Some formulas for the central trinomial and Motzkin numbers

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Abstract. We prove two new formulas for the central trinomial coefficients and the Motzkin numbers.

Let c_n denote the nth central trinomial coefficient, defined as the coefficient of x^n in the expansion of $(1+x+x^2)^n$, or more combinatorially as the number of planar paths starting at $(0,0)$ and ending at $(n,0)$, whose allowed steps are $(1,0),(1,1),(1,-1)$. Let m_n denote the nth Motzkin number, defined as the number of such planar paths which do not descend below the x-axis. The first few c_n's are $1,3,7,19,51,...$, and the first few m_n's are $1,2,4,9,21,....$ We prove:

Theorem.

$$m_n = \sum_{k=\lceil (n+2)/3\rceil}^{\lfloor (n+2)/2\rfloor} \frac{(3k-2)!}{(2k-1)!(n+2-2k)!(3k-n-2)!} \tag{1}$$

$$c_n = (-1)^{n+1} + 2n \sum_{k=\lceil n/3\rceil}^{\lfloor n/2\rfloor} \frac{(3k-1)!}{(2k)!(n-2k)!(3k-n)!} \tag{2}$$

It is interesting to compare these formulas with some of the other known formulas for m_n and c_n (see [2]):

$$m_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n!}{k!(k+1)!(n-2k)!}$$

$$m_n = \sum_{k=0}^{n} \frac{(-1)^{n+k}\ n!\,(2k+2)!}{k!\,((k+1)!)^2\,(k+2)(n-k)!}$$

$$c_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n!}{(k!)^2(n-2k)!}$$

$$c_n = \frac{1}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor} \frac{3^k(2n-2k)!}{k!(n-k)!(n-2k)!}$$

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Formulas such as (1) and (2) can be proven automatically by computer, using the methods and software of Petkovšek, Wilf and Zeilberger [1]. We offer an independent, non-automatic proof that involves a certain symmetry idea which might lead to the discovery of other such identities. Two simpler auxiliary identities used in the proof are also automatically verifiable and shall not be proved.

Proof of (1). Our proof uses a variant of the generating function for the numbers m_n, namely (see [2])

$$f(x) = \frac{1 - x + \sqrt{1 + 2x - 3x^2}}{2} = 1 - x^2 + \sum_{n=3}^{\infty} (-1)^{n+1} m_{n-2} x^n$$

f satisfies $f(0) = 1, f(1) = 0$ and is decreasing on $[0, 1]$. Another property of f that will be essential in the proof, is that it satisfies

$$f(x)^2 - f(x)^3 = x^2 - x^3, \qquad 0 \leq x \leq 1, \tag{3}$$

as can easily be verified. A simple corollary of this is that $f(f(x)) = x$ for $x \in [0, 1]$.

Next, define

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} (x^2 - x^3)^k$$

Since on $[0, 1]$, the maximal value attained by $x^2 - x^3$ is $4/27$ (at $x = 2/3$), by Stirling's formula the series is seen to converge everywhere on $[0, 1]$, to a function $g(x)$ which is (real-)analytic except at $x = 2/3$. We now expand $g(x)$ in powers of $1 - x$; all rearrangement operations are permitted by absolute convergence:

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k} (1-x)^k =$$

$$= \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} (1-x)^k \sum_{j=0}^{k} \binom{2k}{j} (-1)^j (1-x)^j =$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=\lceil n/3 \rceil}^{n} \binom{2k}{n-k} (-1)^{n+k} \frac{2(3k-2)!}{(2k)!(k-1)!} \right) (1-x)^n = 1 - x,$$

where the last equality follows from the automatically verifiable (see [1]) identity

$$\sum_{k=\lceil n/3\rceil}^{n} \frac{(-1)^k(3k-2)!}{(k-1)!(n-k)!(3k-n)!} = 0, \qquad n > 1$$

We have shown that $g(x) = 1 - x$ near $x = 1$. But since $g(x)$ is defined as a function of $x^2 - x^3$, by (3) it follows that $g(f(x)) = g(x)$, and therefore near $x = 0$ we have

$$g(x) = g(f(x)) = 1 - f(x) = x^2 + \sum_{n=3}^{\infty}(-1)^n m_{n-2} x^n$$

Now to prove (1), we expand $g(x)$ into powers of x, again using easily justifiable rearrangement operations

$$g(x) = \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k}(1-x)^k =$$

$$= \sum_{k=1}^{\infty} \frac{2(3k-2)!}{(2k)!(k-1)!} x^{2k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j =$$

$$= \sum_{n=2}^{\infty} \left((-1)^n \sum_{k=\lceil n/3\rceil}^{\lfloor n/2\rfloor} \frac{(3k-2)!}{(2k-1)!(n-2k)!(3k-n)!} \right) x^n$$

Equating coefficients in the last two formulas gives (1). ■

Proof of (2). We use a similar idea, this time using instead of the function $f(x)$ the function $-\log f(x)$, which generates a sequence related to c_n. Since the generating function for c_n is well known (see [2]) to be $1/\sqrt{1-2x-3x^2}$, it is easy to verify that

$$\frac{f'(x)}{f(x)} = \sum_{n=0}^{\infty} \frac{(-1)^n c_{n+1} - 1}{2} x^n$$

and therefore

$$-\log f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n c_n + 1}{2n} x^n$$

Now define the function

$$h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!}(x^2 - x^3)^k$$

which again converges for all $x \in [0,1]$ to a function which is analytic except at $x = 2/3$. Expanding $h(x)$ into powers of $1 - x$ gives

$$h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!}(1-x)^k \sum_{j=0}^{2k} \binom{2k}{j} (-1)^j (1-x)^j =$$

$$= \sum_{n=1}^{\infty} \left(\sum_{k=\lceil n/3 \rceil}^{n} \binom{2k}{n-k} (-1)^{n-k} \frac{(3k-1)!}{k!(2k)!} \right) (1-x)^n =$$

$$= \sum_{n=1}^{\infty} \frac{(1-x)^n}{n} = -\log x,$$

again making use of a verifiable identity [1], namely that

$$(-1)^n \sum_{k=\lceil n/3 \rceil}^{n} \frac{(-1)^k (3k-1)!}{k!(n-k)!(3k-n)!} = \frac{1}{n}, \qquad n \geq 1$$

So $h(x) = -\log x$ near $x = 1$, and therefore because of the symmetry property (3) we have that $h(x) = -\log f(x)$ near $x = 0$. Expanding $h(x)$ in powers of x near $x = 0$ gives

$$-\log f(x) = h(x) = \sum_{k=1}^{\infty} \frac{(3k-1)!}{k!(2k)!} x^{2k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j x^j =$$

$$= \sum_{n=2}^{\infty} \left((-1)^n \sum_{k=\lceil n/3 \rceil}^{\lfloor n/2 \rfloor} \frac{(3k-1)!}{(2k)!(n-2k)!(3k-n)!} \right) x^n$$

Equating coefficients with our previous expansion of $h(x)$ gives (2). ■

References

[1] M. Petkovšek, H. S. Wilf, D. Zeilberger, $A = B$, A. K. Peters, 1996.

[2] N. J. A. Sloane, editor (2003), The On-Line Encyclopedia of Integer Sequences, `http://www.research.att.com/~njas/sequences/`, sequences `A002426`, `A001006`.

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