x)}{u-x}du$$

The second step. Having proved $g_n(x) \to g(x)$ for almost all $x \in [a,b]$, we now finish the proof by showing that the g_n are themselves uniformly integrable. Let $x \in [A,B] \subset [a,b]$ as before. The estimates derived above imply that for some constants k_1, k_2 (depending on the diagram ω),

$$g_n(x) \le k_1(x-A)^{-M} \cdot (B-x)^{-M} e^{k_2(s_n(x)\vee t_n(x))/2^n}$$

(If $A \ne a$ and $B \ne b$, then $p_n(x)$ are uniformly bounded by a constant and $q_n(x)$ is bounded by the above expression; if $A = a$ or $B = b$, then it is the $p_n(x)$ that contributes the factor $(x-A)^{-M}$(or, respectively, $(B-x)^{-M}$), which disappears from the bound on $q_n(x)$.)

To show that this sequence of functions is uniformly integrable on $[A,B]$, we shall show that it is bounded in $\mathrm{L}_p[A,B]$ for some $p > 1$. In fact, $((x-A)(B-x))^{-M}$ is in $\mathrm{L}_p[A,B]$ for $1 \le p < M^{-1}$, in particular for $p_0 = (1+M^{-1})/2 > 1$. Let $q_0 = p_0/(p_0-1)$, and let $\epsilon > 0$ such that $p_1 = (1+\epsilon)p_0 < M^{-1}$. If we show that the sequence of functions $\exp(k_2(s_n(x)\vee t_n(x))/2^n)$ is uniformly bounded in L_q for any $q \ge 1$, and thus in particular for $(1+\epsilon)q_0$, then by Hölder's inequality it will follow that the product of the two expressions, which majorizes $g_n(x)$, is bounded in $\mathrm{L}_{1+\epsilon}$ and thus uniformly integrable. And indeed:

$$\int_A^B \exp(k_2(s_n(x)\vee t_n(x))/2^n)^q dx =$$

$$= \sum_{j=1}^{\infty} e^{k_2\cdot q\cdot j/2^n} \cdot |\{x \in [A,B] : s_n(x)\vee t_n(x) = j\}| \le$$

$$\le 2(B-A)\cdot \sum_{j=1}^{\infty} e^{k_2\cdot q\cdot j/2^n} \cdot 2^{-j},$$

and this is finite (and decreasing in n, thus bounded) after some initial value $n = n_0(q)$.

5.2. The interior transition measure

We now calculate the density of the interior transition measure of ω. The calculation is quite similar to the one in the previous subsection, as well as the various proofs of convergence. Therefore, we shall only write explicitly the calculation of the limiting density for piecewise linear diagrams. We use the same notation as in section 5.1.1.: ω is a piecewise linear diagram, ω_n is the sequence of approximating rectangular diagrams defined using the equipartition points x_k. ν_n is the interior transition measure of ω_n, and ν'_n is the absolutely continuous version of it whereby the probability of each y_k is dispersed uniformly over the interval $[x_k, x_{k+1}]$. $g_n(x)$ is the density of ν'_n. We calculate: Let $x \in (A, B) \subset [a, b]$, and let $0 \leq k < n$ such that $x_{N+k} < x < x_{N+k+1}$, then

$$g_n(x) = \left(\frac{B-A}{n}\right)^{-1} \cdot \nu(y_{N+k}) =$$

$$= \frac{2}{A(\omega_n)} \cdot \left(\frac{B-A}{n}\right)^{-1} (x_{N+k+1} - y_{N+k}) \cdot (y_{N+k} - x_{N+k}) \cdot$$

$$\cdot \prod_{i<N+k} \left(1 + \frac{y_i - x_i}{y_k - y_i}\right) \prod_{i>N+k+1}^{n} \left(1 + \frac{x_i - y_{i-1}}{y_{i-1} - y_k}\right) =$$

$$= \left[\frac{2}{A(\omega_n)} \cdot \prod_{i<N} \left(1 + \frac{y_i - x_i}{y_k - y_i}\right) \cdot \prod_{i>N+n} \left(1 + \frac{x_i - y_{i-1}}{y_{i-1} - y_k}\right)\right] \cdot$$

$$\cdot \left[\left(\frac{B-A}{n}\right)^{-1} (x_{N+k+1} - y_{N+k}) \cdot (y_{N+k} - x_{N+k}) \cdot\right.$$

$$\left. \cdot \prod_{i=N}^{N+k-1} \left(1 + \frac{y_i - x_i}{y_k - y_i}\right) \prod_{i=N+k+2}^{N+n} \left(1 + \frac{x_i - y_{i-1}}{y_{i-1} - y_k}\right)\right]$$

Again we treat the two parenthesized expressions separately; the first one converges to

$$\frac{2}{A(\omega)} \cdot \exp\left(\int_a^A \frac{(1+\omega'(x))/2}{x-u} du + \int_B^b \frac{(1-\omega'(x))/2}{u-x} du\right).$$

The second one is

$$\left(\frac{B-A}{n}\right)^{-1} \cdot \frac{1-\omega'(x)}{2} \cdot \frac{B-A}{n} \cdot \frac{1+\omega'(x)}{2} \cdot \frac{B-A}{n} \cdot$$

$$\cdot \prod_{i=0}^{k-1} \left(1 + \frac{(1+\omega'(x))/2}{i}\right) \cdot \prod_{i=1}^{n-k-1} \left(1 + \frac{(1-\omega'(x))/2}{i}\right) \sim$$

$$\Gamma\left(\frac{1+\omega'(x)}{2}\right)^{-1} \cdot \Gamma\left(\frac{1-\omega'(x)}{2}\right)^{-1} \cdot$$

$$\cdot \left(\frac{(B-A)k}{n}\right)^{(1+\omega'(x))/2} \cdot \left(\frac{(B-A)(n-k)}{n}\right)^{(1-\omega'(x))/2} \sim$$

$$\frac{1}{\pi}\cos(\pi\omega'(x)/2)\cdot(x-A)^{(1+\omega'(x))/2}\cdot(B-x)^{(1-\omega'(x))/2}$$

Altogether we have

$$\lim_{n\to\infty} g_n(x) = \frac{2}{\pi\cdot A(\omega)}\cos(\pi\omega'(x)/2)\cdot(x-A)^{(1+\omega'(x))/2}\cdot(B-x)^{(1-\omega'(x))/2}\cdot$$

$$\cdot\exp\left(\int_a^A \frac{(1+\omega'(x))/2}{x-u}du + \int_B^b \frac{(1-\omega'(x))/2}{u-x}du\right)$$

Now as before, rearranging the terms and letting A and B tend to x from above and below gives (1b) for piecewise linear diagrams.

6. Proof of Theorem 4 and the inversion formulas

Our main goal in this section is to prove Theorem 3, that gives the shape of the diagram ω in terms of the density of the transition measures of ω. We start by proving Theorem 4, which contains the essential computational idea behind the proof. We then proceed with the proof of Theorem 3, where again, the basic idea is to approximate the diagram by rectangular diagrams, and the transition measures by atomic measures. There will be two approximation steps. First, we treat the case of densities which are step functions with finitely many values. Next we approximate an arbitrary density satisfying the assumptions of Theorem 3 by such step functions. The details are given only for the exterior walk case (Theorem 3a). The proof of Theorem 3b follows the same reasoning, where the uses of formula (2) and Theorem 5 are replaced by the their respective analogues, formula (7) and Theorem 6.

6.1. Proof of Theorem 4

Recall that $p_n(t) = t(t-1)(t-2)...(t-n)$, and $(k+\lambda_{n,k})_{k=0}^{n-1}$ are the roots of p_n'. Let $0 < x < 1$, and denote $k = \lfloor x\cdot n\rfloor$. Then $k+\lambda_{n,k}$ is the root of the equation

$$\frac{p_n'(x)}{p_n(x)} = \sum_{j=0}^{n}\frac{1}{x-j} = 0$$

in the interval $(k, k+1)$. In other words we have

$$\sum_{j=0}^{k}\frac{1}{\lambda_{n,k}+k-j} - \sum_{j=k+1}^{n}\frac{1}{-\lambda_{n,k}+j-k} = 0,$$

or, transforming the indices,

$$\sum_{j=0}^{k}\frac{1}{\lambda_{n,k}+j} - \sum_{j=0}^{n-k-1}\frac{1}{(1-\lambda_{n,k})+j} = 0.$$

By the classical relations

$$(13) \qquad \sum_{j=0}^{m} \frac{1}{u+j} = -\frac{\Gamma'(u)}{\Gamma(u)} + \log(m) + o(1)_{m\to\infty}$$

$$(14) \qquad -\frac{\Gamma'(u)}{\Gamma(u)} + \frac{\Gamma'(1-u)}{\Gamma(1-u)} = \pi\cot(\pi u)$$

(equivalent to (11) and (12), respectively) the latter equation transforms to

$$\pi\cot(\pi\lambda_{n,k}) = \log\left(\frac{n-k-1}{k}\right) + o(1)_{n\to\infty} = \log\left(\frac{1-x}{x}\right) + o(1)$$

■

6.2. The inversion formula for the exterior walk

6.2.1. Step functions

The notation, and the techniques of approximation, are much like in the previous sections. Let $g(x)$, the density of the exterior transition measure μ of a diagram ω, be a step function, taking on finitely many strictly positive values on $[a,b]$. Thus, $g(x)$ is a mixture of uniform densities on each of the segments where $g(x)$ is constant. We approximate this transition measure by the corresponding mixture of discrete uniform measures: For each n, divide as before each (maximal) segment where g is constant into n equal parts. If $[A,B]$ is one such segment, let $a = x_0 < x_1 < x_2 < ... < x_{l\cdot n}$ be these division points, and let $A = x_N < x_{N+1} < ... < x_{N+n} = B$ be the division points inside the interval $[A,B]$. (There is an implicit dependence on n here, and as before we suppress it for convenience of notation.) Define the measure μ_n as the discrete measure, concentrated on the x_k, and giving to the point x_k the measure $\mu([x_k, x_{k+1}])$. Let, for each n, ω_n be the (rectangular) diagram corresponding to the discrete measure μ_n. Let $y_0 < y_1 < y_2 < ... < y_{l\cdot n-1}$ be the sequence of maxima of the diagram ω_n. For each $k = 0,1,2,...,n-1$, since $x_{N+k} < y_{N+k} < x_{N+k+1}$, we can write $y_{N+k} = x_{N+k} + \lambda_{N+k}\cdot(B-A)/n$ for some $0 < \lambda_{N+k} < 1$.

It is clear that $\mu_n \to \mu$ weakly as $n\to\infty$, and thus by Theorem 5, $\omega_n \to \omega$ uniformly on $[a,b]$. We now proceed to calculate the limit of ω_n, by calculating the limit of the λ_{N+k}. By (2), the y_i are the roots of the equation $\sum_k \frac{\mu_n(x_k)}{x-x_k} = 0$. For y_{N+k}, we write this as

$$\sum_{j=N}^{N+k} \frac{\mu_n(x_j)}{(N+k-j+\lambda_{N+k})\frac{B-A}{n}} - \sum_{j=N+k+1}^{N+n} \frac{\mu_n(x_j)}{(j-(N+k)-\lambda_{N+k})\frac{B-A}{n}} =$$

$$= \sum_{j=1}^{N-1} \frac{\mu_n(x_j)}{x_j - y_{N+k}} + \sum_{j=N+n+1}^{l\cdot n} \frac{\mu_n(x_j)}{x_j - y_{N+k}}$$

The RHS is $\int_{[a,A]\cup[B,b]} \frac{g(u)du}{u-y_{N+k}} + o_\epsilon(1)$ as $n \to \infty$, the $o_\epsilon(1)$ being uniformly small for all values of k between ϵn and $(1-\epsilon)n$. The LHS can be rewritten as

$$g(y_{N+k}) \left(\sum_{j=0}^{k} \frac{1}{j+\lambda_{N+k}} - \sum_{j=0}^{N-k-1} \frac{1}{j+(1-\lambda_{N+k})} \right).$$

As in the proof of Theorem 4 above, we may use (13) and (14) to transform this expression as

$$g(y_{N+k}) \left(\pi \cot(\pi\lambda_{N+k}) - \log\left(\frac{B-y_{N+k}}{y_{N+k}-A} \right) \right) + o_\epsilon(1)$$

(with the same uniformity property). A further rearrangement of the terms, similar to that done in the previous sections, leads to the equation

$$\lambda_{N+k} = \frac{1}{\pi}\text{arccot}\left[\frac{1}{\pi} \left(\log\left(\frac{b-y_{N+k}}{y_{N+k}-a} \right) + \frac{1}{g(y_{N+k})} \int_a^b \frac{g(u)-g(y_{N+k})}{u-x} du \right) \right] + o_\epsilon(1).$$

Now, for $x \in [A,B)$, let $k=k(n)$ such that $x_{N+k} \le x < x_{N+k+1}$, then

$$\omega_n(x) - \omega(A) = \sum_{j=N}^{N+k-1} (y_j - x_j) - \sum_{j=N+1}^{N+k} (x_j - y_{j-1}) + O\left(\frac{1}{n}\right) =$$

$$= \sum_{j=N}^{N+k-1} \frac{\lambda_j(B-A)}{n} - \sum_{j=N}^{N+k-1} \frac{(1-\lambda_j)(B-A)}{n} + O\left(\frac{1}{n}\right) =$$

$$= \frac{2(B-A)}{n} \sum_{j=N}^{N+k-1} \lambda_j - \frac{k(B-A)}{n} + O\left(\frac{1}{n}\right) =$$

$$= \frac{2(B-A)}{n} \left(\sum_{j=N+\epsilon n}^{N+k-1)n} \lambda_j + O(\epsilon) \right) - \frac{k(B-A)}{n} + O\left(\frac{1}{n}\right) =$$

$$= -(x-A) + \frac{2}{\pi} \int_A^x \text{arccot}\left[\frac{1}{\pi} \left(\log\left(\frac{b-t}{t-a} \right) + \frac{1}{g(t)} \int_a^b \frac{g(u)-g(t)}{u-t} du \right) \right] dt +$$
$$+ O\left(\frac{1}{n}\right) + o_\epsilon(1) + O(\epsilon)$$

Which finishes the proof, since ϵ was arbitrary.

6.2.2. Piecewise smooth functions

We now present the final approximation step required to finish the proof of Theorem 3a. Let $\omega \in \mathcal{D}[a,b]$ be such that its exterior transition measure is absolutely continuous, with a density $g(x)$ that is piecewise-continuously-differentiable, has bounded derivative, and is bounded away from 0. We approximate $g(x)$ by a sequence $g_n(x)$ of step functions constructed by the method specified in Lemma 3. Let $\omega_n \in \mathcal{D}[a,b]$ be the diagram whose exterior transition measure is $g_n(x)dx$. Since $g_n(x) \to g(x)$ for almost all $x \in [a,b]$, we have $\mu_n \to \mu$ weakly and thus by Theorem 5, $\omega_n \to \omega$ uniformly on $[a,b]$. By Lemma 3, and by the fact that the inverse formula holds for the g_n, we have for almost all $t \in [a,b]$

$$\lim_{n\to\infty} \omega_n'(t) = -1 + \frac{2}{\pi}\text{arccot}\left[\frac{1}{\pi}\left(\log\left(\frac{b-t}{t-a}\right) + \frac{1}{g(t)}\int_a^b \frac{g(u)-g(t)}{u-t}du\right)\right].$$

(It is here that the boundedness assumptions on g are used.) Therefore, we have for each $x \in [a,b]$ the chain of equalities

$$\omega(x) = \lim_{n\to\infty}\omega_n(x) = \lim_{n\to\infty}\int_a^x \omega_n'(t)dt = \int_a^x \lim_{n\to\infty}\omega_n'(t)dt$$

So we have shown that

$$\omega(x) = \int_a^x \left\{-1 + \frac{2}{\pi}\text{arccot}\left[\frac{1}{\pi}\left(\log\left(\frac{b-t}{t-a}\right) + \frac{1}{g(t)}\int_a^b \frac{g(u)-g(t)}{u-t}du\right)\right]\right\}dt,$$

and this implies that for almost all x

$$\omega'(x) = -1 + \frac{2}{\pi}\text{arccot}\left[\frac{1}{\pi}\left(\log\left(\frac{b-x}{x-a}\right) + \frac{1}{g(x)}\int_a^b \frac{g(u)-g(x)}{u-x}du\right)\right],$$

as was claimed.

7. Other formulas

We gather in this section some more integral formulas that arise out of the study of the hook walks.

7.1. Unrotated diagrams

We describe the hook walks and formulas (1a) and (1b) for the original continual diagrams, described simply as increasing functions f on some interval $[a,b]$ such that $f(a) = 0$. We call such a function an *unrotated diagram* on $[a,b]$, and denote the set of such diagrams by $\mathcal{U}[a,b]$. For each unrotated diagram f, there corresponds a diagram $\omega \in \mathcal{D}[A,B]$ related to f by

$$t = \frac{x+f(x)}{\sqrt{2}}, \qquad \omega(t) = \frac{f(x)+b-x}{\sqrt{2}}, \tag{15}$$

where $A = a$ (for concreteness) and $B = (b+f(b))/\sqrt{2}$. The domain of f is defined as the set

$$D_f = \{(x,y) : a \le x \le b, 0 \le y \le f(x)\}$$

and the dual domain is

$$D'_f = \{(x,y) : a \le x \le b, f(x) \le y \le f(b)\}$$

The interior hook of a point $(x,y) \in D_f$ is the set

$$\{(x',y') \in D_f : (x' \le x \text{ and } y' = y) \text{ or } (x' = x \text{ and } y' \ge y)\}$$

and the exterior hook of a point $(x,y) \in D'_f$ is

$$\{(x',y') \in D'_f : (x' = x \text{ and } y' \le y) \text{ or } (x' \ge x \text{ and } y' = y)\}$$

The interior and exterior hook walks are now defined exactly as before. The interior and exterior transition measures are the distributions of the x-coordinate of the limiting point of the walks.

If $f \in \mathcal{U}[a,b]$ is continuous, piecewise twice-continuously-differentiable, with bounded second derivative and first derivative bounded away from 0 and ∞, then the corresponding ω is in $\mathcal{S}[A,B]$ and we may use the change of variables (15) to calculate the density of the transition measures of f. We have:

$$dt = \frac{1+f'(x)}{\sqrt{2}}dx, \qquad \omega'(t) = \frac{d\omega/dx}{dt/dx} = \frac{(f'(x)-1)/\sqrt{2}}{(f'(x)+1)/\sqrt{2}} = 1 - \frac{2}{1+f'(x)}$$

We leave to the reader to verify:

Theorem 7. If f satisfies the above conditions, then:

(6a) The density of the exterior transition measure for f is equal to

$$\frac{1}{\pi}(1+f'(x)) \cdot \sin\left(\frac{\pi}{1+f'(x)}\right) \cdot$$
$$\cdot (x-a+f(x))^{-\frac{f'(x)}{1+f'(x)}} \cdot (b-x+f(b)-f(x))^{-\frac{1}{1+f'(x)}} \cdot$$
$$\cdot \exp\left(\int_a^b \frac{1}{u-x+f(u)-f(x)} \cdot \frac{f'(u)-f'(x)}{1+f'(x)}du\right)$$

(6b) The density of the interior transition measure for f is equal to

$$\frac{1}{\pi}(1+f'(x))\cdot\sin\left(\frac{\pi}{1+f'(x)}\right)\cdot$$

$$\cdot(x-a+f(x))^{-\frac{1}{1+f'(x)}}\cdot(b-x+f(b)-f(x))^{-\frac{f'(x)}{1+f'(x)}}\cdot$$

$$\cdot\exp\left(-\int_a^b\frac{1}{u-x+f(u)-f(x)}\cdot\frac{f'(u)-f'(x)}{1+f'(x)}du\right)$$

In particular, for such f we have

$$\int_a^b f(x)dx =$$

$$=\int_a^b\left\{\frac{1}{\pi}(1+f'(x))\cdot\sin\left(\frac{\pi}{1+f'(x)}\right)\cdot\right.$$

$$\cdot(x-a+f(x))^{\frac{f'(x)}{1+f'(x)}}\cdot(b-x+f(b)-f(x))^{\frac{1}{1+f'(x)}}\cdot$$

$$\left.\cdot\exp\left[-\int_a^b\frac{1}{u-x+f(u)-f(x)}\cdot\frac{f'(u)-f'(x)}{1+f'(x)}du\right]\right\}dx$$

$$\pi=\int_a^b\left[(1+f'(x))\cdot\sin\left(\frac{\pi}{1+f'(x)}\right)\cdot\right.$$

$$\cdot(x-a+f(x))^{-\frac{f'(x)}{1+f'(x)}}\cdot(b-x+f(b)-f(x))^{-\frac{1}{1+f'(x)}}\cdot$$

$$\left.\cdot\exp\left(\int_a^b\frac{1}{u-x+f(u)-f(x)}\cdot\frac{f'(u)-f'(x)}{1+f'(x)}du\right)\right]dx$$

7.2. The abstract definition of the transition measures

Equations (3) and (8) may be thought of as an abstract, nonconstructive way of defining the transition measures of a diagram. Equipped with our formulas for the densities of the transition measures, we may substitute them into (3) and (8), respectively, to obtain some more integration identities:

Theorem 8. Let $\omega\in\mathcal{S}[a,b]$. Then for $x\notin[a,b]$:

(7a)

$$\exp\left(\frac{1}{2}\int_a^b\frac{\omega'(t)-\operatorname{sgn}(t)}{t-x}dt\right)=$$

$$=\int_a^b\left[\frac{1}{\pi}\cos(\pi\omega'(t)/2)\cdot(t-a)^{-(1+\omega'(t))/2}\cdot(b-t)^{-(1-\omega'(t))/2}\cdot\right.$$

$$\cdot \exp\left(\frac{1}{2}\int_a^b \frac{\omega'(u)-\omega'(t)}{u-t}du\right)\cdot \frac{1}{1-t/x}\Bigg] dt$$

(7b)

$$\exp\left(-\frac{1}{2}\int_a^b \frac{\omega'(t)-\mathrm{sgn}(t)}{t-x}dt\right) =$$

$$= 1 - \frac{z(\omega)}{x} - \int_a^b \Bigg[\frac{1}{\pi}\cos(\pi\omega'(t)/2)\cdot (t-a)^{(1+\omega'(t))/2}\cdot (b-t)^{(1-\omega'(t))/2}\cdot$$

$$\exp\left(-\frac{1}{2}\int_a^b \frac{\omega'(u)-\omega'(t)}{u-t}du\right)\cdot \frac{1}{x(x-t)}\Bigg] dt$$

Note that letting $x \to \infty$ in these equations gives (2a) and (2b).

7.3. Relations between walks

So far, we have only considered the two kinds of hook walks, one of which leaves from the corner of the dual domain of the diagram, and the other from a uniformly chosen point in the domain of the diagram. Having calculated the density for the transition measure of exterior corner walks, it is not difficult to transform it into a formula for the density of the transition measure of an interior walk leaving from an arbitrary point in the domain. We do this for unrotated diagrams: Let $f \in \mathcal{U}[a,b]$ be continuous, piecewise twice-continuously-differentiable, with bounded second derivative and first derivative bounded away from 0. First, by replacing f with f^{-1} in Theorem 7 we may obtain a formula for the density of the transition measure of the *interior corner* walk. Next, a simple scaling transforms this to a formula for the density of the transition measure for any interior walk starting from a point (s,t) in the domain. This density, which we denote by $g_{s,t}(x)$, is given by

$$(f^{-1}(t) < x < s) \quad g_{s,t}(x) = \frac{1}{\pi}(1+f'(x))\sin\left(\frac{\pi}{1+f'(x)}\right)\cdot$$

$$\cdot \left(x - f^{-1}(t) + f(x) - t\right)^{-\frac{1}{1+f'(x)}} \cdot (s - x + f(s) - f(x))^{-\frac{f'(x)}{1+f'(x)}}\cdot$$

$$\cdot \exp\left(-\int_{f^{-1}(t)}^{s} \frac{1}{u-x+f(u)-f(x)}\cdot\frac{f'(u)-f'(x)}{1+f'(x)}du\right)$$

We can now write some equations that describe some of the interrelations between the different walks: The first equation expresses the defining fact that each step of the walk goes from the current point to a point in the hook of the current point, chosen uniformly. Thus, the densities $g_{s,t}$ must satisfy

Theorem 9.

$$g_{s,t}(x) = \frac{1}{s - f^{-1}(t) + f(s) - t}\left(\int_x^s g_{v,t}(x)dv + \int_t^{f(x)} g_{s,v}(x)dv\right)$$

(This equation is equal in content, but not in form, to eq. (4.3.5) of Kerov (1993).)

The second equation expresses the fact that the uniform interior walk is really a mixture of all the walks with different given starting points (s,t) , with respect to the normalized area measure $dsdt/A(f)$. This implies the identity

Theorem 10.

$$\frac{1}{\pi A(f)}(1 + f'(x)) \sin\left(\frac{\pi}{1 + f'(x)}\right) \cdot$$

$$\cdot (x - a + f(x))^{\frac{f'(x)}{1+f'(x)}} \cdot (b - x + f(b) - f(x))^{\frac{1}{1+f'(x)}} \cdot$$

$$\cdot \exp\left(-\int_a^b \frac{1}{u - x + f(u) - f(x)} \cdot \frac{f'(u) - f'(x)}{u - x} du\right) =$$

$$= \int_x^b \int_0^{f(x)} g_{s,t}(x) \frac{dt\ ds}{A(f)}$$

Which, after cancelling identical terms on both sides, becomes

$$(x - a + f(x))^{\frac{f'(x)}{1+f'(x)}} \cdot (b - x + f(b) - f(x))^{\frac{1}{1+f'(x)}} \cdot$$

$$\cdot \exp\left(-\int_a^b \frac{1}{u - x + f(u) - f(x)} \cdot \frac{f'(u) - f'(x)}{u - x} du\right) =$$

$$\int_x^b \int_0^{f(x)} \Bigg[\left(x - f^{-1}(t) + f(x) - t\right)^{-\frac{1}{1+f'(x)}} \cdot (s - x + f(s) - f(x))^{-\frac{f'(x)}{1+f'(x)}} \cdot$$

$$\cdot \exp\left(-\int_{f^{-1}(t)}^s \frac{1}{u - x + f(u) - f(x)} \cdot \frac{f'(u) - f'(x)}{u - x} du\right) \Bigg] dt\ ds$$

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