Abstract

In two previous papers, the study of partitions with short sequences has been developed both for its intrinsic interest and for a variety of applications. The object of this paper is to extend that study in various ways. First, the relationship of partitions with no consecutive integers to a theorem of MacMahon and mock theta functions is explored independently. Secondly, we derive in a succinct manner a relevant definite integral related to the asymptotic enumeration of partitions with short sequences. Finally, we provide the generating function for partitions with no sequences of length $K$ and part exceeding $N$.

1 Introduction

In his classic two volume work, Combinatory Analysis [5], P.A. MacMahon devotes Chapter IV of Volume 2 to “Partitions Without Sequences”. His object in this chapter is to make a thorough study of partitions in which no consecutive integers (i.e. integers that differ by 1) occur. He concludes this chapter with what we will call MacMahon’s Theorem.

**Theorem 1.1.** The number of partitions of an integer $N$ into parts $\neq \pm 1 \pmod{6}$ equals the number of partitions of $N$ with no consecutive integers as summands and no ones.

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For example, for \( n = 10 \), the first set of partitions is 10, 8 + 2, 6 + 4, 6 + 2 + 2, 4 + 4 + 2, 4 + 3 + 3, 4 + 2 + 2 + 2, 3 + 3 + 2 + 2, 2 + 2 + 2 + 2 + 2;\) the second set is 10, 8 + 2, 7 + 3, 6 + 3, 6 + 4, 6 + 2 + 2, 5 + 5, 4 + 4 + 2, 4 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 2. The fact that each set of partitions has the same number of elements (in this case 9), is MacMahon’s assertion.

In two previous papers [2, 4], MacMahon’s ideas have been generalized to the consideration of partitions in which sequences of consecutive integers have been restricted to contain fewer than \( k \) terms (MacMahon only dealt with \( k = 2 \)).

In Section 2 of this paper we shall explore in detail various aspects of MacMahon’s work in [5; Vol. II, Ch. IV]. In Section 3 we discuss the generalization to partitions without \( k \) consecutive parts: First, we obtain a new and simplified proof of the Holroyd-Liggett-Romik definite integral that was used in [4] to obtain results on the asymptotic enumeration of these classes of partitions. Secondly, we strengthen the results of [2] by obtaining a double series representation of the generating function for partitions in which each part is \( \leq N \) and sequences of consecutive integers have length less than \( k \). Finally, Section 4 contains some remarks on a probabilistic interpretation of the mock theta function \( \chi(q) \) studied by Ramanujan.

# 2 Investigation of MacMahon’s theorem

We begin with some definitions.

**Definition 2.1.** Let

\[
g_n = \text{the number of partitions of } n \text{ with no two consecutive parts},
\]

\[
h_n = \text{the number of partitions of } n \text{ with no two consecutive parts and no 1’s},
\]

\[
G_2(q) = \sum_{n=0}^{\infty} g_n q^n, \quad (2.1)
\]

\[
H_2(q) = \sum_{n=0}^{\infty} h_n q^n, \quad (2.2)
\]

\[
\chi(q) = \sum_{n=0}^{\infty} \prod_{j=1}^{n} \frac{q^{2j}}{(1 - q^j + q^{2j})}, \quad (2.3)
\]
where $\chi(q)$ is one of the third-order mock theta function studied by Ramanujan [6]; p. 354.

2.1 A bijective proof of Theorem 1.1

Proof. By passing to the conjugate partition, the number of partitions of $n$ with no 1’s and no two consecutive parts is clearly seen to be equal to the number of partitions of $n$ not containing any part exactly once. Here is a bijection between the set $C_n$ of partitions of $n$ not containing any part exactly once, and the set $B_n$ of partitions of $n$ into parts congruent to 0, 2, 3, 4 mod 6: If $n = \sum_{k=1}^{\infty} k r_k$ is a partition in $C_n$ ($r_k$ is the multiplicity of $k$, or the number of parts equal to $k$ in the partition), $r_k \in \{0, 2, 3, 4, \ldots \}$, then each $r_k$ can be written uniquely as $r_k = s_k + t_k$, where $s_k \in \{0, 3\}$ and $t_k \in \{0, 2, 4, 6, 8, \ldots \}$. Define a partition $n = \sum_{j=1}^{\infty} j b_j$ by

\[
\begin{align*}
    b_{6k+1} &= 0 \quad (k = 0, 1, 2, 3, \ldots) \\
    b_{6k+5} &= 0 \\
    b_{6k+2} &= \frac{1}{2} t_{3k+1} \\
    b_{6k+4} &= \frac{1}{2} t_{3k+2} \\
    b_{6k+3} &= \frac{1}{3} s_{2k+1} + t_{6k+3} \\
    b_{6k+6} &= \frac{1}{3} s_{2k+2} + t_{6k+6}
\end{align*}
\]

This partition is in $B_n$, and it is not difficult to check that any partition in $B_n$ is obtained in this way from a unique partition in $C_n$. \hfill \Box

2.2 A $q$-series for $G_2(q)$

We give a simplified proof of the following $q$-series representation for $G_2(q)$, which was stated in [2, eq. (4.2)]:

Theorem 2.2.

\[
G_2(q) = 1 + \sum_{n=1}^{\infty} q^n \prod_{j=1}^{n-1} \frac{(1 - q^j + q^{2j})}{\prod_{j=1}^{n} (1 - q^j)}. \tag{2.4}
\]
Proof. Again by passing to the conjugate partition, we see that $g_n$ is the number of partitions of $n$ where all the parts except possibly the largest part do not appear exactly once.

Write (2.4) as

$$G_2(q) = 1 + \sum_{n=1}^{\infty} \left[ \frac{q^n}{1-q^n} \prod_{j=1}^{n-1} \frac{1 - q^j + q^{2j}}{1-q^j} \right].$$

(2.6)

The coefficient of $q^N$ in the $n$-th summand on the right-hand side is equal to the number of partitions of $N$ with maximal part $n$, where no part except possibly the largest part appears exactly once. So the coefficient of $q^N$ in the entire sum on the right-hand side is exactly $g_N$. \qed

2.3 The MacMahon-Fine identity

In [2], it was shown that a combination of identities due to MacMahon [5]; Vol. II, p. 52, and Fine [3]; p. 57 show that

$$G_2(q) = H_2(q)\chi(q).$$

(2.5)

This identity can be given the following combinatorial interpretation:

Theorem 2.3. For each integer $n \geq 1$ and $0 \leq k \leq \sqrt{n}$, let $f_{n,k}$ be the number of partitions of $n - k^2$ in which no part which is greater than $k$ appears exactly once. Then for each $n \geq 1$,

$$g_n = \sum_{k=0}^{\lfloor \sqrt{n} \rfloor} f_{n,k}. \quad (2.6)$$

Proof. From the remark at the beginning of the proof of Theorem 2.2, we can write

$$H_2(q) = \prod_{j=1}^{\infty} (1 + q^{2j} + q^{3j} + q^{4j} + \ldots) = \prod_{n=1}^{\infty} \frac{1 - q^j + q^{2j}}{1-q^j}, \quad (2.7)$$
(this is an alternative way to prove Theorem 1.1). Now combining (2.5) and (2.7) and the definition of $\chi(q)$ gives

$$G_2(q) = \sum_{k=0}^{\infty} q^{k^2} \left( \prod_{j=1}^{k} \frac{1}{1-q^j} \right) \cdot \left( \prod_{j=k+1}^{\infty} \frac{1-q^j + q^{2j}}{1-q^j} \right).$$

The coefficients of $q^n$ in the left- and right-hand side of this equation are clearly the left and right-hand sides of (2.6), respectively.

A natural question is whether Theorem 2.3 has a simple combinatorial explanation.

3 Partitions without $k$ consecutive parts

3.1 The Holroyd-Liggett-Romik integral

In [4], the following result concerning the asymptotic enumeration of partitions without $k$ consecutive parts was proved:

**Theorem 3.1 (Holroyd, Liggett and Romik, [4]).** Let $p_k(n)$ denote the number of partitions of $n$ not containing $k$ consecutive parts. Then for each fixed $k > 1$, we have as $n \to \infty$

$$p_k(n) = e^{(1+o(1))c_k \sqrt{n}},$$

where

$$c_k = \pi \sqrt{\frac{2}{3} \left( 1 - \frac{2}{k(k+1)} \right)}.$$

The proof of this result relies on a special case of the following family of definite integrals, also proved in [4]: For every $0 < a < b$, a decreasing function $f_{a,b} : [0,1] \to [0,1]$ can be defined by $f_{a,b}(0) = 1, f_{a,b}(1) = 0$ and $f_{a,b}(x)^a - f_{a,b}(x)^b = x^a - x^b$ in between. In the simplest case $f_{1,2} - f_{1,2}^2 = x - x^2$, we have $f_{1,2}(x) = 1 - x$. Then we have:

**Theorem 3.2 (Holroyd, Liggett and Romik, [4]).**

$$\int_0^1 \frac{1 - \log f_{a,b}(x)}{x} dx = \frac{\pi^2}{3ab}.$$
We give here a new and shorter proof of this result. We remark that the proof given in [4], while considerably more complicated, seems to contain more interesting information, see [7].

**Proof.** The integral in the theorem can be interpreted as a double integral:

\[ I_{a,b} := \int_0^1 \frac{-\log f_{a,b}(x)}{x} \, dx = \int_0^1 \frac{dx}{x} \int_{f_{a,b}(x)}^1 \frac{dy}{y} = \int \int_D \frac{dx\,dy}{xy}, \]

where \( D \) is a symmetric domain bounded below by \( y^a - y^b = x^a - x^b \), above by \( y = 1 \), and to the right by \( x = 1 \). Bisect it along its symmetry axis \( y = x \) and substitute \( y = xt, dy = xdt \) to get

\[ I_{a,b} = 2 \int \int_{D'} \frac{dx\,dt}{xt}, \]

where \( D' \) is bounded below by \( x^{b-a} = (1 - t^a)/(1 - t^b) \), above by \( t = 1 \), and to the right by \( x = 1 \). Integrating \( x \) we get

\[ I_{a,b} = \frac{2}{b - a} \int_0^1 \log \left( \frac{1 - t^b}{1 - t^a} \right) \frac{dt}{t}. \]

Finally, if we split the logarithm in two and substitute \( x = t^b \) in the first integral and \( x = t^a \) in the second, the desired result is obtained.

\[ I_{a,b} = \frac{2}{b - a} \left( -\frac{1}{b} + \frac{1}{a} \right) \int_0^1 \log(1 - x) \frac{dx}{x} = \frac{\pi^2}{3ab}. \]

\[ \Box \]

### 3.2 The restricted generating function

We must now substantially extend the definitions that appear at the beginning of Section 2.

Let

\[ g_{m,n}(k,N) = \begin{cases} \text{the number of partitions of } n \text{ into } m \text{ parts in which} \\ \text{each part is } \leq N \text{ and there is no string of parts forming} \\ \text{a sequence of consecutive integers of length } k, \end{cases} \]

\[ G_k(N; x, q) = \sum_{m,n=0}^\infty g_{m,n}(k,N)x^m q^n. \]
We note in passing that with regard to the definitions in Section 2,

\[ g_n = \sum_{m \geq 0} g_{m,n}(2, \infty), \]

and

\[ G_2(q) = G_2(\infty; 1, q). \]

In [2; eq. [2.5]], it was proven that

\[ G_k(\infty; x, q) = \frac{1}{(xq; q)_{\infty}} \sum_{r,s \geq 0} \frac{(-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}}}{(q^k; q^k)_s(q^{k+1}; q^{k+1})_r}, \tag{3.1} \]

where

\[ (A; q)_t = (1 - A)(1 - Aq) \ldots (1 - Aq^{t-1}), \quad (A; q)_0 = 1. \]

Our object here is to prove the following result for \( G_k(N; x, q) \) which reduces to (3.1) when \( N \to \infty \).

**Theorem 3.3.**

\[ G_k(N; x, q) = \frac{1}{(xq; q)_N} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+1}{2}} \left[ N - kr - ks - r + 1 \right]_s \left[ N - kr - ks \right]_r, \tag{3.2} \]

where

\[ \left[ \begin{array}{c} A \\ B \end{array} \right]_t = \begin{cases} 0 & \text{if } B < 0 \text{ or } B > A, \\ \frac{(q^t; q^{t})_A}{(q^t; q^{t})_B(q^{t-t})_{A-B}} & \text{for } 0 \leq B \leq A. \end{cases} \]

**Proof.** We begin by noting that there is a defining recurrence for \( G_k(N; x, q) \). Namely,

\[ G_k(N; x, q) = \begin{cases} \frac{1}{(xq;q)_N} & \text{if } 0 \leq N < k, \\ G_k(N-1; x, q) + & \\ \sum_{i=1}^{k-1} \frac{x^i q^{N+(N-1)+\ldots+(N-i+1)} G_k(N-i-1; x, q)}{(1-xq^N)(1-xq^{N-1})\ldots(1-xq^{N-i+1})} & \end{cases} \tag{3.3} \]
This last assertion is easily verified as follows. If \( N < k \), then there can be no sequences of \( k \) consecutive integers among the parts. Hence for \( N < k \), all partitions with parts \( \leq N \) must be included and the generating function in this case is

\[
\frac{1}{(xq;q)_N}
\]
as asserted.

To establish the bottom line of (3.3), we note that among the partitions generated by \( G_k(N;x,q) \) there are some in which \( N \) does not appear as a part. These are generated by \( G_k(N-1;x,q) \). If \( N \) does appear as a part, it then lies in a sequence of consecutive integers of maximal length \( i \) where \( 1 \leq i < k \). The portion of such partitions containing only parts in \([N - i + 1, N]\) is generated by

\[
x^i q^{N+(N-1)+\cdots+(N-i+1)} (1 - xq^N)(1 - xq^{N-1}) \cdots (1 - xq^{N-i+1})
\]
and all other parts must be \( < N - i \), and consequently are generated by \( G_k(N - i - 1;x,q) \). Hence the right-hand side of (3.3) generates precisely those partitions generated by \( G_k(N;x,q) \) thus establishing the second line of (3.3).

We now define

\[
S(k;N) = (xq;q)_N G_k(N;x,q).
\]
Consequently \( S(k;N) \) is uniquely determined by the recurrence

\[
S(k,N) = \begin{cases} 
1, & \text{if } 0 \leq N < k, \\
\sum_{i=1}^{k-1} x^i q^{N+(N-1)+\cdots+(N-i+1)} (1 - xq^{N-i}) S(k,N-i-1). & \end{cases}
\]

We now define

\[
\sigma(k,N) = \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{(k+1)(r+s)^2+(k+1)(r+1)}
\]

\[
\times \left[ N - kr - ks - r + 1 \right]_k \left[ N - kr - ks \right]_k. \tag{3.6}
\]

We wish to show that \( S(k,N) = \sigma(k,N) \) in order to complete the proof of this theorem. To do this we need only show that \( \sigma(k,N) \) also satisfies the defining recurrence (3.5).
Immediately we see that if $N < k$, then the only non-vanishing term of the double sum in (3.6) occurs for $s = r = 0$. Hence

$$\sigma(k, N) = 1 \quad \text{if } 0 \leq N < k.$$  

We shall prove the following equivariant recurrence for $\sigma(k, N)$ when $n \geq k$:

$$\sum_{i=0}^{k-1} x^i q^{N_i - \binom{i}{2}} \times (\sigma(k, N - i) - \sigma(k, N - i - 1)) + x^k q^{kn - \binom{k}{2}} \sigma(k, N - k) = 0. \quad (3.7)$$
We now simplify the left-hand side of (3.7).

\[
\sum_{i=0}^{k-1} x^i q^{N_i - \binom{i}{2}} (\sigma(k, N - i) - \sigma(k, N - i - 1))
\]

\[
= \sum_{i=0}^{k-1} x^i q^{N_i - \binom{i}{2}} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+s+1}{2}}
\times \left\{ \begin{array}{c}
q^{(N-i-kr-ks-r+1-s)} \left[ N - kr - ks - r - i \right]_k \left[ N - kr - ks - i \right]_k \\
q^{(k+1)(N-i-kr-ks-r)} \left[ N - kr - ks - r - i \right]_k \left[ N - kr - ks - 1 - i \right]_k \\
\end{array} \right\}
\]

\[
= \sum_{i=0}^{k-1} x^i q^{N_i - \binom{i}{2}} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+s+1}{2}}
\times \left\{ \begin{array}{c}
q^{(N-k(r+s)-r+i+1-s-i)} \left[ N - kr - ks - r \right]_k \left[ N - kr - ks - i \right]_k \\
q^{(k+1)(N-k(r+s)-r+i)} \left[ N - kr - ks - r \right]_k \left[ N - kr - ks - i - 1 \right]_k \\
\end{array} \right\}
\]

(having replaced \(s\) by \(s+i\) and \(r\) by \(r-i\))

\[
= \sum_{i=0}^{k-1} q^{N_i - \binom{i}{2}} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+s+1}{2}}
\times \left\{ \begin{array}{c}
q^{(N-k(r+s)-r-s+1)} \left[ N - kr - ks - r \right]_k \left[ N - i - kr - ks \right]_k \\
+ \sum_{i=1}^{k} q^{N(i-1)-\binom{i-1}{2}} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2+(k+1)\binom{r+s+2}{2}}
\times \left\{ \begin{array}{c}
q^{(k+1)(N-kr-ks-r)} \left[ N - kr - ks - r \right]_k \left[ N - kr - ks \right]_k \\
\end{array} \right\}
\end{array} \right\}
\]

(having replaced \(i\) by \(i-1\) in the second sum.)

Now examination of the exponents on \(x\) and \(q\) reveals that each term in the second sum for \(1 \leq i \leq k-1\) is the negative of each term in the first sum. Hence all that remains after cancellation is the term \(i = 0\) in the first sum and the term \(i = k\) in the second.
Hence
\[
\sum_{i=0}^{k-1} x^i q^{N_i - (i)} (\sigma(k, N - i) - \sigma(k, N - i - 1))
= \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)(r+1) + k(N-(k+1)(r+s)+1)}
\times \left[ N - kr - ks - r \atop s - 1 \right]_k \left[ N - k - kr - ks \atop r \right]_{k+1}
\]
\[
+ q^{N(k-1)-(k-1)\binom{1}{2}} \sum_{r,s \geq 0} (-1)^{s+k-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)(r+1)}
\times q^{(k+1)(N-kr-ks-r)} \left[ N - k - kr - ks \atop s + k - 1 \right]_k \left[ N - k - kr - ks \atop r - k \right]_{k+1}
:= S_1 + S_2
\]  
(3.8)

Let us now define
\[
S_3 := x^k q^{N+(N-1)+\ldots+(N-k-1)} \sigma(k, N - k)
\]  
(3.9)
\[
= x^k q^{N-(N-k)} \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s)^2 + (k+1)(r+1)}
\times \left[ N - k - kr - ks - r + 1 \atop s \right]_k \left[ N - k - kr - ks \atop r \right]_{k+1}
\]
\[
= q^{N-(N-k)} \sum_{r,s \geq 0} (-1)^{s-1} x^{ks+(k+1)r} q^{\binom{k+1}{2}(r+s-1)^2 + (k+1)(r+1)}
\times \left[ N - k - kr - ks - r + 1 \atop s - 1 \right]_k \left[ N - k - kr - ks \atop r \right]_{k+1}
\]
(3.10)

(where we have replaced \(s\) by \(s - 1\)).

In order to complete the proof of the recurrence (3.7) for \(\sigma(k, n)\) we need only show that
\[
S_1 + S_2 = -S_3.
\]
Now

\[ S_1 + S_3 = \sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} \left[ \frac{N - kr - ks}{r} \right]_{k+1} \]

\[ \left\{ \frac{q^{(k+1)(r+s)^2+(k+1)(r^2)} + k(N-(k+1)(r+s)+1)}{s-1} \right\}_k \]

\[ -q^{kN-(\frac{k}{2})+(\frac{k+1}{2})(r+s-1)^2+(k+1)(r^2)} \left[ \frac{N - kr - ks - r + 1}{s-1} \right]_k \]

\[ = -\sum_{r,s \geq 0} (-1)^s x^{ks+(k+1)r} \left[ \frac{N - kr - ks}{r} \right]_{k+1} q^{kN-(\frac{k}{2})+(\frac{k+1}{2})(r+s-1)^2+(k+1)(r^2)} \]

\times q^{k(N-kr-ks-r-s+2)} \left[ \frac{N - kr - ks - r}{s-2} \right]_k \]

(by [1; eq. (3.3.3), p. 35])

\[ -\sum_{r,s \geq 0} (-1)^{s+k+1} x^{ks+(k+1)r} \left[ \frac{N - k - kr - ks}{r - k} \right]_{k+1} \]

\times q^{kN-(\frac{k}{2})+(\frac{k+1}{2})(r+s)^2+(k+1)(r^2)}

\times q^{k(N-k(r+s+1)-(r+s+1)+2)} \left[ \frac{N - kr - ks - r}{s + k - 1} \right]_k \]

\[ = -S_2. \]

Thus \( S_1 + S_2 = -S_3 \); so the desired recurrence is established for \( \sigma(k,n) \). Consequently \( S(k,n) = \sigma(k,n) \) for all \( k \geq 1, n \geq 0 \) which is the result to be proved.

\[ \square \]

4 Further remarks

4.1 A probabilistic interpretation of \( \chi(q) \)

The mock theta function \( \chi(q) \) has an interpretation in terms of conditional probabilities in some probability space. Let \( 0 < q < 1 \), and let \( C_1, C_2, \ldots \) be a sequence of independent events with probabilities

\[ P(C_n) = 1 - q^n, \quad n = 1, 2, 3 \ldots \]
Define events $A$ and $B$ by

$$A = \bigcap_{n=1}^{\infty} (C_n \cup C_{n+1}),$$
$$B = \bigcap_{n=2}^{\infty} (C_n \cup C_{n+1}).$$

**Theorem 4.1.** The following relations hold:

$$P(A|B) = (1 - q)\chi(q),$$
$$P(C_1|A) = 1/\chi(q).$$

**Proof.** Let

$$F(q) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n}.$$  

Holroyd, Liggett and Romik [4] proved that

$$P(A) = \frac{G_2(q)}{F(q)},$$

and by a similar argument it follows that

$$P(B) = \frac{H_2(q)}{(1 - q)F(q)}.$$  

Then, using (2.5):

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{(1 - q)G_2(q)}{H_2(q)} = (1 - q)\chi(q),$$
$$P(C_1|A) = \frac{P(C_1 \cap A)}{P(A)} = \frac{P(C_1 \cap B)}{P(A)} = \frac{P(C_1)P(B)}{P(A)}$$
$$= \frac{(1 - q)H_2(q)/(1 - q)F(q)}{G_2(q)/F(q)} = 1/\chi(q).$$

Incidentally, since probabilities are between 0 and 1, we get that for $0 < q < 1$,

$$\chi(q) < \frac{1}{1 - q}.$$  

\[\square\]
References


GEORGE ANDREWS
DEPARTMENT OF MATHEMATICS
PENN SYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PA 16802, USA
andrews@math.psu.edu

HENRIK ERIKSSON
KTH, COMPUTER SCIENCE
SE-10044 STOCKHOLM, SWEDEN
henrik@nada.kth.se

FEDOR PETROV
PETERSBURG DEPARTMENT OF THE RUSSIAN ACADEMY OF SCIENCES
fedorpetrov@mail.ru
DAN ROMIK
DEPARTMENT OF STATISTICS
367 EVANS HALL
UNIVERSITY OF CALIFORNIA
BERKELEY, CA 94720-3860, USA
romik@stat.berkeley.edu