Limit shapes for random square Young tableaux

Boris Pittel [1]
Department of Mathematics
Ohio State University
Columbus, Ohio 43210
email: bgp@math.ohio-state.edu

Dan Romik
Department of Mathematics
The Weizmann Institute of Science
Rehovot 76100, Israel

Abstract. Our main result is a limit shape theorem for the two-dimensional surface defined by a uniform random $n \times n$ square Young tableau. The analysis leads to a calculus of variations minimization problem that resembles the minimization problems studied by Logan-Shepp, Vershik-Kerov, and Cohn-Larsen-Propp. We solve this problem by developing a general technique for solving variational problems of this kind. An extension to rectangular Young tableaux is also given.

We also apply the main result to show that the location of a particular entry in the tableau is in the limit governed by a semicircle distribution, and to the study of *extremal Erdös-Szekeres permutations*, namely permutations of the numbers $1, 2, \ldots, n^2$ whose longest monotone subsequence is of length n.

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1 Introduction

1.1 Random square Young tableaux

In this paper, we study the large-scale asymptotic behavior of uniform random Young tableaux chosen from the set of tableaux of square shape. Recall that a *Young diagram* is a graphical representation of a partition $\lambda : \lambda(1) \geq \lambda(2) \geq \ldots \geq \lambda(k)$ of $n = \sum \lambda_i$ as an array of *cells*, where row i has λ_i cells. For a Young diagram λ (we will often identify a partition with its Young diagram), a *Young tableau* of shape λ is a filling of the cells of λ with the numbers $1, 2, \ldots, n$ such that the numbers along every row and column are increasing.

A square Young tableau is a Young tableau whose shape is an $n \times n$ square Young diagram. The number of such tableaux is known by the hook formula of Frame-Thrall-Robinson (see (6) below) to be

$$\frac{(n^2)!}{[1 \cdot (2n-1)][2 \cdot (2n-2)]^2 [3 \cdot (2n-3)]^3 \ldots [(n-1)(n+1)]^{n-1}\ n^n}.$$

A square tableau $T = (t_{i,j})_{i,j=1}^n$ can be depicted geometrically as a three-dimensional stack of cubes over the two-dimensional square $[0,n] \times [0,n]$, where $t_{i,j}$ cubes are stacked over the square $[i-1,i] \times [j-1,j] \times \{0\}$. Alternatively, the function $(i,j) \to t_{i,j}$ can be thought of as the graph of the (non-continuous) surface of the upper envelope of this stack. By rescaling the $n \times n$ square to a square of unit sides, and rescaling the heights of the columns of cubes so that they are all between 0 and 1, one may consider the family of square tableaux as $n \to \infty$. This raises the natural question, whether the shape of the stack for a *random* $n \times n$ square tableau exhibits some asymptotic behavior as $n \to \infty$. The answer is given by the following theorem, and is illustrated in Figure 1.

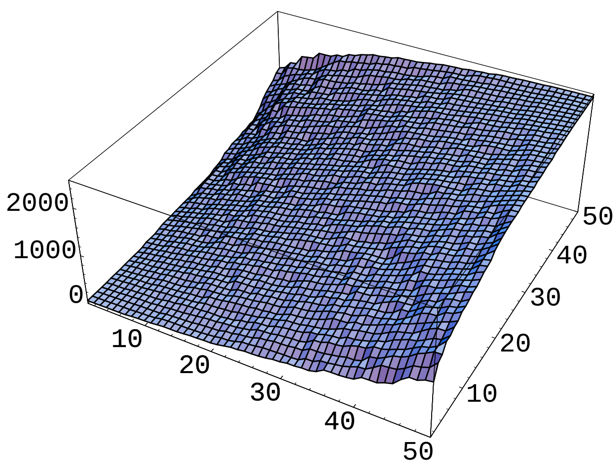
Theorem 1. Let $\mathcal{T}_n$ be the set of $n \times n$ square Young tableaux, and let $\mathbb{P}_n$ be the uniform probability measure on $\mathcal{T}_n$. Then for the function $L : [0,1] \times [0,1] \to [0,1]$ defined below, we have:

(i) *Uniform convergence to the limit shape:* for all $\epsilon > 0$,

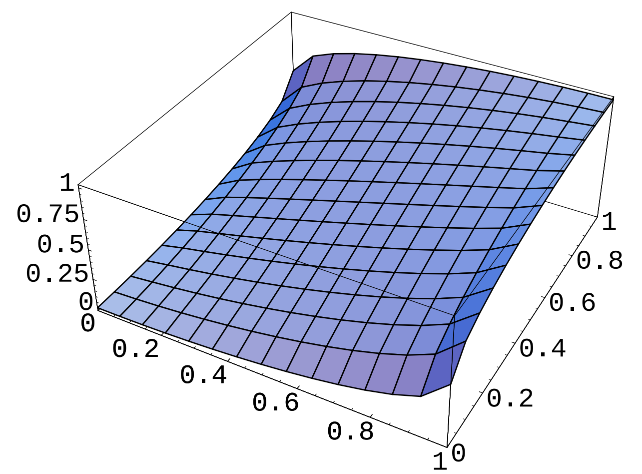
$$\mathbb{P}_n \left(T \in \mathcal{T}_n : \max_{1 \leq i,j \leq n} \left| \frac{1}{n^2} t_{i,j} - L\left(\frac{i}{n}, \frac{j}{n}\right) \right| > \epsilon \right) \xrightarrow[n\to\infty]{} 0.$$

(ii) *Rate of convergence in the interior of the square:* for all $\epsilon > 0$,

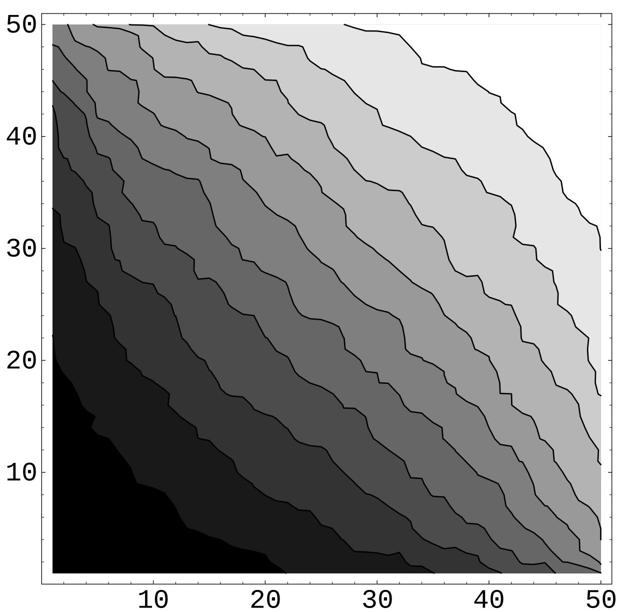
$$\mathbb{P}_n \Bigg(T \in \mathcal{T}_n : \max_{\substack{1 \leq i,j \leq n \\ \min(ij,(n-i)(n-j)) > n^{3/2+\epsilon}}} \left| \frac{1}{n^2} t_{i,j} - L\left(\frac{i}{n}, \frac{j}{n}\right) \right| > \frac{1}{n^{(1-\epsilon)/2}} \Bigg) \xrightarrow[n\to\infty]{} 0.$$



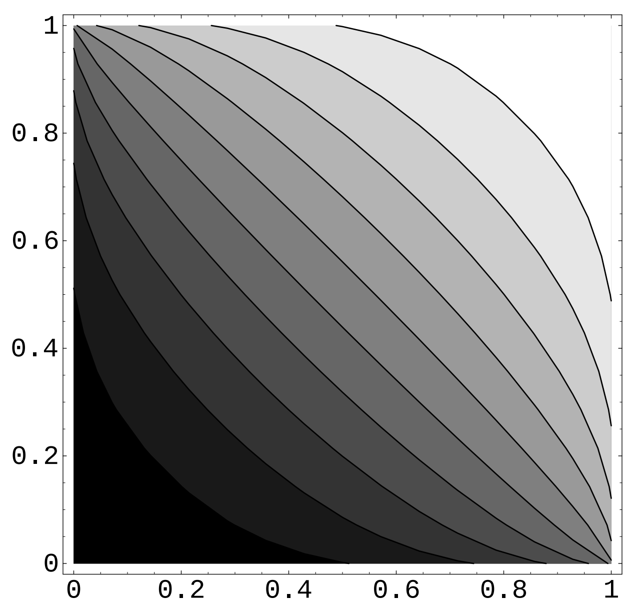
(a) 3D plot of simulated tableau



(b) The limit surface $L(x, y)$



(c) Contour plot of simulated tableau



(d) Contour plot of L

Figure 1: A simulated 50×50 random tableau and the limit surface

Definition of L. We call the function L the *limit surface of square Young tableaux.* It is defined by the implicit equation

$$x+y=\frac{2}{\pi}(x-y)\tan^{-1}\left(\frac{(1-2L(x,y))(x-y)}{\sqrt{4L(x,y)(1-L(x,y))-(x-y)^2}}\right)$$
$$+\frac{2}{\pi}\tan^{-1}\left(\frac{\sqrt{4L(x,y)(1-L(x,y))-(x-y)^2}}{1-2L(x,y)}\right)$$

for $0\le y\le 1-x\le 1$, together with the reflection property

$$L(x,y)=1-L(1-x,1-y)$$

(where $\tan^{-1}$ is the arctangent function). It is more natural to describe L in terms of its level curves $\{L(x,y)=\alpha\}$. First, introduce the *rotated coordinate system*

$$u=\frac{x-y}{\sqrt{2}},\qquad v=\frac{x+y}{\sqrt{2}}. \tag{1}$$

In the $u-v$ plane, the square $[0,1]\times[0,1]$ transforms into the rotated square

$$\diamond=\{(u,v)\in\mathbb{R}^2 : |u|\le\sqrt{2}/2, |u|\le v\le\sqrt{2}-|u|\}.$$

Now define the one-parameter family of functions $(g_\alpha)_{0\le\alpha\le 1}$ given by

$$g_\alpha : [-\sqrt{2\alpha(1-\alpha)},\sqrt{2\alpha(1-\alpha)}]\to\mathbb{R},$$

$$g_\alpha(u)=\begin{cases}\frac{2}{\pi}u\tan^{-1}\left(\frac{(1-2\alpha)u}{\sqrt{2\alpha(1-\alpha)-u^2}}\right)+\frac{\sqrt{2}}{\pi}\tan^{-1}\left(\frac{\sqrt{2(2\alpha(1-\alpha)-u^2)}}{1-2\alpha}\right) & 0\le\alpha<\frac{1}{2},\\ -\frac{2}{\pi}u\tan^{-1}\left(\frac{(2\alpha-1)u}{\sqrt{2\alpha(1-\alpha)-u^2}}\right)-\frac{\sqrt{2}}{\pi}\tan^{-1}\left(\frac{\sqrt{2(2\alpha(1-\alpha)-u^2)}}{2\alpha-1}\right)+\sqrt{2} & \frac{1}{2}<\alpha\le 1,\\ \frac{\sqrt{2}}{2} & \alpha=\frac{1}{2}.\end{cases} \tag{2}$$

Then in the rotated coordinate system, the surface $\bar{L}(u,v)=L(x(u,v),y(u,v))$ can be described as the surface whose level curves $\{\bar{L}(u,v)=\alpha\}$ are exactly the curves $\{v=g_\alpha(u)\}$. That is,

$$\{(u,v)\in\diamond : \bar{L}(u,v)=\alpha\}=\{(u,v)\in\diamond : |u|\le\sqrt{2\alpha(1-\alpha)}, v=g_\alpha(u)\}.$$

This is illustrated in Figure 2. It is straightforward to check that the curves $v=g_\alpha(u)$ do not intersect, and so define a surface [2].

[2] See equation (65) in section 3.4.

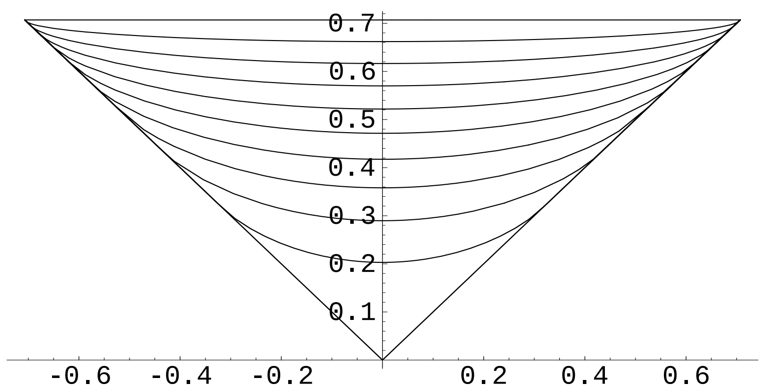


Figure 2: The curves $v = g_\alpha(u)$ for $\alpha = 0.05,\ 0.1,\ 0.15,\ 0.2,\ \ldots\ ,\ 0.5$

Note some special values of $L(x, y)$ which can be computed explicitly:

$$\begin{aligned} L(t,0) &= L(0,t) = \frac{1-\sqrt{1-t^2}}{2}, \\ L(t,1) &= L(1,t) = \frac{1+\sqrt{2t-t^2}}{2}, \\ L(t,t) &= \frac{1-\cos(\pi t)}{2}. \end{aligned}$$

The approach in proving Theorem 1 is the variational approach. Namely, we identify the large-deviation rate functional of the level curves of the random surface defined by the tableau, then analyze the functional and find its minimizers. This will give Theorem 1(ii), with the rate of convergence following from classical norm estimates for some integral operators. The treatment of the boundary of the square, required for Theorem 1(i), turns out to be more delicate, and will require special arguments.

The variational problem which we solve resembles the variational problems studied by Logan-Shepp [14], Vershik-Kerov [19, 20] and Cohn-Larsen-Propp [6]. In these previous studies, it was fairly straightforward to verify that the proposed solution to the variational problem was indeed the solution, but finding that solution was very difficult and required deep insights and some guesswork. One notable feature of our solution, which we believe to be of broader interest beyond its application to the problem of square Young tableaux, is that we develop a general technique for systematically solving variational problems of this kind without having to guess the solution. This may prove useful in dealing with similar problems.

1.2 Location of particular entries

Theorem 1 identifies the approximate value of the entry of a typical square tableau in a given location in the square. A dual outlook is to ask where a given value k will appear in the square tableau, since all the values between 1 and n^2 appear exactly once. These questions are almost equivalent. Indeed, if k is approximately $\alpha \cdot n^2$, then Theorem 1 predicts that with high probability the entry k will appear in the vicinity of the level curve $\{L(x,y) = \alpha\}$ (the fact that this actually follows from Theorem 1 is a simple consequence of the monotonicity property of the tableau along rows and columns). However, one may ask a more detailed question about the limiting distribution of the location of the entry k on the level curve. It turns out that its u-coordinate has approximately the semicircle distribution. This is made precise in the following theorem.

Theorem 2. For a tableau $T \in \mathcal{T}_n$ and $1 \leq k \leq n^2$, denote by $(i(T,k), j(T,k))$ the location of the entry k in T, and denote $X(T,k) = i(T,k)/n,\ Y(T,k) = j(T,k)/n$. Let $0 < \alpha < 1$, let k_n be a sequence of integers such that $k_n/n^2 \xrightarrow[n\to\infty]{} \alpha$, and for each n let T_n be a uniform random tableau in $\mathcal{T}_n$. Then as $n \to \infty$, the random vector $(X(T_n,k_n), Y(T_n,k_n))$ converges in distribution to the random vector

$$(X_\alpha, Y_\alpha) := \left(\frac{V_\alpha + U_\alpha}{\sqrt{2}}, \frac{V_\alpha - U_\alpha}{\sqrt{2}} \right),$$

where U_α is a random variable with density function

$$f_{U_\alpha}(u) = \frac{\sqrt{2\alpha(1-\alpha) - u^2}}{\pi\alpha(1-\alpha)} \mathbf{1}_{[-\sqrt{2\alpha(1-\alpha)}, \sqrt{2\alpha(1-\alpha)}]}(u) \tag{3}$$

and $V_\alpha = g_\alpha(U_\alpha)$.

1.3 Extremal Erdös-Szekeres permutations

The famous Erdös-Szekeres theorem states that a permutation of $1, 2, \ldots, n^2$ must have either an increasing subsequence of length n or a decreasing subsequence of length n. This can be proved using the pigeon-hole principle, but also follows from the RSK correspondence using the observation that a Young diagram of area n^2 must have either width or height at least n.

For the width and height of a Young diagram of area n^2 to be *exactly* n, the diagram must be a square. From the RSK correspondence it thus follows that to each permutation of $1, 2, \ldots, n^2$ whose longest increasing subsequence and longest decreasing subsequence have length exactly n, there correspond a pair of square $n \times n$ Young tableaux. Such a permutation has the minimal possible length of a longest *monotone* subsequence, and it seems appropriate to term such permutations *extremal Erdös-Szekeres permutations* (we are not aware of any previous references to these permutations, aside from a brief mention in [13], exercise 5.1.4.9).

As an application of our limit shape result, we will prove the following result on the length of the longest increasing subsequence when just an initial segment of a random extremal Erdös-Szekeres permutation is read.

Theorem 3. For each n, let π_n be a uniform random extremal Erdös-Szekeres permutation of $1, 2, \dots, n^2$. For $1 \le k \le n^2$, Let $l_{n,k}$ be the length of the longest increasing subsequence in the sequence $\pi_n(1), \pi_n(2), \dots, \pi_n(k)$. Denote $\alpha = k/n^2$, and $\alpha_0 = n^{-2/3+\epsilon}$. Then for any $\epsilon > 0$, and $\omega(n) \to \infty$ however slowly,

$$\max_{\alpha_0 \le k/n^2 \le 1/2} \mathbb{P}(|l_{n,k} - 2\sqrt{\alpha(1-\alpha)}n| > \alpha_0^{1/2}\omega(n)n) \xrightarrow[n\to\infty]{} 0.$$

Thus the random fluctuations of $l_{n,k}$ around $2\sqrt{\alpha(1-\alpha)}n$ are not likely to be of order substantially larger than $n^{2/3}$.

1.4 Random rectangular Young tableaux

The methods which we will use to prove Theorems 1, 2, and 4 work equally well for rectangular Young tableaux, in the limit when the size of the rectangle grows and its relative proportions tend to a limiting value $\theta > 0$. For each possible value θ of the ratio between the sides of the rectangle, there is a limiting surface L_θ for random rectangular Young tableaux. Analogously to the square tableaux, the rectangular $n_1 \times n_2$ tableaux can be viewed as the result of applying the RSK algorithm to a permutation of $\{1, \dots, n_1 n_2\}$ with the property that the lengths of the longest increasing and the longest decreasing subsequences are exactly equal n_1 and n_2 (by the Erdös-Szekeres theorem, the two lengths cannot be simultaneously below n_1 and n_2, respectively).

Let $\theta > 0$. We may assume that $\theta \le 1$, otherwise exchange the two sides of the rectangle. Define $L_\theta : [0,1] \times [0,\theta] \to [0,1]$, the *limit surface of rectangular tableaux with side ratio* θ, as follows. For each $0 < \alpha < 1$, the α-level curve $\{(x,y) : L_\theta(x,y) = \alpha\}$ is given in rotated $u - v$ coordinates by

$$\{(u, h_{\theta,\alpha}(u)) : -\beta_1 \le u \le \beta_2\},$$

where

$$\begin{aligned}
\overline{\beta} &= \sqrt{2\theta\alpha(1-\alpha)}, \\
\beta_1 &= \overline{\beta} - \alpha(1-\theta)\sqrt{2}/2, \qquad \beta_2 \;=\; \overline{\beta} + \alpha(1-\theta)\sqrt{2}/2, \\
h_{\theta,\alpha}(u) &= \theta\sqrt{2}/2 \pm (\beta_1 - \theta\sqrt{2}/2) + \frac{2\overline{\beta}}{\pi}\Bigg[\pm(-\xi-\gamma_1)\tan^{-1}\sqrt{\frac{(1-\xi)(\gamma_1-1)}{(1+\xi)(\gamma_1+1)}} \\
&\quad +(\xi-\gamma_2)\tan^{-1}\sqrt{\frac{(1+\xi)(\gamma_2-1)}{(1-\xi)(\gamma_2+1)}} \\
&\quad +\frac{1}{2}\left(\sin^{-1}\xi + \frac{\pi}{2}\right)\frac{1-\theta}{\sqrt{2}\,\overline{\beta}} \pm \frac{\pi}{2}(\gamma_1-1)\Bigg], \qquad 0<\alpha\le\frac{1}{2}, \\
\pm &= \begin{cases} + & 0<\alpha\le\theta/(1+\theta), \\ - & \theta/(1+\theta)<\alpha\le 1/2, \end{cases} \\
\xi &= \frac{u-\alpha(1-\theta)\sqrt{2}/2}{\overline{\beta}}, \qquad u\in[-\beta_1,\beta_2], \\
\gamma_1 &= \frac{\alpha+\theta(1-\alpha)}{\sqrt{2}\,\overline{\beta}}, \qquad \gamma_2 \;=\; \frac{\theta\alpha+1-\alpha}{\sqrt{2}\,\overline{\beta}}, \\
h_{\theta,\alpha}(u) &= (1+\theta)\sqrt{2}/2 - h_{\theta,1-\alpha}((1-\theta)\sqrt{2}/2-u), \qquad \frac{1}{2}<\alpha<1,
\end{aligned}$$

see Figure 3.

Theorem 4. For integers $n,m>0$, let $\mathcal{T}_{n,m}$ be the set of tableaux whose shape is an $n\times m$ rectangular diagram, and let $\mathbb{P}_{n,m}$ be the uniform probability measure on $\mathbb{P}_{n,m}$. If $T=(t_{i,j})_{i,j}\in\mathcal{T}_{n,m}$, define the rescaled tableau surface of T as the function $\tilde{S}_T:[0,1)\times[0,m/n)\to[0,1]$ given by

$$\tilde{S}_T(x,y) = \frac{1}{nm}t_{\lfloor nx\rfloor+1,\lfloor ny\rfloor+1}.$$

Let $0<\theta\le 1$. If m_n is a sequence of integers such that $m_n/n\to\theta$ as $n\to\infty$, then for all $\epsilon>0,\ x\in[0,1), y\in[0,\theta)$,

$$\mathbb{P}_{n,m_n}(T\in\mathcal{T}_{n,m_n}:|\tilde{S}_T(x,y)-L_\theta(x,y)|>\epsilon)\xrightarrow[n\to\infty]{}0.$$

In the main the argument parallels the proof of Theorem 1. We encourage the motivated reader to go through the necessary computations.

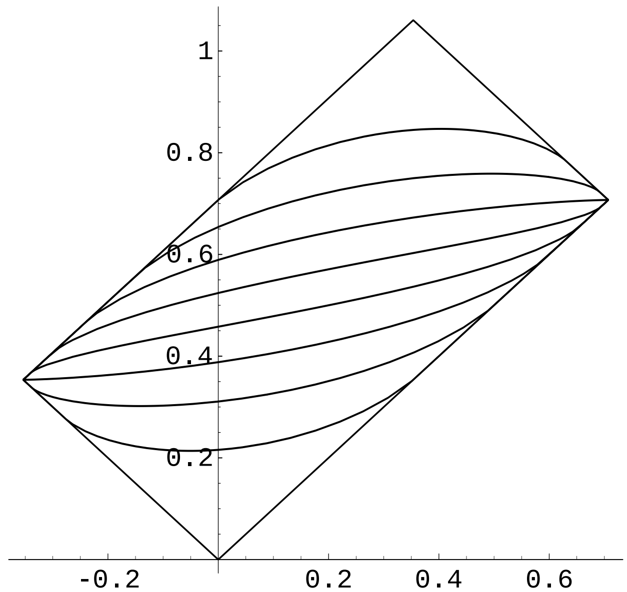


Figure 3: The curves $h_{\theta,\alpha}$ for $\theta = 0.5$, $\alpha = k/9$, $k = 1, 2, \ldots, 8$.

1.5 Random square Young tableaux and Plancherel measure

As a final introductory note, we remark that perhaps the true importance of the uniform random square Young tableaux model studied in this paper is best seen in connection with the well-studied model of Plancherel measure. The formula (7), which is the starting point of our analysis, is a natural analogue of, and indeed a deformation of, the formula $d(\lambda_0)^2/|\lambda_0|!$ for Plancherel measure, in the sense that fixing k and letting $n \to \infty$ yields Plancherel measure in the limit. Therefore the ideas in this paper may be applicable beyond the immediate interest of the main results themselves, and one might hope that using the connection between random square Young tableaux and Plancherel measure, new insights to both models may be gained. For more information on Plancherel measure, see the papers [1, 3, 4, 5, 11, 12, 14, 19, 20].

1.6 Organization of the paper

The remainder of the paper is organized as follows: In the next section, we present the variational approach to the limit surface of random square Young tableaux, based on the hook formula of Frame-Thrall-Robinson. The level curves of L appear as minimizers of a certain functional. This leads to a proof of Theorem 1 in the interior of the square, except for the explicit identification of L. Section 3 is dedicated to the derivation of the explicit formula for the minimizer.

In section 4, we complete the proof of Theorem 1, treating the more delicate case of the boundary of the square, and prove Theorem 3. In section 5, we discuss the *hook walk* of Greene-Nijenhuis-Wilf and the concept of the *co-transition measure* of a Young diagram. Using the explicit formulas for the co-transition measure derived in [16], we compute the co-transition measure of the level curves g_α, proving Theorem 2. In section 6 we mention some open problems.

2 A variational problem for random square tableaux

2.1 A large-deviation principle

One may consider a tableau $T \in \mathcal{T}_n$ as a path in the *Young graph* of all Young diagrams, starting with the empty diagram, and leading up to the $n \times n$ square diagram, where each step is of adding one box to the diagram. Identify T with this sequence $\lambda_T^0 = \phi \subset \lambda_T^1 \subset \lambda_T^2 \subset \ldots \subset \lambda_T^{n^2} = \square_n$ of diagrams. (λ_T^k is simply the sub-diagram of the square comprised of those boxes where the value of the entry of T is $\leq k$.) Theorem 1 is then roughly equivalent, in a sense that will be made precise later, to the statement that for each $1 \leq k \leq n^2 - 1$, the rescaled shape of λ_T^k for a random $T \in \mathcal{T}_n$ resembles the *sub-level set*

$$\{(x, y) \in [0, 1]^2 : L(x, y) \leq k/n^2\}$$

of L, with probability $1 - o(1)$ as $n \to \infty$. It is this approach that leads to the large-deviation principle. Namely, we can estimate the probability that the sub-diagram λ_T^k has a given shape:

Lemma 1. For $T \in \mathcal{T}_n$, denote as before $\lambda_T^0 \subset \ldots \subset \lambda_T^{n^2}$ the path in the Young graph defined by T, and for each $0 \leq k \leq n^2$, let $\lambda_T^k : \lambda_T^k(1) \geq \lambda_T^k(2) \geq \ldots \geq \lambda_T^k(n)$ be the lengths of the columns of λ_T^k (some of them may be 0). For any Young diagram $\lambda : \lambda(1) \geq \lambda(2) \geq \ldots \geq \lambda(n)$ whose graph lies within the $n \times n$ square, define the function $f_\lambda : [0, 1] \to [0, 1]$ by

$$f_\lambda(x) = \frac{1}{n} \lambda(\lceil nx \rceil). \tag{4}$$

(Note that this depends implicitly on n.) Let $0 \le k \le n^2$, and let $\alpha = k/n^2$. Then for any given diagram $\lambda_0 \subseteq \Box_n$ with area k, we have

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : \lambda_T^k = \lambda_0\right) = \exp\Bigg(- (1+o(1))n^2(I(f_{\lambda_0}) + H(\alpha) + C)\Bigg) \qquad (5)$$

as $n \to \infty$, where

$$\begin{aligned} C &= \frac{3}{2} - 2\log 2, \\ H(\alpha) &= -\alpha\log(\alpha) - (1-\alpha)\log(1-\alpha), \\ I(g) &= \int_0^1\int_0^1 \log|g(x) - y + g^{-1}(y) - x|dy\,dx, \\ g^{-1}(y) &= \inf\{x \in [0,1] : g(x) \le y\}. \end{aligned}$$

The $o(1)$ is uniform over all λ_0 and all $0 \le k \le n^2$.

Proof. For a Young diagram $\lambda : \lambda(1) \ge \lambda(2) \ge \ldots \ge \lambda(l)$ of area m, denote by $d(\lambda)$ the number of Young tableaux of shape λ (also known as the *dimension* of λ, as it is known to be equal to the dimension of a certain irreducible representation corresponding to λ of the symmetric group of order m). Recall the *hook formula* of Frame-Thrall-Robinson [8], which says that $d(\lambda)$ is given by

$$d(\lambda) = \frac{m!}{\prod_{(i,j)\in\lambda} h_{i,j}}, \qquad (6)$$

where the product is over all boxes (i,j) in the diagram, and $h_{i,j}$ is the *hook number* of a box, given by

$$\begin{aligned} h_{i,j} &= \lambda(i) - j + \lambda'(j) - i + 1 \\ &= 1 + \text{number of boxes either to the right of, or below } (i,j) \end{aligned}$$

(and where λ' is the conjugate partition to λ.) Then we have [3]

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : \lambda_T^k = \lambda_0\right) = \frac{d(\lambda_0)d(\Box_n \setminus \lambda_0)}{d(\Box_n)}, \qquad (7)$$

where $d(\Box_n \setminus \lambda_0)$ means the number of fillings of the numbers $1, \ldots, n^2 - k$ in the cells of the skew-Young diagram $\Box_n \setminus \lambda_0$ that are monotonically *decreasing* along rows and columns. This is because $\Box_n \setminus \lambda_0$ can be thought of as an ordinary diagram, when viewed from the opposite corner of the square. The number of square tableaux whose k-th subtableau has shape λ_0 is simply the number of tableaux of shape λ_0, times the number of fillings of the

[3]Note to the reader: this is probably the most important formula in the paper!

numbers $k+1, k+2, \ldots, n^2$ in the cells of $\square_n \setminus \lambda_0$ that are monotonically increasing along rows and columns – and these are of course isomorphic to tableaux of shape $\square_n \setminus \lambda_0$, by replacing each entry i with n^2+1-i.

Take minus the logarithm of (7) and divide by n^2, using (6). The right-hand side becomes

$$a+b+c-d := \frac{1}{n^2}\log\left(\frac{(n^2)!}{k!(n^2-k)!}\right) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\lambda(i)}\log(\lambda(i)-j+\lambda'(j)-i+1)$$

$$+\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=\lambda(i)+1}^{n}\log(j-\lambda(i)+i-\lambda'(j)+1) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\log(2n-i-j+1).$$

By Stirling's formula, we have $a = n^{-2}\log\binom{n^2}{k} = H(\alpha)+o(1)$, with the required uniformity in k. The other summands look like Riemann sums of double integrals. Indeed, we claim that

$$\begin{aligned} b &= \int_0^1\int_0^{f_{\lambda_0}(x)}\log\left(f_{\lambda_0}(x)-y+f_{\lambda_0}^{-1}(y)-x\right)dy\,dx + \frac{k}{n^2}\log n + o(1), \\ c &= \int_0^1\int_{f_{\lambda_0}(x)}^1\log\left(y-f_{\lambda_0}(x)+x-f_{\lambda_0}^{-1}(y)\right)dy\,dx + \frac{n^2-k}{n^2}\log n + o(1), \\ d &= \int_0^1\int_0^1\log(2-x-y)dy\,dx + \log n + o(1) = C + \log n + o(1), \end{aligned}$$

which on summing and exponentiating would give the lemma. Let us prove, for example, the first of these equations. Write

$$\begin{aligned} b &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\lambda(i)}\log(\lambda(i)-j+\lambda'(j)-i+1) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{\lambda(i)}\log\left(\frac{\lambda(i)-j+\lambda'(j)-i+1}{n}\right) + \frac{k}{n^2}\log n. \end{aligned}$$

Fix $1 \le i \le n$ and $1 \le j \le \lambda(i)$. Denote $h = (\lambda(i)-j+\lambda'(j)-i+1)/n$. Approximate $n^{-2}\log h$ in the above sum by the double integral

$$Q := \int_{(i-1)/n}^{i/n}\int_{(j-1)/n}^{j/n}\log\left(f_{\lambda_0}(x)-y+f_{\lambda_0}^{-1}(y)-x\right)dy\,dx.$$

A change of variables transforms this (check the definition of f_{λ_0}) into

$$Q = \int_{-1/2n}^{1/2n}\int_{-1/2n}^{1/2n}\log(x+y+h)dx\,dy.$$

Note that h may take the values $1/n, 2/n, \ldots, (2n-1)/n$. If $h = 1/n$, then integrating we get

$$Q = -\frac{\log n}{n^2} + n^{-2} \int_0^1 \int_0^1 \log(u+v)du\,dv = \frac{\log h}{n^2} + O(n^{-2}).$$

If $h \geq 2/n$, by the integral mean value theorem, we have for some $\eta \in [-1, 1]$,

$$Q = \frac{\log(h + \eta n^{-1})}{n^2} = \frac{\log h}{n^2} + O((n^3 h)^{-1}).$$

Clearly then the last estimate holds for $h = 1/n$ as well. The sum of the remainders over all $1 \leq i \leq n$, $1 \leq j \leq \lambda(i)$ is of order

$$n^{-2} \sum_{(i,j)\in\lambda_0} \frac{1}{h_{i,j}} \leq n^{-2} \sum_{m=1}^{2n-1} \frac{a(m)}{m},$$

where

$$a(m) := \#\{(i,j) \in \lambda_0 : h_{i,j} = m\}.$$

Clearly $a(m) \leq n$, since each row i of λ_0 contains at most one cell (i, j) with $h_{i,j} = m$. This gives that the sum of the remainders is of order

$$n^{-2} \sum_{m=1}^{2n-1} \frac{n}{m} = O\left(\frac{\log n}{n}\right),$$

which is indeed $o(1)$. ■

2.2 Two formulations of the variational problem

Lemma 1 says, roughly, that the exponential order of the probability that a random square tableau T has a given k-subtableau shape, where k is approximately $\alpha \cdot n^2$, is given by the value of the functional I on the boundary g of the shape, plus some terms depending only on α. Following the well-known methodology of large deviation theory, the natural next step is to identify the global minimum of I over the appropriate class of functions, or in other words to find the *most likely* shape for the α-level set. If we can prove that there is a unique minimum, and identify it, that will be a major step towards proving Theorem 1. So we have arrived at the following variational problem.

Variational problem 1. For each $0 < \alpha < 1$, any weakly decreasing function $f : [0,1] \to [0,1]$ such that $\int_0^1 f(x)dx = \alpha$ is called *α-admissible*. Find the unique α-admissible function that minimizes the functional

$$I(f) = \int_0^1 \int_0^1 \log|f(x) - y + f^{-1}(y) - x|dy\,dx.$$

We now simplify the form of the functional I, by first rotating the coordinate axes by 45 degrees, and then reparametrizing the square by the "hook coordinates" – an idea used in [19], [20], [14]. Let u, v be the rotated coordinates as in (1). Given an α-admissible function $f : [0,1] \to [0,1]$, there corresponds to it a function $g : [-\sqrt{2}/2, \sqrt{2}/2] \to [0, \sqrt{2}]$, such that

$$y = f(x) \iff v = g(u)$$

(see Figure 4). The class of α-admissible functions translates to those functions $g : [-\sqrt{2}/2, \sqrt{2}/2] \to [0, \sqrt{2}]$ that are 1-Lipschitz, and satisfy $g(-\sqrt{2}/2) = g(\sqrt{2}/2) = \sqrt{2}/2$ and

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (g(u) - |u|) du = \alpha. \tag{8}$$

We continue to call such functions α-admissible. We call a function admissible if it is α-admissible for some $0 \le \alpha \le 1$.

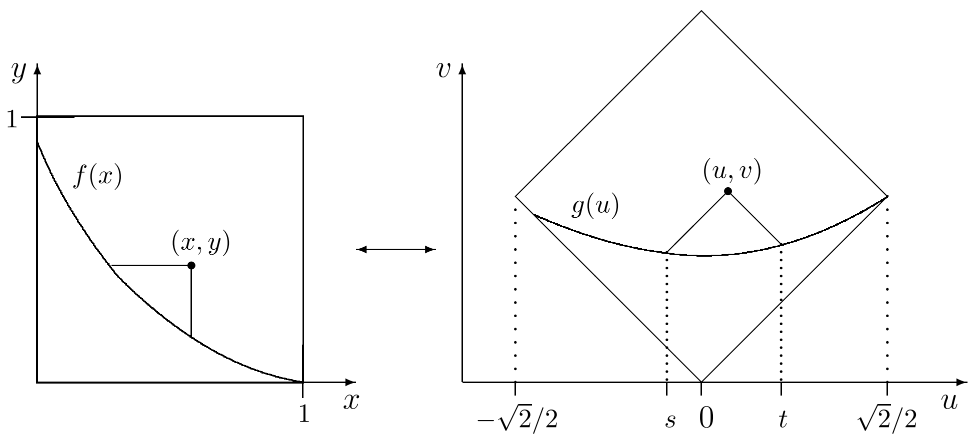


Figure 4: The rotated graph and the hook coordinates s, t

To derive the new form of the functional, write

$$I(f) = I_1(f) + I_2(f) := \int_0^1 \int_0^{f(x)} \log(h_f(x,y))dy\,dx + \int_0^1 \int_{f(x)}^1 \log(h_f(x,y))dy\,dx,$$

where $h_f(x,y)$ is the *hook function* of f,

$$h_f(x,y) = |f(x) - y + f^{-1}(y) - x|.$$

Now, set

$$J(g) = J_1(g) + J_2(g) := I_1(f) + I_2(f),$$

where f and g are rotated versions of the same graph as in Figure 4. Then

$$J_2(g) = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{g(u)}^{\sqrt{2}-|u|} \log h_f(x,y) dv\, du.$$

Reparametrize this double integral by the *hook coordinates* s and t,

$$s = \frac{f^{-1}(y) - y}{\sqrt{2}}, \qquad t = \frac{x - f(x)}{\sqrt{2}}$$

(see Figure 4). The Lipschitz property ensures that this transformation is one-to-one from the region

$$\{(u,v) : -\sqrt{2}/2 \le u \le \sqrt{2}/2,\ g(u) \le v \le \sqrt{2} - |u|\}$$

onto the region

$$\Delta = \{(s,t) : -\sqrt{2}/2 \le s \le t \le \sqrt{2}/2\}.$$

Therefore the integral transforms as

$$J_2(f) = \iint_\Delta \log\left(\sqrt{2}(t-s)\right) \left|\frac{\partial(u,v)}{\partial(s,t)}\right| ds\, dt.$$

It remains to compute the Jacobian $\partial(u,v)/\partial(s,t)$. An easy computation gives (see [19], [20], [14])

$$\frac{\partial(u,v)}{\partial(s,t)} = \frac{1}{2}(1-g'(s))(1+g'(t)).$$

(This can be viewed geometrically as follows: draw on the u-axis in Figure 4 the two intervals $[s, s+ds]$, $[t, t+dt]$. The set of points in the square for which the hook coordinates fall inside the two intervals is approximately a rectangle with sides $(1-g'(s))/\sqrt{2}$ and $(1+g'(t))/\sqrt{2}$.) So

$$J_2(g) = \frac{1}{2} \iint_\Delta \log\left(\sqrt{2}(t-s)\right) (1-g'(s))(1+g'(t)) ds\, dt.$$

A similar computation for J_1, using "lower" instead of "upper" hook coordinates, shows that

$$J_1(g) = \frac{1}{2} \iint_\Delta \log\left(\sqrt{2}(t-s)\right) (1+g'(s))(1-g'(t)) ds\, dt.$$

This gives

$$\begin{aligned} J(g) &= \frac{1}{2} \iint_\Delta \log\left(\sqrt{2}(t-s)\right) \left[(1-g'(s))(1+g'(t)) + (1+g'(s))(1-g'(t))\right] ds\, dt \\ &= \frac{1}{2} \iint_\Delta \log\left(\sqrt{2}(t-s)\right) \left(2 - 2g'(s)g'(t)\right) ds\, dt \\ &= -\frac{1}{2} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \log|t-s| \cdot g'(s)g'(t) ds\, dt + \log 2 - \frac{3}{2}. \end{aligned}$$

We can now state a reformulation of the original variational problem.

Variational problem 2. For each $0 < \alpha < 1$, a function $g : [-\sqrt{2}/2, \sqrt{2}/2] \to [0, \sqrt{2}]$ is called α-admissible if: $g(-\sqrt{2}/2) = g(\sqrt{2}/2) = \sqrt{2}/2$; g is 1-Lipschitz; and $\int_{-\sqrt{2}/2}^{\sqrt{2}/2}(g(u) - |u|)du = \alpha$. Find the unique α-admissible function that minimizes the functional

$$K(g) = -\frac{1}{2}\int_{\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)g'(t)\log|s-t|dsdt. \tag{9}$$

2.3 Deduction of Theorem 1(ii)

In the next section, we prove the following theorem.

Theorem 5. For each $0 < \alpha < 1$, let $\tilde{g}_\alpha$ be the unique extension of g_α (defined in (2)) to an α-admissible function, namely

$$\tilde{g}_\alpha(u) = \begin{cases} g_\alpha(u) & |u| \le \sqrt{2\alpha(1-\alpha)} \\ |u| & \sqrt{2\alpha(1-\alpha)} \le |u| \le \sqrt{2}/2 \end{cases}$$

for $0 < \alpha \le 1/2$, and

$$\tilde{g}_\alpha(u) = \begin{cases} g_\alpha(u) & |u| \le \sqrt{2\alpha(1-\alpha)} \\ \sqrt{2} - |u| & \sqrt{2\alpha(1-\alpha)} \le |u| \le \sqrt{2}/2 \end{cases}$$

for $1/2 < \alpha < 1$. Then:

(i) $\tilde{g}_\alpha$ is the unique solution to Variational problem 2;

(ii) $K(\tilde{g}_\alpha) = -H(\alpha) + \log 2$;

(iii) For any α-admissible function g we have

$$K(g) \ge K(\tilde{g}_\alpha) + K(g - \tilde{g}_\alpha).$$

Assuming this as proven, our goal is now to prove Theorem 1. At the beginning of this section, we claimed that Theorem 1 was equivalent to the statement that the subtableau λ_T^k has shape approximately described by the region bounded under the graph of the level curve $\{L = k/n^2\}$ (which in rotated coordinates is given by the curve $v = \tilde{g}_\alpha(u)$, where $\alpha = k/n^2$). We shall now make precise the sense in which this is true, and see how this follows from the fact that $\tilde{g}_\alpha$ is the minimizer.

For a continuous function $p : [-\sqrt{2}/2, \sqrt{2}/2] \to \mathbb{R}$, define its supremum norm

$$||p||_\infty = \max_{u \in [-\sqrt{2}/2, \sqrt{2}/2]} |p(u)|.$$

Lemma 2. *K is continuous in the supremum norm on the space of admissible functions.*

Proof. Consider the symmetric bilinear form

$$\langle g, h\rangle = -\frac{1}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)h'(t)\log|s-t|ds\,dt \tag{10}$$

defined whenever g and h are almost everywhere differentiable functions on $[-\sqrt{2}/2, \sqrt{2}/2]$ with bounded derivative. We show that $\langle\cdot,\cdot\rangle$ is continuous in the supremum norm with respect to any of its arguments, when restricted to the set of 1-Lipschitz functions; this will imply the lemma, since $K(g) = \langle g, g\rangle$. Write (10) more carefully as

$$\langle g, h\rangle = -\frac{1}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)\cdot \lim_{\epsilon\searrow 0}\left[\int_{-\sqrt{2}/2}^{s-\epsilon} h'(t)\log(s-t)dt + \int_{s+\epsilon}^{\sqrt{2}/2} h'(t)\log(t-s)dt\right]ds.$$

For $s \in (-\sqrt{2}/2, \sqrt{2}/2)$ which is a point of differentiability of h, integration by parts gives

$$\begin{aligned}
&\int_{-\sqrt{2}/2}^{s-\epsilon} h'(t)\log(s-t)dt + \int_{s+\epsilon}^{\sqrt{2}/2} h'(t)\log(t-s)dt = \\
&= h(t)\log(s-t)\Big|_{t=-\sqrt{2}/2}^{t=s-\epsilon} - \int_{-\sqrt{2}/2}^{s-\epsilon}\frac{h(t)}{t-s}dt + h(t)\log(t-s)\Big|_{t=s+\epsilon}^{t=\sqrt{2}/2} - \int_{s+\epsilon}^{\sqrt{2}/2}\frac{h(t)}{t-s}dt \\
&= h\left(\frac{\sqrt{2}}{2}\right)\log\left(\frac{\sqrt{2}}{2}-s\right) - h\left(-\frac{\sqrt{2}}{2}\right)\log\left(\frac{\sqrt{2}}{2}+s\right) \\
&\qquad + (h(s-\epsilon)-h(s+\epsilon))\log\epsilon - \int_{[-\sqrt{2}/2,s-\epsilon]\cup[s+\epsilon,\sqrt{2}/2]}\frac{h(t)}{t-s}dt \\
&\qquad \xrightarrow[\epsilon\searrow 0]{} h\left(\frac{\sqrt{2}}{2}\right)\log\left(\frac{\sqrt{2}}{2}-s\right) - h\left(-\frac{\sqrt{2}}{2}\right)\log\left(\frac{\sqrt{2}}{2}+s\right) - \pi\tilde{h}(s),
\end{aligned}$$

where $\tilde{h}$ is the Hilbert transform of h, defined by the principal value integral

$$\tilde{h}(s) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{h(t)}{t-s}dt$$

(think of h as a function on $\mathbb{R}$ which is 0 outside $[-\sqrt{2}/2, \sqrt{2}/2]$.) Going back to (10), this gives

$$\begin{aligned}
\langle g, h\rangle = &-\frac{1}{2}h\left(\frac{\sqrt{2}}{2}\right)\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)\log\left(\frac{\sqrt{2}}{2}-s\right)ds \\
&+\frac{1}{2}h\left(\frac{-\sqrt{2}}{2}\right)\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)\log\left(\frac{\sqrt{2}}{2}+s\right)ds \\
&+\frac{\pi}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)\tilde{h}(s)ds.
\end{aligned}\tag{11}$$

Now recalling that the Hilbert transform is an isometry on $L_2(\mathbb{R})$ (see [18], Theorem 90), and using the fact that

$$\left| \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \log\left(\frac{\sqrt{2}}{2} \pm s\right) ds \right| = \frac{2-\log 2}{\sqrt{2}} < 1,$$

this implies that for 1-Lipschitz functions g, h_1, h_2,

$$\begin{aligned} |\langle g, h_1 - h_2\rangle| &\leq ||h_1 - h_2||_\infty + \frac{\pi}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} |\tilde{h}_1(s) - \tilde{h}_2(s)| ds \\ &\leq ||h_1 - h_2||_\infty + 2^{1/4}\frac{\pi}{2}\left(\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\tilde{h}_1(s) - \tilde{h}_2(s)\right)^2 ds\right)^{1/2} \\ &\leq ||h_1 - h_2||_\infty + 2^{1/4}\frac{\pi}{2}\left(\int_{\mathbb{R}} \left(\tilde{h}_1(s) - \tilde{h}_2(s)\right)^2 ds\right)^{1/2} \\ &= ||h_1 - h_2||_\infty + 2^{1/4}\frac{\pi}{2}\left(\int_{-\sqrt{2}/2}^{\sqrt{2}/2} (h_1(s) - h_2(s))^2 ds\right)^{1/2} \\ &\leq \left(1 + 2^{1/2}\frac{\pi}{2}\right) ||h_1 - h_2||_\infty. \end{aligned}$$

■

We have another use for (11). Let f be a Lipschitz function on $[-\sqrt{2}/2, \sqrt{2}/2]$ that satisfies $f(\pm\sqrt{2}/2) = 0$. Denote by

$$F[f](x) = \int_{\mathbb{R}} f(t)e^{-ixt}dt$$

the Fourier transform of a function f. Recall the well-known formulas

$$\begin{aligned} F[\tilde{f}](x) &= i \cdot \operatorname{sgn} x \cdot F[f](x), \\ F[f'](x) &= i \cdot x \cdot F[f](x), \\ \int_{\mathbb{R}} f_1(t)\overline{f_2(t)}dt &= \frac{1}{2\pi}\int_{\mathbb{R}} F[f_1](x)\overline{F[f_2](x)}dx. \end{aligned}$$

Then, by (11)

$$\begin{aligned} K(f) = \langle f, f\rangle = \frac{\pi}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} f'(s)\tilde{f}(s)ds \\ = \frac{1}{4}\int_{\mathbb{R}} F[f'](x)\overline{F[\tilde{f}](x)}dx = \frac{1}{4}\int_{\mathbb{R}} |x| \cdot |F[f](x)|^2 dx. \end{aligned} \tag{12}$$

We note as a lemma an important consequence of this identity which we shall need later on.

Lemma 3. If f is a Lipschitz function with $f(\pm\sqrt{2}/2) = 0$ as above, then $K(f) \geq 0$, and $K(f) = 0$ only if $f \equiv 0$. ■

Lemma 3 will be used in the next section to easily deduce uniqueness of the minimizer. In fact, Theorem 5 gives all the necessary information to prove a non-quantitative version of Theorem 1, i.e. without the rate-of-convergence estimates. However, we can do better, by noting that Theorem 5(iii), together with the representation (12), can be used to give quantitative estimates for the rate of convergence in Theorem 1. We prove the following strengthening of Lemma 3:

Lemma 4. For every $r \in (2, 3)$, there exists a constant $c = c(r) > 0$ such that for all 2-Lipschitz functions $f : [-\sqrt{2}/2, \sqrt{2}/2] \to \mathbb{R}$ that satisfy $f(\pm\sqrt{2}/2) = 0$, we have

$$K(f) \geq c||f||_\infty^r.$$

Proof. Had the power of $|x|$ in (12) been 2, $K(f)$ would have been equal to 1/4 times the squared L_2-norm of $xF[f](x) = F[f'](x)$. Having $|x|$ in (12) invites the conclusion that instead we are dealing with the squared L_2-norm of $f^{(1/2)}(x)$, the *fractional* derivative of f of order 1/2.

To see that this is indeed the case, and to use the full power of such an interpretation of $K(f)$, let us recall the corresponding definitions. For $\alpha \in (0, 1)$, the fractional derivative $f^{(\alpha)}(x)$ of order α is defined by

$$f^{(\alpha)}(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(x) - f(x-t)}{t^{1+\alpha}} dt. \tag{13}$$

The integral exists as $f(x)$ is Lipschitz and bounded. Clearly $f^{(\alpha)}(x) \equiv 0$ for $x \leq -\sqrt{2}/2$. Then

$$\begin{aligned} F[f^{(\alpha)}](x) &= \int_{\mathbb{R}} e^{-ixt} f^{(\alpha)}(t) dt \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{1 - e^{-ix\tau}}{\tau^{1+\alpha}} d\tau \cdot F[f](x) = (ix)^\alpha F[f](x), \end{aligned} \tag{14}$$

where

$$(ix)^\alpha := \begin{cases} |x|^\alpha \exp(i\alpha\pi/2), & x > 0, \\ |x|^\alpha \exp(-i\alpha\pi/2), & x < 0. \end{cases}$$

Indeed, setting

$$z^{1+\alpha} = |z| \exp(i(1+\alpha)\theta)), \quad \text{if } z = |z|e^{i\theta}, \quad \theta \in (-\pi, \pi),$$

we have

$$\int_0^\infty \frac{1-e^{-ix\tau}}{\tau^{1+\alpha}}d\tau = (ix)^\alpha \int_0^{i\infty} \frac{1-e^{-z}}{z^{1+\alpha}}dz = (ix)^\alpha \int_0^\infty \frac{1-e^{-\tau}}{\tau^{1+\alpha}}d\tau$$
$$= (ix)^\alpha \frac{1}{\alpha}\int_0^\infty \tau^{-\alpha}e^{-\tau}d\tau = (ix)^\alpha \frac{\Gamma(1-\alpha)}{\alpha}.$$

In particular, for $\alpha = 1/2$, we get from (14) that

$$\left|F[f^{(1/2)}](x)\right|^2 = |x| \cdot |F[f](x)|^2,$$

whence, by (14) and isometry of the Fourier transform,

$$K(f) = \frac{1}{4}\int_{\mathbb{R}} |x| \cdot |F[f](x)|^2 dx = \frac{\pi}{2}|f^{(1/2)}(x)|^2. \tag{15}$$

The fractional integration operator, inverse to that in (13), is known to be given by

$$f(x) = (I_\alpha f^{(\alpha)})(x), \qquad (I_\alpha h)(x) := \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-t)^{\alpha-1}h(t)dt. \tag{16}$$

As a check, the Fourier transform of the RHS is

$$\frac{1}{\Gamma(\alpha)}F[f^{(\alpha)}](x)\int_0^\infty \tau^{\alpha-1}e^{-ix\tau}d\tau = (ix)^{-\alpha}F[f^{(\alpha)}](x) = F[f](x).$$

By Theorem 383 in [10], for $p > 1$ and

$$0 < \alpha < \frac{1}{p}, \qquad q = \frac{p}{1-\alpha p},$$

I_α maps L_p into L_q, and is bounded. That is, there exists a constant $c(p) > 0$ such that

$$||I_\alpha h||_q \le c(p)||h||_p. \tag{17}$$

Introduce $\psi(x) = f^{(\alpha)}(x)\mathbf{1}_{(-\infty,\sqrt{2}/2]}(x)$, so that ψ is supported by $[-\sqrt{2}/2, \sqrt{2}/2]$. According to (16),

$$(I_\alpha \psi)(x) = f(x), \qquad x \le \sqrt{2}/2.$$

So, using (17) and monotonicity of the L_s-averages, we have

$$||f||_q \le ||I_\alpha\psi||_q \le c(p)||\psi||_p \le c_1(p)||\psi||_2 \le c_2(p)||f^{(\alpha)}||_2, \quad c_1(p) := (\sqrt{2})^{1/p-1/2}c(p).$$

In light of (15), for $\alpha = 1/2$ we obtain then

$$||f||_q^2 \le c_2(p)K(f), \quad c_2(p) := \frac{2}{\pi}c_1(p)^2, \quad \left(p \in (1,2),\ q = \frac{p}{1-p/2}\right). \tag{18}$$

Let $x_0 \in (-\sqrt{2}/2, \sqrt{2}/2)$ be such that $|f(x_0)| = ||f||_\infty$. Since f is 2-Lipschitz,

$$|f(x)| \geq ||f||_\infty - 2|x - x_0|, \quad |x - x_0| \leq \frac{||f||_\infty}{2}.$$

Then

$$||f||_q^2 \geq \left(2\int_0^{||f||_\infty/2} (||f||_\infty - 2y)^q dy\right)^{2/q} = \frac{||f||_\infty^{2(q+1)/q}}{(q+1)^{2/q}},$$

so, using (18), we conclude that, for an absolute constant $c^*(p,q) > 0$,

$$K(f) \geq c^*(p)||f||_\infty^{2(q+1)/q}.$$

It remains to observe that

$$\frac{2(q+1)}{q} = 1 + \frac{2}{p}$$

can be made arbitrarily close to 2 from above by selecting p sufficiently close to 2 from below. This completes the proof. ■

Theorem 6. For a Young diagram λ whose graph lies within the $n \times n$ square, let $g_\lambda(u)$ be the rotated coordinate version of the function $f_\lambda(x)$ defined in (4). Denote $\alpha = k/n^2$. Then for all $2 < r < 3$, there are constants $c = c(r) > 0, C = C(r) > 0$ such that for any $\epsilon > 0$ and for any n,

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{1 \leq k \leq n^2-1} ||g_{\lambda_T^k} - \tilde{g}_\alpha||_\infty > \epsilon\right) \leq C \exp(3n - c\,\epsilon^r n^2). \tag{19}$$

Consequently, with probability subexponentially close to 1, for all k the supnorm distance between $g_{\lambda_T^k}$ and $\tilde{g}_\alpha$, $(\alpha = k/n^2)$, does not exceed $n^{-1/2+\delta}$, $(\delta > 0)$.

Proof. Let $p(m)$ be the number of partitions of an integer m. It is known that for all m, $p(m) \leq \exp(\pi\sqrt{2m/3})$ (see [2], Theorem 14.5). Fix n, $1 \leq k \leq n^2 - 1$, $\epsilon > 0$. Using Lemma 1,

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : ||g_{\lambda_T^k} - \tilde{g}_\alpha||_\infty > \epsilon\right) = \sum_{\substack{\lambda_0 \subseteq \square_n \text{ of area } k \\ ||g_{\lambda_0} - \tilde{g}_\alpha||_\infty > \epsilon}} \mathbb{P}_n\left(T \in \mathcal{T}_n : \lambda_T^k = \lambda_0\right)$$

$$\leq p(k) \sup_{\substack{\lambda_0 \subseteq \square_n \text{ of area } k \\ ||g_{\lambda_0} - \tilde{g}_\alpha||_\infty > \epsilon}} \exp\left(-(1+o(1))n^2(K(g_{\lambda_0}) + H(\alpha) - \log 2)\right). \tag{20}$$

Let λ_0 be a diagram contained in $\square_n$ of area k, such that $||g_{\lambda_0} - \tilde{g}_\alpha||_\infty > \epsilon$. Since g_{λ_0} is α-admissible, using Theorem 5 and Lemma 4 we have

$$K(g_{\lambda_0}) + H(\alpha) - \log 2 \geq K(g_{\lambda_0} - \tilde{g}_\alpha) > c(r)||g_{\lambda_0} - \tilde{g}_\alpha||_\infty^r \geq c(r)\epsilon^r.$$

Combining this with (20) and with the above remark on the number of partitions of an integer gives that for n larger than some absolute initial bound,

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : ||g_{\lambda_T^k} - \tilde{g}_\alpha||_\infty > \epsilon\right) \leq \exp(2.8\sqrt{\alpha}n - cn^2\epsilon^r).$$

Taking the union bound over all $1 \leq k \leq n^2 - 1$ gives (19). ■

Lemma 5. For each $(x, y) \in (0, 1) \times (0, 1)$, let (u, v) be their rotated coordinates as in (1). Let $\alpha_0 = L(x, y)$, so that $|u| < \sqrt{2\alpha_0(1-\alpha_0)}$ and $v = \tilde{g}_{\alpha_0}(u)$. There exist absolute constants $c_1, c_2 > 0$ such that if we set

$$\sigma(x, y) = \min(xy, (1-x)(1-y)),$$

$$d(x, y) = c_1\sqrt{\sigma(x, y)}, \quad \Delta(x, y) = c_2\sigma^2(x, y),$$

we will have that for all $0 < \alpha < 1$ and $\delta < \Delta(x, y)$, if $|\tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u)| < \delta \cdot d(x, y)$ then $|\alpha - \alpha_0| < \delta$.

Proof. Since $\tilde{g}_\alpha(u)$ increases with α, it suffices to prove existence of two absolute constants $\gamma_1, \gamma_2 > 0$ such that

$$|\tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u)| \geq \gamma_1\sigma^{1/2}(x, y)|\alpha - \alpha_0|, \quad \text{if } |\alpha - \alpha_0| \leq \gamma_2\sigma(x, y).$$

Because of the symmetry property $\tilde{g}_{1-\alpha}(u) = \sqrt{2} - \tilde{g}_\alpha(u)$, we may assume that $x + y \leq 1$, or equivalently that $\alpha_0 \leq 1/2$.

To prove the above claim, we note the following inequalities. Notice first that

$$\sqrt{2\alpha_0(1-\alpha_0)} \geq v \implies \alpha_0 \geq \frac{1 - \sqrt{1-2v^2}}{2}.$$

Likewise, $\alpha^{(-)}$ that corresponds to the lowest point (u, u) is given by

$$\alpha^{(-)} = \frac{1 - \sqrt{1-2u^2}}{2}.$$

and we see that

$$\alpha_0 - \alpha^{(-)} \geq \frac{\sqrt{1-2u^2} - \sqrt{1-2v^2}}{2} = \frac{v^2 - u^2}{\sqrt{1-2u^2} + \sqrt{1-2v^2}} \geq \frac{v^2 - u^2}{2} = xy. \tag{21}$$

(21) says that decreasing α_0 by $x_0 y_0$ gives us a feasible α, for which $(u, \tilde{g}_\alpha(u))$ lies between (u, v) and the lowest point (u, u), such that $u \le \sqrt{2\alpha(1-\alpha)}$.

Let us estimate from above $\tilde{g}_\alpha(u)$ for $\alpha \in [\alpha^{(-)}, \alpha_0]$. From (65) it follows that

$$\frac{\partial \tilde{g}_\alpha(u)/\partial\alpha}{\sqrt{\tilde{g}_\alpha(u)^2 - u^2}} \ge c$$

for some absolute constant $c > 0$. (Indeed, $2\alpha(1-\alpha) = \beta^2(\alpha) \ge \tilde{g}_\alpha(u)^2$.) Integrating from $\alpha \in [\alpha^{(-)}, \alpha_0]$ and exponentiating, we obtain

$$\frac{\tilde{g}_{\alpha_0}(u) + \sqrt{\tilde{g}_{\alpha_0}(u)^2 - u^2}}{\tilde{g}_\alpha(u) + \sqrt{\tilde{g}_\alpha(u)^2 - u^2}} \ge \exp(c(\alpha_0 - \alpha)),$$

or equivalently

$$\frac{\tilde{g}_\alpha(u) - \sqrt{\tilde{g}_\alpha(u)^2 - u^2}}{\tilde{g}_{\alpha_0}(u) - \sqrt{\tilde{g}_{\alpha_0}(u)^2 - u^2}} \ge \exp(c(\alpha_0 - \alpha)).$$

Consequently

$$\sqrt{\tilde{g}_\alpha(u)^2 - \tilde{g}_{\alpha_0}(u)^2} \le \cosh(c(\alpha_0 - \alpha))\sqrt{\tilde{g}_{\alpha_0}(u)^2 - u^2} - \sinh(c(\alpha_0 - \alpha))\tilde{g}_{\alpha_0}(u),$$

or

$$\tilde{g}_\alpha(u)^2 \le \left[\cosh(c(\alpha_0 - \alpha))\tilde{g}_{\alpha_0}(u) - \sinh(c(\alpha - \alpha_0))\sqrt{\tilde{g}_{\alpha_0}(u)^2 - u^2}\right]^2,$$

so that

$$\tilde{g}_\alpha(u) \le \cosh(c(\alpha_0 - \alpha))\tilde{g}_{\alpha_0}(u) - \sinh(c(\alpha - \alpha_0))\sqrt{\tilde{g}_{\alpha_0}(u)^2 - u^2}.$$

Consequently, for some constants $c_i > 0$,

$$\begin{aligned} \tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u) &\le -c_3(\alpha_0 - \alpha)[(v^2 - u^2)^{1/2} - c_4(\alpha_0 - \alpha)v] \\ &= -c_5(\alpha_0 - \alpha)[(xy)^{1/2} - c_6(\alpha_0 - \alpha)(x + y)] \\ &\le -c_7(\alpha_0 - \alpha)(xy)^{1/2}, \end{aligned}$$

provided that

$$\alpha_0 - \alpha \le c_8 \frac{(xy)^{1/2}}{x + y}.$$

¿From (21) we know that we can go below α_0 by xy at least. Pick $\rho = \min(1, c_8/3)$; then the last inequality holds for $\alpha_0 - \alpha \le \rho xy$, and we have

$$\tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u) \le -c_7(\alpha_0 - \alpha)(xy)^{1/2}, \qquad \alpha \in [\alpha_0 - \rho xy, \alpha_0]. \tag{22}$$

Now for $\alpha_0 \le \alpha \le 1/2$ we know that

$$\frac{\partial \tilde{g}_\alpha(u)}{\partial \alpha} \ge c_9\sqrt{v^2 - u^2} = c_{10}(xy)^{1/2},$$

so that

$$\tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u) \geq c_{10}(xy)^{1/2}(\alpha - \alpha_0). \tag{23}$$

By symmetry, for $1/2 \leq \alpha \leq 1 - \alpha_0$,

$$\tilde{g}_\alpha(u) - \tilde{g}_{1-\alpha_0} \leq -c_{10}((1-\alpha_0) - \alpha)((1-x)(1-y))^{1/2}, \tag{24}$$

and, for $1 - \alpha_0 \leq \alpha \leq 1 - \alpha_0 + \rho(1-x)(1-y)$,

$$\tilde{g}_\alpha(u) - \tilde{g}_{\alpha_0}(u) \geq c_7(\alpha - (1-\alpha_0))((1-x)(1-y))^{1/2}. \tag{25}$$

The inequalities (22), (23), (24), (25) prove the claim with $\gamma_1 = \min\{c_7, c_{10}\}$ and $\gamma_2 = \rho$.
■

Proof of Theorem 1(ii). We now prove Theorem 1(ii), the part of Theorem 1 that deals with the interior of the square. The treatment of the boundary of the square is more delicate and is deferred to section 4, being essentially equivalent to Theorem 3.

Fix $1 \leq i, j \leq n$ such that

$$\min(ij, (n-i)(n-j)) > n^{3/2+\epsilon}. \tag{26}$$

Let (u, v) be the rotated coordinates corresponding to $(x, y) = (i/n, j/n)$. Let $\alpha_0 = L(i/n, j/n)$, so that $v = \tilde{g}_{\alpha_0}(u)$ and $|u| < \sqrt{2\alpha_0(1-\alpha_0)}$. For each tableau $T = (t_{i,j})_{1 \leq i,j \leq n} \in \mathcal{T}_n$ let $k_T = t_{i,j}$, and let $\beta_T = k_T/n^2$. Note that k_T is an integer representing the smallest s such that λ_T^s contains the box (i, j). This implies that $v = g_{\lambda_T^{k_T}}(u)$. Apply Lemma 5 with $(x, y) = (i/n, j/n)$ and $\delta = n^{-(1-\epsilon)/2}$. Note that because of (26), for n large we have $\delta < \Delta(x, y)$ as required. Then, making use of Theorem 6 we get

$$\begin{aligned}
\mathbb{P}_n\left(T \in \mathcal{T}_n : \left|\frac{1}{n^2}t_{i,j} - L\left(\frac{i}{n}, \frac{j}{n}\right)\right| > \frac{1}{n^{(1-\epsilon)/2}}\right) &= \mathbb{P}_n\Big(T \in \mathcal{T}_n : |\beta_T - \alpha_0| > \frac{1}{n^{(1-\epsilon)/2}}\Big) \\
\text{(by Lemma 5)} \quad &\leq \mathbb{P}_n\Big(T \in \mathcal{T}_n : |\tilde{g}_{\beta_T}(u) - \tilde{g}_{\alpha_0}(u)| > \frac{d(i/n, j/n)}{n^{(1-\epsilon)/2}}\Big) \\
&= \mathbb{P}_n\Big(T \in \mathcal{T}_n : \Big|g_{\lambda_T^{k_T}}(u) - \tilde{g}_{\beta_T}(u)\Big| > \frac{d(i/n, j/n)}{n^{(1-\epsilon)/2}}\Big) \\
\text{(by Theorem 6 with } r = 2+\epsilon) \quad &\leq C\exp\left(3n - cn^2\left(\frac{d(i/n, j/n)}{2n^{(1-\epsilon)/2}}\right)^{2+\epsilon}\right) \\
\text{(for } n \text{ large, by (26))} \quad &\leq C'\exp(-c'n^{3/2}).
\end{aligned}$$

Taking the union bound over all $1 \leq i, j \leq n$ satisfying (26) gives the result. ■

3 Solution of the variational problem

3.1 Preliminaries

In this section, we prove Theorem 5. We actually *derive* the explicit formula for the minimizer using methods of the calculus of variations and the theory of singular (Cauchy-type) integral equations. Our derivation makes only one a priori assumption (obtained by educated guesswork and later verified by computer simulations) on the graphical form that the minimizer would take, and so is in a sense more systematic than the analogous treatments in the fundamental papers [14], [19], [20], where the solutions are brilliantly guessed using the properties of the Hilbert transform. We believe that our technique may prove useful in the treatment of similar problems in the future.

First, observe that because of symmetry, we need only treat the case $\alpha \leq 1/2$; the mapping $g \to \sqrt{2} - g$ takes the set of α-admissible functions bijectively onto the set of $(1-\alpha)$-admissible functions, and has the property that $K(\sqrt{2} - g) = K(g)$.

Next, observe that for $\alpha = 1/2$ the assertion is immediate, because of Lemma 3.

We prove another fact that follows from general considerations, before turning to the derivation of the minimizer.

Lemma 6. For any $0 < \alpha < 1$, the functional K has a unique α-admissible minimizer.

Proof. The functional K is continuous on the space of α-admissible functions, and is bounded below by Lemma 3. By the Arzela-Ascoli theorem, the space of α-admissible functions is compact in the topology induced by the supremum norm (since the admissible functions are uniformly bounded and equicontinuous). Therefore K has a minimizer. To prove that the minimizer is unique, let h_1 and h_2 be two distinct α-admissible minimizers. Then $\tilde{h} = (h_1+h_2)/2$ is also an α-admissible function, and $g = (h_1-h_2)/2 \not\equiv 0$, $g(\pm\sqrt{2}/2) = 0$. So, using the parallelogram identity and Lemma 3,

$$K(\tilde{h}) = \frac{1}{2}K(h_1) + \frac{1}{2}K(h_2) - K(g) < \min_{h \text{ is } \alpha\text{-admissible}} K(h),$$

a contradiction. ■

3.2 The derivation

We now proceed with the derivation of the minimizer, which we shall denote $h = h_\alpha$. The dependence on α will be suppressed except where it is required. For the rest of this section, α will be a fixed value in $(0, 1/2)$, unless stated otherwise.

First, note that, under the condition $h(\pm\sqrt{2}/2) = \sqrt{2}/2$, the α-condition $\int_{-\sqrt{2}/2}^{\sqrt{2}/2}(h(u)-|u|)du = \alpha$ is equivalent to

$$-\int_{-\sqrt{2}/2}^{\sqrt{2}/2} uh'(u)du = \alpha - \frac{1}{2}. \tag{27}$$

We now formulate a sufficient condition for h to be a minimizer. It is based on a standard recipe of the calculus of variations, the Lagrange formalism. We form the Lagrange function

$$\mathcal{L}(h,\lambda) = K(h) - \lambda \int_{-\sqrt{2}/2}^{\sqrt{2}/2} uh'(u)du$$

and require that, for some λ, h_α be a local minimum point of $\mathcal{L}(h,\lambda)$ in the convex set of functions h subject to all the restrictions except the α-condition (27). To be sure, we ought to include into the function a term λ' times the integral of h', since h must meet another constraint

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} h'(u)du = 0. \tag{28}$$

We chose not to, since – in the square case – even without this constraint $h'(u)$ will turn out to be odd anyway. Since $\mathcal{L}(h,\lambda)$ depends explicitly on h' alone, we get the equations for the sufficient condition in a simple-minded manner, by taking partial derivatives of $\mathcal{L}$ with respect to $h'(s)$, $s \in (-\sqrt{2}/2, \sqrt{2}/2)$, and paying attention only to the constraint $-1 \le h'(s) \le 1$. The resulting "complementary slackness" conditions are

$$w(s) := -\int_{-\sqrt{2}/2}^{\sqrt{2}/2} h'(t)\log|s-t|dt - \lambda s \quad \text{is} \quad \begin{cases} = 0, & \text{if } -1 < h'(s) < 1, \\ \ge 0, & \text{if } h'(s) = -1, \\ \le 0, & \text{if } h'(s) = 1. \end{cases} \tag{29}$$

Lemma 7. If h is an α-admissible function that, for some $\lambda \in \mathbb{R}$, satisfies (29) for all $s \in (-\sqrt{2}/2, \sqrt{2}/2)$ for which $h'(s)$ is defined, then h is a minimizer.

Proof. If g is a 1-Lipschitz function on $[-\sqrt{2}/2, \sqrt{2}/2]$, then (29) implies that $(g'(s) - h'(s))w(s) \ge 0$ for all s for which this is defined, so

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} g'(s)w(s)ds \ge \int_{-\sqrt{2}/2}^{\sqrt{2}/2} h'(s)w(s)ds.$$

If g is α-admissible, by (27) this can be written as

$$\begin{aligned} 2\langle h,g\rangle + \alpha - \frac{1}{2} &= 2\langle h,g\rangle - \lambda\int_{-\sqrt{2}/2}^{\sqrt{2}/2} sg'(s)ds \\ &\ge 2\langle h,h\rangle - \lambda\int_{-\sqrt{2}/2}^{\sqrt{2}/2} sh'(s)ds = 2\langle h,h\rangle + \alpha - \frac{1}{2}, \end{aligned}$$

which shows that

$$\langle h, g\rangle \geq \langle h, h\rangle.$$

Therefore, by Lemma 3 applied to the function $g - h$,

$$\langle g, g\rangle = \langle h, h\rangle + 2\langle h, g-h\rangle + \langle g-h, g-h\rangle \geq \langle h, h\rangle,$$

so h is a minimizer. ■

We are about to prove part (i) of Theorem 5, namely that $h = \tilde{g}_\alpha$ is the minimizer. Assuming this, note that in the above proof we actually showed that

$$\langle g, g\rangle \geq \langle \tilde{g}_\alpha, \tilde{g}_\alpha\rangle + \langle g - \tilde{g}_\alpha, g - \tilde{g}_\alpha\rangle,$$

which is precisely the claim of part (iii) of Theorem 5. So it remains to prove parts (i) and (ii).

Our challenge now is to determine an admissible h that meets the conditions (29). Now look at Figure 1(c) with your head tilted 45 degrees to the right. Based on the shape of the level curves, we make the following assumption: For some $\beta = \beta(\alpha) \in (0, \sqrt{2}/2)$,

$$h'(s) \quad \text{is} \begin{cases} = -1, & \text{if } -\sqrt{2}/2 < s < -\beta, \\ \in (-1,1), & \text{if } -\beta < s < \beta, \\ = +1, & \text{if } \beta < s < \sqrt{2}/2. \end{cases} \tag{30}$$

Substituting this into (29) gives that for $-\beta < s < \beta$,

$$\begin{aligned} -\int_{-\beta}^{\beta} h'(t)\log|s-t|dt &= \lambda s - \int_{-\sqrt{2}/2}^{\beta} \log(s-t)dt + \int_{\beta}^{\sqrt{2}/2} \log(t-s)dt \\ &= \lambda s + (\sqrt{2}/2 - s)\log(\sqrt{2}/2 - s) - (\sqrt{2}/2 + s)\log(\sqrt{2}/2 + s) \\ &\quad + (\beta + s)\log(\beta + s) - (\beta - s)\log(\beta - s) \end{aligned} \tag{31}$$

Assume that $h'(s)$ is continuously differentiable on $(-\beta, \beta)$. Differentiate (31), to obtain

$$-\int_{-\beta}^{\beta} \frac{h'(t)}{s-t}dt = \lambda + \log\frac{\beta^2 - s^2}{\frac{1}{2} - s^2}, \tag{32}$$

where the left-hand side is a principal value integral.

In the theory of integral equations this is known as an airfoil equation. Solving it is tantamount to inverting a Hilbert transform on a finite interval. Fortunately for us, it can be solved! The following theorem appears in [7], Section 3.2, p. 74. (See also [15], Section 9.5.2.)

Theorem 7. The general solution of the airfoil equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{g(y)}{y-x}\,dx = f(x), \quad |x|<1,$$

with the integral understood in the principal value sense, and $f(x)$ satisfying a Hölder condition, is given by

$$g(x) = \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\frac{\sqrt{1-y^2}f(y)}{x-y}\,dy + \frac{c}{\sqrt{1-x^2}}. \tag{33}$$

Applying Theorem 7 to (32), we get the equation

$$h'(s) = \frac{1}{\pi^2(\beta^2-s^2)^{1/2}}\int_{-\beta}^{\beta}(\beta^2-t^2)^{1/2}\left(\lambda+\log\frac{\beta^2-t^2}{\frac{1}{2}-t^2}\right)\frac{dt}{s-t} + \frac{c}{(\beta^2-s^2)^{1/2}}. \tag{34}$$

Here the integral is again in the sense of principal value, and the equation must hold for some value of c.

We evaluate the integral in (34). Consider the contribution of the λ-term first. Substituting $t=\beta\sin x$ and later $u=\tan x/2$, we get

$$\begin{aligned}\int_{-\beta}^{\beta}\frac{(\beta^2-t^2)^{1/2}}{s-t}dt &= \beta\int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{s/\beta-\sin x}dx\\ &= \beta\int_{-\pi/2}^{\pi/2}(s/\beta+\sin x)dx + \beta\left(1-(s/\beta)^2\right)\int_{-\pi/2}^{\pi/2}\frac{dx}{s/\beta-\sin x}\\ &= \pi s + \frac{2\beta\left(1-(s/\beta)^2\right)}{s/\beta}\int_{-1}^{1}\frac{du}{u^2-2(\beta/s)u+1}.\end{aligned}$$

For $|s|<\beta$, the denominator in the last integral has two real roots, $u_1\in(-1,1)$ and $u_2\notin(-1,1)$. A simple computation shows that the principal value of this integral at $u=u_1$ is zero. So

$$\int_{-\beta}^{\beta}\frac{(\beta^2-t^2)^{1/2}}{s-t}dt = \pi s, \qquad s\in(-\beta,\beta). \tag{35}$$

Turn to the log-part of the integral in (34). Substituting $t=\tau\beta$, $s=v_1\beta$, $(2\beta^2)^{-1}=v_2^2$, we see that

$$\int_{-\beta}^{\beta}\frac{(\beta^2-t^2)^{1/2}}{s-t}\log\frac{\beta^2-t^2}{\frac{1}{2}-t^2}dt = \beta[I(s/\beta,\sqrt{2}/(2\beta)) - I(-s/\beta,\sqrt{2}/(2\beta))], \tag{36}$$

where

$$I(\xi,\gamma) = \int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{\xi-\eta}\log\frac{1+\eta}{\gamma+\eta}d\eta, \qquad \xi\in[-1,1],\gamma\geq 1,$$

is evaluated in the following lemma.

Lemma 8.

$$I(\xi,\gamma) = \pi\Bigg[1-\gamma+\sqrt{\gamma^2-1}-\xi\log\left(\gamma+\sqrt{\gamma^2-1}\right) \\ -2\sqrt{1-\xi^2}\tan^{-1}\sqrt{\frac{(\gamma-1)(1-\xi)}{(\gamma+1)(1+\xi)}}\,\Bigg]. \tag{37}$$

Proof. Notice that $I(\xi,1)=0$, and, for $x>1$,

$$\begin{aligned}\frac{\partial I(\xi,x)}{\partial x} &= -\int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{(\xi-\eta)(x+\eta)}d\eta \\ &= -\frac{1}{x+\xi}\left[\int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{\xi-\eta}d\eta+\int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{x+\eta}d\eta\right] \\ &= -\frac{\pi\xi}{x+\xi}-\frac{1}{x+\xi}\int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{x+\eta}d\eta, \end{aligned} \tag{38}$$

see (35). Substituting $\eta=\sin t$, $(t\in[-\pi/2,\pi/2])$, and then $t=2\tan^{-1}u$, $(u\in[-1,1])$, we evaluate

$$\begin{aligned}\int_{-1}^{1}\frac{(1-\eta^2)^{1/2}}{x+\eta}d\eta &= (xt+\cos t)|_{\pi/2}^{\pi/2}+(1-x^2)\int_{-\pi/2}^{\pi/2}\frac{dt}{x+\sin t} \\ &= \pi x+2(1-x^2)\int_{-1}^{1}\frac{du}{x(1+u^2)+2u} = \\ &= \pi x-2(x^2-1)^{1/2}\left[\tan^{-1}\sqrt{\frac{x+1}{x-1}}+\tan^{-1}\sqrt{\frac{x-1}{x+1}}\right] \\ &= \pi(x-(x^2-1)^{1/2}). \end{aligned} \tag{39}$$

Combining this with (38), we obtain

$$\frac{\partial I(\xi,x)}{\partial x} = -\pi+\frac{\pi(x^2-1)^{1/2}}{x+\xi}.$$

We integrate this equation from $x=1$ to $x=\gamma>1$, and use the substitutions $x=\cosh t$, $t\in[0,t_0]$, with

$$t_0=\operatorname{arccosh}\gamma=\log\left(\gamma+(\gamma^2-1)^{1/2}\right),$$

and then $u=e^t$, $u\in[1,u_0]$, with

$$u_0=e^{t_0}=\gamma+(\gamma^2-1)^{1/2}.$$

We have

$$
\begin{aligned}
I(\xi,\gamma) &= -\pi(\gamma-1)+\pi\int_0^{t_0}\frac{\sinh^2 t}{\cosh t+\xi}dt \\
&= -\pi(\gamma-1)+\pi\left[(\sinh t-\xi t)|_0^{t_0}+2(\xi^2-1)\int_0^{u_0}\frac{du}{u^2+2\xi u+1}\right]. \qquad (40)
\end{aligned}
$$

Here

$$
(\sinh t-\xi t)|_0^{t_0} = (\gamma^2-1)^{1/2}-\xi\log\left(\gamma+(\gamma^2-1)^{1/2}\right), \qquad (41)
$$

and the last integral equals

$$
\begin{aligned}
\frac{1}{\sqrt{1-\xi^2}}\tan^{-1}\frac{u+\xi}{(1-\xi^2)^{1/2}}\Bigg|_1^{u_0} &= \frac{1}{\sqrt{1-\xi^2}}\tan^{-1}\frac{(u_0-1)(1-\xi^2)^{1/2}}{1-\xi^2+(u_0+\xi)(1+\xi)} \\
&= \frac{1}{\sqrt{1-\xi^2}}\tan^{-1}\frac{u_0-1}{u_0+1}\sqrt{\frac{1+\xi}{1-\xi}} = \frac{1}{\sqrt{1-\xi^2}}\tan^{-1}\sqrt{\frac{(\gamma-1)(1-\xi)}{(\gamma+1)(1+\xi)}}. \qquad (42)
\end{aligned}
$$

Combining (40), (41), (42) gives (37). ■

Now from (35), (36) and (37) we get

$$
\begin{aligned}
h'(s) = {}& \frac{c}{(\beta^2-s^2)^{1/2}}+\frac{s}{\pi(\beta^2-s^2)^{1/2}}\left(\lambda-2\log\frac{1+\sqrt{1-2\beta^2}}{\sqrt{2}\beta}\right) \\
&+\frac{2}{\pi}\left(\tan^{-1}\sqrt{\frac{(\gamma-1)(1+\xi)}{(\gamma+1)(1-\xi)}}-\tan^{-1}\sqrt{\frac{(\gamma-1)(1-\xi)}{(\gamma+1)(1+\xi)}}\right), \qquad (43)
\end{aligned}
$$

with $\xi = s/\beta$, $\gamma = \sqrt{2}/(2\beta)$, or, after some simplification,

$$
\begin{aligned}
h'(s) \quad = \quad & \frac{c}{(\beta^2-s^2)^{1/2}}+\frac{s}{\pi(\beta^2-s^2)^{1/2}}\left(\lambda-2\log\frac{1+\sqrt{1-2\beta^2}}{\sqrt{2}\beta}\right) \\
&+\frac{2}{\pi}\tan^{-1}\frac{(1-2\beta^2)^{1/2}s}{(\beta^2-s^2)^{1/2}}.
\end{aligned}
$$

We now observe that the only values of c and λ for which the right-hand side is bounded as $s\nearrow\beta$, $s\searrow-\beta$, and therefore has a chance of being the derivative of an α-admissible function, are

$$
c=0, \qquad \lambda = 2\log\frac{1+\sqrt{1-2\beta^2}}{\sqrt{2}\beta}. \qquad (44)
$$

Therefore we get

$$
h'(s) = \frac{2}{\pi}\tan^{-1}\frac{(1-2\beta^2)^{1/2}s}{(\beta^2-s^2)^{1/2}}. \qquad (45)
$$

Note that $h'(s) \in (-1,1)$. We have determined $h'(s)$, except the value of $\beta = \beta(\alpha)$ such that h is α-admissible, i.e., satisfies (27). Rewrite (27) as

$$\int_{-\beta}^{\beta} sh'(s)ds = \alpha - \beta^2. \tag{46}$$

Besides evaluating this last integral, to compute $h(s)$ explicitly we will need $\int_{-\beta}^{s} h'(u)du$. To this end, integrating the first arctangent-of-radical function in (43) on the interval $[-1,\xi]$, $(\xi \in (-1,1])$, we get

$$\int_{-1}^{\xi} \tan^{-1}\sqrt{\frac{(\gamma-1)(1+\eta)}{(\gamma+1)(1-\eta)}}d\eta$$
$$= \xi \tan^{-1}\sqrt{\frac{(\gamma-1)(1+\eta)}{(\gamma+1)(1-\eta)}} - \frac{\sqrt{\gamma^2-1}}{2}\int_{-1}^{\xi} \frac{\eta\, d\eta}{(\gamma-\eta)\sqrt{1-\eta^2}}. \tag{47}$$

Substituting in the last integral $\eta = \sin t$, and then $u = \tan t$, we transform it into

$$\begin{aligned}
&-t_0 - \frac{\pi}{2} + \gamma\int_{-\pi/2}^{t_0} \frac{dt}{\gamma - \sin t} && [t_0 = \sin^{-1}\xi] \\
&= -t_0 - \frac{\pi}{2} + 2\int_{-1}^{u_0} \frac{du}{1+u^2-2u/\gamma} && [u_0 = \tan t_0/2] \\
&= -t_0 - \frac{\pi}{2} + \frac{2\gamma}{\sqrt{\gamma^2-1}}\left(\tan^{-1}\frac{u_0-\gamma^{-1}}{\sqrt{1-\gamma^{-2}}} + \tan^{-1}\frac{1+\gamma^{-1}}{\sqrt{1-\gamma^{-2}}}\right) \\
&= -t_0 - \frac{\pi}{2} + \frac{2\gamma}{\sqrt{\gamma^2-1}}\tan^{-1}\left(\frac{1+u_0}{1-u_0}\sqrt{\frac{\gamma-1}{\gamma+1}}\right);
\end{aligned} \tag{48}$$

here

$$\frac{1+u_0}{1-u_0} = \frac{1+\tan t_0/2}{1-\tan t_0/2} = \frac{1+\sin t_0}{\cos t_0} = \frac{1+\xi}{\sqrt{1-\xi^2}} = \sqrt{\frac{1+\xi}{1-\xi}}. \tag{49}$$

From (47), (48), (49) we obtain

$$\int_{-1}^{\xi} \tan^{-1}\sqrt{\frac{(\gamma-1)(1+\eta)}{(\gamma+1)(1-\eta)}}d\eta$$
$$= (\xi-\gamma)\tan^{-1}\sqrt{\frac{(\gamma-1)(1+\xi)}{(\gamma+1)(1-\xi)}} + \frac{\sqrt{\gamma^2-1}}{2}\left(\sin^{-1}\xi + \frac{\pi}{2}\right). \tag{50}$$

In a similar fashion

$$\int_{-1}^{1} \eta \tan^{-1}\sqrt{\frac{(1+\eta)(\gamma-1)}{(1-\eta)(\gamma+1)}}d\eta = \frac{\pi}{4}(1-\gamma^2+\gamma\sqrt{\gamma^2-1}), \tag{51}$$

and the integral in the negative arctangent in (43) is obviously given by (51) as well. Using (43) and (51), we see that the α-condition (46) is equivalent to

$$\beta^2(\gamma^2 - \gamma\sqrt{\gamma^2-1}) = \alpha \iff 1 - 2\alpha = \sqrt{1-2\beta^2},$$

the latter being possible only if $\alpha < 1/2$. In that case

$$\beta = \sqrt{2\alpha(1-\alpha)}. \tag{52}$$

Consequently, see (44),

$$\lambda = \log\frac{1-\alpha}{\alpha}, \tag{53}$$

and, see (45),

$$h'(s) = \frac{2}{\pi}\tan^{-1}\left(\frac{(1-2\alpha)s}{\sqrt{2\alpha(1-\alpha)-s^2}}\right), \qquad s \in (-\sqrt{2\alpha(1-\alpha)}, \sqrt{2\alpha(1-\alpha)}). \tag{54}$$

Furthermore, denoting the integral in (50) by $J(\xi,\gamma)$, we easily get

$$\begin{aligned} h(s) &= \beta + \int_{-\beta}^{s} h'(t)dt = \beta(1 + J(\xi,\gamma) + J(-\xi,\gamma) - J(1,\gamma)) \\ &= \frac{2}{\pi}s\tan^{-1}\left(\frac{(1-2\alpha)s}{\sqrt{2\alpha(1-\alpha)-s^2}}\right) + \frac{\sqrt{2}}{\pi}\tan^{-1}\left(\frac{\sqrt{2(2\alpha(1-\alpha)-s^2)}}{1-2\alpha}\right). \end{aligned} \tag{55}$$

We have derived a formula for a candidate minimizer, which we now recognize as the function $\tilde{g}_\alpha$ that we defined in section 2. To be sure, this function was determined so as to meet the ramifications of *some* of the constraints. However, looking at (54), we see that $-1 < h'(s) < 1$ for $s \in (-\beta, \beta)$, so h is indeed 1-Lipschitz, even though so far we haven't paid attention to this constraint! Furthermore, since $h'(s)$ is odd, the constraint (28) is met automatically, and it is the reason why we were able to satisfy the boundary constraints $h(-\sqrt{2}/2) = h(\sqrt{2}/2) = \sqrt{2}/2$. Also, we determined β from the requirement that h should satisfy (46), which under these boundary conditions is equivalent to the α-condition. We conclude that, at the very least, $\tilde{g}_\alpha$ meets all the constraints, thus is α-admissible.

By Lemma 7, to prove that $\tilde{g}_\alpha$ is the minimizer, it only remains to prove that $\tilde{g}_\alpha$ satisfies the conditions (29). By (32), $w'(s) \equiv 0$ for $|s| < \beta$. And $w(0) = 0$ as $h'(t)$ is odd. So $w(s) \equiv 0$ for $|s| < \beta$, hence the first condition in (29) is met. As for the remaining conditions, by (anti)symmetry, it suffices to check, say, the third condition, namely that

$$F(s,\alpha) := -\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \tilde{g}'_\alpha(t)\log|s-t|dt - \lambda(\alpha)s \le 0, \qquad \beta(\alpha) \le s \le \sqrt{2}/2.$$

Fix $0 < s \le \sqrt{2}/2$, and let $\hat{\alpha} = (1 - \sqrt{1-2s^2})/2$, so that $\beta(\hat{\alpha}) = s$. Clearly, because of the first condition in (29), $F(s, \hat{\alpha}) = 0$. To finish the proof, we will now show that $\partial F(s,\alpha)/\partial\alpha > 0$ for $0 < \alpha < \hat{\alpha}$. By (31),

$$\frac{\partial F(s,\alpha)}{\partial\beta} = -\int_{-\beta}^{\beta} \frac{\partial \tilde{g}'_\alpha(t,\alpha)}{\partial\beta} \log|s-t|dt - s\frac{\partial\lambda}{\partial\beta}. \tag{56}$$

Using (45) and simplifying gives

$$\frac{\partial \tilde{g}'_\alpha(t)}{\partial\beta} = -\frac{2}{\pi\beta(1-2\beta^2)^{1/2}} \cdot \frac{t}{(\beta^2-t^2)^{1/2}}.$$

Since $\beta'(\alpha) = (1-2\beta^2)^{1/2}/\beta$, (56) becomes

$$\frac{\partial F(s,\alpha)}{\partial\alpha} = \frac{2}{\pi\beta^2}\int_{-\beta}^{\beta} \frac{t\log|s-t|}{(\beta^2-t^2)^{1/2}}dt + \frac{s}{(1-\alpha)\alpha}.$$

Here the integral equals

$$-(\beta^2-t^2)^{1/2} - \log|s-t|\Big|_{-\beta}^{\beta} - \int_{-\beta}^{\beta}\frac{(\beta^2-t^2)^{1/2}}{s-t}dt = -\pi(s-(s^2-\beta^2)^{1/2}),$$

see (37). Therefore

$$\begin{aligned}
\frac{\partial F(s,\alpha)}{\partial\alpha} &= -\frac{2}{\beta^2}\left(s-(s^2-\beta^2)^{1/2}\right) + \frac{s}{(1-\alpha)\alpha} \\
&= s\left(\frac{1}{(1-\alpha)\alpha} - \frac{2}{\beta^2}\right) + \frac{2}{\beta^2}(s^2-\beta^2)^{1/2} \\
&= \frac{2}{\beta^2}(s^2-\beta^2)^{1/2} > 0.
\end{aligned}$$

3.3 Direct computation of $K(\tilde{g}_\alpha)$

Our next goal in this section is to show that $K(\tilde{g}_\alpha) = -H(\alpha) + \log 2$. There are two ways to do this. First, looking at the proof of Theorem 6, we see that we may repeat the arguments of that proof (without assuming the value of $K(\tilde{g}_\alpha)$ as in that proof) to deduce that the value M_α of $K(\tilde{g}_\alpha) + H(\alpha) - \log 2$ *must* be 0. For, if it were greater than 0, then, denoting $k = \lfloor \alpha n^2 \rfloor$, we would have

$$\begin{aligned}
1 = \mathbb{P}_n\left(T \in \mathcal{T}_n\right) &= \sum_{\lambda_0 \text{ of area } k} \mathbb{P}_n\Big(T \in \mathcal{T}_n : \lambda_T^k = \lambda_0\Big) \\
&\le p(n^2)\exp\left(-(1+o(1))n^2 M_{k/n^2}\right) \xrightarrow[n\to\infty]{} 0
\end{aligned}$$

(since M_α is obviously continuous in α.) On the other hand, if $M_\alpha < 0$, then for some sufficiently large n, we would have for some diagram λ_0 of area $\lfloor \alpha n^2 \rfloor$ contained in $\square_n$, that $K(g_{\lambda_0}) + H(\alpha) - \log 2 < 0$ (take a diagram for which g_{λ_0} approximates $\tilde{g}_\alpha$, and use Lemma 2). But this again implies a contradiction:

$$1 \geq \mathbb{P}_n\left(T \in \mathcal{T}_n : \lambda_T^{\lfloor \alpha n^2 \rfloor} = \lambda_0\right) = \exp\left(-(1+o(1))n^2(K(g_{\lambda_0}) + H(\alpha) - \log 2)\right) > 1.$$

These last remarks notwithstanding, we find it worthwhile to compute $K(\tilde{g}_\alpha)$ directly, if only to thoroughly test our derivation of $\tilde{g}_\alpha$, and to show that all the integrals involved can be evaluated explicitly.

For $h = \tilde{g}_\alpha$, rewrite (27) as

$$-\int_{-\sqrt{2}/2}^{\sqrt{2}/2} u(h'(u) - \operatorname{sgn} u)du = \alpha.$$

Using this, multiply both sides of (29) by $(h'(s) - \operatorname{sgn} s)$ and integrate, obtaining

$$K(h) = -\frac{\lambda\alpha}{2} - \frac{1}{2}\int_{-\sqrt{2}/2}^{\sqrt{2}/2} h'(t)\Big[2t\log|t| - (t+\sqrt{2}/2)\log|t+\sqrt{2}/2| \\ - (t-\sqrt{2}/2)\log|t-\sqrt{2}/2|\Big]dt, \tag{57}$$

where we found before that $\lambda = \log((1-\alpha)/\alpha)$. Denote

$$S(t) = 2t\log|t| - (t+\sqrt{2}/2)\log|t+\sqrt{2}/2| - (t-\sqrt{2}/2)\log|t-\sqrt{2}/2|,$$

and set

$$K_1(h) = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} h'(t)S(t)dt,$$

so that $K(h) = -\lambda\alpha/2 - K_1(h)/2$. Just like (56),

$$\frac{\partial K_1(h_\alpha)}{\partial\beta} = \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\partial h'_\alpha(t)}{\partial\beta} S(t)dt = \frac{2}{\pi\beta(1-2\beta^2)^{1/2}}\int_{-\beta}^{\beta} \frac{-t}{(\beta^2-t^2)^{1/2}} S(t)dt \\ = -\frac{2}{\pi\beta(1-2\beta^2)^{1/2}}\int_{-\beta}^{\beta}(\beta^2-t^2)^{1/2}\Big[2\log|t| - \log|t+\sqrt{2}/2| - \log|t-\sqrt{2}/2|\Big]dt. \tag{58}$$

Denote

$$E(s,\beta) = \int_{-\beta}^{\beta}(\beta^2-t^2)^{1/2}\log|t-s|dt,$$

so that

$$\frac{\partial K_1(h_\alpha)}{\partial\beta} = 2E(0,\beta) - E(-\sqrt{2}/2,\beta) - E(\sqrt{2}/2,\beta). \tag{59}$$

By (35) and (37),

$$\begin{aligned}\frac{\partial E(s,\beta)}{\partial s} &= \int_{-\beta}^{\beta} \frac{(\beta^2-t^2)^{1/2}}{s-t}dt \\ &= \begin{cases} \pi s & |s|<\beta, \\ \pi(\operatorname{sgn} s)\left(|s|-(s^2-\beta^2)^{1/2}\right), & \beta<|s|<\sqrt{2}/2. \end{cases}\end{aligned} \tag{60}$$

Therefore

$$2E(0,\beta) = E(\beta,\beta)+E(-\beta,\beta)+\pi\int_{\beta}^{0} sds+\pi\int_{-\beta}^{0} sds = E(\beta,\beta)+E(-\beta,\beta)-\pi\beta^2. \tag{61}$$

Likewise

$$E(-\sqrt{2}/2,\beta)+E(\sqrt{2}/2,\beta) = E(-\beta,\beta)+E(\beta,\beta)+2\pi\int_{\beta}^{\sqrt{2}/2}\left(s-(s^2-\beta^2)^{1/2}\right)ds, \tag{62}$$

where

$$\begin{aligned}\int_{\beta}^{\sqrt{2}/2}(s^2-\beta^2)^{1/2}ds &= \frac{1}{2}\left[s(s^2-\beta^2)^{1/2}-\beta^2\log\left(s+(s^2-\beta^2)^{1/2}\right)\right]\Bigg|_{\beta}^{\sqrt{2}/2} \\ &= \frac{1}{2}\left(\frac{1-2\alpha}{2}-\alpha(1-\alpha)\log\frac{1-\alpha}{\alpha}\right).\end{aligned} \tag{63}$$

So, using $\beta = (2\alpha(1-\alpha))^{1/2}$,

$$E(-\sqrt{2}/2,\beta)+E(\sqrt{2}/2,\beta) = E(-\beta,\beta)+E(\beta,\beta)+\pi\left(-\beta^2+\alpha+\alpha(1-\alpha)\log\frac{1-\alpha}{\alpha}\right),$$

and, combining this relation with (61), we simpify (59) to

$$\frac{\partial K_1(h_\alpha)}{\partial\beta} = -\pi\left(\alpha+\alpha(1-\alpha)\log\frac{1-\alpha}{\alpha}\right).$$

So, by (58)

$$\frac{\partial K_1(h_\alpha)}{\partial\alpha} = \frac{\partial K_1(h_\alpha)}{\partial\beta}\cdot\frac{(1-2\beta^2)^{1/2}}{\beta} = \frac{1}{1-\alpha}+\log\frac{1-\alpha}{\alpha}.$$

Since $h'_\alpha \equiv 0$ at $\alpha = 1/2$, we have $K_1(h) = 0$ at $\alpha = 1/2$. Hence

$$\begin{aligned}K_1(h_\alpha) = \int_{1/2}^{\alpha}\left(\frac{1}{1-x}+\log\frac{1-x}{x}\right)dx &= -\log(1-\alpha)-2\log 2 \\ &\quad -(1-\alpha)\log(1-\alpha)-\alpha\log\alpha,\end{aligned} \tag{64}$$

which gives finally for $K(h_\alpha)$

$$K(h) = \alpha\log\alpha+(1-\alpha)\log(1-\alpha)+\log 2 = -H(\alpha)+\log 2.$$

The proof of Theorem 5 is complete. ■

3.4 The parametric family $\tilde{g}_\alpha$

The minimality proof in section 3.2 relied on the possibility to consider simultaneously the whole family of variational problems, and thus to differentiate the minimizer $\tilde{g}_\alpha$ with respect to α. Moreover, to reveal a little secret, we anticipated the formula (53) for the Lagrange multiplier λ. According to a general (semiformal) recipe of the calculus of variations (more specifically, mathematical programming), we knew that this λ, dual to the α-condition, should be equal to $dK(\tilde{g}_\alpha)/d\alpha$, which we have proved to be correct. The advantages of this approach of varying the parameter α go even deeper than that. It will turn out that the partial derivative of the minimizer $g_\alpha(\cdot)$ with respect to α is the key to the distributional properties of the random tableau. Using the formula for the minimizer, we compute easily that

$$\frac{\partial \tilde{g}_\alpha(u)}{\partial \alpha} = \begin{cases} 0 & \sqrt{2\alpha(1-\alpha)} < |u| \le \sqrt{2}/2 \\ \frac{\sqrt{2\alpha(1-\alpha)-u^2}}{\pi\alpha(1-\alpha)} & |u| \le \sqrt{2\alpha(1-\alpha)} \end{cases} \tag{65}$$

For each α, direct integration reveals that $\partial \tilde{g}_\alpha(u)/\partial\alpha$ is a probability density function, i.e.

$$\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\partial \tilde{g}_\alpha(u)}{\partial \alpha}\, du = 1.$$

(In fact, it is the density of the semicircle distribution, and it will play a prominent role later – see section 5.) This observation is in perfect harmony with the fact that $\tilde{g}_\alpha$ satisfies the α-condition, thus providing a partial check of our computations. Indeed

$$\begin{aligned} \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\tilde{g}_\alpha(u) - |u|)\, du &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} (\tilde{g}_\alpha(u) - \tilde{g}_0(u))\, du \\ &= \int_{-\sqrt{2}/2}^{\sqrt{2}/2} \left(\int_0^\alpha \frac{\partial \tilde{g}_s(u)}{\partial s} ds \right) du = \int_0^\alpha \left(\int_{-\sqrt{2}/2}^{\sqrt{2}/2} \frac{\partial \tilde{g}_s(u)}{\partial s} du \right) ds \\ &= \int_0^\alpha 1\, ds = \alpha. \end{aligned}$$

Had we been presented with the minimizer $\tilde{g}_\alpha$ "out of the blue", this would have been the least computational way to prove its α-admissibility.

4 The boundary of the square

4.1 Proof of Theorem 3

In this section, we prove Theorem 3. As was remarked in section 1.3, the RSK correspondence induces a correspondence between extremal Erdös-Szekeres permutations π of

$1, 2, \ldots, n^2$ and pairs $T_1, T_2 \in \mathcal{T}_n$ of square tableaux. By the well known result of Schensted [17], in this correspondence the length $l_{n,k}$ of the longest increasing subsequence in $\pi(1), \pi(2), \ldots, \pi(k)$ is equal to the length $\lambda^k_{T_1}(1)$ of the first row of $\lambda^k_{T_1}$. So the distribution of $l_{n,k}$ under a uniform random choice of extremal Erdös-Szekeres permutation π is equal to the distribution of the length of the first row of λ^k_T in a uniform random square tableau $T \in \mathcal{T}_n$. Denoting for the remainder of this section $\alpha = \alpha(k) = k/n^2$, we can therefore reformulate Theorem 3 as stating that

$$\max_{\alpha_0 \le k/n^2 \le 1/2} \mathbb{P}_n \left(T \in \mathcal{T}_n : \left| \lambda^k_T(1) - 2\sqrt{\alpha(1-\alpha)}n \right| > \alpha_0^{1/2} \omega(n) n \right) \xrightarrow[n\to\infty]{} 0. \qquad (66)$$

Theorem 6 looks as if it might imply (66). In fact, it only implies a lower bound on $\lambda^k_T(1)$. The reason is that $g_{\lambda^k_T}$ can be very close in the supremum norm to $\tilde{g}_\alpha$ (as is known to happen with high probability by Theorem 6), while $n^{-1}\lambda^k_T(1)$ might still be much larger than $2\sqrt{\alpha(1-\alpha)}$ (see (68) below).

Lemma 9. Let $\alpha_0 = n^{-2/3+\epsilon}$, $\delta = n^{-1/3(1-\epsilon)}$, $\epsilon \in (0, 2/3)$. Then

$$\mathbb{P}_n \left(T \in \mathcal{T}_n : \min_{\alpha_0 \le \alpha \le 1/2} (\lambda^k_T(1) - 2\sqrt{\alpha(1-\alpha)}n) \le -\delta n \right) = O(n^{-b}) \qquad (67)$$

for every $b > 0$.

Proof. We use the notation of Theorem 6. The length of the first row $\lambda^k_T(1)$ can be extracted from the rotated coordinate graph $g_{\lambda^k_T}$ using the following relation:

$$\frac{1}{n}\lambda^k_T(1) = \sqrt{2}\inf\left\{ u \in [0, \sqrt{2}/2] : g_{\lambda^k_T}(u) = u \right\}. \qquad (68)$$

It follows from (54) that, uniformly for $\alpha \in [\alpha_0, 1/2]$ and $|u| < \sqrt{2\alpha(1-\alpha)}$,

$$|\partial \tilde{g}_\alpha(u)/\partial u - 1| = \frac{2}{\pi}\tan^{-1}\frac{\sqrt{2\alpha(1-\alpha)-u^2}}{(1-2\alpha)|u|} \ge c(\sqrt{2\alpha(1-\alpha)} - |u|)^{1/2},$$

$c > 0$ being an absolute constant. Consequently, for $\alpha \in [\alpha_0, 1/2]$,

$$\tilde{g}_\alpha(\sqrt{2\alpha(1-\alpha)} - \delta) - (\sqrt{2\alpha(1-\alpha)} - \delta) \ge c\delta^{3/2}.$$

So if $T \in \mathcal{T}_n$ has the property that, for some k in question,

$$\lambda^k_T(1) - 2\sqrt{\alpha(1-\alpha)}n < -\delta n,$$

then by (68),

$$\begin{aligned} ||g_{\lambda^k_T} - \tilde{g}_\alpha||_\infty &\ge \sup\{|g_{\lambda^k_T}(u) - \tilde{g}_\alpha(u)| : \sqrt{2\alpha(1-\alpha)} - \delta < u < \sqrt{2\alpha(1-\alpha)}\} \\ &= \sup\{\tilde{g}_\alpha(u) - u : \sqrt{2\alpha(1-\alpha)} - \delta < u < \sqrt{2\alpha(1-\alpha)}\} \ge c\delta^{3/2}. \end{aligned}$$

So, by Theorem 6 with $\epsilon := c\delta^{3/2}$,

$$\begin{aligned}
&\mathbb{P}_n\left(T \in \mathcal{T}_n : \min_{\alpha_0 \le \alpha \le 1/2} (\lambda_T^k(1) - 2\sqrt{\alpha(1-\alpha)}n) \le -\delta n\right) \\
&\qquad \le \mathbb{P}_n\left(T \in \mathcal{T}_n : \max_{\alpha_0 \le \alpha \le 1/2} ||g_{\lambda_T^k} - \tilde{g}_\alpha||_\infty \ge c\delta^{3/2}\right) \\
&\qquad \le \exp(3n - \hat{c}n^2\delta^{3r/2}) \le \exp(3n - \hat{c}n^{2-r(1-\epsilon)/2}) \xrightarrow[n\to\infty]{} 0,
\end{aligned}$$

provided that we choose a feasible r, i. e. $r \in (2,3)$, such that $r < 2(1-\epsilon)^{-1}$. ■

To prove the upper bound and thus conclude the proof of Theorem 3, it suffices to prove an upper bound for the *expected value* of λ_T^k, namely that, for $\alpha_0 \le \alpha \le 1/2$,

$$\mathbb{E}_n\left[\lambda_T^k(1)\right] \le 2\sqrt{\alpha(1-\alpha)}n + O(\alpha_0^{1/2}n), \tag{69}$$

where $\mathbb{E}_n$ denotes expectation with respect to the probability measure $\mathbb{P}_n$. Indeed, choosing $\omega(n) \to \infty$ however slowly, we bound

$$\begin{aligned}
&\mathbb{P}_n\left(T \in \mathcal{T}_n : \lambda_T^k(1) \ge 2\sqrt{\alpha(1-\alpha)}n + \alpha_0^{1/2}\omega(n)n\right) \\
&\text{(by Markov's inequality)} \quad \le \quad (\alpha_0^{1/2}\omega(n)n)^{-1}\mathbb{E}_n\left[\max(0, \lambda_T^k(1) - 2\sqrt{\alpha(1-\alpha)}n)\right] \\
&\text{(by Lemma 9, for any } b > 0) \quad \le \quad (\alpha_0^{1/2}\omega(n)n)^{-1}\left(\mathbb{E}_n\left[\lambda_T^k(1) - 2\sqrt{\alpha(1-\alpha)} + \delta n\right] + O(n^{1-b})\right) \\
&\qquad = \quad O((\alpha_0^{1/2}n + \delta n)/(\alpha_0^{1/2}\omega(n)n)) = O(\omega(n)^{-1}).
\end{aligned}$$

Write

$$\lambda_T^k(1) = \sum_{j=1}^k I_{n,j},$$

where $I_{n,j} = \lambda_T^j(1) - \lambda_T^{j-1}(1) =$ indicator of the event that λ_T^j is obtained from λ_T^{j-1} by adding a box to the first row. Let $p_{n,j} = \mathbb{E}_n(I_{n,j})$.

Lemma 10. In the notation of Lemma 9, as $n \to \infty$,

$$p_{n,j} \le \frac{n^2 - 2j}{n\sqrt{j(n^2-j)}} + O(\delta n(n^2 - 2j + 1)^{-1}),$$

uniformly for $\alpha_0 \le j/n^2 \le 1/2$.

Proof. Let $\mathcal{Y}_{n,j}$ be the set of Young diagrams of area j contained in the $n \times n$ square. For a diagram $\lambda \in \mathcal{Y}_{n,j}$, denote by $\text{next}(\lambda)$ the diagram obtained from λ by adding a box to the first row. Then, conditioning $I_{n,j}$ on the shape λ_T^{j-1}, we write

$$\begin{aligned} p_{n,j} &= \mathbb{P}_n\left(\lambda_T^j = \text{next}(\lambda_T^{j-1})\right) = \sum_{\lambda \in \mathcal{Y}_{n,j-1}} \frac{d(\lambda) d(\square_n \setminus \text{next}(\lambda))}{d(\square_n)} \\ &= \sum_{\lambda \in \mathcal{Y}_{n,j-1}} \frac{d(\text{next}(\lambda)) d(\square_n \setminus \text{next}(\lambda))}{d(\square_n)} \cdot \frac{d(\lambda)}{d(\text{next}(\lambda))} \end{aligned}$$

This is nearly an average over $\mathcal{Y}_{n,j}$ with respect to the measure (7); in fact, slightly less, since not any $\lambda' \in \mathcal{Y}_{n,j}$ is of the form $\text{next}(\lambda)$ for some $\lambda \in \mathcal{Y}_{n,j-1}$. It follows from the convexity of the function $x \to x^2$ that

$$\begin{aligned} p_{n,j}^2 &\leq \sum_{\lambda \in \mathcal{Y}_{n,j-1}} \frac{d(\text{next}(\lambda)) d(\square_n \setminus \text{next}(\lambda))}{d(\square_n)} \cdot \left(\frac{d(\lambda)}{d(\text{next}(\lambda))}\right)^2 \\ &= \sum_{\lambda \in \mathcal{Y}_{n,j-1}} \frac{d(\lambda) d(\square_n \setminus \lambda)}{d(\square_n)} \cdot \frac{d(\lambda) d(\square_n \setminus \text{next}(\lambda))}{d(\text{next}(\lambda)) d(\square_n \setminus \lambda)}. \end{aligned} \tag{70}$$

We now note the amusing identity

$$\frac{d(\lambda) d(\square_n \setminus \text{next}(\lambda))}{d(\text{next}(\lambda)) d(\square_n \setminus \lambda)} = \frac{n^2 - \lambda(1)^2}{j(n^2 - j + 1)}, \qquad (\lambda \in \mathcal{Y}_{n,j-1}) \tag{71}$$

which follows from writing out the hook products for $d(\cdot)$ in (6) and observing cancellation of almost all the factors - see Figure 5. Here is a proof of (71). Clearly the only hook lengths influenced by this operation are of the cells in the first row and the $(\lambda(1)+1)$-th column. In particular,

$$\frac{d(\lambda)}{d(\text{next}(\lambda))} = \frac{1}{j} \prod_{i=1}^{\lambda(1)} \frac{\lambda(1) - i + 1 + \lambda'(i)}{\lambda(1) - i + \lambda'(i)};$$

here $\lambda'(i)$ is the number of cells in the i-th column of λ. Clearly the fraction factors "telescope" on each subinterval of $[1, \lambda(1)]$ where $\lambda'(i)$ is constant. Let $[i_1, i_2]$ be such a (maximal) subinterval. Maximality implies that $(i_2, \lambda'(i_2))$ is a corner of λ, and that $(\lambda'(i_1)+1, i_1)$ is a corner of $\square_n \setminus \text{next}(\lambda)$. Then

$$\prod_{i=1}^{\lambda(1)} \frac{\lambda(1) - i + 1 + \lambda'(i)}{\lambda(1) - i + \lambda'(i)} = \frac{\lambda(1) - i_1 + 1 + \lambda'(i_1)}{\lambda(1) - i_2 + \lambda'(i_2)} = \frac{h_{\square_n \setminus \text{next}(\lambda)}(\lambda'(i_1)+1, \lambda(1)+1)}{h_\lambda(1, u_2)}$$

where, say, $h_\lambda(u,v)$ denotes the hook length for a cell $(u,v) \in \lambda$. Multiplying these fractions for all such subintervals $[i_1, i_2]$, we get

$$\frac{d(\lambda)}{d(\text{next}(\lambda))} = \frac{1}{j} \left(\prod_{(u,v) \in \text{corners}(\lambda)} f(u,v) \right)^{-1} \cdot \left(\prod_{(u,v) \in \text{corners}(\square_n \setminus \lambda)} g(u,v) \right). \tag{72}$$

Here corners(μ) is the corner set of a diagram μ; $f(u,v)$ is the hook length of a cell in the first row of λ whose vertical leg ends at the corner $(u,v) \in \text{corners}(\lambda)$; $g(u,v)$ is the hook length of a cell in $\square_n \setminus \text{next}(\lambda)$ from the $(\lambda(1)+1)$-th column whose horizontal arm ends at the corner $(u,v) \in \text{corners}(\square_n \setminus \lambda)$. Next, considering separately the first row cells $(1,v)$, $v > \lambda(1)$, the top $\lambda'(1)$ cells in the $(\lambda(1)+1)$-th column, and finally the bottom $n - \lambda'(1)$ cells in that column, we obtain

$$\frac{d(\square_n \setminus \text{next}(\lambda))}{d(\square_n \setminus \lambda)} = \frac{n - \lambda(1)}{n^2 - j + 1} \cdot \prod_{k=2}^{\lambda'(1)} \frac{\lambda(1) - \lambda(k) + k}{\lambda(1) - \lambda(k) + k - 1} \cdot \frac{\lambda(1) + n}{\lambda(1) + \lambda'(1)}. \tag{73}$$

Here, analogously to the $d(\lambda)/d(\text{next}(\lambda))$ case,

$$\begin{aligned} &\frac{1}{\lambda(1) + \lambda'(1)} \prod_{k=2}^{\lambda'(1)} \frac{\lambda(1) - \lambda(k) + k}{\lambda(1) - \lambda(k) + k - 1} \\ &\qquad = \left(\prod_{(u,v) \in \text{corners}(\lambda)} f(u,v) \right) \cdot \left(\prod_{(u,v) \in \text{corners}(\square_n \setminus \lambda)} g(u,v) \right)^{-1}. \end{aligned} \tag{74}$$

Putting (72), (73), (74) together gives (71).

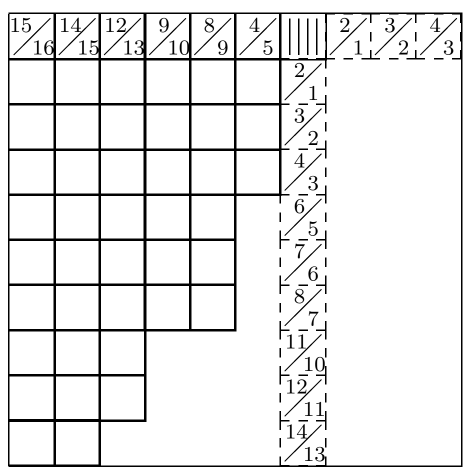


Figure 5: Illustration of (71) for $\lambda = (6,6,6,6,5,5,5,3,3,2)$: The numbers in the cells are the hook lengths before and after the new cell is added.

Combining (70) and (71) gives that

$$p_{n,j}^2 \le \mathbb{E}_n\left[\frac{n^2 - \lambda_T^{j-1}(1)^2}{j(n^2-j+1)}\right] \tag{75}$$

By Lemma 9, we may write

$$\mathbb{E}_n(\lambda_T^{j-1}(1)) \ge \frac{2\sqrt{j(n^2-j)}}{n} - \delta n,$$

($\delta = n^{-(1-\epsilon)/3}$), for all $j/n^2 \in [\alpha_0, 1/2]$. So, using $\mathbb{E}_n^2[\lambda_T^{j-1}(1)] \le \mathbb{E}[(\lambda_T^{j-1}(1))^2]$,

$$p_{n,j}^2 \le \frac{(n^2-2j)^2}{n^2 \cdot j(n^2-j)} + \frac{4\delta}{\sqrt{j(n^2-j)}},$$

or, using $(1+z)^{1/2} \le 1 + z/2$ for $j < n^2/2$,

$$p_{n,j} \le \frac{n^2-2j}{n\sqrt{j(n^2-j)}} + O(\delta n(n^2-2j+1)^{-1}).$$

The estimate holds for $j = n^2/2$ as well, since $\delta^{1/2}n^2 \to \infty$. ■

Note that (75) implies in particular the rough bound

$$p_{n,j} \le \frac{n}{\sqrt{j(n^2-j+1)}},$$

valid for all $j \le n^2$. Now, to complete the proof of Theorem 3, we use this bound for $j \le \alpha_0 n^2$ and Lemma 10 for $j > \alpha_0 n^2$. First

$$\mathbb{E}_n\left[\lambda_T^k(1)\right] = \sum_{j \le \alpha_0 n^2} p_{n,j} + \sum_{\alpha_0 n^2 < j \le k} p_{n,j} = \Sigma_1 + \Sigma_2.$$

Here

$$\Sigma_1 \le 2 \sum_{j \le \alpha_0 n^2} j^{-1/2} = O(n\alpha_0^{1/2}),$$

and

$$\Sigma_2 \le \sum_{\alpha_0 n^2 < j \le k} \frac{n^2-2j}{n\sqrt{j(n^2-j)}} + O(\delta n \log n).$$

The last sum is bounded above by

$$n\int_{\alpha_0 - n^{-2}}^{\alpha} \frac{1-2t}{\sqrt{t(1-t)}}\,dt = 2n\sqrt{\alpha(1-\alpha)} + O(n\alpha_0^{1/2}).$$

Therefore, since $\alpha_0^{1/2} \gg \delta \log n$,

$$\mathbb{E}_n[\lambda_T^k] \le 2n\sqrt{\alpha(1-\alpha)} + O(n\alpha_0^{1/2}).$$

So (69) follows. Theorem 3 is proved. ■

4.2 Proof of Theorem 1(i)

With our enhanced understanding of the distribution of $\lambda_T^k(1)$, we may now prove Theorem 1(i). First we show that for individual boundary points, the tableau approaches the limit surface. Fix (x, y) on the boundary of the square. By symmetry, we may assume that $y = 0, 0 < x < 1$. Let $\epsilon > 0$. Denote

$$\alpha = L(x,0) = \frac{1-\sqrt{1-x^2}}{2},$$

so that $x = 2\sqrt{\alpha(1-\alpha)}$. For any tableau $T \in \mathcal{T}_n$, denote $k_T = t_{\lfloor nx \rfloor+1,1}$, and let $\beta_T = k_T/n^2$. We want to show that with high probability, $|\beta_T - \alpha|$ is small. Note that k_T is an integer representing the smallest j for which $\lambda_T^j > nx$. Therefore $nx \le \lambda_T^{k_T}(1) < nx+1$, or

$$\left|\lambda_T^{k_T}(1) - x\right| \le \frac{1}{n} \tag{76}$$

The function $f(t) := L(t,0) = (1-\sqrt{1-t^2})/2$ is monotonically increasing and uniformly continuous on $[0,1]$. Choose a $\delta > 0$ such that $|t-t'| < \delta$ implies $|f(t) - f(t')| < \epsilon/3$. Choose numbers $0 = a_0 < a_1 < a_2 < \ldots < a_N = 1/2$ such that $a_{i+1} - a_i < \epsilon/3$, $i = 0,1,2,\ldots,N-1$. Denote $x_i = f^{-1}(a_i) = 2\sqrt{a_i(1-a_i)}$.

Let $T \in \mathcal{T}_n$ be a tableau that satisfies

$$\left|\frac{1}{n}\lambda_T^{\lfloor a_i n^2 \rfloor}(1) - x_i\right| < \frac{\delta}{2}, \qquad (i = 1,2,\ldots,N) \tag{77}$$

(this happens with high probability, by (66)). Let $0 \le i < N$ be such that $a_i \le \beta_T < a_{i+1}$. Then clearly

$$x_i - \frac{\delta}{2} < \frac{1}{n}\lambda_T^{\lfloor a_i n^2 \rfloor}(1) \le \frac{1}{n}\lambda_T^{k_T}(1) \le \frac{1}{n}\lambda_T^{\lfloor a_{i+1} n^2 \rfloor}(1) < x_{i+1} + \frac{\delta}{2} \tag{78}$$

Combining this with (76) we get, for $n > 2/\delta$,

$$x_i - \delta < x < x_{i+1} + \delta.$$

Therefore

$$a_i - \frac{\epsilon}{3} < \alpha = f(x) < a_{i+1} + \frac{\epsilon}{3},$$

and, since also $a_i \le \beta_T < a_{i+1}$ and $a_{i+1} - a_i < \epsilon/3$, we get

$$|\beta_T - \alpha| < \epsilon.$$

Summarizing, we have shown that

$$\mathbb{P}_n\left(T \in \mathcal{T}_n : |\beta_T - \alpha| < \epsilon\right) \ge \mathbb{P}_n\left(T \in \mathcal{T}_n : \forall\, i = 1,2,\ldots,N, \quad \left|\frac{1}{n}\lambda_T^{\lfloor a_i n^2 \rfloor}(1) - x_i\right| < \frac{\delta}{2}\right) \xrightarrow[n\to\infty]{} 1. \tag{79}$$

Theorem 1(i) now follows easily. It is enough to say that, because of the monotonicity of the tableau $t_{i,j}$ as a function of i and j, and the monotonicity of the limit surface function L, given $\epsilon > 0$ we can find finitely many points $(x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N) \in [0,1] \times [0,1] \setminus \{(0,0), (1,0), (0,1), (1,1)\}$ such that the event inclusion

$$\left\{ T \in \mathcal{T}_n : \max_{1 \le i,j \le n} \left| \frac{1}{n^2} t_{i,j} - L\left(\frac{i}{n}, \frac{j}{n}\right) \right| > \epsilon \right\} \subseteq \bigcup_{i=1}^{N} \left\{ T \in \mathcal{T}_n : \left| \frac{1}{n^2} t_{\lfloor n x_i \rfloor + 1, \lfloor n y_i \rfloor + 1} - L(x_i, y_i) \right| > \frac{\epsilon}{10} \right\} \tag{80}$$

holds. But now, the $\mathbb{P}_n$-probability of each of the individual events in this union tends to 0 as $n \to \infty$ – because of Theorem 1(ii) for the points (x_i, y_i) in the interior of the square (using the continuity of the function L), and because of (79) for the points on the boundary. ■

5 The hook walk and the cotransition measure of a diagram

In this section, we study the location of the k-th entry in the random tableau $T \in \mathcal{T}_n$, when $k \approx \alpha \cdot n^2$. The idea is to condition the distribution of the location of the k-th entry on the shape λ_T^k of the k-th subtableau of T. Given the shape λ_T^k, the distribution of the location of the k-th entry is exactly the so-called *cotransition measure* of λ_T^k (see below). We know from Theorem 6 that with high probability, the rescaled shape of λ_T^k is approximately described in rotated coordinates by the level curve $v = \tilde{g}_\alpha(u)$. Romik [16] showed that the cotransition measure is a continuous functional on the space of continual Young diagrams, and derived an explicit formula for the probability density of its u-coordinate. By substituting the level curve $\tilde{g}_\alpha$ in the formula from [16], we will get exactly the semicircle density (3), proving Theorem 2.

Let $\lambda : \lambda(1) \ge \lambda(2) \ge \ldots \ge \lambda(m) > 0$ be a Young diagram with $k = |\lambda| = \sum_i \lambda(i)$ cells. A cell $c = (i,j) \in \lambda$ $(1 \le i \le m, 1 \le j \le \lambda(i))$ is called a *corner* cell if removing it leaves a Young diagram $\lambda \setminus c$, or in other words if $j = \lambda(i)$ and $(i = m$ or $\lambda(i) > \lambda(i+1))$. If T is a Young tableau of shape λ, let $c_{\max}(T)$ be the cell containing the maximal entry k in T. Obviously $c_{\max}(T)$ is a corner cell of λ.

The *cotransition measure* of λ is the probability measure μ_λ on corner cells of λ, which assigns to a corner cell c measure

$$\mu_\lambda(c) = \frac{d(\lambda \setminus c)}{d(\lambda)} \tag{81}$$

(with $d(\lambda)$ as in (6).) This is a probability measure, since one may divide up the $d(\lambda)$ tableaux of shape λ according to the value of $c_{\max}(T)$; for any corner cell c, there are precisely $d(\lambda \setminus c)$ tableaux for which $c_{\max}(T) = c$. In other words μ_λ describes the distribution of $c_{\max}(T)$, for a uniform random choice of a tableau T of shape λ.

It is fascinating that there exists a simple algorithm to sample from μ_λ. This is known as the *hook walk* algorithm of Greene-Nijenhuis-Wilf, and it can be described as follows: Choose a cell $c = (i, j) \in \lambda$ uniformly among all k cells. Now execute a random walk, replacing at each step the cell c with a new cell c', where c' is chosen uniformly among all cells which lie either to the right of, or (exclusive or) below c. The walk terminates when a corner cell is reached, and it can be shown [9] that the probability of reaching c is given by (81). Figure 6 shows a Young diagram, its corner cells and a sample hook walk path.

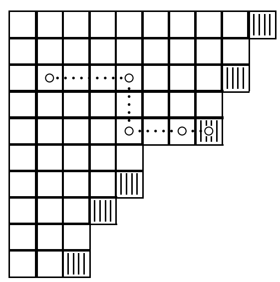


Figure 6: A Young diagram, its corners and a hook walk path

Now consider a sequence $\lambda_n : \lambda_n(1) \geq \lambda_n(2) \geq \lambda_n(3) \geq \ldots$ of Young diagrams for which, under suitable scaling, the shape converges to some limiting shape described by a continuous function. More precisely, let $f_{\lambda_n}(x)$ be as in (4), and let g_{λ_n} be its rotated coordinate version. Let $f_\infty : [0, \infty) \to [0, \infty)$ be a weakly decreasing function, and let g_∞ be its rotated coordinate version, a 1-Lipschitz function. In this more general setting, think of g_{λ_n} and g_∞ as functions defined on all $\mathbb{R}$. Assume that: there exists an $M > 0$ such that $f_\infty(x) = 0$ for $x \geq M$, and on $[0, M]$ f is twice continuously differentiable, and its derivative is bounded away from 0 and ∞ (equivalently, for some $K < 0 < K'$, $g_\infty(u) = |u|$ for $u \notin (K, K')$, and g is twice continuously differentiable in $[K, K']$ with derivative bounded awaw from -1 and 1). Finally, assume that

$$||g_{\lambda_n} - g_\infty||_\infty \xrightarrow[n\to\infty]{} 0.$$

For any n, let (I_n, J_n) be a μ_{λ_n}-distributed random vector. Let $X_n = I_n/n, Y_n = J_n/n$. We paraphrase results from [16].

Theorem 8. (Romik [16], Theorems 1(b), 6) As $n \to \infty$, (X_n, Y_n) converges in distribution to the random vector

$$(X, Y) := \left(\frac{V+U}{2}, \frac{V-U}{2}\right),$$

where $V = g_\infty(U)$ and U is a random variable on $[K, K']$ with density function

$$\phi_U(x) = \frac{2}{\pi A} \cos\left(\frac{\pi g'_\infty(x)}{2}\right) \sqrt{(x-K)(K'-x)} \exp\left(\frac{1}{2}\int_K^{K'} \frac{g'_\infty(u)}{x-u} du\right), \qquad (82)$$

with

$$A = \int_0^M f_\infty(x)dx = \int_K^{K'} (g_\infty(u) - |u|)du$$

and the integral in the exponential being a principal value integral.

Proof of Theorem 2. We may assume that $0 < \alpha < 1/2$. The proof of Theorem 2 now consists of an observation, a remark, and a computation.

The observation is that the distribution of the location of the k_n-th entry in a random tableau $T \in \mathcal{T}_n$ is the distribution of the maximal entry in the shape $\lambda_T^{k_n}$ of the k_n-th subtableau of T. Because by Theorem 6, this shape (suitably rescaled and rotated) converges in probability to $\tilde{g}_\alpha$ (Theorem 2 assumes $k_n/n^2 \to \alpha$), we may apply Theorem 8 and conclude that Theorem 2 is true with a density for U_α given by taking $g_\infty = \tilde{g}_\alpha$, $A = \alpha$, $-K = K' = \sqrt{2\alpha(1-\alpha)}$ in (82).

The remark is that the above is not quite true, since $\tilde{g}_\alpha$ does not satisfy the assumptions of Theorem 8! The problem is that

$$-\lim_{\epsilon \searrow 0} \tilde{g}'_\alpha(-\sqrt{2\alpha(1-\alpha)} + \epsilon) = \lim_{\epsilon \searrow 0} \tilde{g}'_\alpha(\sqrt{2\alpha(1-\alpha)} - \epsilon) = 1,$$

so the derivative is not bounded away from -1 and 1. However, since this only happens near the two boundary points, going over the computations in [16] shows that this is not a problem, and the formula (82) is still valid in this case [4].

The computation is the verification that (82) gives the semicircle distribution (3) under the above substitutions. We compute, using (32) and the identity $\cos(\tan^{-1} v) =$

[4]Alternatively, one may use the less explicit formula (8) from [16], which is valid even without the assumption that g'_∞ is bounded away from ± 1, to verify directly that (3) is the cotransition measure of $\tilde{g}_\alpha$.

$(1+v^2)^{-1/2}$:

$$\begin{aligned}
\frac{2}{\pi A} &= \frac{2}{\pi\alpha} \\
\sqrt{(x-K)(K'-x)} &= \sqrt{2\alpha(1-\alpha)-x^2} \\
\exp\left(\frac{1}{2}\int_K^{K'} \frac{\tilde{g}_\alpha(u)}{x-u}du\right) &= \sqrt{\frac{\alpha}{1-\alpha}} \cdot \sqrt{\frac{\frac{1}{2}-x^2}{2\alpha(1-\alpha)-x^2}}, \\
\cos\left(\frac{\pi\tilde{g}'_\alpha(x)}{2}\right) &= \cos\left(\tan^{-1}\frac{(1-2\alpha)x}{\sqrt{2\alpha(1-\alpha)-x^2}}\right) \\
&= \left(1+\frac{(1-4\alpha(1-\alpha))x^2}{2\alpha(1-\alpha)-x^2}\right)^{-1/2} = \frac{\sqrt{2\alpha(1-\alpha)-x^2}}{2\sqrt{\alpha(1-\alpha)}\sqrt{\frac{1}{2}-x^2}}
\end{aligned}$$

Multiplying the above expressions gives

$$\phi_U(x) = \frac{1}{\pi\alpha(1-\alpha)}\sqrt{2\alpha(1-\alpha)-x^2}, \qquad |x| \le \sqrt{2\alpha(1-\alpha)},$$

as claimed. ■

6 Open problems

We conclude with some open problems.

- **Gaussian fluctuations.** Prove a central limit theorem for the fluctuations of $g_{\lambda_T^{\lfloor \alpha n^2 \rfloor}}$ around $\tilde{g}_\alpha$, and for the fluctuations of the cotransition measure of $\lambda_T^{\lfloor \alpha n^2 \rfloor}$ around the semicircle distribution, in the spirit of [11].
- **Limiting distribution of $l_{n,k}(\pi)$.** Find a scaling sequence a_n and a distribution function F such that, in the notation of Theorem 3,

$$\frac{l_{n,\lfloor \alpha n^2 \rfloor} - 2\sqrt{\alpha(1-\alpha)}n}{a_n} \xrightarrow[n\to\infty]{\text{in distribution}} F.$$

- **Limit surface for random Young tableaux of given shape.** Prove a limit surface theorem for random Young tableaux of other shapes. In general, one can consider any decreasing function $f : [0,\infty) \to [0,\infty)$ such that $\int_0^\infty f(x)dx = 1$ as a *continual Young diagram*, i.e. as a limit of the rescaled graphs of a sequence of Young diagrams of increasing sizes. We conjecture that for each such continual diagram f, there should exist a limit surface L_f, defined on the domain

$$D_f := \{(x,y) : x \ge 0,\ 0 \le y \le f(x)\}$$

bounded between the x-axis and the graph of f, that describes the asymptotic behavior of almost all random Young tableaux of shape approximated by f.

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