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JEU DE TAQUIN DYNAMICS ON INFINITE YOUNG TABLEAUX AND SECOND CLASS PARTICLES

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We study an infinite version of the "*jeu de taquin*" sliding game, which can be thought of as a natural measure-preserving transformation on the set of infinite Young tableaux equipped with the Plancherel probability measure. We use methods from representation theory to show that the Robinson–Schensted–Knuth (RSK) algorithm gives an isomorphism between this measure-preserving dynamical system and the one-sided shift dynamics on a sequence of independent and identically distributed random variables distributed uniformly on the unit interval. We also show that the jeu de taquin paths induced by the transformation are asymptotically straight lines emanating from the origin in a random direction whose distribution is computed explicitly, and show that this result can be interpreted as a statement on the limiting speed of a second-class particle in the *Plancherel-TASEP* particle system (a variant of the Totally Asymmetric Simple Exclusion Process associated with Plancherel growth), in analogy with earlier results for second class particles in the ordinary TASEP.

1. Introduction.

1.1. *Overview*: *Jeu de taquin on infinite Young tableaux*. The goal of this paper is to study in a new probabilistic framework a combinatorial process that is well known to algebraic combinatorialists and representation theorists. This process is known as the *jeu de taquin* (literally "teasing game") or sliding game. Its remarkable properties have been studied since its introduction in a seminal paper by Schützenberger (1977). Its main importance is as a tool for studying the combinatorics of permutations and Young tableaux, especially with regards to the Robinson–Schensted–Knuth (RSK) algorithm, which is a fundamental object

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of algebraic combinatorics. However, the existing jeu de taquin theory deals exclusively with the case of *finite* permutations and tableaux. A main new idea of the current paper is to consider the implications of "sliding theory" for *infinite* tableaux. As the reader will discover below, this will lead us to some important new insights into the asymptotic theory of Young tableaux, as well as to unexpected new connections to ergodic theory and to well-known random processes of contemporary interest in probability theory, namely the Totally Asymmetric Simple Exclusion Process (TASEP), the corner growth model and directed last-passage percolation.

Our study will focus on a certain measure-preserving dynamical system, that is, a quadruple $\mathfrak{J} = (\Omega, \mathcal{F}, \mathsf{P}, J)$, where $(\Omega, \mathcal{F}, \mathsf{P})$ is a probability space and $J : \Omega \to \Omega$ is a measure-preserving transformation. The sample space Ω will be the set of *infinite Young tableaux*; the probability measure P will be the *Plancherel measure*, and the measure-preserving transformation J will be the *jeu de taquin map*. To define these concepts, we need to first recall some basic notions from combinatorics.

1.2. *Basic definitions.*

1.2.1. *Young diagrams and Young tableaux.* Let $n \geq 1$ be an integer. An *integer partition* (or just *partition*) of n is a representation of n in the form $n = \lambda(1) + \lambda(2) + \cdots + \lambda(k)$, where $\lambda(1) \geq \cdots \geq \lambda(k) > 0$ are integers. Usually the vector $\lambda = (\lambda(1), \dots, \lambda(k))$ is used to denote the partition. We denote the set of partitions of n by $\mathbb{Y}_n$ (where we also define $\mathbb{Y}_n$ for $n = 0$ as the singleton set consisting of the "empty partition," denoted by $\varnothing$), and the set of all partitions by $\mathbb{Y} = \bigcup_{n=0}^{\infty} \mathbb{Y}_n$. If $\lambda \in \mathbb{Y}_n$ we call n the *size* of λ and denote $|\lambda| = n$.

Given a partition $\lambda = (\lambda(1), \dots, \lambda(k))$ of n, we associate with it a *Young diagram*, which is a diagram of k left-justified rows of unit squares (also called boxes or cells) in which the jth row has $\lambda(j)$ boxes. We use the French convention of drawing the Young diagrams from the bottom up; see Figure 1. Since Young diagrams are an equivalent way of representing integer partitions, we refer to a Young diagram interchangeably with its associated partition.

The set $\mathbb{Y}$ of Young diagrams forms in a natural way the vertex set of a directed graph called the *Young graph* (or *Young lattice*), where we connect two diagrams

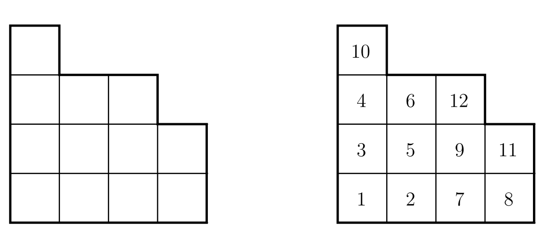

FIG. 1. *The Young diagram $\lambda = (4, 4, 3, 1)$ and a Young tableau of shape λ.*

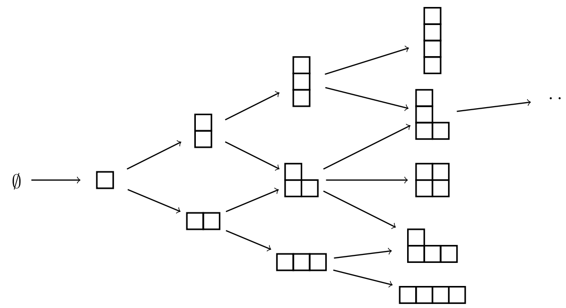

FIG. 2. *The Young graph.*

λ, ν by a directed edge if $|\nu| = |\lambda| + 1$ and ν can be obtained from λ by the addition of a single box; see Figure 2. We denote the adjacency relation in this graph by $\lambda \nearrow \nu$.

Given a Young diagram λ of size n, an *increasing tableau* of shape λ is a filling of the boxes of λ with some distinct real numbers $x_1, \dots, x_n$ such that the numbers along each row and column are in increasing order. A *Young tableau* (also called *standard Young tableau* or *standard tableau*) of shape λ is an increasing tableau of shape λ where the numbers filling it are exactly $1, \dots, n$. The set of standard Young tableaux of shape λ will be denoted by SYT_λ. One useful way of thinking about these objects is that a Young tableau t of shape λ encodes (bijectively) a path in the Young graph

$$\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_n = \lambda \tag{1}$$

starting with the empty diagram and ending at λ. The way the encoding works is that the kth diagram λ_k in the path is the Young diagram consisting of these boxes of λ which contain a number $\le k$. Going in the opposite direction, given the path (1) one can reconstruct the Young tableau by writing the number k in a given box if that box was added to λ_{k-1} to obtain λ_k. The Young tableau t constructed in this way is referred to as the *recording tableau* of the sequence (1).

1.2.2. *Plancherel measure.* Denote by f^λ the number of standard Young tableaux of shape λ. It is well known that

$$n! = \sum_{\lambda \in \mathbb{Y}_n} (f^\lambda)^2,$$

a fact easily explained by the RSK algorithm [Fulton (1997), page 52]. Thus, if we define a measure P_n on $\mathbb{Y}_n$ by setting

$$\mathsf{P}_n(\lambda) = \frac{(f^\lambda)^2}{n!} \qquad (\lambda \in \mathbb{Y}_n), \tag{2}$$

then P_n is a probability measure. The measure P_n is called the *Plancherel measure* of order n. From the viewpoint of representation theory, one can argue that this is one of the most natural probability measures on $\mathbb{Y}_n$ since it corresponds to taking a random irreducible component of the left-regular representation of the symmetric group S_n, which is one of the most natural and important representations; see Section 4.4 below.

Another well-known fact is that the Plancherel measures of all different orders can be coupled to form a Markov chain

$$\varnothing = \Lambda_0 \nearrow \Lambda_1 \nearrow \Lambda_2 \nearrow \cdots, \tag{3}$$

where each Λ_n is a random Young diagram distributed according to P_n. This is done by defining the conditional distribution of Λ_{n+1} given Λ_n using the following transition rule:

$$\operatorname{Prob}(\Lambda_{n+1} = \nu | \Lambda_n = \lambda) = \begin{cases} \dfrac{f^\nu}{(n+1) f^\lambda}, & \text{if } \lambda \nearrow \nu, \\ 0, & \text{otherwise,} \end{cases} \tag{4}$$

for each $\lambda \in \mathbb{Y}_n, \nu \in \mathbb{Y}_{n+1}$. The fact that the right-hand side of (4) defines a valid Markov transition matrix and that the push-forward of the measure P_n under this transition rule is P_{n+1} is explained by Kerov (1999), where the process $(\Lambda_n)_{n=0}^{\infty}$ has been called the *Plancherel growth process* [see also Romik (2014), Section 1.19]. Here, we shall think of the same process in a slightly different way by looking at the recording tableau associated with the chain (3). Since this is now an infinite path in the Young graph, the recording tableau is a new kind of object which we call an *infinite Young tableau*. This is defined as an infinite matrix $t = (t_{i,j})_{i,j=1}^{\infty}$ of natural numbers where each natural number appears exactly once and the numbers along each row and column are increasing. Graphically, an infinite Young tableau can be visualized, similarly as before, as a filling of the boxes of the "infinite Young diagram" occupying the entire first quadrant of the plane by the natural numbers. We use the convention that the numbering of the boxes follows the Cartesian coordinates, that is, $t_{i,j}$ is the number written in the box (i, j) which is in the ith column and jth row, with the rows and columns numbered by the elements of the set $\mathbb{N} = \{1, 2, \dots\}$ of the natural numbers. Denote by Ω the set of infinite Young tableaux.

We remark that the usual (i.e., noninfinite) Young tableaux are very useful in the *representation theory of the symmetric groups*: one can find a very natural base of the appropriate representation space which is indexed by Young tableaux [Ceccherini-Silberstein, Scarabotti and Tolli (2010)]. Thus, it should not come as a surprise that infinite tableaux are very useful for studying harmonic analysis on the *infinite symmetric group* S_∞; see Vershik and Kerov (1981).

Now, just as finite Young tableaux are in bijection with paths in the Young graph leading up to a given Young diagram, the infinite Young tableaux are similarly in bijection with those infinite paths in the Young graph starting from the empty diagram that have the property that any box is eventually included in some diagram of the path. We call an infinite tableau corresponding to such an infinite path the *recording tableau* of the path, similarly to the case of finite paths. Thus, under this bijection the Plancherel growth process (3) can be interpreted as a random infinite Young tableau; that is, a probability measure on the set Ω of infinite Young tableaux, equipped with its natural measurable structure, namely, the minimal σ-algebra $\mathcal{F}$ of subsets of Ω such that all the coordinate functions $t \mapsto t_{i,j}$ are measurable. (Note that the Plancherel growth process almost surely has the property of eventually filling all the boxes—for example, this follows trivially from Theorem 3.1 below.)

We denote this probability measure on $(\Omega, \mathcal{F})$ by P, and refer to it as the *Plancherel measure of infinite order*, or (where there is no risk of confusion) simply *Plancherel measure*.

1.2.3. *Jeu de taquin.* Given an infinite Young tableau $t = (t_{i,j})_{i,j=1}^{\infty} \in \Omega$, define inductively an infinite up-right lattice path in $\mathbb{N}^2$

$$\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t), \dots, \tag{5}$$

where $\mathbf{p}_1(t) = (1, 1)$, and for each $k \geq 2$, $\mathbf{p}_k = (i_k, j_k)$ is given by

$$\mathbf{p}_k = \begin{cases} (i_{k-1} + 1, j_{k-1}), & \text{if } t_{i_{k-1}+1, j_{k-1}} < t_{i_{k-1}, j_{k-1}+1}, \\ (i_{k-1}, j_{k-1} + 1), & \text{if } t_{i_{k-1}+1, j_{k-1}} > t_{i_{k-1}, j_{k-1}+1}. \end{cases} \tag{6}$$

That is, one starts from the corner box of the tableau and starts traveling in unit steps to the right and up, at each step choosing the direction among the two in which the entry in the tableau is smaller. We refer to the path (5) defined in this way as the *jeu de taquin path* of the tableau t. This is illustrated in Figure 3(a).

We now use the jeu de taquin path to define a new infinite tableau $s = J(t) = (s_{i,j})_{i,j=1}^{\infty}$, using the formula

$$s_{i,j} = \begin{cases} t_{\mathbf{p}_{k+1}} - 1, & \text{if } (i, j) = \mathbf{p}_k \text{ for some } k, \\ t_{i,j} - 1, & \text{otherwise.} \end{cases} \tag{7}$$

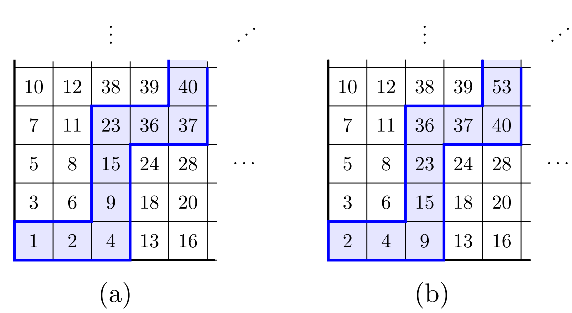

FIG. 3. (a) *A part of an infinite Young tableau t. The highlighted boxes form the beginning of the jeu de taquin path* $\mathbf{p}(t)$. (b) *The outcome of "sliding" of the boxes along the highlighted jeu de taquin path. The outcome of the jeu de taquin transformation $J(t)$ is obtained by subtracting* 1 *from all the entries.*

The mapping $t \mapsto s = J(t)$ defines a transformation $J : \Omega \to \Omega$, which we call the *jeu de taquin map*. In words, the way the transformation works is by removing the box at the corner, then sliding the second box of the jeu de taquin path into the space left vacant by the removal of the first box, and continuing in this way, successively sliding each box along the jeu de taquin path into the space vacated by its predecessor. At the end, one subtracts 1 from all entries to obtain a new array of numbers. It is easy to see that the resulting array is an infinite Young tableau: the definition of the jeu de taquin path guarantees that the sliding is done in such a way that preserves monotonicity along rows and columns. For an example, compare Figure 3(a) and 3(b).

The above construction is a generalization of the construction of Schützenberger (1977) who introduced it for finite Young tableaux. Schützenberger's jeu de taquin turned out to be a very powerful tool of algebraic combinatorics and the representation theory of symmetric groups; in particular, it is important in studying combinatorics of words, the Robinson–Schensted–Knuth (RSK) correspondence and the Littlewood–Richardson rule; see Fulton (1997) for an overview.

1.2.4. *An infinite version of the Robinson–Schensted–Knuth algorithm*. Next, we consider an infinite version of the *Robinson–Schensted–Knuth (RSK) algorithm* which can be applied to an infinite sequence $(x_1, x_2, x_3, \dots)$ of distinct real numbers.[3] This infinite version was considered in a more general setup by Kerov and Vershik (1986) [the finite version of the algorithm, summarized here, is discussed in detail by Fulton (1997)]. The algorithm performs an inductive computation, reading the inputs $x_1, x_2, \dots$ successively, and at each step applying a so-called *insertion step* to its previous computed output together with the next input x_n.

The insertion step, given an increasing tableau P_{n-1} and a number x_n produces a new increasing tableau P_n whose shape λ_n is obtained from λ_{n-1} by the addition of a single box. The new tableau P_n is computed by performing a succession of *bumping steps* whereby x_n is inserted into the first row of the diagram (as far to the right as possible so that the row remains increasing and no gaps are created), bumping an existing entry from the first row into the second row, which results in an entry of the second row being bumped to the third row, and so on, until finally the entry being bumped settles down in an unoccupied position outside the diagram λ. An example is shown in Figure 4.

For each $n \geq 0$, after inserting the first n inputs $x_1, \dots, x_n$ the algorithm produces a triple (λ_n, P_n, Q_n), where $\lambda_n \in \mathbb{Y}_n$ is a Young diagram with n boxes, P_n is an increasing tableau of shape λ_n containing the numbers $x_1, \dots, x_n$, and Q_n is a standard Young tableau of shape λ_n. The shapes satisfy $\lambda_{n-1} \nearrow \lambda_n$, that is, at each step one new box is added to the current shape, with the tableau Q_n being simply

[3] Actually, this is an infinite version of a special case of RSK that predates it and is known as the *Robinson–Schensted algorithm*, but we prefer to use the RSK mnemonic due to its convenience and familiarity to a large number of readers.

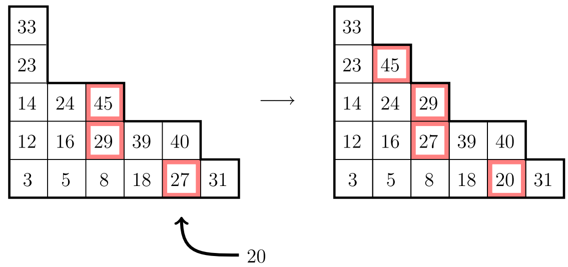

FIG. 4. *Example of an insertion step. The highlighted boxes indicate the locations of bumped entries.*

the recording tableau of the path $\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n$. The tableau P_n is the information that will be acted upon by the next insertion step, and is called the *insertion tableau*. We will refer to λ_n as the RSK *shape associated to* $(x_1, \dots, x_n)$.

In this infinite version of the algorithm, we shall assume that $x_1, x_2, \dots$ are such that the infinite Young graph path $\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \dots$ can be encoded by an infinite recording tableau Q_∞ (i.e., we assume that every box in the first quadrant eventually gets added to some λ_n). For our purposes, the information in the insertion tableaux P_n will not be needed, so we simply discard it, and define the (*infinite*) RSK *map* by

$$\mathrm{RSK}(x_1, x_2, \dots) = Q_\infty.$$

1.3. *The main results*. We are now ready to state our main results.

1.3.1. *The jeu de taquin path*. Our first result concerns the asymptotic behavior of the jeu de taquin path. For a given infinite tableau $t \in \Omega$, we define $\Theta = \Theta(t) \in [0, \pi/2]$ by

$$\big(\cos\Theta(t), \sin\Theta(t)\big) = \lim_{k\to\infty} \frac{\mathbf{p}_k(t)}{\|\mathbf{p}_k(t)\|}$$

whenever the limit exists, and in this case refer to Θ as the *asymptotic angle* of the jeu de taquin path.

THEOREM 1.1 (Asymptotic behavior of the jeu de taquin path). *The jeu de taquin path converges* P*-almost surely to a straight line with a random direction. More precisely, we have*

$$\mathsf{P}\left[\lim_{k\to\infty} \frac{\mathbf{p}_k}{\|\mathbf{p}_k\|} \textit{ exists}\right] = 1.$$

Under the Plancherel measure P, *the asymptotic angle* Θ *is an absolutely continuous random variable on* $[0, \pi/2]$ *whose distribution has the following explicit description*:

$$\Theta \stackrel{\mathcal{D}}{=} \Pi(W), \tag{8}$$

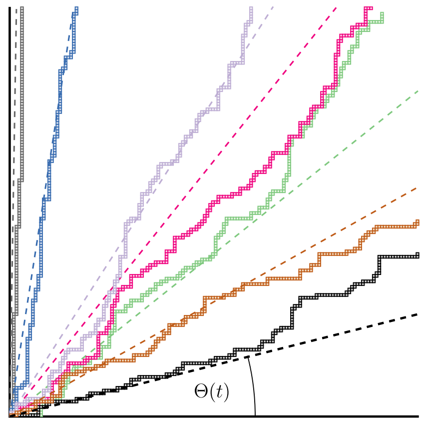

FIG. 5. *Several simulated paths of jeu de taquin and (dashed lines) their asymptotes.*

where W is a random variable distributed according to the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$ on $[-2, 2]$, that is, having density given by

$$\mathcal{L}_{\mathrm{SC}}(dw) = \frac{1}{2\pi}\sqrt{4-w^2}\,dw \qquad (|w| \le 2), \tag{9}$$

and $\Pi(\cdot)$ is the function

$$\Pi(w) = \frac{\pi}{4} - \cot^{-1}\left[\frac{2}{\pi}\left(\sin^{-1}\left(\frac{w}{2}\right) + \frac{\sqrt{4-w^2}}{w}\right)\right] \qquad (-2 \le w \le 2).$$

Figure 5 shows simulation results illustrating the theorem. Figure 6 shows a plot of the density function of Θ. Note that the definition of the distribution of Θ has a more intuitive geometric description; see Section 3.3 for the details.

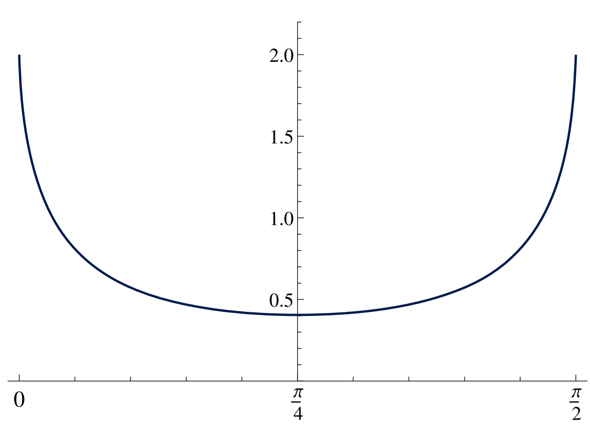

FIG. 6. *A plot of the density function of Θ. The density is bounded but is heavily skewed, with most of the probability concentrated near the ends of the interval $[0, \pi/2]$.*

1.3.2. *The Plancherel-TASEP interacting particle system.* One topic that we will explore in more detail later is an analogy between Theorem 1.1 and a result of Ferrari and Pimentel (2005) on competition interfaces in the corner growth model. Furthermore, this result is essentially a reformulation of previous results of Ferrari and Kipnis (1995) and Mountford and Guiol (2005) on the limiting speed of second class particles in the *Totally Asymmetric Simple Exclusion Process (TASEP)*; similarly, our Theorem 1.1 affords a reinterpretation in the language of interacting particle systems, involving a variant of the TASEP which we call the *Plancherel-TASEP particle system*. We find this reinterpretation to be just as interesting as the result above. However, because of the complexity of the necessary background, and to avoid making this introductory section excessively long, we formulate this version of the result here without explaining the meaning of the terminology used, and defer the details and further exploration of this connection to Section 7. We encourage the reader to visit the discussion in that section to gain a better appreciation of the context and importance of the result.

THEOREM 1.2 (The second class particle trajectory). *For $n \geq 0$, let $X(n)$ denote the location at time n of the second-class particle in the Plancherel-TASEP interacting particle system. The limit*

$$W = \lim_{n\to\infty} \frac{X(n)}{\sqrt{n}}$$

exists almost surely and is a random variable distributed according to the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$.

The limiting random variable W can be thought of as an asymptotic speed parameter for the second-class particle. Namely, if one considers for each $n \geq 1$ the scaled trajectory functions

$$\widehat{X}_n(t) = \frac{X(\lfloor nt \rfloor)}{\sqrt{n}} \qquad (t > 0), \tag{10}$$

then Theorem 1.2 can be reformulated as saying that as $n \to \infty$, almost surely the trajectory will follow asymptotically one of the curves in the one-parameter family $(\alpha\sqrt{t})_{-2\leq\alpha\leq 2}$, where the parameter α is random and chosen according to the distribution $\mathcal{L}_{\mathrm{SC}}$. If one reparameterizes time by replacing t with t^2 (which is arguably a more natural parameterization—see the discussion in Section 7.5), we get the statement that the limiting trajectory of the second-class particle is asymptotically a straight line with slope α. This is analogous to the result of Mountford and Guiol (2005), where the process is the ordinary TASEP and the limiting speed of the second-class particle has the uniform distribution $U(-1, 1)$ on the interval $[-1, 1]$.

1.3.3. *The jeu de taquin dynamical system.* It is worth pointing out that the jeu de taquin applied to an infinite tableau $t \in \Omega$ produces two interesting pieces of information: the jeu de taquin path (5):

$$\mathbf{p}(t) = \big(\mathbf{p}_1(t), \mathbf{p}_2(t), \ldots\big),$$

and another infinite tableau $J(t) \in \Omega$. This setup naturally raises questions about the iterations of the jeu de taquin map

$$t, J(t), J\big(J(t)\big), \ldots$$

or, in other words, about the dynamical system $\mathfrak{J} = (\Omega, \mathcal{F}, \mathsf{P}, J)$, which we call the *jeu de taquin dynamical system*. The following result shows that this is indeed a very natural point of view.

THEOREM 1.3 (Measure preservation and ergodicity). *The dynamical system $\mathfrak{J} = (\Omega, \mathcal{F}, \mathsf{P}, J)$ is measure-preserving and ergodic.*

We believe the part of the above result concerning the measure-preservation may be known to experts in the field, though we are not aware of a reference to it in print. The second part concerning ergodicity is new.

The next result sheds light on the behavior of the jeu de taquin dynamical system $\mathfrak{J}$, by showing that it has probably the simplest possible structure one could hope for, namely, it is isomorphic to an i.i.d. shift.

THEOREM 1.4 (Isomorphism to an i.i.d. shift map). *Let* $\mathfrak{S} = ([0,1]^{\mathbb{N}}, \mathcal{B}, \mathrm{Leb}^{\otimes\mathbb{N}}, S)$ *denote the measure-preserving dynamical system corresponding to the* (*one-sided*) *shift map on an infinite sequence of independent random variables with the uniform distribution* $U(0,1)$ *on the unit interval* $[0,1]$. *That is*, $\mathrm{Leb}^{\otimes\mathbb{N}} = \prod_{n=1}^{\infty}(\mathrm{Leb})$ *is the product of Lebesgue measures on* $[0,1]$, $\mathcal{B}$ *is the product σ-algebra on* $[0,1]^{\mathbb{N}}$, *and* $S : [0,1]^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ *is the shift map*, *defined by*

$$S(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

Then the mapping $\mathrm{RSK} : [0,1]^{\mathbb{N}} \to \Omega$ *is an isomorphism between the measure-preserving dynamical systems $\mathfrak{J}$ and $\mathfrak{S}$.*

Note that such a complete characterization of the highly nontrivial measure-preserving system $\mathfrak{J}$ may open up many possibilities for additional applications. We hope to explore these possibilities in future work. Furthermore, in contrast to many structure theorems in ergodic theory that show isomorphism of complicated dynamical systems to i.i.d. shift maps via an abstract existential argument that does not provide much insight into the nature of the isomorphism, here the isomorphism is a completely explicit, familiar and highly structured mapping—the RSK algorithm.

Note also that RSK is defined on the set of sequences $(x_1, x_2, \ldots)$ which satisfy the assumption mentioned in Section 1.2.4. This is known (see again Theorem 3.1 below) to be a set of full measure with respect to $\mathrm{Leb}^{\otimes\mathbb{N}}$.

Theorem 1.4 above encapsulates several separate claims: first, that the Plancherel measure P is the push-forward of the product measure $\mathrm{Leb}^{\otimes\mathbb{N}}$ under the mapping RSK; this is easy and well known (see Lemma 2.2 below). Second, that RSK is a factor map (also known as homomorphism) of measure-preserving dynamical systems. This is the statement that

$$J \circ \mathrm{RSK} = \mathrm{RSK} \circ S, \tag{11}$$

that is, the following diagram commutes:

$$\begin{array}{ccc} [0,1]^{\mathbb{N}} & \xrightarrow{S} & [0,1]^{\mathbb{N}} \\ \Big\downarrow \mathrm{RSK} & & \Big\downarrow \mathrm{RSK} \\ \Omega & \xrightarrow{J} & \Omega \end{array}$$

This is somewhat nontrivial but follows from known combinatorial properties of the RSK algorithm and jeu de taquin in the finite setting. Finally, the hardest part is the claim that this factor map is in fact an isomorphism. It is also the most surprising: recall that in the infinite version of the RSK map we discarded all the information contained in the insertion tableaux $(P_n)_{n=1}^{\infty}$. In the finite version of RSK, the insertion tableau is essential to inverting the map, so how can we hope to invert the infinite version without this information? It turns out that Theorem 1.1 plays an essential part: the asymptotic direction of the jeu de taquin path provides the key to inverting RSK in our "infinite" setting. This is explained next.

1.3.4. *The inverse of infinite* RSK. The secret to inversion of infinite RSK is as follows. We will show in a later section (see Theorem 5.2 below) that the limiting direction Θ of the jeu de taquin path is a function of only the first input X_1 in the sequence of i.i.d. uniform random variables $X_1, X_2, \ldots$ to which the RSK factor map is applied. Moreover, this function is an explicit (and invertible) function. This gives us the key to inverting the map $\mathrm{RSK}(\cdot)$ and, therefore, proving the isomorphism claim, since, if we can recover X_1 from the infinite tableau T, then by iterating the map J and using the factor property we can similarly recover the successive inputs $X_2, X_3, \ldots,$ etc. Thus, we get the following explicit description of the inverse RSK map.

THEOREM 1.5 (The inverse of infinite RSK). *The inverse mapping* RSK^{-1}: $\Omega \to [0,1]^{\mathbb{N}}$ *is given* P-*almost surely by*

$$\mathrm{RSK}^{-1}(t) = \big[F_\Theta\big(\Theta_1(t)\big), F_\Theta\big(\Theta_2(t)\big), F_\Theta\big(\Theta_3(t)\big), \ldots\big],$$

where we denote $\Theta_k = \Theta \circ J^{k-1}$ (*this refers to functional iteration of* J *with itself* $k-1$ *times*), *and where* $F_\Theta(s) = \mathsf{P}(\Theta \le s)$ *is the cumulative distribution function of the asymptotic angle* Θ.

Note that one particular consequence of this theorem, which taken on its own, already makes for a rather striking statement, is the fact that under the measure P, the sequence of asymptotic angles $(\Theta_k)_{k=1}^{\infty}$ obtained by iteration of the map J as above is a sequence of independent and identically distributed random variables. The full statement of the theorem can be interpreted as the stronger fact, which seems all the more surprising, that this i.i.d. sequence is actually related in a simple way (via coordinate-wise application of the monotone increasing function F_Θ^{-1}) to the original sequence of i.i.d. $U(0,1)$ random variables fed as input to the RSK algorithm. As a referee pointed out to us, an earlier clue to this type of isomorphism phenomenon can be found in the context of RSK applied to random words over a finite alphabet; see O'Connell (2003), O'Connell and Yor (2002) and the remark in Section 8.2.

1.4. *Overview of the paper.* We have described our main results, but the rest of the paper also contains additional results of independent interest. The plan of the paper is as follows. In Section 2, we recall some additional facts from the combinatorics of Young tableaux, which we use to pick some of the low-hanging fruit in our theory of infinite jeu de taquin, namely, the proof of Theorem 1.3 and the fact that RSK is a factor map, and as preparation for the more difficult proofs. In Section 3, we prove a weaker version of Theorem 1.1 that shows convergence in distribution (instead of almost sure convergence) of the direction of the jeu de taquin path to the correct distribution. This provides additional intuition and motivation, since this weaker result is much easier to prove than Theorem 1.1.

Next, we attack Theorem 1.1, which conceptually is the most difficult part of the paper. Here, we apply methods from the representation theory of the symmetric group. The necessary background is developed in Section 4, where a key technical result is proved (this is the only part of the paper where representation theory is used, and it may be skipped if one is willing to assume the validity of this technical result). This result is used in Section 5 to prove two additional results which are of independent interest (especially to readers interested in asymptotic properties of random Young tableaux) but which we did not elaborate on in this Introduction. We refer to these results as the *asymptotic determinism of* RSK and *asymptotic determinism of jeu de taquin*.

With the help of these results, Theorems 1.1, 1.4 and 1.5 are then proved in Section 6.

Section 7 is then dedicated to exploring the connection between our results and the theory of interacting particle systems. In particular, we study in depth the point of view in which a "lazy" version of the jeu de taquin path is reinterpreted as encoding the trajectory of a second-class particle in the Plancherel-TASEP particle system, and consider how our results are analogous to results discussed in the papers of Ferrari and Kipnis (1995), Ferrari and Pimentel (2005), Mountford and Guiol (2005) in connection with the TASEP and the closely related *corner growth model* (also known under the name *directed last passage percolation*). This analogy is one of the main "inspirational" forces of the paper, so the reader interested in this point of view may want to read this section before the more technical proofs in the sections preceding it.

Finally, Section 8 mentions some additional directions related to the ideas explored in this paper that we plan to discuss in future work.

1.5. *Notation*. Throughout the paper, we use the following notational conventions: the letters μ, λ, ν will generally be used to denote deterministic Young diagrams, and capital Greek letters such as Λ, Π will be used to denote random Young diagrams. Similarly, lower case letters such as t, s may be used to denote a deterministic Young tableau, and T will denote a random one. The normalized semicircle distribution (9) (on $[-2, 2]$, which is the case when its variance is 1 and its even moments are the Catalan numbers) will always be denoted by $\mathcal{L}_{\mathrm{SC}}$. A generic context-dependent probability will be denoted by $\mathrm{Prob}(\cdot)$, and expectation by $\mathbb{E}$; the symbol P will be reserved for Plancherel measure on the space Ω of infinite Young tableaux. Other notation will be introduced as needed in the appropriate place.

2. Elementary properties of jeu de taquin and RSK. In this section, we recall some standard facts about Young tableaux, and use them to prove the easier parts of the results described in the introduction (measure preservation, ergodicity and the factor map property). We also start building some additional machinery that will be used later to attack the more difficult claims about the asymptotics of the jeu de taquin path and the invertibility of RSK.

2.1. *Finite version of jeu de taquin*. Let $\lambda \in \mathbb{Y}_n$ for some $n \geq 1$. To each Young tableau t of shape λ, there is associated a finite jeu de taquin path $(1, 1) = \mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m$ defined analogously to (6) except that the path terminates at the last place it visits inside the diagram λ, and for the purposes of interpreting the formula (6) we consider $t_{i,j} = \infty$ for positions outside λ. We can similarly define a finite jeu de taquin map j that takes a tableau t of shape λ and returns a tableau $s = j(t)$ of shape μ for some $\mu \in \mathbb{Y}_{n-1}$ satisfying $\mu \nearrow \lambda$. This is defined by the same formula as (7), with the shape μ being formed from λ by removing the last box of the jeu de taquin path.

LEMMA 2.1. *For any $\lambda \in \mathbb{Y}_n$, denote by*

$$j_\lambda : \mathrm{SYT}_\lambda \to \bigsqcup_{\mu : \mu \nearrow \lambda} \mathrm{SYT}_\mu$$

the restriction of the finite jeu de taquin map j to SYT_λ. Then j_λ is a bijection.

PROOF. This is a standard fact; see Fulton (1997), page 14. The idea is that given the tableau $s = j(t)$ and the shape λ, one can recover t by performing a "reverse sliding" operation, starting from the unique cell in the difference $\lambda \setminus \mu$. $\square$

From the lemma, it follows that the preimage $j^{-1}(t)$ of a tableau t of shape λ contains one element for each ν for which $\lambda \nearrow \nu$, namely

$$j^{-1}(t) = \{ j_\nu^{-1}(t) : \nu \in \mathbb{Y}, \lambda \nearrow \nu \}. \tag{12}$$

2.2. *Measure preservation.* We now prove that the jeu de taquin map J preserves the Plancherel measure P, which is the easier part of Theorem 1.3. The proof requires verifying that the identity

$$\mathsf{P}(J^{-1}(E)) = \mathsf{P}(E) \tag{13}$$

holds for any event $E \in \mathcal{F}$. We shall do this for a family of cylinder sets of a certain form, defined as follows. If $\lambda = (\lambda(1), \dots, \lambda(k)) \in \mathbb{Y}_n$ and $s = (s_{i,j})_{1 \le i \le k, 1 \le j \le \lambda(i)}$ is a Young tableau of shape λ [where $s_{i,j}$ is our notation for the entry written in the box in position (i, j)], we define the event $E_s \in \mathcal{F}$ by

$$E_s = \{ t = (t_{i,j})_{i,j=1}^{\infty} \in \Omega | t_{i,j} = s_{i,j} \text{ for all } 1 \le i \le k, 1 \le j \le \lambda(i) \}. \tag{14}$$

The family of sets of the form E_s clearly generates $\mathcal{F}$ and is a π-system, so by a standard fact from measure theory [Durrett (2010), Theorem A.1.5, page 345], it will be enough to check that (13) holds for E_s.

Note that if s is the recording tableau of the path $\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \dots \nearrow \lambda_n = \lambda$ in the Young graph, then in the language of the Plancherel growth process (3), E_s corresponds to the event that

$$\{\Lambda_k = \lambda_k \text{ for } 0 \le k \le n\}.$$

Therefore, it is easy to see from (4) that

$$\mathsf{P}(E_s) = \frac{f^\lambda}{n!}, \tag{15}$$

since when multiplying out the transition probabilities in (4) one gets a telescoping product.

On the other hand, let us compute $\mathsf{P}(J^{-1}(E_s))$. From (12), we see that $J^{-1}(E_s)$ can be decomposed as the disjoint union

$$J^{-1}(E_s) = \bigsqcup_{\nu : \lambda \nearrow \nu} E_{j_\nu^{-1}(s)}.$$

Applying (15) to each summand, we see that

$$\mathsf{P}(J^{-1}(E_s)) = \sum_{\nu \in \mathbb{Y}_{n+1}, \lambda \nearrow \nu} \frac{f^\nu}{(n+1)!},$$

and this is equal to $f^\lambda/n! = \mathsf{P}(E_s)$ by the well-known relation

$$(n+1)f^\lambda = \sum_{\nu:\lambda\nearrow\nu} f^\nu$$

[see equation (7) in Greene, Nijenhuis and Wilf (1984); note that this relation also explains why (4) is a valid Markov transition rule]. So, (13) holds for the event E_s, as claimed.

2.3. RSK *and Plancherel measure*. The following lemma is well known [see, e.g., Kerov and Vershik (1986)], and can be used as an equivalent alternative definition of Plancherel measure. We include its proof for completeness.

LEMMA 2.2. *Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with the $U(0,1)$ distribution. The random infinite Young tableau*

$$T = \mathrm{RSK}(X_1, X_2, \ldots)$$

is distributed according to the Plancherel measure P. *In other words*, P *is the push-forward of the product measure* $\mathrm{Leb}^{\otimes\mathbb{N}}$ (*defined in Theorem* 1.4) *under the mapping* $\mathrm{RSK}:[0,1]^{\mathbb{N}} \to \Omega$.

PROOF. Let P' be the distribution measure of T. Let $\lambda = (\lambda(1), \ldots, \lambda(k)) \in \mathbb{Y}_n$ for some $n \geq 1$ and let $s = (s_{i,j})_{1\leq i\leq k, 1\leq j\leq\lambda(i)}$ be a Young tableau of shape λ. Then the event $\{T \in E_s\}$ [with E_s as in (14)] can be written equivalently as $\{Q_n = s\}$, where Q_n is the recording tableau part of the RSK algorithm output (P_n, Q_n) corresponding to the first n inputs $(X_1, \ldots, X_n)$. Note that Q_n is dependent only on the order structure of the sequence $X_1, \ldots, X_n$; this order is a uniformly random permutation in the symmetric group S_n, and by the properties of the RSK correspondence,

$$\mathrm{Prob}(Q_n = s) = f^\lambda/n!, \tag{16}$$

since there are f^λ possibilities to choose the insertion tableau P_n, each of them corresponding to a single permutation among the $n!$ possibilities. Therefore, we have that

$$\mathsf{P}'(E_s) = \mathrm{Prob}(T \in E_s) = \mathrm{Prob}(Q_n = s) = \frac{f^\lambda}{n!} = \mathsf{P}(E_s).$$

Since this is true for any Young tableau s, and the events E_s form a π-system generating $\mathcal{F}$, it follows that the measures P' and P coincide. □

2.4. RSK *is a factor map*. We now prove (11). We need the following result which concerns RSK and jeu de taquin in the finite setup; see Sagan (2001), Proposition 3.9.3, for a proof.

LEMMA 2.3 [Schützenberger (1963)]. *Let $x_1, \dots, x_n$ be distinct numbers. Let Q_n be the recording tableau associated by* RSK *to $(x_1, x_2, \dots, x_n)$ and let $\widetilde{Q}_{n-1}$ be the recording tableau associated to $(x_2, x_3, \dots, x_n)$. Then*

$$\widetilde{Q}_{n-1} = j(Q_n),$$

where j is the finite version of the jeu de taquin map.

Let $(x_1, x_2, \dots) \in [0, 1]^{\mathbb{N}}$ be a sequence for which the infinite tableau RSK$(x_1, x_2, \dots) = Q_\infty$ is defined. In the notation of the lemma, Q_∞ is the unique infinite tableau that "projects down" to the sequence of finite recording tableaux Q_n (in the sense that deleting all entries $> n$ gives Q_n). The sequence of recording tableaux $\widetilde{Q}_{n-1} = j(Q_n)$ of $(x_2, \dots, x_n)$ for $n \geq 1$ also determines a unique infinite tableau $\widetilde{Q}_\infty$ with the same projection property, which is therefore the recording tableau of $(x_2, x_3, \dots) = S(x_1, x_2, \dots)$. Because j is a finite version of J, it is easy to see that this implies $J(Q_\infty) = \widetilde{Q}_\infty$, which is the relation (11) for the input $(x_1, x_2, \dots)$.

Note that (11) also implies that the measure-preserving system $\mathfrak{J}$ is ergodic, since a factor of an ergodic system is ergodic [Silva (2008), page 119]. So, we have finished proving Theorem 1.3.

2.5. *Monotonicity properties of* RSK. We will identify the set of boxes of an infinite Young tableau with $\mathbb{N}^2$. We introduce a partial order on $\mathbb{N}^2$ as follows:

$$(x_1, y_1) \preceq (x_2, y_2) \quad \iff \quad x_1 \leq x_2 \text{ and } y_1 \geq y_2.$$

If $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_k)$ are finite sequences we denote by

$$\mathbf{ab} = (a_1, \dots, a_n, b_1, \dots, b_k)$$

their concatenation. Also, if b is a number we denote by

$$\mathbf{a}b = (a_1, \dots, a_n, b)$$

the sequence $\mathbf{a}$ appended by b, etc.

For a finite sequence $\mathbf{a} = (a_1, \dots, a_n)$ we denote by $\operatorname{Ins}(\mathbf{a}) \in \mathbb{N}^2$ the last box which was inserted to the Young diagram by the RSK algorithm applied to the sequence $\mathbf{a}$. In other words, it is the box containing the biggest number in the recording tableau associated to $\mathbf{a}$.

LEMMA 2.4. *Assume that the elements of the sequence $\mathbf{a} = (a_1, \dots, a_l)$ and b, b' are distinct numbers and $b < b'$. Then we have the relations*:

(a) $\operatorname{Ins}(\mathbf{a}b) \prec \operatorname{Ins}(\mathbf{a}bb')$;

(b) $\mathrm{Ins}(\mathbf{a}b') \succ \mathrm{Ins}(\mathbf{a}b'b)$;
(c) $\mathrm{Ins}(\mathbf{a}b) \preceq \mathrm{Ins}(\mathbf{a}b')$;
(d) $\mathrm{Ins}(\mathbf{a}b') \preceq \mathrm{Ins}(\mathbf{a}bb')$.

PROOF. Parts (a) and (b) are slightly weaker versions of the "Row Bumping Lemma" in Fulton [(1997), page 9]. The remaining parts (c) and (d) follow using a similar argument of comparing the "bumping routes." □

Note that part (a) [resp., part (b)] in the lemma above implies that if a sequence $\mathbf{a} = (a_1, \dots, a_n)$ is arbitrary and $\mathbf{b} = (b_1, \dots, b_k)$ is increasing (resp., decreasing), and $\square_1, \dots, \square_{n+k}$ are the boxes of the RSK shape associated to the concatenated sequence $\mathbf{ab}$, written in the order in which they were added (i.e., $\square_j$ being the box containing the entry j in the recording tableau), then $\square_{n+1} \prec \cdots \prec \square_{n+k}$ (resp., $\square_{n+1} \succ \cdots \succ \square_{n+k}$). Part (c) shows that the function $z \mapsto \mathrm{Ins}(\mathbf{a}z)$ is weakly increasing with respect to the order $\preceq$.

2.6. *Symmetries of* RSK. For a box $(i, j) \in \mathbb{N}^2$ we denote by $(i, j)^t = (j, i)$ the transpose box, obtained under the mirror image across the axis $x = y$. For a Young diagram $\lambda \in \mathbb{Y}_n$ the transposed diagram $\lambda^t \in \mathbb{Y}_n$ is obtained by transposing all boxes of the original Young diagram. In the following lemma, we recall some of the well-known symmetry properties of the RSK algorithm.

LEMMA 2.5. *Let $x_1, \dots, x_n$ be a sequence of distinct elements and let λ be the corresponding* RSK *shape. Then*:

(a) *the* RSK *shape associated to the sequence $x_n, x_{n-1}, \dots, x_1$ is equal to λ^t*;
(b) *the* RSK *shape associated to the sequence $1 - x_1, 1 - x_2, \dots, 1 - x_n$ is equal to λ^t*;
(c) *the* RSK *shape associated to the sequence $1 - x_n, 1 - x_{n-1}, \dots, 1 - x_1$ is equal to λ*.

PROOF. Claim (c) follows from (a) and (b), which are both immediate consequences of Greene's Theorem [Stanley (1999), Theorem A1.1.1]. □

3. The limit shape and the semicircle transition measure.

3.1. *The limit shape of Plancherel-random diagrams*. In what follows, the limit shape theorem for Plancherel-distributed random Young diagrams, due to Logan and Shepp (1977) and Vershik and Kerov (1977, 1985) (that was instrumental in the solution of the famous Ulam problem on the asymptotics of the maximal increasing subsequence length in a random permutation), will play a key role, so we recall its formulation.

Given a Young diagram $\lambda = (\lambda(1), \dots, \lambda(k)) \in \mathbb{Y}_n$, we identify it with the subregion

$$A_\lambda = \bigcup_{1 \le i \le k, 1 \le j \le \lambda(i)} [i-1, i] \times [j-1, j] \tag{17}$$

of the first quadrant of the plane. Transform this region by introducing the coordinate system

$$u = x - y, \qquad v = x + y$$

(the so-called Russian coordinates) rotated by 45 degrees and stretched by the factor $\sqrt{2}$ with respect to the (x, y) coordinates. In the (u, v)-coordinates, the region A_λ now has the form

$$A_\lambda = \{(u, v) : -\lambda'(1) \le u \le \lambda(1), |u| \le v \le \phi_\lambda(u)\},$$

where $\lambda'(1) = k$ is the number of parts of λ, and ϕ_λ is a piecewise linear function on $[-\lambda'(1), \lambda(1)]$ with slopes $\phi'_\lambda = \pm 1$. We extend ϕ_λ to be defined on all of $\mathbb{R}$ by setting $\phi_\lambda(u) = |u|$ for $u \notin [-\lambda'(1), \lambda(1)]$, as illustrated in Figure 7. The function ϕ_λ, called *profile* of λ, is a useful way to encode the shape of the diagram λ.

We can also consider a scaled version of ϕ_λ given by

$$\tilde{\phi}_\lambda(u) = \frac{1}{\sqrt{n}} \phi_\lambda(\sqrt{n} u).$$

This scaling leads to a diagram with constant area (equal to 2, in this coordinate system), and is naturally suitable for dealing with asymptotic questions about the shape λ.

The following version of the limit shape theorem with an explicit error estimate is a slight variation of the one given by Vershik and Kerov (1985) [it follows from the numerical estimates in Section 3 of that paper by modifying some parameters in an obvious way; see also Romik (2014), Chapter 1].

THEOREM 3.1 (The limit shape of Plancherel-random Young diagrams). *Define the function $\Omega_* : \mathbb{R} \to [0, \infty)$ by*

$$\Omega_*(u) = \begin{cases} \frac{2}{\pi}\left[u \sin^{-1}\left(\frac{u}{2}\right) + \sqrt{4 - u^2}\right], & \text{if } -2 \le u \le 2, \\ |u|, & \text{otherwise}. \end{cases} \tag{18}$$

Let $\varnothing = \Lambda_0 \nearrow \Lambda_1 \nearrow \Lambda_2 \nearrow \dots$ denote the Plancherel growth process as in (3). Then there exists a constant $C > 0$ such that for any $\varepsilon > 0$, we have

$$\operatorname{Prob}\Big(\sup_{u \in \mathbb{R}} |\tilde{\phi}_{\Lambda_n}(u) - \Omega_*(u)| > \varepsilon\Big) = O(e^{-C\sqrt{n}}) \qquad \text{as } n \to \infty.$$

See Figure 8 for an illustration of the profile of a typical Plancherel-random diagram shown together with the limit shape.

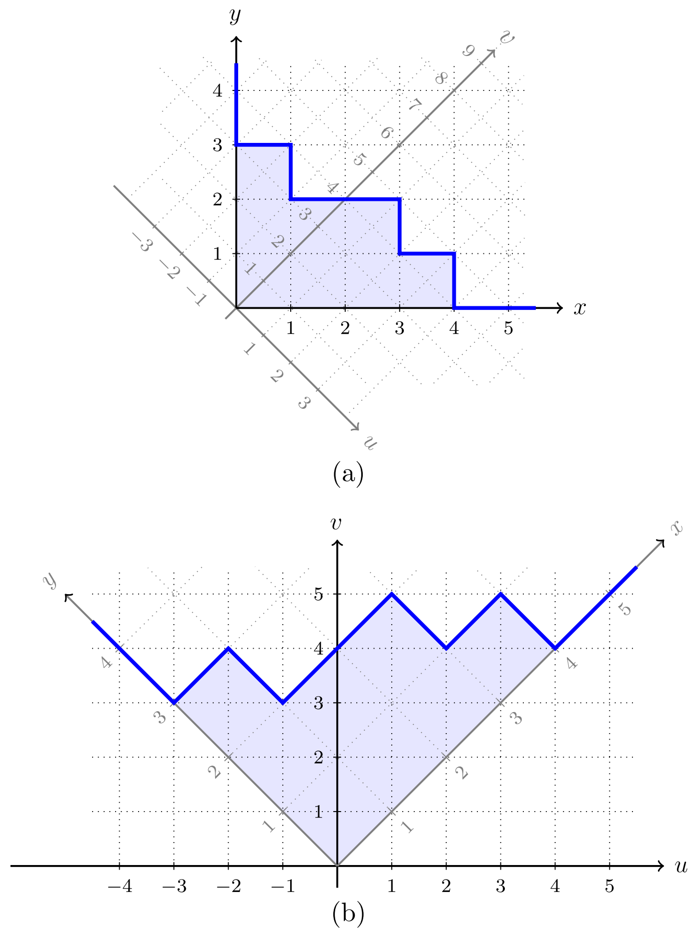

FIG. 7. *A Young diagram* $\lambda = (4, 3, 1)$ *shown in* (a) *the French*, *and* (b) *the Russian convention*. *The solid line represents the profile* ϕ_λ *of the Young diagram*. *The coordinate system* (u, v) *corresponding to the Russian convention and the coordinate system* (x, y) *corresponding to the French convention are shown*.

3.2. *The transition measure*. Next, we recall the concept of the *transition measure* of Young diagrams and its extension to smooth shapes, developed by Kerov (1993, 1999) see also [Romik (2004)]. For a Young diagram $\lambda \in \mathbb{Y}_n$, this is defined simply as the probability measure on the set of diagrams $\nu \in \mathbb{Y}_{n+1}$ such that $\lambda \nearrow \nu$ (or equivalently on the set of boxes that can be attached to λ to form a new Young diagram) given by (4). Kerov observed that as a sequence of diagrams approaches in the scaling limit a smooth shape (in a sense similar to that of the limit shape theorem above), the transition measures also converge, and thus depend continuously, in an appropriate sense, on the shape. For the limit shape Ω_*, which is the only one we will need to consider, the transition measure (in this limiting sense) is

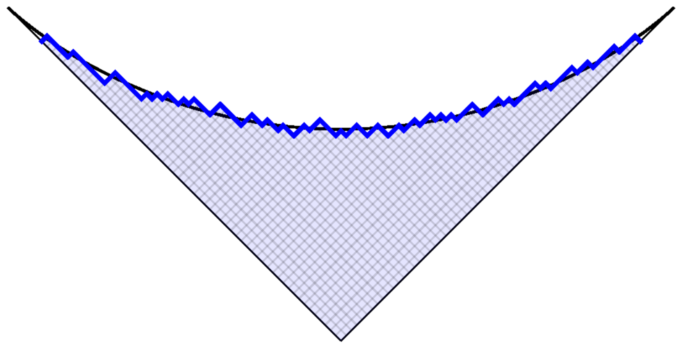

FIG. 8. *The limit shape $v = \Omega_*(u)$ superposed with the (rescaled) profile $\tilde{\phi}_{\Lambda_n}$ of a simulated Plancherel-distributed random Young diagram of order $n = 1000$.*

the semicircle distribution. The precise result, paraphrased slightly to bring it to a form suitable for our application, is as follows.

THEOREM 3.2 (Transition measure of P_n-random Young diagrams). *For each $n \geq 1$, denote by $\mathbf{d}_n = (a_n, b_n)$ the random position of the box that was added to the random Young diagram Λ_{n-1} in (3) to obtain Λ_n. Then we have the convergence in distribution*

$$\frac{1}{\sqrt{n}}(a_n - b_n, a_n + b_n) \xrightarrow{\mathcal{D}} (U, V) \qquad \text{as } n \to \infty, \tag{19}$$

where U is a random variable with the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$ on $[-2, 2]$, and $V = \Omega_*(U)$. In other words, in the (u, v)-coordinates, the position of the box added according to the transition measure (4) has in the limit a u-coordinate distributed according to the semicircle distribution and its v-coordinate is related to its u-coordinate by the function Ω_*.

PROOF. This follows immediately by combining Theorem 3.1 with the fact that the transition measure of the curve Ω_* is $\mathcal{L}_{\mathrm{SC}}$, and the fact that the mapping taking a continual Young diagram to its transition measure is continuous in the uniform norm (with the weak topology on measures on $\mathbb{R}$). For the proofs of these facts, refer to Kerov (1993, 1999) [see also Romik (2004)]. □

3.3. *Weak asymptotics for the jeu de taquin path.* As an application of these ideas, we prove the convergence in distribution of the directions along the jeu de taquin path in the infinite Plancherel-random tableau. This is a weaker version of Theorem 1.1 that identifies the distribution (8) but does not include the fact that the jeu de taquin path is asymptotically a straight line. It will be convenient to work with a modified version of the jeu de taquin path in which time is reparameterized to correspond more closely to the Plancherel growth process (3). We call this the *natural parameterization* of the jeu de taquin path. To define it, let $\mathbf{q}_n = \mathbf{p}_{K(n)}$ denote the position of the last box in the jeu de taquin path contained in the diagram Λ_n, that is, $K(n)$ is the maximal number k such that $t_{\mathbf{p}_k}$, the tableau entry in position $\mathbf{p}_k$, is $\leq n$. The reparameterized sequence $(\mathbf{q}_n)_{n\geq 1}$ is simply a slowed-down or "lazy" version of the jeu de taquin path: as n increases, it either jumps to its right or up if in the Plancherel growth process a box was added in one of those two positions, and stays put at other times.

THEOREM 3.3. *Let T be a Plancherel-random infinite Young tableau with a naturally-parameterized jeu de taquin path $(\mathbf{q}_n)_{n=1}^{\infty}$. We have the convergence in distribution*

$$\frac{\mathbf{q}_n}{\|\mathbf{q}_n\|} \xrightarrow{\mathcal{D}} (\cos\Theta, \sin\Theta) \qquad \text{as } n \to \infty,$$

where Θ is the random variable defined by (8).

To show this, we need the following lemma, which also gives one possible explanation for why the slowed-down parameterization may be considered natural (another explanation, related to the "second-class particle" interpretation, is suggested in Section 7).

LEMMA 3.4. *For any fixed $n \geq 1$, we have the equality in distribution*

$$\mathbf{q}_n \stackrel{\mathcal{D}}{=} \mathbf{d}_n.$$

PROOF. Let $X_1, \ldots, X_n$ be i.i.d. $U(0,1)$ random variables. Let Λ_n be the Young diagram associated by RSK to the sequence $(X_1, X_2, \ldots, X_n)$ and let $\widetilde{\Lambda}_{n-1}$ be the Young diagram associated to $(X_2, X_3, \ldots, X_n)$. From Lemmas 2.2 and 2.3, we get that

$$\mathbf{q}_n \stackrel{\mathcal{D}}{=} \Lambda_n \setminus \widetilde{\Lambda}_{n-1}.$$

Let $(Y_1, \ldots, Y_n) = (1 - X_n, 1 - X_{n-1}, \ldots, 1 - X_1)$. In this way, $Y_1, \ldots, Y_n$ are i.i.d. $U(0,1)$ random variables, and thus the path in the Young graph $\varnothing = M_0 \nearrow \cdots \nearrow M_n$ corresponding to the sequence via RSK is distributed according to the Plancherel measure. It follows that

$$\mathbf{d}_n \stackrel{\mathcal{D}}{=} M_n \setminus M_{n-1}.$$

Applying Lemma 2.5(c) for the sequence $(X_1, \ldots, X_n)$ and for the sequence $(X_2, \ldots, X_n)$, we get however that $M_n = \Lambda_n$ and $M_{n-1} = \widetilde{\Lambda}_{n-1}$, which completes the proof. $\square$

PROOF OF THEOREM 3.3. Define random angles $(\theta_n)_{n=1}^{\infty}$ by

$$\mathbf{d}_n = (a_n, b_n) = \|\mathbf{d}_n\|(\cos\theta_n, \sin\theta_n),$$

where $0 \le \theta_n \le \pi/2$ for $n \ge 1$. By Lemma 3.4, it is enough to show that $\theta_n \xrightarrow{\mathcal{D}} \Theta$, or equivalently that

$$\cot(\pi/4 - \theta_n) \xrightarrow{\mathcal{D}} \cot(\pi/4 - \Theta) = \frac{2}{\pi}\left(\sin^{-1}\left(\frac{W}{2}\right) + \frac{\sqrt{4 - W^2}}{W}\right), \tag{20}$$

where $W \sim \mathcal{L}_{\mathrm{SC}}$ as in Theorem 1.1. But note that

$$\cot(\pi/4 - \theta_n) = \frac{a_n + b_n}{a_n - b_n},$$

the ratio of the v- and u- coordinates of $\mathbf{d}_n$, since the $\pi/4$ term corresponds exactly to the angle of rotation between (x, y) and (u, v) coordinates. So, by (19), $\cot(\pi/4 - \theta_n) \xrightarrow{\mathcal{D}} V/U = \Omega_*(U)/U$, where a_n, b_n, V and U are defined in Theorem 3.2, and it is easy to see from the definition of $\Omega_*(\cdot)$ in (18) that this is exactly the distribution appearing on the right-hand side of (20). □

Note that the proof above gives a simple geometric characterization of the distribution of the limiting random angle Θ. Namely, in the Russian coordinate system we choose a random vector (U, V) that lies on the limit shape by drawing U from the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$, and taking $V = \Omega_*(U)$. The random variable Θ is the angle subtended between the ray $\{u = v > 0\}$ (which corresponds to the positive x-axis) and the ray pointing from the origin to (U, V).

4. Plactic Littlewood–Richardson rule, Jucys–Murphy elements and the semicircle distribution.

4.1. *Pieri growth.* Our goal in this section will be to prove a technical result that we will need for the proofs of Theorems 1.1 and 1.5. The result concerns a particular way of growing a Plancherel-random Young diagram of order n by k additional boxes. We refer to this type of growth as *Pieri growth*, because of its relation to the Pieri rule from algebraic combinatorics. This is defined as follows. Fix $n, k \ge 1$, and consider the following way of generating a pair $\Lambda_n \subset \Gamma_{n+k}$ of random Young diagrams, where $\Lambda_n \in \mathbb{Y}_n$ and $\Gamma_{n+k} \in \mathbb{Y}_{n+k}$: first, take a sequence $A_1, \dots, A_n$ of i.i.d. random variables with the $U(0, 1)$ distribution, and define Λ_n as the RSK shape associated with the input sequence $A_1, \dots, A_n$ (so, Λ_n is distributed according to the Plancherel measure P_n of order n). Next, take a sequence $B_1, \dots, B_k$ of i.i.d. random variables with the $U(0, 1)$ distribution, *conditioned to be in increasing order* [i.e., the vector $(B_1, \dots, B_k)$ is chosen uniformly at random from the set $\{(b_1, \dots, b_k) : 0 \le b_1 \le \cdots \le b_k \le 1\}$], then let Γ_{n+k} be the RSK shape associated with the concatenated sequence $(A_1, \dots, A_n, B_1, \dots, B_k)$.

Let $\nu \in \mathbb{Y}_k$ be a Young diagram with k boxes or, more generally, let $\nu = \lambda \setminus \mu$ (for $\lambda \in \mathbb{Y}_{n+k}$, $\mu \in \mathbb{Y}_n$) be a skew Young diagram with k boxes. Let

$$\square_1 = (i_1, j_1), \qquad \square_2 = (i_2, j_2), \qquad \dots, \qquad \square_k = (i_k, j_k)$$

denote the positions of its boxes (arranged in some arbitrary order). For each $1 \leq \ell \leq k$, we will call $u_\ell = i_\ell - j_\ell$ the *u-coordinate* of the box $\square_\ell$. (In the literature, such a u-coordinate is usually called the *content* of $\square_\ell$, but in order to avoid notational collisions with the content of a box of a Young tableau, we decided not to use this term in this meaning.) The sequence $(u_1, \dots, u_k)$ of the u-coordinates of the boxes of ν will turn out to be very useful.

THEOREM 4.1. *For each n, k, let $u_1, \dots, u_k$ be the u-coordinates of the boxes of $\Gamma_{n+k} \setminus \Lambda_n$, where the Pieri growth pair $\Lambda_n \subset \Gamma_{n+k}$ is defined above. Let $m_{n,k}$ denote the empirical measure of the u-coordinates $u_1, \dots, u_k$ (scaled by a factor of $n^{-1/2}$), given by*

$$m_{n,k} = \frac{1}{k} \sum_{\ell=1}^{k} \delta_{n^{-1/2} u_\ell},$$

where for a real number x, symbol δ_x denotes a delta measure concentrated at x. Let $k = k(n)$ be a sequence such that $k = o(\sqrt{n})$ as $n \to \infty$. Then as $k \to \infty$, the random measure $m_{n,k}$ converges weakly in probability to the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$, and furthermore, for any $\varepsilon > 0$ and any $u \in \mathbb{R}$ we have the estimate

$$\operatorname{Prob}\big(\big|F_{m_{n,k}}(u) - F_{\mathrm{SC}}(u)\big| > \varepsilon\big) = O\left(\frac{1}{k} + \frac{k}{\sqrt{n}}\right) \qquad \text{as } n \to \infty,$$

where F_{SC} denotes the cumulative distribution function of the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$, and $F_{m_{n,k}}$ denotes the cumulative distribution function of $m_{n,k}$.

In order to prove this result, we will apply the "plactic" version of the Littlewood–Richardson rule (Theorem 4.2) which, roughly speaking, says that the probabilistic behavior of the RSK shape associated to a concatenation of two random sequences with prescribed RSK shapes coincides with the probabilistic behavior of a random irreducible component of a certain representation of the symmetric group. In this way, the quantities describing the probabilistic properties of the random probability measure $m_{n,k}$ can be calculated by the machinery of representation theory, and specifically the Jucys–Murphy elements. We present the necessary tools below.

4.2. *The symmetric group and its representation theory.* Let $n, k \geq 1$ be given. In the following, we will view S_n as the group of permutations of the set $\{1, \dots, n\}$, S_k as the group of permutations of the set $\{n+1, \dots, n+k\}$ and S_{n+k} as the group of permutations of $\{1, \dots, n+k\}$. In this way $S_n \times S_k$ is identified with the subgroup of S_{n+k} consisting of those permutations of $\{1, \dots, n+k\}$ which leave the sets $\{1, \dots, n\}$ and $\{n+1, \dots, n+k\}$ invariant. In this article, we will consider only the groups which have one of the above forms. We review below some basic facts from representation theory, tailored for this particular setup.

For a representation $\rho : G \to \operatorname{End} W$ of some finite group G, we define its *normalized character*

$$\chi^W(g) = \frac{\operatorname{Tr} \rho(g)}{(\text{dimension of } W)} \qquad \text{for } g \in G.$$

The group algebra $\mathbb{C}(G)$ can be alternatively viewed as the algebra of functions $\{f : G \to \mathbb{C}\}$; as multiplication we take the convolution of functions. For any element $f \in \mathbb{C}[G]$ of the group algebra, we will denote by $\chi^W(f)$ the extension of the character by linearity:

$$\chi^W(f) = \sum_{g \in G} f(g) \chi^W(g).$$

For a modern approach to the representation theory of symmetric groups, we refer to the monograph of Ceccherini-Silberstein, Scarabotti and Tolli (2010). There is a bijective correspondence between the set of (equivalence classes of) irreducible representations of the symmetric group S_n and the set $\mathbb{Y}_n$ of Young diagrams with n boxes. We denote by V^λ the irreducible representation $\rho^\lambda : S_n \to \operatorname{End} V^\lambda$ which corresponds to $\lambda \in \mathbb{Y}_n$. The dimension of the space V^λ is equal to f^λ, the number of standard Young tableaux of shape λ. We use the shorthand notation χ^λ for the corresponding character χ^{V^λ}.

Two representations of the symmetric groups will play a special role in the following. The *trivial representation* $V_{S_k}^{\text{trivial}}$ of S_k is the one for which the vector space $V_{S_k}^{\text{trivial}}$ is one-dimensional and any group element $g \in S_k$ acts on it trivially by identity. The corresponding character

$$\chi_{S_k}^{\text{trivial}}(g) = 1$$

is constantly equal to 1. The trivial representation is irreducible and corresponds to the Young diagram (k) which has only one row; in other words $V_{S_k}^{\text{trivial}} = V^{(k)}$. The *regular representation* $V_{S_n}^{\text{regular}}$ of S_n is the one for which the vector space $V_{S_n}^{\text{regular}} = \mathbb{C}(S_n)$ is just the group algebra and the action is given by multiplication from the left. The corresponding character

$$\chi_{S_n}^{\text{regular}}(g) = \delta_e(g) = \begin{cases} 1, & \text{if } g = e, \\ 0, & \text{otherwise,} \end{cases}$$

is equal to the delta function at the group unit.

4.3. *Isomorphism between $\mathcal{C}(\mathbb{Y}_n)$ and $Z\mathbb{C}(S_n)$.* For a Young diagram $\lambda \in \mathbb{Y}_n$, we define

$$q_\lambda = \frac{(f^\lambda)^2}{n!} \chi^\lambda.$$

The elements $(q_\lambda : \lambda \in \mathbb{Y}_n)$ form a linear basis of the center $Z\mathbb{C}[S_n]$ of the group algebra. They form a commuting family of orthogonal projections, in other words

$$q_\lambda q_\mu = \begin{cases} q_\lambda, & \text{if } \lambda = \mu, \\ 0, & \text{otherwise,} \end{cases}$$

which shows that

$$(f : \mathbb{Y}_n \to \mathbb{C}) \mapsto \sum_{\lambda \in \mathbb{Y}_n} f(\lambda) q_\lambda \in Z\mathbb{C}(S_n)$$

is an isomorphism between the commutative algebra $\mathcal{C}(\mathbb{Y}_n)$ of functions on $\mathbb{Y}_n$ (with pointwise addition and multiplication) and the center $Z\mathbb{C}(S_n)$ of the symmetric group algebra. Thanks to this isomorphism any $f \in \mathcal{C}(\mathbb{Y}_n)$ can be identified with an element of the center $Z\mathbb{C}(S_n)$ which for simplicity will be denoted by the same symbol.

The inverse isomorphism associates to $f \in Z\mathbb{C}(S_n)$ a function on Young diagrams which is explicitly given by

$$\lambda \mapsto \chi^\lambda(f). \tag{21}$$

4.4. *The random Young diagram associated to a representation.* For a representation W of the symmetric group S_n we consider its decomposition into irreducible components:

$$W = \bigoplus_{\lambda \in \mathbb{Y}_n} m_\lambda V^\lambda, \tag{22}$$

where $m_\lambda \in \mathbb{N} \cup \{0\}$ denotes the multiplicity. The representation W induces a probability measure on $\mathbb{Y}_n$ given by

$$\mathbb{P}_W(\lambda) = \frac{m_\lambda(\text{dimension of } V^\lambda)}{(\text{dimension of } W)} \qquad \text{for } \lambda \in \mathbb{Y}_n.$$

In other words, the representation W of S_n gives rise to a random Young diagram Λ with n boxes; we will say that Λ is the *random Young diagram associated to the representation* W. The probability of λ is proportional to the total dimension of all irreducible components of W which are of type $[\lambda]$. Alternatively, we can select some linear basis $e_1, \dots, e_l$ of the vector space W in such a way that each basis vector e_i belongs to one of the summands in (22). With the uniform measure we randomly select a basis vector e_i; this vector corresponds to a Young diagram Λ which has the desired distribution.

This choice of probability measure on $\mathbb{Y}_n$ has an advantage that the corresponding expected value of random variables has a very simple representation-theoretic interpretation. Namely, for $f \in \mathcal{C}(\mathbb{Y}_n)$ [which under the identification from Section 4.3 can be seen as $f \in Z\mathbb{C}(S_n)$], it is immediate from the definitions that

$$\mathbb{E}_W f(\Lambda) = \chi^W(f), \tag{23}$$

where $\mathbb{E}_W$ denotes the expectation with respect to the measure $\mathbb{P}_W$.

An important example is the case when $W = V_{S_n}^{\text{regular}}$ is the regular representation of the symmetric group; then the corresponding probability distribution on $\mathbb{Y}_n$ is the Plancherel measure (2).

4.5. *Outer product and Littlewood–Richardson coefficients.* If V is a representation of S_n and W is a representation of S_k we denote by

$$V \circ W = (V \otimes W) \uparrow_{S_n \times S_k}^{S_{n+k}}$$

their *outer product*. It is a representation of S_{n+k} which is induced from the tensor representation $V \otimes W$ of the Cartesian product $S_n \times S_k$.

There are several equivalent ways to define Littlewood–Richardson coefficients but for the purposes of this article it will be most convenient to use the following one. For Young diagrams $\lambda \in \mathbb{Y}_n$, $\mu \in \mathbb{Y}_k$, $\nu \in \mathbb{Y}_{n+k}$, we define the *Littlewood–Richardson coefficient* $c_{\lambda,\mu}^{\nu}$ as the multiplicity of the irreducible representation $V^\lambda \otimes V^\mu$ of the group $S_n \times S_k$ in the restricted representation $V^\nu \downarrow_{S_n \times S_k}^{S_{n+k}}$.

Equivalently, $c_{\lambda,\mu}^{\nu}$ is equal to the multiplicity of the irreducible representation V^ν in the outer product $V^\lambda \circ V^\mu$. It follows that the random Young diagram associated to the outer product $V^\lambda \circ V^\mu$ has the distribution

$$\mathbb{P}_{V^\lambda \circ V^\mu}(\nu) = \frac{1}{\text{dimension of } V^\lambda \circ V^\mu} c_{\lambda,\mu}^{\nu} f^{\nu}. \tag{24}$$

4.6. *The plactic Littlewood–Richardson rule.* The following result is essentially a reformulation of the usual form of the plactic Littlewood–Richardson rule [Fulton (1997), Chapter 5].

THEOREM 4.2. *Let the Young diagrams $\lambda \in \mathbb{Y}_n$, $\mu \in \mathbb{Y}_k$ be fixed. Let $\mathbf{A} = (A_1, \dots, A_n) \in [0,1]^n$ and $\mathbf{B} = (B_1, \dots, B_k) \in [0,1]^k$ be random sequences sampled according to the product of Lebesgue measures, conditioned so that λ, respectively μ, is the* RSK *shape associated to* $\mathbf{A}$, *respectively* $\mathbf{B}$. *Then the distribution of the* RSK *shape associated to the concatenated sequence* $\mathbf{AB}$ *coincides with the distribution* (24) *of the random Young diagram associated to the representation* $V^\lambda \circ V^\mu$.

PROOF. Let $\mathcal{A} = [0,1]$ be the alphabet (linearly ordered set) of the numbers from the unit interval. For the purpose of the following definition, we consider $\mathrm{RSK}_n : \mathcal{A}^n \to \mathbb{Y}_n$ as a map which to words of length n associates the corresponding RSK shape. For a Young diagram $\lambda \in \mathbb{Y}_n$, we define the formal linear combination

$$\widetilde{S}_\lambda = \frac{n!}{(f^\lambda)^2} \sum_{\substack{\mathbf{A}=(A_1,\dots,A_n)\in\mathcal{A}^n, \\ \mathrm{RSK}_n(\mathbf{A})=\lambda}} \mathbf{A}$$

of all words for which the RSK shape is equal to λ. This formal linear combination can be alternatively viewed as a function $\widetilde{S}_\lambda : \mathcal{A}^n \to \mathbb{R}$; then it becomes a density of a probability measure on $\mathcal{A}^n$. This measure is the probability distribution of a random sequence $\mathbf{A}$ with the uniform distribution on $\mathcal{A}^n$, conditioned to have the RSK shape equal to λ.

There are f^λ possible choices of a recording tableau of shape λ. It follows that the plactic class corresponding to a given insertion tableau of shape λ consists of f^λ elements of $\mathcal{A}^n$. Therefore, the embedding of $\mathcal{A}^n$ into the plactic monoid maps $\widetilde{S}_\lambda$ to $\frac{n!}{f^\lambda} S_\lambda$, where S_λ is the plactic Schur polynomial, defined as

$$S_\lambda = \sum_{\text{shape}(P)=\lambda} P,$$

where the sum runs over all increasing tableaux P of shape λ and with the entries in the alphabet $\mathcal{A}$.

We now use one of the forms of the plactic Littlewood–Richardson rule [Fulton (1997), page 63], which says that for arbitrary $\lambda \in \mathbb{Y}_n$, $\mu \in \mathbb{Y}_k$, we have that

$$S_\lambda S_\mu = \sum_{\nu \in \mathbb{Y}_{n+k}} c^\nu_{\lambda,\mu} S_\nu,$$

where the product is taken in the plactic monoid. Therefore,

$$\widetilde{S}_\lambda \widetilde{S}_\mu = \frac{1}{\binom{n+k}{k} f^\lambda f^\mu} \sum_{\nu \in \mathbb{Y}_{n+k}} c^\nu_{\lambda,\mu} f^\nu \widetilde{S}_\nu. \tag{25}$$

If we interpret $\widetilde{S}_\lambda$ and $\widetilde{S}_\mu$ as densities of probability measures on $\mathcal{A}^n$ and $\mathcal{A}^k$, respectively, and as a product we take concatenation of sequences, then $\widetilde{S}_\lambda \widetilde{S}_\lambda$ can be interpreted as a density of a probability measure on $\mathcal{A}^{n+k}$. In this way, (25) can be interpreted as follows: the left-hand side in the plactic monoid is equal to the distribution of the RSK shape associated to the concatenated sequence $\mathbf{AB}$. The probability distribution of this RSK shape is given by the coefficients standing at the right-hand side:

$$\operatorname{Prob}\bigl(\mathrm{RSK}_n(\mathbf{AB}) = \nu\bigr) = \frac{1}{\binom{n+k}{k} f^\lambda f^\mu} c^\nu_{\lambda,\mu} f^\nu,$$

which coincides with (24), as required. □

4.7. *Jucys–Murphy elements and u-coordinates of boxes.* We define the *Jucys–Murphy elements* as the elements of the symmetric group algebra

$$X_i = (1, i) + \cdots + (i-1, i) \in \mathbb{C}(S_n)$$

given for each $1 \le i \le n$ by the formal sum of transpositions interchanging the element i with smaller numbers. The following lemma summarizes some fundamental properties of Jucys–Murphy elements [Jucys (1974)].

LEMMA 4.3. *Let* $\lambda \in \mathbb{Y}_n$ *be a Young diagram, and let* $u_1, \dots, u_n$ *be the* u-*coordinates of its boxes. Let* $P(x_1, \dots, x_n)$ *be a symmetric polynomial in* n *variables. Then*:

1. $P(X_1, \dots, X_n) \in \mathbb{C}(S_n)$ *belongs to the center of the group algebra.*
2. *We denote by* $\rho^\lambda : S_n \to V^\lambda$ *the irreducible representation of the symmetric group* S_n *corresponding to the Young diagram* λ; *then the operator* $\rho^\lambda(P(X_1, \dots, X_n))$ *is a multiple of the identity operator, and hence can be identified with a complex number. The value of this number is equal to*

$$\chi^\lambda\big(P(X_1, \dots, X_n)\big) = P(u_1, \dots, u_n).$$

4.8. *Growth of Young diagrams and Jucys–Murphy elements*. This section is devoted to the proof of the following result which will be essential for the proof of Theorem 4.1.

THEOREM 4.4. *We keep the notation from Section* 4.1, *except that the* u-*coordinates of the boxes of* $\Gamma_{n+k} \setminus \Lambda_n$ *will now be denoted by* $u_{n+1}, \dots, u_{n+k}$. *For any symmetric polynomial* $P(x_{n+1}, \dots, x_{n+k})$ *in* k *variables we have*

$$\mathbb{E} P(u_{n+1}, \dots, u_{n+k}) = \big(\chi_{S_n}^{\text{regular}} \otimes \chi_{S_k}^{\text{trivial}}\big)\big(P(X_{n+1}, \dots, X_{n+k}) \downarrow_{S_n \times S_k}^{S_{n+k}}\big), \tag{26}$$

where $F \downarrow_{S_n \times S_k}^{S_{n+k}} \in \mathbb{C}(S_n \times S_k)$ *denotes the restriction of* $F \in \mathbb{C}(S_{n+k})$ *to the subgroup* $S_n \times S_k$.

Before we do this, we show the following technical result.

LEMMA 4.5. *Let* $\lambda \in \mathbb{Y}_n$, $\mu \in \mathbb{Y}_k$ *be given. Let* Γ *be a random Young diagram associated to the outer product* $V^\lambda \circ V^\mu$ *of the corresponding irreducible representations. Let* $u_{n+1}, \dots, u_{n+k}$ *be the* u-*coordinates of the boxes of the skew Young diagram* $\Gamma \setminus \lambda$ (*one can show that always* $\lambda \subseteq \Gamma$). *Then for any symmetric polynomial* $P(x_{n+1}, \dots, x_{n+k})$ *in* k *variables*

$$\mathbb{E} P(u_{n+1}, \dots, u_{n+k}) = \big(\chi^\lambda \otimes \chi^\mu\big)\big(P(X_{n+1}, \dots, X_{n+k}) \downarrow_{S_n \times S_k}^{S_{n+k}}\big).$$

PROOF. This proof is modeled after the proof of Proposition 3.3 in Biane (1998). The regular representation of the symmetric group decomposes as follows:

$$\mathbb{C}(S_{n+k}) = \bigoplus_{\gamma \in \mathbb{Y}_{n+k}} V^\gamma \otimes V^\gamma \tag{27}$$

as an $S_{n+k} \times S_{n+k}$-module. The image of the projection $q_\lambda \otimes q_\mu \in \mathbb{C}(S_n \times S_k)$ acting from the left on the decomposition (27) is equal to

$$\begin{aligned} (q_\lambda \otimes q_\mu)\mathbb{C}(S_{n+k}) &= \bigoplus_{\gamma \in \mathbb{Y}_{n+k}} c_{\lambda,\mu}^\gamma (V^\lambda \otimes V^\mu) \otimes V^\gamma \\ &= (V^\lambda \otimes V^\mu) \otimes \bigoplus_{\gamma \in \mathbb{Y}_{n+k}} c_{\lambda,\mu}^\gamma V^\gamma, \end{aligned} \tag{28}$$

which we view as a $(S_n \times S_k) \times S_{n+k}$-module and where the multiplicity $c^{\gamma}_{\lambda,\mu} \in \mathbb{N} \cup \{0\}$ is the Littlewood–Richardson coefficient. It follows that if we view (28) as a (right) S_{n+k}-module, the distribution of a random Young diagram associated to it coincides with the distribution of a random Young diagram Γ associated to the outer product $V^\lambda \circ V^\mu$.

Assume that $F \in \mathbb{C}(S_{n+k})$ commutes with the projection $q_\lambda \otimes q_\mu$ and furthermore that F acts from the left on (28) as follows: on the summand corresponding to $\gamma \in \mathbb{Y}_{n+k}$ it acts by multiplication by some scalar which we will denote by $F(\gamma)$. From the above discussion, it follows that if Γ is a random Young diagram associated to the outer product $V^\lambda \circ V^\mu$ then

$$\mathbb{E}F(\Gamma) = \frac{\operatorname{Tr} F}{(\text{dimension of the image of } q_\lambda \otimes q_\mu)},$$

where for the meaning of the trace $\operatorname{Tr} F$ we view F as acting from the left on (28). The numerator is equal to the trace of $(q_\lambda \otimes q_\mu)F \in \mathbb{C}(S_{n+k})$ which we view this time as acting from the left on the regular representation, thus it is equal to

$$(n+k)!\big[(q_\lambda \otimes q_\mu)F\big](e) = \frac{(n+k)!(f^\lambda)^2(f^\mu)^2}{n!^2k!^2}\big[(\chi^\lambda \otimes \chi^\mu)F\big](e).$$

The last factor on the right-hand side can be written as

$$\begin{aligned}\big[(\chi^\lambda \otimes \chi^\mu)F\big](e) &= \sum_{g \in S_n \times S_k} (\chi^\lambda \otimes \chi^\mu)(g^{-1})F(g) \\ &= \sum_{g \in S_n \times S_k} (\chi^\lambda \otimes \chi^\mu)(g)F(g) = (\chi^\lambda \otimes \chi^\mu)\big(F \downarrow^{S_{n+k}}_{S_n \times S_k}\big),\end{aligned}$$

where we used the fact that the characters of the symmetric groups satisfy $\chi^\gamma(g) = \chi^\gamma(g^{-1})$. Thus,

$$\mathbb{E}F(\Gamma) = C_{\lambda,\mu}(\chi^\lambda \otimes \chi^\mu)\big(F \downarrow^{S_{n+k}}_{S_n \times S_k}\big)$$

for some constant $C_{\lambda,\mu}$ which depends only on λ and μ. In order to calculate the exact value of this constant, we can take $F = \delta_e \in \mathbb{C}(S_{n+k})$ to be the unit of the symmetric group algebra $\mathbb{C}(S_{n+k})$ which therefore corresponds to a function $F : \mathbb{Y}_n \to \mathbb{C}$ which is identically equal to 1. It follows that $C_{\lambda,\mu} = 1$, and thus

$$\mathbb{E}F(\Gamma) = (\chi^\lambda \otimes \chi^\mu)\big(F \downarrow^{S_{n+k}}_{S_n \times S_k}\big). \tag{29}$$

We denote by p_ℓ the power-sum symmetric polynomial

$$p_\ell(x_{n+1}, \dots, x_{n+k}) = \sum_{1 \le i \le k} x^\ell_{n+i}.$$

Let $u_1, \dots, u_n$ be the u-coordinates of the boxes of the Young diagram λ. For a given Young diagram $\gamma \in \mathbb{Y}_{n+k}$ such that $\lambda \subseteq \gamma$ we denote by $u_{n+1}, \dots, u_{n+k}$ the

u-coordinates of the boxes of $\gamma \setminus \lambda$; in this way $u_1, \dots, u_{n+k}$ are the u-coordinates of the boxes of γ. Lemma 4.3 shows that the operator

$$\sum_{1 \le i \le n+k} X_i^\ell \in \mathbb{C}(S_{n+k}) \tag{30}$$

acts from the right on (28) as follows: on the summand corresponding to γ it acts by multiplication by the scalar $\sum_{1 \le i \le n+k} u_i^\ell$. Furthermore, it does not matter if we act from the left or from the right because (30) belongs to the center of $\mathbb{C}(S_{n+k})$, and thus it commutes with the projection $q_\lambda \otimes q_\mu$.

Lemma 4.3 shows that the operator

$$\sum_{1 \le i \le n} X_i^\ell \in \mathbb{C}(S_n) \tag{31}$$

belongs to the center of the symmetric group algebra $\mathbb{C}(S_n)$ therefore it commutes with the projector $q_\lambda \otimes q_\mu \in \mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \subseteq \mathbb{C}(S_{n+k})$. Furthermore, Lemma 4.3 shows that (31) acts from the left on (28) as follows: on any summand it acts by multiplication by the scalar $\sum_{1 \le i \le n} u_i^\ell$. It follows that the difference of (30) and (31)

$$\sum_{1 \le i \le n+k} X_i^\ell - \sum_{1 \le i \le n} X_i^\ell = \sum_{1 \le i \le k} X_{n+i}^\ell = p_\ell(X_{n+1}, \dots, X_{n+k})$$

commutes with $q_\lambda \otimes q_\mu$ and acts on (28) from the left as follows: on the summand corresponding to γ it acts by multiplication by

$$\sum_{1 \le i \le n+k} u_i^\ell - \sum_{1 \le i \le n} u_i^\ell = \sum_{1 \le i \le k} u_{n+i}^\ell = p_\ell(u_{n+1}, \dots, u_{n+k}).$$

Since power-sum symmetric functions generate the algebra of symmetric polynomials, we proved in this way that $P(X_{n+1}, \dots, X_{n+k})$ commutes with $q_\lambda \otimes q_\mu$ and acts on (28) from the left as follows: on the summand corresponding to γ it acts by multiplication by $P(u_{n+1}, \dots, u_{n+k})$. This shows that (29) can be applied to $F = P(X_{n+1}, \dots, X_{n+k})$ which completes the proof. □

PROOF OF THEOREM 4.4. The construction of Pieri growth given in Section 4.1 can be formulated equivalently as follows. First, choose a random Young diagram Λ_n according to the Plancherel measure of order n; in other words Λ_n is a random Young diagram with the distribution corresponding to the left regular representation. Then, conditioned on the event $\Lambda_n = \lambda \in \mathbb{Y}_n$, we take $(A_1, \dots, A_n)$ to be a vector of i.i.d. $U(0,1)$ random variables conditioned to have λ as its associated RSK shape; and then similarly take $(B_1, \dots, B_k)$ to be a vector of i.i.d. $U(0,1)$ random variables conditions to have the single-row diagram (k) as its associated RSK shape.

For $F \in \mathbb{C}(S_n \times S_k)$, we define $(\mathrm{Id} \otimes \chi^{\mathrm{trivial}}_{S_k}) F \in \mathbb{C}(S_n)$ by a partial application of the character $\chi^{\mathrm{trivial}}_{S_k}$ to the second factor as follows:

$$\big[(\mathrm{Id} \otimes \chi^{\mathrm{trivial}}_{S_k}) F\big](g) = \sum_{h \in S_k} \chi^{\mathrm{trivial}}_{S_k}(h) F(g,h) \qquad \text{for } g \in S_n,$$

where we view $(g,h) \in S_n \times S_k$.

Theorem 4.2 shows that if we condition over the event $\Lambda_n = \lambda$ then the distribution of the RSK shape associated to the concatenated sequence $(A_1, \dots, A_n, B_1, \dots, B_k)$ coincides with the distribution of the random Young diagram associated to the representation $V^\lambda \circ V^{\mathrm{trivial}}_{S_k}$. Lemma 4.5 shows that the conditional expected value is given by

$$\begin{aligned} &\mathbb{E}\big(P(u_{n+1}, \dots, u_{n+k}) | \Lambda_n = \lambda\big) \\ &\quad = \big(\chi^\lambda \otimes \chi^{\mathrm{trivial}}_{S_k}\big)\big(P(X_{n+1}, \dots, X_{n+k}) \downarrow^{S_{n+k}}_{S_n \times S_k}\big) \\ &\quad = \chi^\lambda\big((\mathrm{Id} \otimes \chi^{\mathrm{trivial}}_{S_k})\big(P(X_{n+1}, \dots, X_{n+k}) \downarrow^{S_{n+k}}_{S_n \times S_k}\big)\big). \end{aligned} \tag{32}$$

If we view it as a function of $\lambda \in \mathbb{Y}_n$, then (21) shows that it corresponds to the central element

$$\big(\mathrm{Id} \otimes \chi^{\mathrm{trivial}}_{S_k}\big)\big(P(X_{n+1}, \dots, X_{n+k}) \downarrow^{S_{n+k}}_{S_n \times S_k}\big) \in \mathbb{C}(S_n). \tag{33}$$

Let us take the mean value of both sides of (32). The mean value of the left-hand side is equal to the left-hand side of (26). The mean of the right-hand side, by (33) and (23), is equal to the right-hand side of (26). In this way, we showed that equality (26) holds true. □

4.9. *Moments of Jucys–Murphy elements*. For $\alpha \in \mathbb{N}$, we define the appropriate moment of the random measure $m_{n,k}$:

$$M_\alpha = M_\alpha(n,k) = \int_{\mathbb{R}} z^\alpha \, dm_{n,k} = \frac{1}{k} n^{-\alpha/2} \sum_{\ell=1}^{k} u_\ell^\alpha.$$

Notice that M_α is a random variable. In this section, we will find the asymptotics of its first two moments: we will not only calculate the limits but also find the speed at which these limits are obtained since the latter is also necessary for the calculation of the variance $\operatorname{Var} M_\alpha$.

Denote by

$$\gamma_\alpha = \int z^\alpha \, d\mathcal{L}_{\mathrm{SC}} = \begin{cases} C_{\alpha/2}, & \text{if } \alpha \text{ is even}, \\ 0, & \text{if } \alpha \text{ is odd}, \end{cases}$$

the sequence of moments of the semicircle distribution, where $C_m = \frac{1}{m+1}\binom{2m}{m}$ denotes the mth Catalan number. We will prove the following.

THEOREM 4.6. *For each* $\alpha \in \mathbb{N}$, *we have*

$$\mathbb{E} M_\alpha = \gamma_\alpha + O\left(\frac{k}{\sqrt{n}}\right), \tag{34}$$

$$\operatorname{Var} M_\alpha = O\left(\frac{1}{k} + \frac{k}{\sqrt{n}}\right). \tag{35}$$

This kind of calculation is not entirely new; similar calculations already appeared in several papers [Biane (1995, 1998, 2001, Śniady (2006a, 2006b)] in the special case $k = 1$. Our calculation is not very far from the ones mentioned above; in fact, in some aspects it is simpler than some of them since we study a particularly simple character of the symmetric group S_n, namely $\chi_{S_n}^{\text{regular}}$ corresponding to the regular representation.

4.9.1. *The mean value of* M_α. Theorem 4.4 shows that

$$\mathbb{E} M_\alpha = \frac{1}{k} n^{-\alpha/2} \left(\chi_{S_n}^{\text{regular}} \otimes \chi_{S_k}^{\text{trivial}}\right)\left(\sum_{1 \le i \le k} X_{n+i}^\alpha \downarrow_{S_n \times S_k}^{S_{n+k}}\right). \tag{36}$$

The problem is therefore reduced to studying the element

$$\sum_{1 \le i \le k} X_{n+i}^\alpha = \sum_{1 \le i \le k} \sum_{1 \le j_1, \dots, j_\alpha \le n+i-1} (n+i, j_1) \cdots (n+i, j_\alpha) \in \mathbb{C}(S_{n+k}). \tag{37}$$

We say that $\Xi = \{\Xi_1, \dots, \Xi_\ell\}$ is a *set-partition* of some set Z if $\Xi_1, \dots, \Xi_\ell$ are disjoint, nonempty subsets of Z such that $\Xi_1 \cup \dots \cup \Xi_l = Z$. We denote by $|\Xi|$ the number of parts of Ξ, which is equal to ℓ. There is an obvious bijection between set partitions of Z and equivalence relations on Z.

For a given summand contributing to the right-hand side of (37), we define the sets

$$Z_\Sigma = \{\ell \in \{1, \dots, \alpha\} : j_\ell \le n\},$$
$$Z_\Pi = \{\ell \in \{1, \dots, \alpha\} : j_\ell \ge n+1\}.$$

We also define a set-partition Σ of the set Z_Σ which corresponds to the equivalence relation

$$p \sim q \quad \Longleftrightarrow \quad j_p = j_q \qquad \text{for } p, q \in Z_\Sigma.$$

In an analogous way, we define a set-partition Π of the set Z_Π.

It is easy to see that if $1 \le i \le k$, and $j_1, \dots, j_\alpha \le n+i-1$, and $1 \le i' \le k$, and $j_1', \dots, j_\alpha' \le n+i'-1$ are such that the corresponding set-partitions coincide: $\Sigma = \Sigma'$ and $\Pi = \Pi'$ then there exists a permutation $g \in S_n \times S_k$ with the property that $g(n+i) = n+i'$, $g(j_\ell) = j_\ell'$. It follows that the corresponding summands

$$(n+i, j_1) \cdots (n+i, j_\alpha) \quad \text{and} \quad (n+i', j_1') \cdots (n+i', j_\alpha')$$

are conjugate by a permutation $g \in S_n \times S_k$. This implies that the corresponding characters

$$(\chi^{\text{regular}}_{S_n} \otimes \chi^{\text{trivial}}_{S_k})\big((n+i, j_1) \cdots (n+i, j_\alpha) \downarrow^{S_{n+k}}_{S_n \times S_k}\big)$$

are equal. This shows that we can group together summands of (37) according to the corresponding partitions Σ and Π.

The contribution to (36) of any summand corresponding to given set-partitions Π and Σ is equal to zero if $(n+i, j_1) \cdots (n+i, j_\alpha)$ restricted to S_n is not equal to the identity for any representative $i, j_1, \dots, j_\alpha$. Otherwise, the total contribution of all such summands is equal to

$$\frac{1}{k} n^{-\alpha/2} (n)_{|\Sigma|} \bigg(\sum_{1 \le i \le k} (i-1)_{|\Pi|} \bigg) = O\bigg(n^{(2|\Sigma|+|\Pi|-\alpha)/2} \bigg(\frac{k}{\sqrt{n}} \bigg)^{|\Pi|} \bigg), \tag{38}$$

where

$$(m)_\ell = \underbrace{m(m-1)\cdots(m-\ell+1)}_{\ell \text{ factors}}$$

denotes the falling factorial.

Assume that the partition Σ has a singleton $\{\ell\}$. Then it is easy to check that the element $j_\ell \in \{1, \dots, n\}$ is not a fixed point of the product $(n+i, j_1) \cdots (n+i, j_\alpha)$, hence the contribution of such partitions Σ is equal to zero. This means that we can assume that every block of Σ has at least two elements. It follows that $2|\Sigma| + |\Pi| \le |Z_\Sigma| + |Z_\Pi| = \alpha$. On the other hand, from the assumptions it follows that $\frac{k}{\sqrt{n}} = o(1)$. There are the following three (not disjoint) cases.

- Suppose that $2|\Sigma| + |\Pi| \le \alpha$ and $|\Pi| \ge 1$. Then (38) is equal to

$$O\bigg(\frac{k}{\sqrt{n}} \bigg).$$

- Suppose that $2|\Sigma| + |\Pi| \le \alpha - 1$ and $|\Pi| \ge 0$. Then (38) is equal to

$$O\bigg(\frac{1}{n} \bigg).$$

- Suppose that $2|\Sigma| + |\Pi| = \alpha$ and $|\Pi| = 0$; in other words, all blocks of Σ have exactly two elements and the partition Π is empty. Then the left-hand side of (38) is equal to

$$1 + O\bigg(\frac{1}{n} \bigg);$$

  this is the only case when the limit of (38) is nonzero.

The above discussion shows that

$$\mathbb{E}M_\alpha = \mathrm{Const}_\alpha + O\left(\frac{1}{n} + \frac{k}{\sqrt{n}}\right) = \mathrm{Const}_\alpha + O\left(\frac{k}{\sqrt{n}}\right),$$

where Const_α is some constant which depends only on α. In this way, we showed that the limit $\lim \mathbb{E}M_\alpha = \mathrm{Const}_\alpha$ of (36) is the same as in the simpler case $k = 1$, related to a single Jucys–Murphy element. This case was computed explicitly by Biane (1995), who showed that

$$\lim_{n\to\infty} n^{-\alpha/2} \chi^{\mathrm{regular}}_{S_{n+1}}(X^\alpha_{n+1}) = \gamma_\alpha,$$

which implies that $\mathrm{Const}_\alpha = \gamma_\alpha$ and proves (34).

4.9.2. *The second moment of M_α.* We now calculate the second moment of the random variable M_α. We have that

$$\mathbb{E}M_\alpha^2 = \frac{1}{k^2} n^{-\alpha} (\chi^{\mathrm{regular}}_{S_n} \otimes \chi^{\mathrm{trivial}}_{S_k}) \left(\left(\sum_{1\le i_1\le k} X^\alpha_{n+i_1} \cdot \sum_{1\le i_2\le k} X^\alpha_{n+i_2} \right) \downarrow^{S_{n+k}}_{S_n\times S_k} \right), \tag{39}$$

so, similarly as in Section 4.9.1, the problem is reduced to studying the element

$$\begin{aligned}\left(\sum_{1\le i\le k} X^\alpha_{n+i} \right)^2 &= \sum_{1\le i_1,i_2\le k} \; \sum_{1\le j_1,\dots,j_\alpha\le n+i_1-1} \; \sum_{1\le j_{\alpha+1},\dots,j_{2\alpha}\le n+i_2-1} (n+i_1, j_1)\cdots \\ &\qquad \times (n+i_1, j_\alpha) \\ &\qquad \times (n+i_2, j_{\alpha+1})\cdots \\ &\qquad \times (n+i_2, j_{2\alpha}) \in \mathbb{C}(S_{n+k}).\end{aligned}$$

In an analogous way, we define sets $Z_\Sigma, Z_\Pi \subseteq \{1,\dots,2\alpha\}$ and the corresponding partitions Σ and Π. In this case, however, the analysis is more difficult, which comes from the fact that it is possible that $j_\ell = n + i_q$ for some values of ℓ and q. If this happens, then we say that the block of Π which contains ℓ is *special*. We can again group summands according to the corresponding set-partitions Σ, Π (and the information about which of the blocks of Π is special, if any). The detailed analysis follows. Just as before, one can assume that every block of Σ contains at least two elements.

Case 1: $i_1 = i_2$. The total contribution of the summands of this form is just equal to $\frac{1}{k}\mathbb{E}M_{2\alpha}$. We already calculated the asymptotic behavior of such expressions; it is equal to $\frac{1}{k}\gamma_{2\alpha} + O(\frac{1}{\sqrt{n}})$.

Case 2: $i_1 < i_2$. Here, we divide into two subcases.

Case 2A: there exists a special block, that is, $j_\ell = n + i_1$ for some index ℓ. If the contribution is nonzero, then it is nonnegative and bounded from above by

$$\frac{1}{k^2} n^{-\alpha} (n)_{|\Sigma|} \left(\sum_{1 \le i_1 < i_2 \le k} (i_2 - 1)_{|\Pi|-1} \right)$$
$$= O\left(\frac{1}{k} n^{(2|\Sigma|+|\Pi|-2\alpha)/2} \left(\frac{k}{\sqrt{n}} \right)^{|\Pi|} \right)$$
$$= O\left(\frac{1}{k} \right).$$

Case 2B: there is no special block, that is, $j_1, \dots, j_{2\alpha}$ are all different from i_1 and i_2. In this case, we divide into two further subcases.

Case 2B(i): Π is not empty. If the contribution is nonzero, then it is nonnegative and bounded from above by

$$\frac{1}{k^2} n^{-\alpha} (n)_{|\Sigma|} \left(\sum_{1 \le i_1 < i_2 \le k} (i_2 - 1)_{|\Pi|} \right)$$
$$= O\left(n^{(2|\Sigma|+|\Pi|-2\alpha)/2} \left(\frac{k}{\sqrt{n}} \right)^{|\Pi|} \right)$$
$$= O\left(\frac{k}{\sqrt{n}} \right).$$

Case 2B(ii): Π is empty. In this case, the contribution of all such summands to (39) does not depend on i_1 and i_2 and can be written as

$$\frac{\binom{k}{2}}{k^2} n^{-\alpha} \chi_{S_n}^{\text{regular}} \left[\left(X_{n+1}^{\alpha} \downarrow_{S_n}^{S_{n+1}} \right)^2 \right]. \tag{40}$$

From the proof of equation (5.1.2) in Biane (1998), it follows that

$$\lim_{n \to \infty} \chi_{S_n}^{\text{regular}} \left[\left(\frac{1}{n^{\alpha/2}} X_{n+1}^{\alpha} \downarrow_{S_n}^{S_{n+1}} \right)^2 \right]$$
$$= \lim_{n \to \infty} \left[\chi_{S_n}^{\text{regular}} \left(\frac{1}{n^{\alpha/2}} X_{n+1}^{\alpha} \downarrow_{S_n}^{S_{n+1}} \right)^2 \right] = (\gamma_\alpha)^2,$$

and, therefore,

$$\chi_{S_n}^{\text{regular}} \left[\left(\frac{1}{n^{\alpha/2}} X_{n+1}^{\alpha} \downarrow_{S_n}^{S_{n+1}} \right)^2 \right] = (\gamma_\alpha)^2 + O\left(\frac{1}{n} \right).$$

It follows that (40) is equal to

$$\frac{1}{2} (\gamma_\alpha)^2 + O\left(\frac{1}{k} + \frac{1}{n} \right).$$

Case 3: $i_1 > i_2$. This case is analogous to Case 2 above.

To summarize, we have shown that

$$\mathbb{E}M_\alpha^2 = (\gamma_\alpha)^2 + O\left(\frac{1}{k} + \frac{k}{\sqrt{n}}\right).$$

Since $\operatorname{Var} M_\alpha = \mathbb{E}M_\alpha^2 - (\mathbb{E}M_\alpha)^2$, combining this with (34) we get (35), which finishes the proof of Theorem 4.6.

4.10. *Proof of Theorem* 4.1. By Theorem 4.6, we get using Chebyshev's inequality that for any $\varepsilon > 0$ and any $\alpha \in \mathbb{N}$,

$$\operatorname{Prob}\big(|M_\alpha - \gamma_\alpha| > \varepsilon\big) = O\left(\frac{1}{k} + \frac{k}{\sqrt{n}}\right). \tag{41}$$

Furthermore, for each $\varepsilon > 0$ and $u \in \mathbb{R}$ there exists a $\delta > 0$ and an integer $A > 0$ with the property that if m is a probability measure on $\mathbb{R}$ such that its moments (up to order A) are δ-close to the moments of $\mathcal{L}_{\mathrm{SC}}$ then $|F_m(u) - F_{\mathrm{SC}}(u)| < \varepsilon$. If this were not the case, then there would exist a sequence of measures which converges in moments to $\mathcal{L}_{\mathrm{SC}}$ but does not converge weakly to $\mathcal{L}_{\mathrm{SC}}$, which is not possible, since $\mathcal{L}_{\mathrm{SC}}$ is compactly supported and therefore uniquely determined by its moments [Durrett (2010), Section 3.3.5]. So, we get from (41) that for any $u \in \mathbb{R}$,

$$\begin{aligned}\operatorname{Prob}\big(\big|F_{m_{n,k}}(u) - F_{\mathrm{SC}}(u)\big| > \varepsilon\big) &\le \sum_{\alpha=1}^{A} \operatorname{Prob}\big(|M_\alpha - \gamma_\alpha| > \delta\big) \\ &= O\left(\frac{1}{k} + \frac{k}{\sqrt{n}}\right),\end{aligned}$$

which proves the claim.

5. The asymptotic determinism of RSK and jeu de taquin.

5.1. *The asymptotic determinism of* RSK. A key fact which we will need in our proof of Theorems 1.1, 1.4 and 1.5, and which is also of interest by itself, is the following: when applying an RSK insertion step with a fixed input $z \in [0, 1]$ to an existing insertion tableau P_n which is the result of n previous insertion steps involving random inputs which are drawn independently from the uniform distribution $U(0, 1)$, the macroscopic position of the new box that is added to the RSK shape depends asymptotically only on the number z being inserted. We refer to this phenomenon as the *asymptotic determinism of* RSK *insertion*. Its precise formulation is given in the following theorem, whose proof will be our first goal in this section.

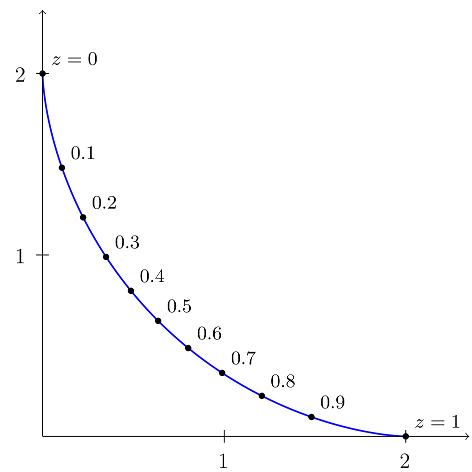

FIG. 9. *The asymptotic position of the new box after RSK insertion as a function of the new input z.*

THEOREM 5.1 (Asymptotic determinism of RSK insertion). *Let*

$$F_{\mathrm{SC}}(t) = F_{\mathcal{L}_{\mathrm{SC}}}(t) = \frac{1}{2} + \frac{1}{\pi}\left(\frac{t\sqrt{4-t^2}}{4} + \sin^{-1}\left(\frac{t}{2}\right)\right) \qquad (-2 \le t \le 2),$$

denote as before the cumulative distribution function of the semicircle distribution $\mathcal{L}_{\mathrm{SC}}$. Fix $z \in [0, 1]$. For each $n \ge 1$, let (Λ_n, P_n, Q_n) be the (random) output of the RSK *algorithm applied to a sequence $X_1, \dots, X_n$ of i.i.d. random variables with distribution $U(0, 1)$, and let*

$$\square_n(z) = (i_n, j_n) = \mathrm{Ins}(X_1, X_2, \dots, X_n, z)$$

denote the random position of the new box added to the shape Λ_n upon applying a further insertion step with the number z as the input. Then we have the convergence in probability

$$n^{-1/2}(i_n - j_n, i_n + j_n) \xrightarrow{\mathrm{P}} \big(u(z), v(z)\big) \qquad \text{as } n \to \infty,$$

where $u(z) = F_{\mathrm{SC}}^{-1}(z)$ and $v(z) = \Omega_*(u(z))$. Moreover, for any $\varepsilon > 0$,

$$\text{(42)} \qquad \mathrm{Prob}\big[\big\|n^{-1/2}(i_n - j_n, i_n + j_n) - \big(u(z), v(z)\big)\big\| > \varepsilon\big] = O\big(n^{-1/4}\big).$$

The asymptotic (rescaled) position of the new box as a function of z is illustrated in Figure 9.

PROOF OF THEOREM 5.1. We consider first the case $z \in \{0, 1\}$: for $z = 0$ the box will be added in the first column and for $z = 1$ the box will be added in the first row, and the question becomes equivalent to the standard problem of finding the asymptotics of the length of the first row and the first column of a Plancherel-distributed random Young diagram, or equivalently of the length of a longest increasing subsequence in a random permutation. The large deviations results in the papers of Deuschel and Zeitouni (1999) and Seppäläinen (1998) immediately imply our claim in that case.

Next, fix $z \in (0, 1)$ and $\varepsilon > 0$. Denote $u = F_{\mathrm{SC}}^{-1}(z) \in (-2, 2)$ and $u' = u + \varepsilon/4 = F_{\mathrm{SC}}^{-1}(z) + \varepsilon/4$. The cumulative distribution function F_{SC} is strictly increasing on $[-2, 2]$; it follows that $F_{\mathrm{SC}}(u') > F_{\mathrm{SC}}(u) = z$. Choose some $\Delta > 0$ in such a way that $z + \Delta < F_{\mathrm{SC}}(u') \le 1$.

Set $k = k(n) = \lceil n^{1/4} \rceil$. Let $\mathbf{X} = (X_1, \dots, X_n) \in [0, 1]^n$ be a sequence of i.i.d. $U(0, 1)$ random variables, and let $\mathbf{Y} = (Y_1, \dots, Y_k) \in [0, 1]^k$ be a sequence of i.i.d. $U(0, 1)$ random variables conditioned to be in increasing order. The RSK shapes Λ_n and Γ_{n+k} associated with the sequences $\mathbf{X}$ and $\mathbf{XY}$, respectively, are a Pieri growth pair as defined in Section 4.1.

Denote $r = \lfloor k(z + \Delta) \rfloor \le k$. Note that, by interpreting the random variables $Y_1, \dots, Y_k$ as the order statistics of k i.i.d. $U(0, 1)$ random variables $Z_1, \dots, Z_k$, we have that

$$\begin{aligned}\operatorname{Prob}(Y_r < z) &= \operatorname{Prob}\bigl(\text{at least } r \text{ of the } Z_j\text{'s are } < z\bigr)\\ &= \sum_{j=r}^{k} \binom{k}{j} z^j (1 - z)^{k-j} = \operatorname{Prob}(S_{k,z} \ge r),\end{aligned}$$

where $S_{k,z}$ is a random variable with a binomial distribution $\operatorname{Binom}(k, z)$. By standard large deviations estimates, we therefore have that for some constant $C > 0$,

$$\operatorname{Prob}(Y_r \le z) = O\bigl(e^{-Ck}\bigr) = O\bigl(e^{-Cn^{1/4}}\bigr) \qquad \text{as } n \to \infty.$$

Let $u_{n+1} \le \cdots \le u_{n+k}$ be the u-coordinates of the boxes of $\Gamma_{n+k} \setminus \Lambda_n$ written in the order in which they were inserted during the application of the RSK algorithm. Assume that the event $\{Y_r > z\}$ occurred; then parts (c) and (d) of Lemma 2.4 imply that

$$\square_n = \operatorname{Ins}(\mathbf{X}z) \preceq \operatorname{Ins}(\mathbf{X}Y_1 \cdots Y_r).$$

It follows that

$$\frac{1}{\sqrt{n}}(i_n - j_n) = \frac{1}{\sqrt{n}}(u\text{-coordinate of } \square_n) \le \frac{1}{\sqrt{n}} u_{n+r}.$$

Now apply Theorem 4.1 to get that

$$\begin{aligned}&\operatorname{Prob}\bigl(\bigl|F_{\mathrm{SC}}(u') - F_{m_{n,k}}(u')\bigr| > F_{\mathrm{SC}}(u') - (z + \Delta)\bigr)\\ &\quad = O\biggl(\frac{1}{k} + \frac{k}{\sqrt{n}}\biggr) = O\bigl(n^{-1/4}\bigr),\end{aligned}$$

where $m_{n,k}$ is the empirical measure of $u_{n+1}, \dots, u_{n+k}$. Outside of this exceptional event, we therefore have that

$$\frac{r}{k} \le z + \Delta \le F_{m_{n,k}}(u'),$$

which, because of the meaning of the empirical measure, implies that

$$\frac{1}{\sqrt{n}} u_{n+r} \le u' = u + \varepsilon/4.$$

To summarize, the above discussion shows that

$$\frac{1}{\sqrt{n}}(i_n - j_n) \le u + \varepsilon/4 \tag{43}$$

holds with probability $\ge 1 - O(n^{-1/4})$. In order to obtain an inequality in the other direction we define $(X'_1, \dots, X'_n) = (1 - X_1, \dots, 1 - X_n)$, and let

$$\Box'_n(z) = (i'_n, j'_n) = \operatorname{Ins}(X'_1, X'_2, \dots, X'_n, 1 - z).$$

Inequality (43) in this setup shows that

$$\frac{1}{\sqrt{n}}(i'_n - j'_n) \le F_{\mathrm{SC}}^{-1}(1 - z) + \varepsilon/4 \tag{44}$$

holds, except on an event with probability $O(n^{-1/4})$. But note that, first, by Lemma 2.5(b), $(i'_n, j'_n) = (j_n, i_n)$, and second, the semicircle distribution is symmetric, which implies that $F_{\mathrm{SC}}^{-1}(1 - z) = -F_{\mathrm{SC}}^{-1}(z)$. So, (44) translates to

$$\frac{1}{\sqrt{n}}(i_n - j_n) \ge u - \varepsilon/4. \tag{45}$$

Combining (43) and (45), we get that

$$\operatorname{Prob}\left[\left|\frac{1}{\sqrt{n}}(i_n - j_n) - u\right| > \varepsilon/4\right] = O(n^{-1/4}).$$

Since the function Ω_* is Lipschitz with constant 1, by appealing to Theorem 3.1 we also get that

$$\operatorname{Prob}\left[\left|\frac{1}{\sqrt{n}}(i_n + j_n) - \Omega_*(u)\right| > \varepsilon/2\right]$$
$$= O(e^{-c\sqrt{n}}) + O(n^{-1/4}) = O(n^{-1/4}).$$

These last two estimates together immediately imply (42). □

5.2. *Asymptotic determinism of jeu de taquin.* We now use the relationship between RSK insertion and jeu de taquin formulated in Lemma 2.3 to deduce from Theorem 5.1 an analogous statement that applies to jeu de taquin, namely the fact that prepending a fixed number $z \in [0, 1]$ to n i.i.d. $U(0, 1)$ random inputs $X_1, \dots, X_n$ causes the jeu de taquin path to exit the RSK shape at a position that is macroscopically deterministic in the limit. We call this property the *asymptotic determinism of jeu de taquin*, and prove it below. In the next section, we will deduce Theorems 1.1 and 1.5 from it.

THEOREM 5.2 (Asymptotic determinism of jeu de taquin). *Let $(X_n)_{n=1}^{\infty}$ be an i.i.d. sequence of random variables with the $U(0,1)$ distribution. Fix $z \in [0,1]$. Let $(\mathbf{q}_n(z))_{n=1}^{\infty}$ be the natural parameterization of the jeu de taquin path associated with the random infinite Young tableau*

$$\mathrm{RSK}(z, X_1, X_2, \ldots),$$

and for each $n \geq 1$ denote $\mathbf{q}_n(z) = (i_n, j_n)$. Then we have the almost sure convergence

$$n^{-1/2}(i_n - j_n, i_n + j_n) \xrightarrow{a.s.} \big(-u(z), v(z)\big) \qquad \text{as } n \to \infty, \tag{46}$$

where $u(z)$ and $v(z)$ are as in Theorem 5.1.

Note that in the setting of Theorem 5.2 it is possible to talk about almost sure convergence, since the random variables are defined on a single probability space.

PROOF OF THEOREM 5.2. For each $n \geq 1$, let $(\Pi_{n+1}, P_{n+1}, Q_{n+1})$ denote the output of the RSK algorithm applied to the input sequence $(z, X_1, \ldots, X_n)$, and let $(\Lambda_n, \widetilde{P}_n, \widetilde{Q}_n)$ denote the output of RSK applied to $(X_1, \ldots, X_n)$. Lemma 2.3 shows that

$$\widetilde{Q}_n = j(Q_{n+1}).$$

An equivalent way of saying this is that the box $\mathbf{q}_n$ is the difference of the RSK shapes Π_{n+1} and Λ_n.

On the other hand, let us see what happens when we reverse the sequences: by Lemma 2.5(a) the RSK shape associated to the sequence $(X_n, \ldots, X_1, z)$ is equal to $(\Pi_{n+1})^t$ and the RSK shape associated to $(X_n, \ldots, X_1)$ is equal to Λ_n^t. Therefore, the box $\mathbf{q}_n^t$ (the reflection of $\mathbf{q}_n$ along the principal diagonal) is the box added to the RSK shape of $X_n, \ldots, X_1$ upon application of a further RSK insertion step with the input z. This is exactly the scenario addressed in Theorem 5.1 [except that the order of $X_1, \ldots, X_n$ has been reversed, but that still gives a sequence of i.i.d. $U(0,1)$ random variables]. Substituting $\mathbf{q}_n^t$ for $\mathbf{d}_n$ in that theorem, we conclude that, for any $\varepsilon > 0$,

$$\mathrm{Prob}\big[\big\|n^{-1/2}(i_n - j_n, i_n + j_n) - \big(-u(z), v(z)\big)\big\| > \varepsilon\big] = O\big(n^{-1/4}\big).$$

This implies a weaker version of (46) with convergence *in probability*. To improve this to almost sure convergence, we will use the Borel–Cantelli lemma, but this requires passing to a subsequence first to get a convergent series. Setting $n_m = m^8$, we get that

$$\sum_{m=1}^{\infty} n_m^{-1/4} \leq \sum_{m=1}^{\infty} m^{-2} < \infty,$$

so from the Borel–Cantelli lemma we get that for any $\varepsilon > 0$, almost surely

$$\big\|n_m^{-1/2}(i_{n_m} - j_{n_m}, i_{n_m} + j_{n_m}) - \big(-u(z), v(z)\big)\big\| < \varepsilon$$

holds for all m large enough. This means that we have the almost sure convergence in (46) along the subsequence $n = n_m$. Finally, note that $n_{m+1}/n_m \to 1$ as $m \to \infty$. It is easy to see that this, together with the fact that the path $(\mathbf{q}_n)_n$ advances monotonically in both the x and y directions, guarantees (deterministically) that convergence along the subsequence implies convergence for the entire sequence. □

6. Proof of Theorems 1.1, 1.4 and 1.5.

PROOF OF THEOREM 1.1. Let $X_1, X_2, \dots$ be an i.i.d. sequence of $U(0,1)$ random variables, and let $(\mathbf{q}_n)_{n=1}^{\infty}$ be the natural parameterization of the jeu de taquin path of the (Plancherel-distributed) RSK image of the sequence, denoting as before $\mathbf{q}_n = (i_n, j_n)$. Conditioning on the value of X_1, the situation is exactly that of Theorem 5.2. By Fubini's theorem, the almost sure convergence in that theorem therefore implies that almost surely (even taking into account the randomness in X_1),

$$\lim_{n\to\infty} n^{-1/2}(i_n - j_n, i_n + j_n) = \big(-u(X_1), v(X_1)\big).$$

It follows in particular that the limit

$$\lim_{n\to\infty} \frac{\mathbf{q}_n}{\|\mathbf{q}_n\|} =: (\cos\Theta, \sin\Theta)$$

exists almost surely, where Θ is the random variable defined by

$$\cot(\pi/4 - \Theta) = \frac{v(X_1)}{-u(X_1)}$$

[as in the proof of Theorem 3.3, the $\pi/4$ comes from the rotation of the (u,v)-coordinate system relative to the standard one]. Since $(\mathbf{q}_n)_n$ is merely a slowed-down version of the original jeu de taquin path $(\mathbf{p}_k)_k$, that is, $\mathbf{q}_n = \mathbf{p}_{K(n)}$ where $K(n) \le n$ for all n and $K(n) \uparrow \infty$ almost surely as $n \to \infty$, it follows also that

$$\frac{\mathbf{p}_k}{\|\mathbf{p}_k\|} \stackrel{\text{a.s.}}{\to} (\cos\Theta, \sin\Theta) \qquad \text{as } k \to \infty.$$

It remains to verify that Θ has the distribution given in (8). This follows from Theorem 3.3, which already identifies the correct distributional limit. To argue a bit more directly, note that by the definition of $u(z)$, the random variable $u(X_1)$ [and hence also $-u(X_1)$] is distributed according to the semicircle distribution on $[-2,2]$, that is, it is equal in distribution to the random variable U from Theorem 3.2. Similarly, $v(X_1) = \Omega_*(-u(X_1))$ is equal in distribution to V from that theorem. So, $\Theta = \frac{\pi}{4} - \cot^{-1}(-v(X_1)/u(X_1))$ is equal in distribution to $\frac{\pi}{4} - \cot^{-1}(V/U)$. This is exactly the random variable whose distribution was shown in the proof of Theorem 3.3 to be given by (8). □

PROOF OF THEOREMS 1.4 AND 1.5. From the discussion in Section 2, we know that there is a measurable subset $A \in \mathcal{B}$ of $[0,1]^{\mathbb{N}}$ with $\mathrm{Leb}^{\otimes\mathbb{N}}(A) = 1$ and such that the map $\mathrm{RSK}\colon A \to \Omega$ is defined on A, and satisfies the homomorphism property (11). To define the inverse homomorphism, let $B \in \mathcal{F}$ be the set

$$B = \left\{ t \in \Omega : \lim_{k\to\infty} \frac{\mathbf{p}_k(t)}{\|\mathbf{p}_k(t)\|} \text{ exists} \right\}.$$

This is the event in Ω on which the random variable Θ from Theorem 1.1 is defined, and we proved that $\mathsf{P}(B) = 1$. Since J is a measure-preserving map, the event

$$C = \bigcap_{n=0}^{\infty} J^{-n}(B) = \{\Theta_n := \Theta \circ J^{n-1} \text{ exists for } n = 1, 2, \ldots\}$$

also satisfies $\mathsf{P}(C) = 1$. On this event, we define a map $\Delta\colon C \to [0,1]^{\mathbb{N}}$ by

$$\Delta(t) = \big(F_\Theta\big(\Theta_1(t)\big), F_\Theta\big(\Theta_2(t)\big), F_\Theta\big(\Theta_3(t)\big), \ldots\big).$$

Clearly, Δ is a measurable function since each of its coordinates is defined in terms of the measurable functions J and Θ on Ω. Now, take some sequence $\mathbf{x} = (x_1, x_2, \ldots) \in A \cap \mathrm{RSK}^{-1}(C)$, and denote $t = \mathrm{RSK}(\mathbf{x})$. Following the argument in the proof of Theorem 1.1 above, we see that $\Theta_1 = \Theta_1(t)$ is related to x_1 via

$$-\cot(\pi/4 - \Theta) = v(x_1)/u(x_1).$$

In particular, Θ is a strictly increasing function of x_1, so, since we also know that the measure $\mathrm{Leb}^{\otimes\mathbb{N}}$ induces the uniform distribution $U(0,1)$ on x_1 and the distribution (8) on Θ, it follows that this functional relation can be alternatively described in terms of the cumulative distribution function of Θ, namely

$$x_1 = F_\Theta(\Theta_1). \tag{47}$$

Now apply the same argument to $J(t)$. By the factor property, we get similarly that $x_2 = F_\Theta(\Theta \circ J(t)) = F_\Theta(\Theta_2(t))$. Continuing in this way, we get that $x_n = F_\Theta(\Theta_n(t))$ for all $n \geq 1$, in other words that

$$\Delta(t) = \mathbf{x},$$

which shows that Δ is inverse to RSK on the set $A \cap \mathrm{RSK}^{-1}(C)$. This completes the proof. □

7. Second class particles. In this section, we take another look at our results on jeu de taquin on infinite Young tableaux, this time from the perspective of the theory of interacting particle systems. As we mentioned briefly in the Introduction, it turns out that there is a very natural and elegant way to reinterpret the results on the jeu de taquin path of a random infinite Young tableau as statements on the behavior of a *second-class particle* in a certain interacting particle system associated with the Plancherel measure, which we call the *Plancherel-TASEP* particle system. This is not only interesting in its own right; it also draws attention to the remarkable similarity of our results to the parallel (and, so far, better-developed) theory of second-class particles in the TASEP.

7.1. *Rost's mapping*. Before introducing the all-important concept of the second-class particle, let us start by recalling a simpler mapping between growth sequences of Young diagrams (i.e., paths in the Young graph) and time-evolutions of a particle system, without the presence of a second-class particle. To our knowledge, this mapping was first described in the classical paper of Rost (1981). Here, the particles occupy a subset of the sites of a lattice—usually taken to be $\mathbb{Z}$, but for our purposes it will be more convenient to imagine the particles as residing in the spaces between the lattice positions, or equivalently on the sites of the shifted (or dual) lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. The mapping can be described as follows: given a Young diagram $\lambda \in \mathbb{Y}$, draw the profile ϕ_λ of λ in the Russian coordinate system, then project each segment of the graph of $\phi_\lambda(u)$ where u ranges over an interval of the form $[m, m+1]$ down to the u-axis. In the particle universe, a segment of slope -1 corresponds to the presence of a particle at the $\mathbb{Z}'$ lattice site $m + \frac{1}{2}$, and a segment of slope $+1$ translates to a vacant site (often referred to as a "hole"), at position $m + \frac{1}{2}$. A site containing a particle is said to be occupied.

It is now easy to see that the allowed transitions of the Young graph (adding a box to a Young diagram λ to get a new diagram ν) correspond to the following "exclusion dynamics" on the particle system: a particle in position $m + \frac{1}{2}$ may jump one step to the right to position $(m+1) + \frac{1}{2}$, and such a jump is only possible if site $(m+1) + \frac{1}{2}$ is currently vacant. This is illustrated in Figure 10. For consistency with the more general theory of exclusion processes, we refer to these transition rules as the *TASE* (*Totally Asymmetric Simple Exclusion*) *rules*.

Note also that the empty Young diagram $\varnothing$ corresponds to an initial state of the particle system wherein the positions $m + \frac{1}{2}$ are occupied for $m < 0$ and vacant for $m \geq 0$. Thus, the mapping we described translates statements about infinite paths on the Young graph starting from the empty diagram to statements about

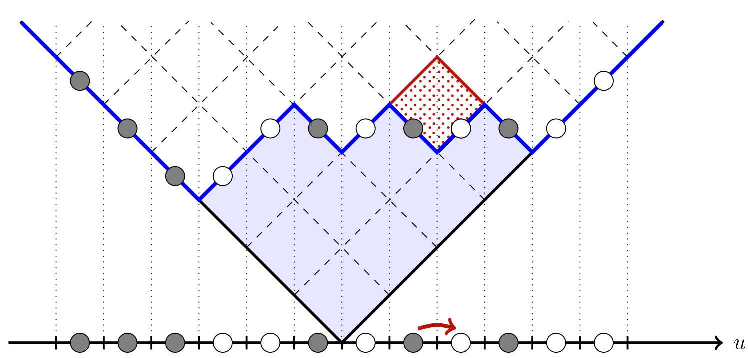

FIG. 10. *A sample configuration of particles on the shifted lattice* $\mathbb{Z}'$ *corresponding via Rost's mapping to the Young diagram* $(4, 3, 2)$. *Particles are depicted by filled circles, empty slots by empty circles. The addition of the dotted box would correspond to a jump of one of the particles one site to the right; such a jump can occur whenever the site to the right of a particle is vacant.*

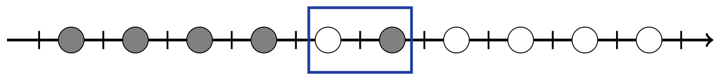

FIG. 11. *The initial state of an enhanced particle system.*

time-evolutions of the particle system starting from this initial state. Note that the mapping is purely combinatorial—we have not imposed any probabilistic structure yet.

7.2. *Enhanced particle systems.* Next, we describe how the structure of the particle system may be enhanced by the addition of a new kind of particle, referred to as a second-class particle, whose behavior is different from that of both ordinary particles and that of holes. Such a particle emerges from an extension of Rost's mapping defined above, described by Ferrari and Pimentel (2005). Consider an infinite path

$$\varnothing = \lambda_0 \nearrow \lambda_1 \nearrow \lambda_2 \nearrow \dots \tag{48}$$

on the Young graph starting from the empty diagram. From the empty diagram, the path always moves to the single-box diagram $\lambda_1 = (1)$. Note that at this point, in the corresponding particle world there is a single pair consisting of a hole lying directly to the left of a particle. Following the terminology of Ferrari and Pimentel, we call this pair the $*$-*pair*, and call the hole on the left side of the pair the $*$-*hole* and the particle on the right the $*$-*particle*. In a picture visualizing this system, we highlight the $*$-pair by drawing a rectangle around it; see Figure 11. We refer to a particle configuration with a $*$-pair as an *enhanced particle configuration*, and call the configuration corresponding to the diagram λ_1 the *initial state*.

Next, introduce dynamics to the enhanced particle system by noting that when the $*$-particle jumps to the right, it swaps with a hole. Thus, in such a transition a triplet of adjacent sites in a "hole–particle–hole" configuration (of which the left-most two sites represent the $*$-pair) becomes a "hole–hole–particle" triplet. Following such a transition, we designate the rightmost two sites of the triplet as the new $*$-pair. In other words, one can say that the $*$-pair has jumped one step to the right, trading places with the hole to its right.

Similarly, another possible transition involving a $*$-pair is when a "particle–hole–particle" triplet, of which the two rightmost positions form a $*$-pair, becomes a "hole–particle–particle" triplet due to the $*$-hole being jumped on by the particle to its left. In this case, following the transition we designate the leftmost two particles as the new $*$-pair, and say the $*$-pair jumped one step to the left. These rules are illustrated in Figure 12.

7.3. *Simplifying the $*$-pair.* We have described the strange-looking rules of evolution of a particle system enhanced by a so-called $*$-pair. One final simplification step will make everything much cleaner and more intuitive. Since a $*$-pair

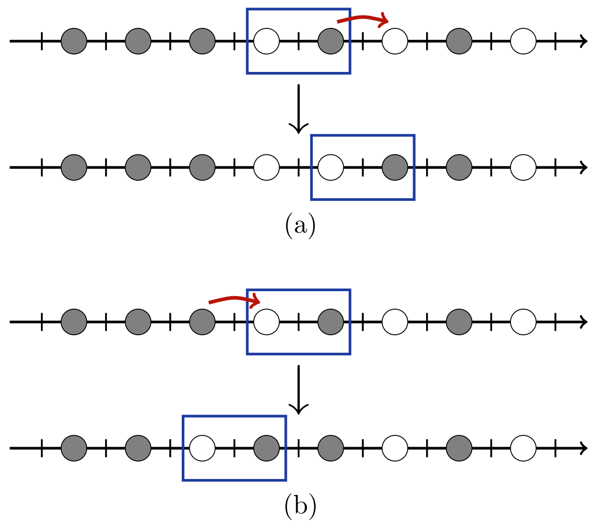

FIG. 12. *Transitions in an enhanced particle system. Transitions not involving the* $*$*-pair obey the usual TASE rules. The possible transitions involving a* $*$*-pair are*: (a) *when the rightmost particle in a* $*$*-pair jumps to the right, the* $*$*-pair also moves to the right*; (b) *when the leftmost hole in a* $*$*-pair is pushed to the left by a particle jumping on it, the* $*$*-pair moves left*.

always consists of a particle with a hole to its right, we may as well consider the pair as occupying a single lattice position, by contracting the two adjacent positions into one, thus effectively "shortening" the lattice by one unit. The result, illustrated in Figure 13, is that now there are three types of sites: those occupied by an "ordinary" particle, holes and a special site representing the $*$-pair, which is

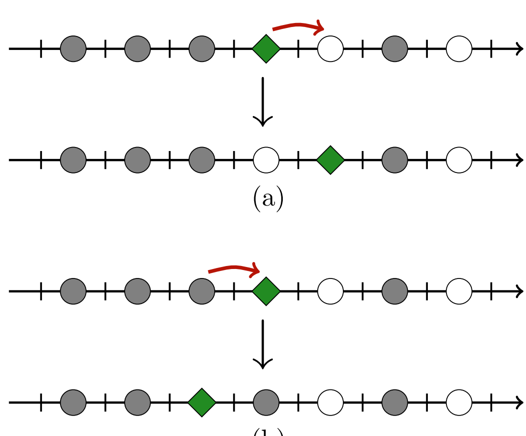

FIG. 13. *A particle system with a second-class particle* (*represented as a diamond*) *and the allowed transitions involving the second-class particle. Other transitions obey the TASE rules.*

of course the second-class particle. In this context, we refer to the ordinary particles as *first-class* particles. Now the transition rules become much more intuitive: a second-class particle (similarly to a first-class particle) can swap with a hole to its right but not with a first-class particle; and it can swap with a first-class particle to its left (which we think of whimsically as the first-class particle "pulling rank" to overtake it, pushing it back to an inferior position in the infinite line of particles—hence the "class" terminology), but not with a hole.

7.4. *The second-class particle and the jeu de taquin path.* Now comes a key observation, which can be understood implicitly from the discussion in Ferrari and Pimentel (2005) by an astute reader, but which (so far as we know) is made here explicitly for the first time. Take an infinite sequence (48) of growing Young diagrams, and assume that it has a recording tableau $t = (t_{i,j})_{i,j=1}^{\infty}$. Let $(\mathbf{q}_n)_{n=1}^{\infty}$ be the jeu de taquin path of the infinite Young tableau t given in the natural time parameterization as defined in Section 3.3, and denote by $(a_n, b_n) = \mathbf{q}_n$ the coordinates of $\mathbf{q}_n$.

PROPOSITION 7.1. *For each $n \geq 1$, let $u(n)$ denote the position at time n of the second-class particle in the particle system associated with the sequence* (48) *via the mapping described in the previous section, where we choose the origin of time and space such that its initial position is $u(0) = 0$. Let $v(n)$ denote the number of times the second-class particle moved up to time n. Then we have*

$$u(n) = a_{n+1} - b_{n+1}, \qquad v(n) = a_{n+1} + b_{n+1} \qquad (n \geq 0).$$

In words, the result says that if one considers the rotated (u, v) coordinate of the natural (also known as "lazy") parameterization of the jeu de taquin path, the sequence of u-coordinates gives the trajectory of the second-class particle, and the v-coordinates parameterize the number of jumps of the second-class particle. In particular, the sequence of u-coordinates of the ordinary (nonlazy) jeu de taquin path $(\mathbf{p}_k)_{k=1}^{\infty}$ can be interpreted as the positions of the second-class particle after its successive jumps, in a time parameterization in which all jumps not involving the second-class particles do not "move the clock."

PROOF OF PROPOSITION 7.1. Denote $u'(n) = a_{n+1} - b_{n+1}$ and $v'(n) = a_{n+1} + b_{n+1}$. We prove by induction on n that $u(n) = u'(n)$, $v(n) = v'(n)$. For $n = 0$, we have $(u'(0), v'(0)) = (u(0), v(0)) = (0, 0)$. For the induction step, it is helpful to go back to the enhanced particle system picture, and consider the position $u(n)$ of the second-class particle at time n to be the midpoint between the positions of the $*$-hole and $*$-particle [this is compatible with the choice of origin for which $u(0) = 0$, since in the initial state the $*$-hole and $*$-particle are

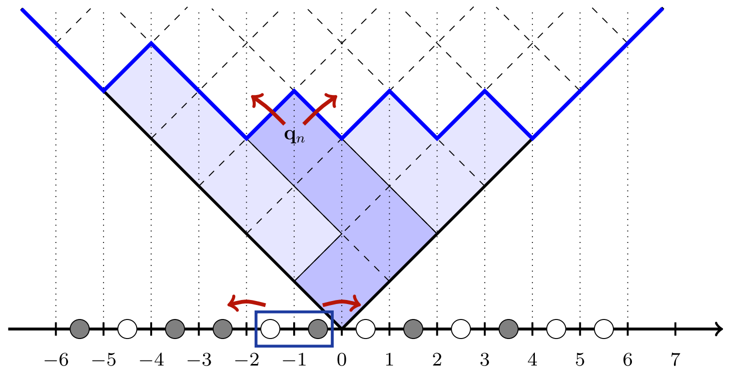

FIG. 14. *The allowed transitions of the $*$-pair in the enhanced particle system and the corresponding effect on the associated Young diagram. A move of the $*$-pair to the left or right corresponds to a north-west or north-east step, respectively, of the jeu de taquin path in Russian coordinates.*

at positions $\pm\frac{1}{2}$]. Now consider possible changes in the vectors $(u(n), v(n))$ and $(u'(n), v'(n))$ when we increment n by 1. For $(u(n), v(n))$, we have that

$$\begin{aligned}(49) \qquad &\big(u(n+1) - u(n), v(n+1) - v(n)\big) \\ &= \begin{cases} (-1, 1), & \text{if the } *\text{-pair moved left at time } n, \\ (1, 1), & \text{if the } *\text{-pair moved right at time } n, \\ (0, 0), & \text{otherwise.} \end{cases}\end{aligned}$$

For $(u'(n), v'(n))$, from the definition of the jeu de taquin path it is easy to see that

$$\begin{aligned}(50) \qquad &\big(u'(n+1) - u'(n), v'(n+1) - v'(n)\big) \\ &= \begin{cases} (-1, 1), & \text{if } \lambda_{n+2} \setminus \lambda_{n+1} = \{(a_{n+1}, b_{n+1} + 1)\}, \\ (1, 1), & \text{if } \lambda_{n+2} \setminus \lambda_{n+1} = \{(a_{n+1} + 1, b_{n+1})\}, \\ (0, 0), & \text{otherwise,} \end{cases}\end{aligned}$$

where λ_m denotes the mth Young diagram in the Young graph path associated with the particle system [recall that time 0 in the enhanced particle system corresponds to the diagram $\lambda_1 = (1)$, not $\lambda_0 = \varnothing$, which explains the discrepancy in the indices on both sides of the equation].

Finally, as Figure 14 illustrates, it is easy to see that each of the three cases in (49) is equivalent to the corresponding case in (50). Thus, we have that

$$\begin{aligned}&\big(u(n+1) - u(n), v(n+1) - v(n)\big) \\ &\qquad = \big(u'(n+1) - u'(n), v'(n+1) - v'(n)\big),\end{aligned}$$

which is just what was needed to complete the induction. □

7.5. *Stochastic models*. We are finally ready to consider *probabilistic* rules for the evolution of a particle system equipped with a second-class particle as described above. Thanks to the mapping taking a path on the Young graph to such a system, it is enough to specify the rules of evolution for a randomly growing family of Young diagrams.

7.5.1. *The Plancherel-TASEP.* Naturally, the first rule we consider is the particle system associated with the Plancherel growth process (or, equivalently, with the Plancherel measure). We call this the *Plancherel-TASEP particle system*; it is the process mentioned in Theorem 1.2 which we formulated in the Introduction without explaining its precise meaning. Finally, we are in a position to prove it.

PROOF OF THEOREM 1.2. By Proposition 7.1, the random variable $X(n)$ in the theorem is simply the u-coordinate $a_{n+1} - b_{n+1}$ of the natural parameterization $\mathbf{q}_n = (a_n, b_n)$ of the jeu de taquin path of a random infinite Young tableau chosen according to Plancherel measure. In the proof of Theorem 1.1 in Section 6, we already saw that after scaling by a factor of $n^{-1/2}$, this random variable converges a.s. to a limiting random variable W having the semicircle distribution. This was exactly the claim to prove. □

7.5.2. *The TASEP.* A second natural and much-studied process is the *Totally Asymmetric Simple Exclusion Processes*, or *TASEP*, introduced by Spitzer (1970) [this is a special case of the much wider family of *exclusion processes*, and we also consider here the TASEP itself with only a specific initial state. For the general theory of such processes, see Liggett (1985)]. Here, we consider the simple random walk on the Young graph starting from the empty diagram $\varnothing$. It is useful to let time flow continuously, so the random walk is a process $(\Pi_t)_{t\geq 0}$ taking values in the Young graph $\mathbb{Y}$, such that, given the state of the walk $\Pi_t = \lambda$ at time t, at subsequent times the walk randomly transitions to each state ν with $\lambda \nearrow \nu$ at an exponential rate of 1. Equivalently, each box in position (i, j) gets added to the randomly growing diagram with an exponential rate of 1, as soon as both the boxes in positions $(i-1, j)$ and $(i, j-1)$ are already included in the shape (where each of these conditions is considered to be satisfied if $i = 1$ or $j = 1$, resp.). This random walk is usually referred to as the *corner growth model*.

A fundamental result for the corner growth model is the following limit shape result, proved by Rost (1981), which is the analogue of Theorem 3.1 for this model.

THEOREM 7.2 (The limit shape of the corner growth model). *Let $A_t = A_{\Pi_t}$ be the planar region associated with the random diagram Π_t as in (17). Define*

$$L = \{(x, y) \in [0, \infty)^2 : \sqrt{x} + \sqrt{y} \leq 1\}.$$

Then for any $\varepsilon > 0$ we have that

$$\operatorname{Prob}[(1-\varepsilon)L \subseteq t^{-1}A_t \subseteq (1+\varepsilon)L] \to 1 \qquad \text{as } t \to \infty.$$

See Figure 15 for an illustration of this result.

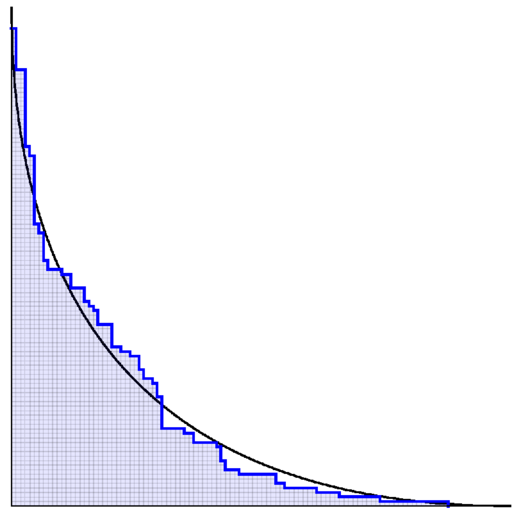

FIG. 15. *A rescaled random Young diagram in the corner growth model and its limit shape. The curved boundary of the limit shape is a rotated parabola, given by the equation* $\sqrt{x}+\sqrt{y}=1$ $(0\le x,y\le 1)$. *In Russian coordinates, it has the equation* $v=\frac{1}{2}(1+u^2)$, $|u|\le 1$.

Next, we can define the TASEP (without a second-class particle) as the continuous-time interacting particle system associated with the corner growth model via Rost's mapping described in Section 7.1. This particle system follows the combinatorial TASE rules described above for the valid particle transitions, but now in addition the probabilistic dynamics governing these transitions are very intuitive rules, namely that each particle can be thought of as having a Poisson "clock" (independent of all others) of times during which it attempts to jump to the right, succeeding if and only if the space to its right is vacant. In other words, in probabilistic language we will say that the resulting process is a Markov process with an infinitesimal generator that can be explicitly written and encapsulates this intuitive interpretation.

Finally, if we add the second-class particle by considering the "enhanced" version of Rost's mapping, we get a richer system following the TASE rules with the additional rules governing transitions involving the second-class particle. And again, the probabilistic laws governing these transitions can be described in the language of Markov processes, or equivalently in terms of each of the first- and second-class particles having a Poisson process of times during which it will attempt to jump.

The following result, proved by Mountford and Guiol (2005), is a precise analogue for the TASEP of Theorem 1.2, and puts our own result in an interesting context.

THEOREM 7.3. *For* $t\ge 0$, *let* $X(t)$ *denote the location at time* t *of the second-class particle in the TASEP with the initial conditions described above. As* $t\to\infty$, *the trajectory of the second-class particle converges almost surely to a straight line with a random speed. More precisely, the limit*

$$U = \lim_{t \to \infty} \frac{X(t)}{t}$$

exists almost surely and is a random variable distributed according to the uniform distribution $U(-1, 1)$.

A weaker version of Theorem 7.3 was proved earlier by Ferrari and Kipnis (1995). It is also worth noting here that the study of trajectories of second-class particles in the TASEP, and some of their higher-order generalizations (e.g., third-class, fourth-class particles, etc.) in the process known as the *multi-species TASEP*, is an active field that has brought to light very interesting results in the last few years; see the recent works by Amir, Angel and Valkó (2011), Angel, Holroyd and Romik (2009), Ferrari and Pimentel (2005), Ferrari, Gonçalves and Martin (2009). The paper by Angel et al. (2007) also studies particle trajectories in the *uniformly random sorting network*, which is an interacting particle system induced by a natural probability measure on Young tableaux that shares some characteristics with Plancherel measure (e.g., the semicircle distribution plays a special role in that context as well). Angel et al. make detailed conjectural predictions about the asymptotic behavior of particle trajectories in that model. It would be interesting to see if some of the techniques used in the current paper may be applicable to the study of these conjectures.

Second-class particles have also been studied recently in connection with *Hammersley's process*, an interacting particle system that is also related to the RSK algorithm and Ulam's problem on longest increasing subsequences. In this setting, a result on the trajectory of second-class particles analogous to Theorem 7.3 was proved by Coletti and Pimentel (2007); see also Cator and Groeneboom (2005, 2006), Cator and Dobrynin (2006) for related results, and Cator and Pimentel (2013) for a recent work considerably generalizing the results of Coletti and Pimentel.

As a final note on the analogy between Theorem 1.2 and Theorem 7.3, we remark that the time parameterization of the Plancherel-TASEP process is somewhat unnatural from the point of view of tracking the second-class particle, and this is what accounts for the scaling $n^{1/2}$ in Theorem 1.2, which causes the second-class particle to appear to slow down over time. As we mentioned briefly in the Introduction, one can argue that it makes more sense to replace the time parameter t in (10) by t^2, leading to particle system dynamics in which changes occur at a constant time scale in each microscopic region (including in the vicinity of the second-class particle). With such a parameterization, the intuitive meaning of Theorem 1.2 becomes more similar to that of Theorem 7.3, namely that the second-class particle moves asymptotically with a limiting speed, which is a random variable whose distribution can be computed [i.e., $U(-1, 1)$ in the case of the TASEP; $\mathcal{L}_{\mathrm{SC}}$ in the case of the Plancherel-TASEP].

7.6. *Competition interfaces in the corner growth model.* In the previous subsections, we reinterpreted the results on the jeu de taquin path of a Plancherel-random infinite Young tableau in terms of the second-class particle in the Plancherel-TASEP particle system. One can also go in the opposite direction, taking Theorem 7.3 above on the behavior of a second-class particle in the TASEP and reformulating it in the language of the corner growth model, or equivalently, infinite Young tableaux. Indeed, such a reformulation of Theorem 7.3 is the central idea in the paper by Ferrari and Pimentel (2005). While the authors of that work do not mention Young tableaux and apparently did not notice the connection to the jeu de taquin path, made explicit in Theorem 7.1 above, they phrased the result in terms of what they call the *competition interface*, which is the boundary separating two competing growth regions in the corner growth model. It is worth recalling this concept, which is interesting in its own right, and noting how it interacts with our point of view.

The idea is as follows. Thinking of the diagram Π_t as a collection of boxes (each represented as a position in $\mathbb{N}^2$), we decompose it into the box $(1,1)$ (assuming t is large enough so that $\Pi_t \neq \varnothing$) together with a union of boxes of two colors

$$\Pi_t = \{(1,1)\} \cup \Pi_t^{\text{green}} \cup \Pi_t^{\text{red}},$$

so that the planar region A_t associated to Π_t is also decomposed into a union of the regions

$$\begin{aligned} A_t &= ([0,1]\times[0,1]) \cup \Bigg(\bigcup_{(i,j)\in\Pi_t^{\text{green}}} [i-1,i]\times[j-1,j] \Bigg) \\ &\quad \cup \Bigg(\bigcup_{(i,j)\in\Pi_t^{\text{red}}} [i-1,i]\times[j-1,j] \Bigg) \\ &=: [0,1]^2 \cup A_t^{\text{green}} \cup A_t^{\text{red}}. \end{aligned}$$

The color of a box $(i,j)\in\Pi_t$ is determined as follows: when the box is added to the randomly growing Young diagram, it is classified as green if $i=1$, red if $j=1$ [except the box $(i,j)=(1,1)$ which has no color]; or, if $i,j\geq 2$ it gets the color of that box among the two boxes $(i,j-1)$, $(i-1,j)$ which was added to the Young diagram at the *later* time. One can think of two competing infections propagating through the first quadrant of the plane, where a box (i,j) becomes infected at an exponential rate 1 after the boxes below it and to its left are already infected; once it is infected the type of the infection (green or red) is decided according to which of the two "infecting" boxes has been infected more recently than the other.

The *competition interface* is defined as the boundary line separating the green and red regions A^t_{green} and A^t_{red}; see Figure 16. As t increases, this line grows by adding straight line segments in the directions $(1,0)$ and $(0,1)$. In fact, the nature of this line is made clear in the following result.

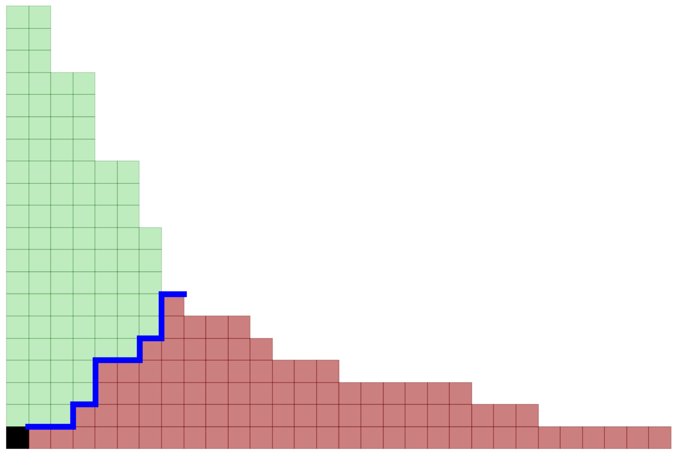

FIG. 16. *Red and green infection regions in the corner growth model and the competition interface.*

PROPOSITION 7.4. *Let* $\emptyset = \nu_0 \nearrow \nu_1 \nearrow \nu_2 \nearrow \dots$ *denote the sequence of Young diagrams that the corner growth model* $(\Pi_t)_{t\geq 0}$ *passes through. The competition interface is the polygonal line connecting the sequence of vertices*

$$(1, 1) = \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \dots$$

given by the jeu de taquin path box positions $(\mathbf{p}_k)_{k=1}^{\infty}$ *associated with the Young graph path* $(\nu_n)_{n=0}^{\infty}$.

PROOF. If at time t the top-right endpoint of the competition interface is in position (a_t, b_t), that means that the Young diagram box indexed by $\mathbb{N}^2$-coordinates (a_t, b_t) is in Π_t but the boxes indexed by $(a_t + 1, b_t)$ and $(a_t, b_t + 1)$ are not in Π_t (see Figure 16 for an example). Assume by induction on $k = a_t + b_t - 1$ that $\mathbf{p}_k = (a_t, b_t)$. The next step $\mathbf{p}_{k+1} - \mathbf{p}_k$ taken by the jeu de taquin path will be $(1, 0)$ or $(0, 1)$ depending on which of the two boxes $(a_t + 1, b_t)$ or $(a_t, b_t + 1)$ will be added to Π_t next; it is easy to see from the definition of the competition interface that its next step will be determined in exactly the same way. $\square$

The analogue of our Theorem 1.1 for the corner growth model is the following result, which is Ferrari and Pimentel's reformulation of Theorem 7.3 in the language of competition interfaces (which, as we observe above, is equivalent to jeu de taquin).

THEOREM 7.5 (Asymptotic behavior of the competition interface). *The competition interface in the corner growth model converges to a straight line with a random direction. More precisely, the limit*

$$(\cos \Phi, \sin \Phi) = \lim_{k\to\infty} \frac{\mathbf{p}_k}{\|\mathbf{p}_k\|}$$

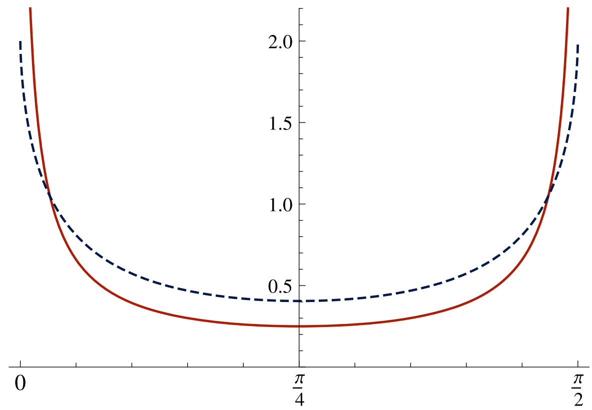

FIG. 17. *A comparison of the density functions of* Θ, *the asymptotic angle of the jeu de taquin path in a Plancherel-random infinite Young tableau* (*dashed*, *dark blue line*) *and of* Φ (*full stroke line*, *in red*), *the asymptotic angle of the competition interface in the corner growth model*, *which can also be interpreted as a jeu de taquin path. The density of* Φ *is unbounded near* 0 *and* $\pi/2$.

exists almost surely. The asymptotic angle Φ *of the competition interface is an absolutely continuous random variable*, *with distribution*

$$\operatorname{Prob}(\Phi \le x) = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} \qquad (0 \le x \le \pi/2).$$

Figure 17 shows a comparison of the density function of Φ with that of Θ, the asymptotic angle of the jeu de taquin path of a Plancherel-random infinite Young tableau.

7.7. *Summary.* In the discussion above, we showed that the jeu de taquin path arises naturally in probabilistic settings which have not been noticed so far and which go beyond its traditional applications to algebraic combinatorics, namely the study of trajectories of second-class particle in interacting particle systems and of the competition interface between two randomly growing regions in the corner growth model. We hope that the reader is convinced that the interplay between the different interpretations and points of view is quite stimulating, and worthy of further study.

8. Additional directions.

8.1. *Asymptotic determinism of RSK and the limit shape of the bumping routes.* In a follow-up paper [Romik and Śniady (2013)], we apply Theorem 5.1 to prove an additional "asymptotic determinism" property with more detailed information on the behavior of RSK insertion in the random setting considered in this paper; namely, we show that the "bumping route" when a deterministic input z is inserted into the insertion tableau P_n (in the notation of Theorem 5.1) converges in the macroscopic scaling to a limiting shape that depends only on z and is given by an explicit formula.

8.2. RSK *and random words in other alphabets*. It is natural to study the properties of RSK applied to an infinite sequence $X_1, X_2, \dots$ of i.i.d. random letters in a more general setup than the one considered in the current paper, that is, with the distribution of the letters being arbitrary. The simplest example is the one in which $X_1, X_2, \dots$ take values in a finite set $[d] = \{1, \dots, d\}$. In this case the random words and the corresponding recording tableaux can be viewed as random walks in $\mathbb{Z}^d$. O'Connell (2003) has shown that, under this identification, RSK coincides with the *generalized Pitman transform* introduced by O'Connell and Yor (2002). The counterparts of some of the results of the current paper have has been proved for the Pitman transform. This topic is studied in a broader context, which also reveals interesting connections with the representation theory of the infinite symmetric group, in another follow-up paper by the second-named author [Śniady (2014)].

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