Identities arising from limit shapes of constrained random partitions

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Abstract. Anatoly Vershik showed how to compute limit shapes for uniform random partitions with constraints on the parts, when the constraints are independent. Here, we develop techniques for dealing with nonindependent constraints. We compute limit shapes for random partitions chosen uniformly from the set of partitions of n into nonconsecutive parts, and from the set of minimal difference 2 partitions of n. This is done in two ways, first using special combinatorial ideas, and second using a probabilistic technique applicable in more general settings. The equivalence of the two methods leads to interesting integral identities.

1. Introduction

The theory of integer partitions has traditionally provided a fertile ground for the interaction of algebraic, combinatorial and analytic ideas. To this rich circle of ideas a new aspect has been added in recent years, that of probability. The theory of random partitions has its historical beginnings with the paper of Erdös and Lehner [4] on the number of parts in random partitions. More recently, Fristedt's [6] convenient representation of random partitions, which he used to obtain new results on the behavior of the parts in a random partition, and the work of Vershik and his collaborators [2,9,10] on the limiting shapes of random partitions, have been major advances in the field.

In this paper, we continue the work of Vershik on the theory of limit shapes of random partitions. We prove limit shape theorems for two classes of partitions satisfying "non-independent" constraints on the parts: partitions not containing two consecutive parts, and partitions with "minimal difference" 2, i.e. not containing repeated parts or two consecutive parts (see below for precise definitions). The non-independence of the constraints means that the technique of Vershik cannot be immediately extended to these cases, and some additional ingenuity is required. We shall derive the limit shapes in two ways: First, by specially-tailored combinatorial ideas which, in each of the two cases, transform

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the constraints into independent ones and thereby allow the computation of the shapes; Second, by introducing a probabilistic technique generalizing Vershik's technique for independent random variables, involving the computation of stationary probabilities of a certain Markov chain. From the equivalence of the two methods, some interesting analytic identities arise.

In section 2 we state the main definitions and results. In section 3 we review Fristedt's conditioning representation for uniform random partitions and generalize it to sub-classes of partitions. In section 4 we prove the limit shape theorem for "ensemble $\mathcal{A}$" of partitions with no two consecutive parts. The method of Markov chains is described and using it, a second formula for the limit shape is derived. In section 5 we use similar ideas, together with a result of Vershik on the limit shape of partitions of n into $t\sqrt{n}$ parts, to compute the limit shape for "ensemble $\mathcal{B}$" of partitions with minimal difference 2 and prove the related identities.

2. Definitions and main results

Recall that a partition λ of an integer n is a sequence $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ...$ of nonnegative integers such that $\sum \lambda_k = n$. The λ_k are called the *parts* or the *summands* of the partition. Alternatively, λ can be described as a sequence $r_1, r_2, r_3, ...$ of nonnegative integers such that $\sum kr_k = n$. r_k is called the multiplicity of k in λ, or the occupation number. The *Young diagram* corresponding to the partition λ is the decreasing function ϕ_λ on $[0, \infty)$ defined by $\phi_\lambda(t) = \lambda_{\lceil t \rceil}$, where $\lceil t \rceil = \inf\{m \in \mathbb{N} : m \geq t\}$ is the ceiling function. (See Fig. 1 - note that the λ_k's represent the *columns* of the diagram and not its rows, in contrast to the tradition in the combinatorics literature; however, since we are interested in the actual shape of the diagram, it seems natural to have the x-axis as the variable axis and the y-axis as the function axis.) The *normalized Young diagram* of λ is the function $\tilde{\phi}_\lambda(t) = \frac{1}{\sqrt{n}}\phi_\lambda(\sqrt{n}t)$, i.e. the Young diagram with both axes contracted by a factor of $\sqrt{n}$ to obtain a diagram of area 1.

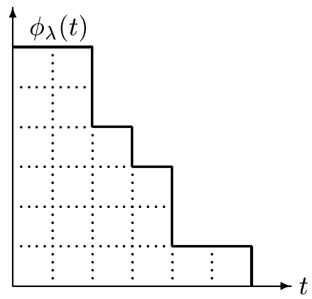

Figure 1: The Young diagram of the partition $21 = 6 + 6 + 4 + 3 + 1 + 1$

Let $\mathcal{P}_n$ be the set of partitions of an integer n. The *ensemble of unrestricted partitions* is the sequence of probability spaces $\mathcal{P} = (\mathcal{P}_n)_{n=1}^{\infty}$, where each $\mathcal{P}_n$ is equipped with the uniform probability measure. A *sub-ensemble* of partitions is a sequence of subsets $\mathcal{E} = (\mathcal{E}_n)_{n=1}^{\infty}$, where each $\mathcal{E}_n \subset \mathcal{P}_n$ and is thought of as a probability space equipped with the uniform probability measure, in other words the measure on $\mathcal{P}_n$ conditioned on the event $\mathcal{E}_n$.

Example 1. The ensemble $\mathcal{R} = (\mathcal{R}_n)_{n=1}^{\infty}$ of restricted partitions, i.e. partitions not containing repeated parts.

When we say that a certain property (depending perhaps on n) holds *for almost all partitions* in a partition ensemble $\mathcal{E} = (\mathcal{E}_n)_{n=1}^{\infty}$, we mean that as $n \to \infty$, the uniform measure on $\mathcal{E}_n$ of the set of partitions satisfying the property, converges to 1.

Example 2: Limit shapes for the ensembles $\mathcal{P}$ and $\mathcal{R}$. The following theorem was proved by Vershik [9]: Let $c = \pi/\sqrt{6}, d = \pi/\sqrt{12}$, and let $t > 0$. Then for almost all partitions λ in $\mathcal{P}$, we have

$$\tilde{\phi}_\lambda(t) \xrightarrow[n\to\infty]{} \Phi(t) := -\frac{1}{c}\log\left(1 - e^{-ct}\right),$$

(Note: this is a slight abuse of notation; the true meaning of this is that for any $\epsilon > 0$, for almost all partitions in $\mathcal{P}$ we have $|\tilde{\phi}_\lambda(t) + \frac{1}{c}\log(1 - e^{-ct})| < \epsilon$.)
And for almost all partitions λ in $\mathcal{R}$ we have

$$\tilde{\phi}_\lambda(t) \xrightarrow[n\to\infty]{} \begin{cases} -\frac{1}{d}\log\left(e^{dt} - 1\right) & 0 < t < \log(2)/d \\ 0 & t > \log(2)/d \end{cases}$$

The graphs of the limiting shapes are shown in Figure 2.

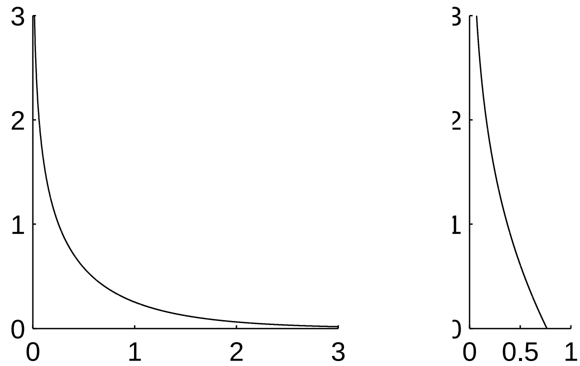

Figure 2: Limit shapes of the ensembles $\mathcal{P}$ and $\mathcal{R}$

We now define the two partition ensembles which are the main objects of interest in this paper:

Ensemble $\mathcal{A} = (\mathcal{A}_n)_{n=1}^{\infty}$: Partitions not containing two consecutive parts, and not containing any parts equal to 1.

Ensemble $\mathcal{B} = (\mathcal{B}_n)_{n=1}^{\infty}$: Partitions not containing repeated parts and not containing two consecutive parts.

(The reason for the restriction on parts equal to 1 in Ensemble $\mathcal{A}$ will become apparent in section 4; however, it is in fact immaterial as far as the limit shape is concerned.)

We now state the main results:

Theorem 1 - the limit shapes. Let $a = \pi/3$, $b = \pi/\sqrt{15}$, $\tau = (1+\sqrt{5})/2$, and let $t > 0$. Then for almost all partitions λ in $\mathcal{A}$, we have

$$\tilde{\phi}_\lambda(t) \xrightarrow[n\to\infty]{} A(t) := \frac{1}{a}\log\left(\frac{1-e^{-at}+e^{-2at}}{1-e^{-at}}\right),$$

and for almost all partitions λ in $\mathcal{B}$ we have

$$\tilde{\phi}_\lambda(t) \xrightarrow[n\to\infty]{} B(t) := \begin{cases} -\frac{1}{b}\log\left(\frac{e^{-2bt}}{1-e^{-bt}}\right) & 0 < t < \log(\tau)/b \\ 0 & t > \log(\tau)/b \end{cases}$$

The graphs of the limiting shapes $A(t)$ and $B(t)$ are shown in Figure 3.

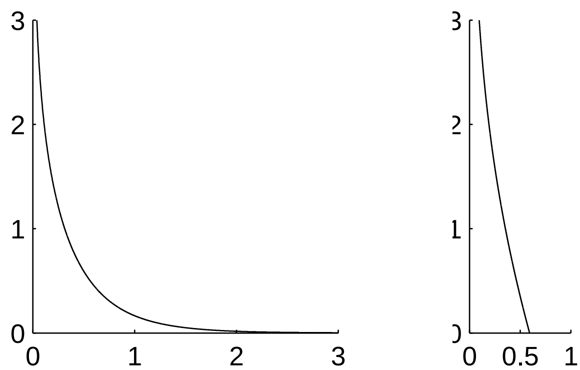

Figure 3: Limit shapes of the ensembles $\mathcal{A}$ and $\mathcal{B}$

In the statement of the above theorem, the expressions for the limiting shape functions have been "cleaned up" to conceal the process whereby these shapes were obtained. In fact, these functions arise as certain indefinite integrals. We will show how thinking about partitions in two different ways can lead to two different expressions for the limit shapes. The fact that these expressions must be equal leads to an interesting identity, which can also be verified independently. We state these identities as a separate theorem:

Theorem 2 - identities arising from the limit shapes. For $0 < p < 1$ define

$$q(p) = p(1-p)^2 \frac{1+p+\sqrt{1+2p-3p^2}}{2\sqrt{1+2p-3p^2}} \left(\frac{1-p+\sqrt{1-2p+3p^2}}{2} \right)^{-3}$$

Then with a, b, and τ as before,

$$A(t) = \int_t^\infty \frac{1}{1-e^{-au}+e^{-2au}} \left(\frac{e^{-2au}}{1-e^{-au}} + e^{-2au} \right) du =$$

$$= \frac{1}{a} \log \left(\frac{1-e^{-at}+e^{-2at}}{1-e^{-at}} \right)$$

$$A^{-1}(t) = \int_t^\infty \frac{1}{1-e^{-au}} q(e^{-au}) du =$$

$$= \frac{1}{2a} \log \left[\frac{1+e^{-at}+\sqrt{1+2e^{-at}-3e^{-2at}}}{2(1-e^{-at})} \right]$$

$$B(t) = \begin{cases} -\frac{1}{b} \log \left(\frac{e^{-2bt}}{1-e^{-bt}} \right) & 0 < t < \log(\tau)/b \\ 0 & t > \log(\tau)/b \end{cases}$$

$$B^{-1}(t) = \int_t^\infty q \left(\frac{e^{-bu}}{1+e^{-bu}} \right) du = -\frac{1}{b} \log \left(\frac{-e^{bt}+\sqrt{e^{2bt}+4e^{bt}}}{2} \right)$$

Remarks. 1. Note that, while verifying the identities in Theorem 2 is fairly straightforward, computing the integrals for $A^{-1}(t)$ and $B^{-1}(t)$ would have been very difficult without knowing the answer in advance! Symbolic integration software such as `Mathematica` did not succeed in solving these integrals, and in fact it was using the alternative expressions for $A(t)$ and $B(t)$ that the answer was obtained, and then verified.
2. The constant $\log(\tau)/b$ appearing in the formula for $B(t)$ represents the fact that almost all minimal difference 2 partitions of n have approximately $(\log(\tau)/b)\sqrt{n}$ parts. This fact (which is used in the proof of Theorem 1) was proved using separate arguments in [8]. The proof involves identities satisfied by the dilogarithm function.

3. Fristedt's representation for random partitions

In this section, we review Fristedt's representation for uniform random partitions, and recall Vershik's approach to proving limit shape theorems using this representation. The idea is to represent the multiplicities (or occupation numbers) in a uniform random partition of n as independent geometric random variables, conditioned on the number being partitioned being equal to n. More precisely, let $0 < x < 1$ be a real parameter, and for $k = 1, 2, 3, \ldots$ define independent random variables $R_1, R_2, R_3, \ldots$ such that $R_k + 1$ has geometric distribution with parameter $1 - x^k$, in other words

$$P_x(R_k = j) = (1 - x^k)x^{kj} \qquad j = 0, 1, 2, \ldots$$

(P_x denotes probability with respect to parameter choice x.) Then $R_1, R_2, R_3, \ldots$ define the multiplicities in a random partition λ of the (random) integer $N = \sum_{k=1}^{\infty} kR_k$. For a given partition $n = 1 \cdot r_1 + 2 \cdot r_2 + 3 \cdot r_3 + \ldots$ (given in terms of the multiplicities), the probability to get this partition in the random model is

$$P_x(R_1 = r_1, R_2 = r_2, R_3 = r_3, \ldots) = \prod_{k=1}^{\infty} P_x(R_k = r_k) = \prod_{k=1}^{\infty} ((1 - x^k)x^{kr_k}) =$$

$$= \frac{x^{\sum_{k \geq 1} kR_k}}{F(x)} = \frac{x^n}{F(x)}, \tag{1}$$

where $F(x) = \sum_{n=0}^{\infty} |\mathcal{P}_n| x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-1}$ is the generating function for unrestricted partitions. This expression is a function only of n (x is fixed), so we get:

Theorem 3. (Fristedt [6]) Let Q_n denote the uniform probability measure on $\mathcal{P}_n$, then for all $\lambda \in \mathcal{P}_n$, we have $Q_n(\lambda) = P_x(\lambda \mid N = n)$.

Theorem 3 extends easily to subclasses of partitions: one must further condition the partition to belong to the subclass. If the subclass can be defined by independent constraints on each of the R_k's, we call the subclass an "independent constraint" subclass. In this case, we will have an equally simple representation of the uniform measure on the subclass as the measure induced by independently choosing random variables R_k, with distribution which is simply the geometric distribution conditioned on the relevant constraint, and finally conditioned on the event $N = n$. For instance, for restricted partitions ("ensemble $\mathcal{R}$") R_k will be a random variable taking the values $0, 1$ with respective probabilities $1/(1 + x^k)$, $x^k/(1 + x^k)$.

It is now fairly easy to describe the Young diagram of the partition in terms of the multiplicities. A slight complication is that one must look at the conjugate partition. Recall that the partition conjugate to the partition $\lambda : \lambda_1 \geq \lambda_2 \geq \ldots \lambda_k$, is the partition $\lambda' : \lambda'_1 \geq \lambda'_2 \geq \ldots \lambda'_m$ defined by

$$\lambda'_j = \#\{i : \lambda_i \geq j\}$$

(graphically, the Young diagram of λ' is the inverse function to the Young diagram of λ.) In terms of the multiplicities $r_1, r_2, \ldots$ of λ we have

$$\lambda'_j = \sum_{k \geq j} r_k.$$

So we have expressed the Young diagram of the conjugate partition λ' in terms of the multiplicities. For the normalized Young diagram, this can be written as

$$\tilde{\phi}_{\lambda'}(t) = \frac{1}{\sqrt{n}} \sum_{k \geq t\sqrt{n}} r_k$$

For a random partition, we have the random young diagram

$$\tilde{\phi}_{\lambda'}(t) = \frac{1}{\sqrt{n}} \sum_{k \geq t\sqrt{n}} R_k$$

So we see that it is simpler to approach the shape of the conjugate diagram. Alternatively, we may work with the multiplicities R'_k of the conjugate diagram and reconstruct from them the Young diagram of the original partition. In Vershik's limit shape theorem for unrestricted partitions these two alternatives are equivalent, since the conjugate of a uniform random unrestricted partition is also a uniform random unrestricted partition. For these partitions, we now fix a special value $x_n = e^{-\pi/\sqrt{6n}}$ of the parameter x. For this choice of parameter, we will show that the event $N = n$ has a relatively high probability, and therefore the limit shape in the conditioned model (i.e. under the measure Q_n) is the same as in the independent model P_{x_n}:

Lemma 1. For parameter choice $x = x_n$, we have as $n \to \infty$

$$P_{x_n}(N = n) = (1 + o(1)) \frac{1}{2 \cdot 6^{1/4} n^{3/4}}$$

and

$$E_{x_n}(\tilde{\phi}_{\lambda'}(t)) = (1 + o(1)) \left(-\frac{1}{c} \log(1 - e^{-ct}) \right) = (1 + o(1)) \cdot \Phi(t)$$

(E denotes expectation; $c = \pi/\sqrt{6}$ as in Example 2.) Furthermore, there exists a function $g(u)$ such that for all $u > 0$, $g(u) > 0$ and

$$P_{x_n}(|\tilde{\phi}_{\lambda'}(t) - \Phi(t)| > u) \leq e^{-g(u)\sqrt{n}}$$

Proof. For the first part, note that $P_{x_n}(N = n)$ is simply a sum over all different partitions of n of the right hand side of (1), so

$$P_{x_n}(N = n) = \frac{|\mathcal{P}_n| x^n}{F(x)}$$

The claim now follows from known asymptotics of $|\mathcal{P}_n|$ and $F(x)$ (see e.g. [7]), namely

$$|\mathcal{P}_n| = (1 + o(1)) \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}, \qquad n \to \infty$$

$$\log F(e^{-s}) = \frac{\pi^2}{6s} + \frac{1}{2} \log s - \frac{1}{2} \log(2\pi) + o(1), \qquad s \searrow 0$$

For the second claim

$$E_{x_n} \tilde{\phi}_{\lambda'}(t) = \frac{1}{\sqrt{n}} \sum_{k \geq t\sqrt{n}} E_{x_n}(R_k) = \frac{1}{\sqrt{n}} \sum_{k \geq t\sqrt{n}} \frac{x_n^k}{1 - x_n^k} =$$

$$= (1+o(1))\int_t^\infty \frac{e^{-cu}}{1-e^{-cu}}du = (1+o(1))\Phi(t)$$

The third claim is proved easily using standard techniques of large deviation theory, and its proof is omitted. ■

Lemma 1 immediately implies Vershik's limit shape result, since

$$Q_n(|\tilde{\phi}_{\lambda'}(t)-\Phi(t)|>u) = P_{x_n}(|\tilde{\phi}_{\lambda'}(t)-\Phi(t)|>u \mid N=n) =$$

$$= \frac{P_{x_n}(|\tilde{\phi}_{\lambda'}(t)-\Phi(t)|>u, N=n)}{N=n} \le \frac{P_{x_n}(|\tilde{\phi}_{\lambda'}(t)-\Phi(t)|>u)}{N=n} \le$$

$$\le \frac{n^{3/4}}{C}e^{-g(u)\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

Note that the precise asymptotics of $P_{x_n}(N)$ are not really required, and one can settle for a much easier-to-prove estimate of the form $P_{x_n}(N=n) \ge 1/n^\alpha$ for some $\alpha > 0$. (The above argument is a much-distilled version of the large-deviations approach to limit shapes of partitions, developed extensively in [2].)

4. Constrained partitions, conjugate partitions and the Markov chain technique

We now adapt Vershik's basic technique to calculate limit shapes for Ensemble $\mathcal{A}$. Conditioning the random partition in Fristedt's model to lie in Ensemble $\mathcal{A}$ will yield, using this method, the limit shape for the partitions *conjugate* to Ensemble $\mathcal{A}$, i.e. the function $A^{-1}(t)$. To get the limit shape of Ensemble $\mathcal{A}$, we must deal with the conjugate partition ensemble! Fortunately, Ensemble $\mathcal{A}$ has a rather simple definition in terms of restrictions on the conjugate of its partitions (this is of course not so much a stroke of good fortune as simply the reason why this paper deals with this particular class of partitions). We define

Ensemble $\mathcal{A}' = (\mathcal{A}'_n)_{n=1}^\infty$: Partitions not containing any part *exactly once.*

It is not difficult to see that a partition λ is in $\mathcal{A}$ if and only if λ' is in $\mathcal{A}'$. (This is the reason why the restriction that $\lambda \in \mathcal{A}$ must contain no 1's was added.) In another stroke of seemingly good fortune, we now observe that the restrictions in ensemble $\mathcal{A}'$ operate separately on each of the multiplicities. In other words, $\mathcal{A}'$ is an independent constraint ensemble and its limit shape (or, rather, the conjugate limit shape $A(t)$) can be computed by Vershik's technique. We first state the conditioned form of Theorem 3 applicable to this case:

Lemma 2. Let Q_n denote the uniform probability measure on $\mathcal{A}_n$. For $0 < x < 1$, let P_x be the measure on partitions in $\mathcal{A}$ induced by choosing random multiplicities $R'_1, R'_2, R'_3, ...$ (of the conjugate partition) such that R'_k is a random variable taking the values $0, 2, 3, 4, ...$ with probabilities

$$P_x(R'_k = j) = \frac{1-x^k}{1-x^k+x^{2k}} \cdot x^{kj} \qquad j = 0, 2, 3, ...$$

Let $N = \sum_k kR'_k$. Then for all $\lambda \in \mathcal{A}_n$, we have $Q_n(\lambda) = P_x(\lambda \mid N = n)$.

Next, we state the constrained partition analogue of Lemma 1:

Lemma 3. For parameter choice $x = x_n := e^{-\pi/3\sqrt{n}}$, we have as $n \to \infty$

$$P_{x_n}(N = n) \geq \frac{C}{n^\alpha}$$

for some constants $C, \alpha > 0$, and

$$E_{x_n}(\tilde{\phi}_\lambda(t)) = (1 + o(1))A(t)$$

There exists a function $h(u) : (0, \infty) \to (0, \infty)$ such that

$$P_{x_n}(|\tilde{\phi}_\lambda(t) - A(t)| > u) \leq e^{-h(u)\sqrt{n}}$$

Proof. For the first part, we again make use of known results on the asymptotics of partition counting functions and partition generating functions. Let $G(x) = \sum_{n=0}^{\infty} |\mathcal{A}_n| x^n$. An argument similar to the one used in the previous section shows that

$$P_{x_n}(N = n) = \frac{|\mathcal{A}_n| x^n}{G(x)}$$

We now use a partition identity due to P. A. MacMahon ([1], p. 14, examples 9,10) to transform $G(x)$ into a more manageable expression:

Theorem 4. (Macmahon) For any $n \geq 1$, $|\mathcal{A}_n|$ is equal to the number of partitions of n into parts which are congruent to 0, 2, 3, or 4 modulo 6. Equivalently, we have the generating function identity

$$G(x) = \prod_{k=0}^{\infty} \frac{1}{(1 - x^{6k+2})(1 - x^{6k+3})(1 - x^{6k+4})(1 - x^{6k+6})}$$

The claim of the Lemma now follows from the results of [1], chapter 6. For the second part, the expectation of R'_k is easily computed to be

$$E_{x_n}(R'_k) = \frac{1}{1 - x_n^k + x_n^{2k}} \left(\frac{x_n^{2k}}{1 - x_n^k} + x_n^{2k} \right)$$

and therefore

$$E_{x_n}(\phi_\lambda(t)) = \frac{1}{\sqrt{n}} \sum_{k \geq t\sqrt{n}} E(R'_k) =$$

$$= (1 + o(1)) \int_t^\infty \frac{1}{1 - e^{-au} + e^{-2au}} \left(\frac{e^{-2au}}{1 - e^{-au}} + e^{-2au} \right) du = (1 + o(1))A(t)$$

The third part is again an easy application of standard techniques in large deviation theory, and is omitted. ∎

The first part of Theorem 1 now follows from Lemma 3 in exactly the same way as the proof of Vershik's theorem in the previous section.

The Markov chain technique

We now explain the technique by which we shall derive the limit shape for the ensemble $\mathcal{A}'$. This requires dealing with the multiplicities $R_1, R_2, ...$ of a random partition in $\mathcal{A}$, which are not independent. Rather, these random variables have the distribution of independent geometric random variables conditioned on the event that no two consecutive R_k's are non-zero.

We shall present the technique in a form that is not fully rigorous. Since our main results are also proven in another way, from a strictly formal point of view this does not pose a problem. However, for the sake of possible future application of these ideas, we wish to stress that each of the steps that follows can be made rigorous using a more detailed analysis. The tools that are needed are the theory of non-homogeneous Markov chains, and the theory of large deviations as explained e.g. in [3] and applied to random partitions in [2].

Recall that, in the analysis of the limit shapes for the unrestricted ensemble $\mathcal{P}$ and for the constrained ensemble $\mathcal{A}$, the first step consisted of identifying the correct value x_n of the parameter required to make the conditioned model behave approximately the same as the non-conditioned one - this is simply the "saddle-point" of the generating function. The next step, which is computationally the crucial one, was to understand the behavior of $E_{x_n}(R_k)$, for $k \approx u\sqrt{n}$. The limit shape was then the integral of this quantity from t to infinity. For unrestricted partitions, $R_k + 1$ was a geometric r.v. with parameter $1 - x_n^k$, with $x_n = e^{-c/\sqrt{n}}$, so we had

$$E_{x_n}(R_k) \approx \frac{e^{-cu}}{1 - e^{-cu}}$$

whereas for ensemble $\mathcal{A}$, we had (with $x_n = e^{-a/\sqrt{n}}$)

$$E_{x_n}(R_k') \approx \frac{1}{1 - e^{-au} + e^{-2au}} \left(\frac{e^{-2au}}{1 - e^{-au}} + e^{-2au} \right)$$

with the saddle point $x_n = e^{-\pi/3\sqrt{n}}$. We now give a heuristic argument explaining why, for ensemble $\mathcal{A}'$, we should expect to have

$$E_{x_n}(R_k) \approx \frac{q(e^{-au})}{1 - e^{-au}}$$

which would explain the expression for $A^{-1}(t)$ given in Theorem 2. First, represent the multiplicities in the form

$$R_k = S_k \cdot T_k$$

where: S_k is a random variable taking the values $0, 1$, and T_k is a random variable taking the values $1, 2, 3, ...$ S_k corresponds to choosing whether the part k will be in the partition, and T_k represents the choice of the number of times this part will appear, if chosen. Clearly, S_k and T_k are independent. T_k will have the geometric distribution with parameter $1 - x_n^k$. The distribution of the S_k's can be described as follows: Take a sequence of independent events $D_1, D_2, D_3, ...$ such that $P(D_k) = x_n^k$, and define the event

$$E = \bigcap_{k=1}^{\infty} (D_k^c \cup D_{k+1}^c) = \text{“no two consecutive } D_k\text{'s occured”}$$

Then

$$\left\{ S_k, \quad k = 1, 2, ... \right\} \stackrel{dist}{=} \left\{ 1_{D_k}, \quad k = 1, 2, ... \middle| \; E \right\}$$

Now,

$$E_{x_n}(R_k) = E_{x_n}(S_k) \cdot E_{x_n}(T_k) \approx (1 - e^{-au})^{-1} E_{x_n}(S_k),$$

and it remains to explain why

$$E_{x_n}(S_k) = P(D_k \mid E) \approx q(e^{-au})$$

Since $P(D_k) \approx e^{-au}$, this leads us to the following natural question: What is the distribution of a sequence of independent events, of probability approximately p, conditioned on the event that no two consecutive events occur? Note that this is a slight simplification of the true situation, since in reality the probabilities $1 - x_n^k$ are changing very slowly. The assumption (which can be fully justified with a little more work) is that conditioning on the event E only affects the sequence D_k “locally”, i.e. if the probabilities of a large number of events near D_k are all approximately e^{-au}, then we can neglect the effect of the change in the probabilities over time. The answer to the simplified question is given in the following

Lemma 4. Let $A_1, A_2, A_3, ..., A_n$ be independent events occurring each with probability p, and let $E_n = \cap_{k=1}^{n-1} (A_k \cup A_{k+1})$. Then if $1 \leq k \leq n$ and both k and $n - k$ are sufficiently large (i.e. the event A_k is “far from the boundary”) then

$$P(A_k \mid E_n) \approx q(p)$$

where $q(p)$ is defined as in Theorem 2:

$$q(p) = p(1-p)^2 \frac{1 + p + \sqrt{1 + 2p - 3p^2}}{2\sqrt{1 + 2p - 3p^2}} \left(\frac{1 - p + \sqrt{1 - 2p + 3p^2}}{2} \right)^{-3}$$

First proof. Define $e_n = P(E_n)$. By conditioning on A_{n-1}, A_n, we have the recurrence

$$e_n = (1-p)e_{n-1} + p(1-p)e_{n-2}$$

together with the initial conditions $e_0 = e_1 = 1$. Therefore it can easily be verified that

$$e_n = \left(\frac{1 + p + \sqrt{1 + 2p - 3p^2}}{2\sqrt{1 + 2p - 3p^2}} \right) \left(\frac{1 - p + \sqrt{1 + 2p - 3p^2}}{2} \right)^n +$$

$$+\left(\frac{-1-p+\sqrt{1+2p-3p^2}}{2\sqrt{1+2p-3p^2}}\right)\left(\frac{1-p-\sqrt{1+2p-3p^2}}{2}\right)^n$$

For large n we have simply

$$e_n \approx \left(\frac{1+p+\sqrt{1+2p-3p^2}}{2\sqrt{1+2p-3p^2}}\right)\left(\frac{1-p+\sqrt{1+2p-3p^2}}{2}\right)^n =: C(p)\cdot\lambda(p)^n$$

Now $P(A_k \mid E_n)$ can be represented in terms of independent events, as follows:

$$P(A_k \mid E_n) = \frac{P(A_k \cap E_n)}{P(E_n)} =$$

$$= \frac{P\left(A_k \cap A_{k-1}^c \cap A_{k+1}^c \cap \left(\bigcap_{j=1}^{k-3}(A_j^c \cup A_{j+1}^c)\right) \cap \left(\bigcap_{j=k+2}^{n-1}(A_j^c \cup A_{j+1}^c)\right)\right)}{P(E_n)} =$$

$$= \frac{p(1-p)^2 e_{k-2}\, e_{n-k-1}}{e_n} \approx p\,(1-p)^2 C(p)\cdot\lambda(p)^{-3} = q(p)$$

■

Second proof. We sketch a proof of Lemma 4 which is more conceptual, and which explains why the technique is related to Markov chains. Such a conceptual outlook will simplify the use of the technique in more general settings.

One may think of the original sequence of events $A_1, A_2, \ldots$ as a stationary Markov chain, whose state space is pairs of events. That is, there are 4 states: "00", "01", "10", and "11", where if at time n the chain is in state "$\alpha\beta$" ($\alpha, \beta \in \{0,1\}$), then α signifies whether event A_n occurred (i.e. $\alpha = 1_{A_n}$) and β signifies whether event A_{n+1} occurred.

The transition matrix of this Markov chain is

$$\begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \end{pmatrix} \end{array}$$

Conditioning on the event E_n is the same as conditioning the Markov chain never to go into the state "11". The behavior of such a "forbidden state" Markov chain can be computed. We state this in more general terms as a lemma; the proof is an easy application of standard methods of Markov chains, see e.g. [5], ch. 15.

Lemma 5. (Forbidden state Markov chains) Let $A = (a_{ij})_{i,j \in S}$ be the transition matrix of a Markov chain with finite state space S. Let $F \subset S$ be a set of forbidden states, such that the sub-matrix $A' = (a_{ij})_{i,j \in S \setminus F}$ is indecomposable and aperiodic, i.e. there exists a k such that the entries $a_{ij}^{(k)}$

of A'^k are all non-zero. Let $X_1, X_2, ..., X_n$ be a random sample of the Markov chain, with any initial distribution. Let E be the event

$$E = \bigcap_{j=1}^{n} \{X_j \notin F\}$$

Let m be a fixed positive integer, and let $1 < k < n$ such that k and $n - k$ are large. Then the distribution of

$$\left\{ X_k, X_{k+1}, X_{k+2}, ..., X_{k+m} \middle| E \right\}$$

is approximately the same as the distribution of a random sample $Y_1, Y_2, ..., Y_{m-1}$ from the Markov chain, with stationary initial distribution, whose transition matrix is $B = (b_{ij})_{i,j \in S \setminus F}$, defined as follows:

$$b_{ij} = \frac{a_{ij} v_j}{\lambda v_i}$$

Where: λ is the Perron-Frobenius eigenvalue of the matrix $(a_{ij})_{i,j \in S \setminus F}$, and $(v_i)_{i \in S \setminus F}$ is its associated eigenvector.

To finish the second proof of Lemma 4, all that remains is to compute the transition matrix for the conditioned Markov chain, and compute its stationary probabilities. The desired quantity $P(A_k \mid E_n)$ is the stationary probability of the state "10". The computation may easily be carried through to give the stated result. ■

5. Ensemble $\mathcal{B}$

In this section, we finish the proof of Theorem 1 by computing the limit shape of ensemble $\mathcal{B}$. The computation using the Markov chain technique is similar to the one in the last section, and is omitted. The approach that will be presented relies on the following simple observation: A minimal difference 2 partition $\lambda : \lambda_1 > \lambda_2 > ... > \lambda_k$ of n can be converted into an unrestricted partition $\mu : \mu_1 > \mu_2 > ... > \mu_k$ of $n - k(k-1)$, defined by $\mu_i = \lambda_i - 2(k-i)$, $i = 1, 2, ..., k$. This mapping establishes a bijection between the minimal difference 2 partitions of n into k parts and unrestricted partitions of $n - k(k-1)$ into k parts. Geometrically, this can be thought of as taking the Young diagram of the partition and deleting a triangular array of squares, of slope -2, that hits the x-axis at the $(k-1)$-th part. Phrasing this in the language of probability, we may say that even though minimal difference 2 partitions are *not* defined using independent constraints on the multiplicities of the parts, the multiplicities are *conditionally independent, given the number of parts.* It is this observation that will enable us an alternative method for computing the limit shape. Two facts are needed: one is Vershik's result on the limit shape of unrestricted partitions of n into $s\sqrt{n}$ parts, and the other is a result on the number of parts in a typical minimal difference 2 partition:

Theorem 5. As $n \to \infty$, almost all partitions of n in ensemble $\mathcal{B}$ have $(1+o(1))(\sqrt{15}\log((1+\sqrt{5})/2)/\pi)\cdot\sqrt{n}$ parts.

Proof. See [8]. ■

Recall that the dilogarithm function is defined by

$$\mathrm{Li}_2(x) = \int_0^x -\log(1-t)\frac{dt}{t} = \sum_{m=1}^{\infty}\frac{x^m}{m^2}, \qquad 0 \le x \le 1$$

Define functions $y : (0,\infty) \to (0,1)$, $\quad z : (0,\infty) \to (0,\pi/\sqrt{6})$ by the implicit equations

$$s^2\mathrm{Li}_2(y(s)) = \log^2(1-y(s))$$
$$z(s) = \sqrt{\mathrm{Li}_2(y(s))}$$

Theorem 6 - the limit shape of partitions into $s\sqrt{n}$ parts. (Vershik [9], see also [10]) Fix $s>0$. Let $\mathcal{P}^s = (\mathcal{P}^s_n)_{n=1}^{\infty}$ be the partition ensemble of partitions of n into $k = \lfloor s\sqrt{n}\rfloor$ parts. Then for almost all partitions λ in $\mathcal{P}^s$, we have

$$\tilde{\phi}_\lambda(t) \xrightarrow[n\to\infty]{} F_s(t) := \begin{cases} -\frac{1}{z(s)}\log\left(\frac{1-e^{-z(s)t}}{y(s)}\right) & 0<t<s \\ 0 & t>s \end{cases}$$

The ground is now set for the computation of the limit shape $B(t)$ in Theorem 1. From the above discussion, it follows that $B(t)$ can be built up from the triangular graph $2(s_0-t)$, together with a scaled version of the graph $F_{s_0/\sqrt{1-s_0^2}}$, where $b=\pi/\sqrt{15}$, $\quad \tau=(1+\sqrt{5})/2$ as in Theorem 1, and $s_0=\log(\tau)/b$. More precisely,

$$B(t) = 0, \qquad t > s_0$$

whereas for $0<t<s_0$, we have

$$B(t) = 2\,(s_0-t) + \sqrt{1-s_0^2}\;F_{s_0/\sqrt{1-s_0^2}}\left(\frac{t}{\sqrt{1-s_0^2}}\right)$$

Now, using the special values

$$y\left(s_0/\sqrt{1-s_0^2}\right) = \tau^{-2}, \qquad z\left(s_0/\sqrt{1-s_0^2}\right) = \sqrt{b^2-\log^2(\tau)}$$

this can easily be simplified to arrive at the expression given in Theorem 1. ■

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