Stirling's Approximation for $n!$: the Ultimate Short Proof?

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For two real sequences x_n and y_n, we write $x_n \sim y_n$ if $\lim_{n\to\infty} x_n/y_n = 1$. Stirling's approximation is $n! \sim \sqrt{2\pi n}(n/e)^n$. We first prove that $n! \sim C\sqrt{n}(n/e)^n$ for some nonzero constant C, then give a short proof that $C = \sqrt{2\pi}$.

Lemma 1. The limit $C = \lim_{n\to\infty} e^n n!/n^{n+1/2}$ exists.

Proof. Denote as usual $\lfloor x \rfloor = \max\{n \in \mathbf{Z} : n \le x\}$ and $\{x\} = x - \lfloor x \rfloor$. Then

$$\begin{aligned}
\log n! &= \sum_{k=1}^{n} \log k = \sum_{k=1}^{n} \int_1^k \frac{dx}{x} = \int_1^n \frac{n - \lfloor x \rfloor}{x} dx \\
&= \int_1^n \frac{n + \frac{1}{2} + (\{x\} - \frac{1}{2}) - x}{x} dx = (n + \tfrac{1}{2}) \log n - n + 1 + \int_1^n \frac{\{x\} - \frac{1}{2}}{x} dx \\
&= (n + \tfrac{1}{2}) \log n - n + 1 + \int_1^\infty \frac{\{x\} - \frac{1}{2}}{x} dx + o(1),
\end{aligned}$$

since $\int_1^t (\{x\} - \frac{1}{2}) dx$ is bounded and $1/x$ goes to 0 as $x \to \infty$. Thus

$$C = \exp\left(1 + \int_1^\infty \frac{\{x\} - \frac{1}{2}}{x} dx\right) > 0 \qquad \blacksquare$$

This is one of the standard proofs; it is a simple example of Euler-Maclaurin summation. Also note that it follows that $\binom{2n}{n} \sim \sqrt{2} \cdot 2^{2n}/C\sqrt{n}$.

Lemma 2. Let f be an $(n+1)$-times continuously-differentiable function on $\mathbf{R}$. Then for all $x \in \mathbf{R}$, $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + \cdots + f^{(n)}(0)x^n/n! + R_n(x)$, where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t)(x-t)^n dt$$

Proof. Induction on n, using integration by parts. $\blacksquare$

We now apply Lemma 2 with $f(x) = (1+x)^{2n+1}$ to calculate $R_n(1)/2^{2n+1}$:

$$\begin{aligned}
\frac{1}{2^{2n+1}} R_n(1) &= \frac{1}{2^{2n+1}} \cdot \frac{1}{n!} \int_0^1 (2n+1)(2n)\cdots(n+1)(1+t)^n (1-t)^n dt \\
&= \frac{2\binom{2n}{n}}{2^{2n+1}} (n + \tfrac{1}{2}) \int_0^1 (1-t^2)^n dt = \frac{\binom{2n}{n}\sqrt{n}}{2^{2n}} (1 + \frac{1}{2n}) \int_0^{\sqrt{n}} (1 - \frac{u^2}{n})^n du \\
&\xrightarrow[n\to\infty]{} \frac{\sqrt{2}}{C} \int_0^\infty e^{-u^2} du = \frac{\sqrt{2}}{C} \cdot \frac{\sqrt{\pi}}{2}
\end{aligned}$$

The convergence of the integrals is justified by the fact that $0 \le (1 - u^2/n)^n \le e^{-u^2}$ in the domain of integration, and $(1 - u^2/n)^n \to e^{-u^2}$ uniformly on compacts. On the other hand, $R_n(1)/2^{2n+1} = \sum_{n<k\le 2n+1} \binom{2n+1}{k}/2^{2n+1} = \frac{1}{2}$. Therefore $C = \sqrt{2\pi}$, as claimed.

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