Read This First

The combinatorics of fully packed loops and Razumov-Stroganov conjectures
A *Mathematica*-based presentation

- **Initialization code (for geeks)**
- **Instructions (for everyone)**

The combinatorics of fully packed loops and Razumov-Stroganov conjectures

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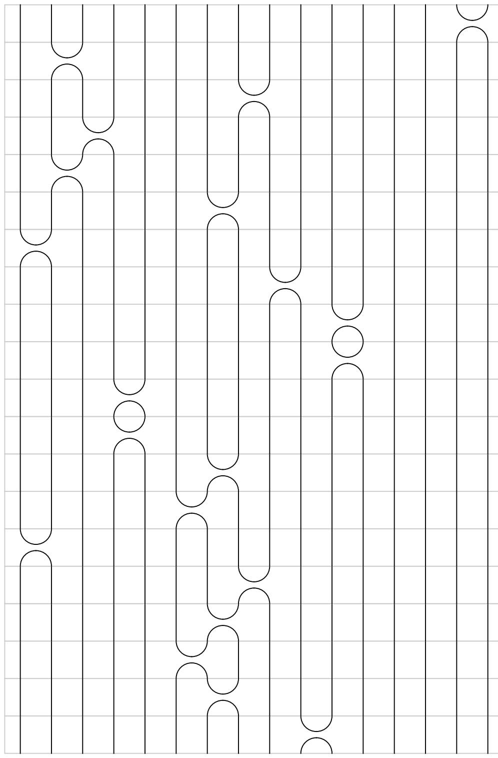
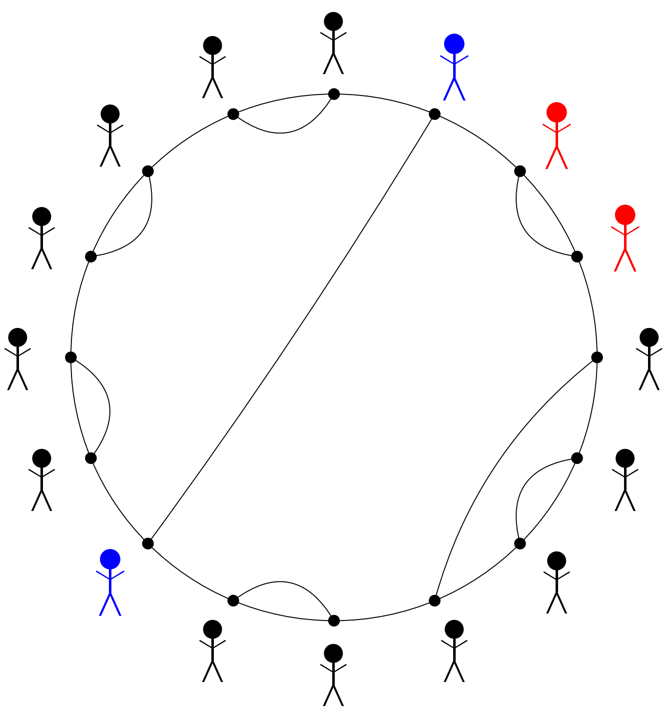




Download this *Mathematica* notebook at https://www.math.ucdavis.edu/~romik/data/uploads/talks/fpl.nb
PDF version at https://www.math.ucdavis.edu/~romik/data/uploads/talks/fpl.pdf

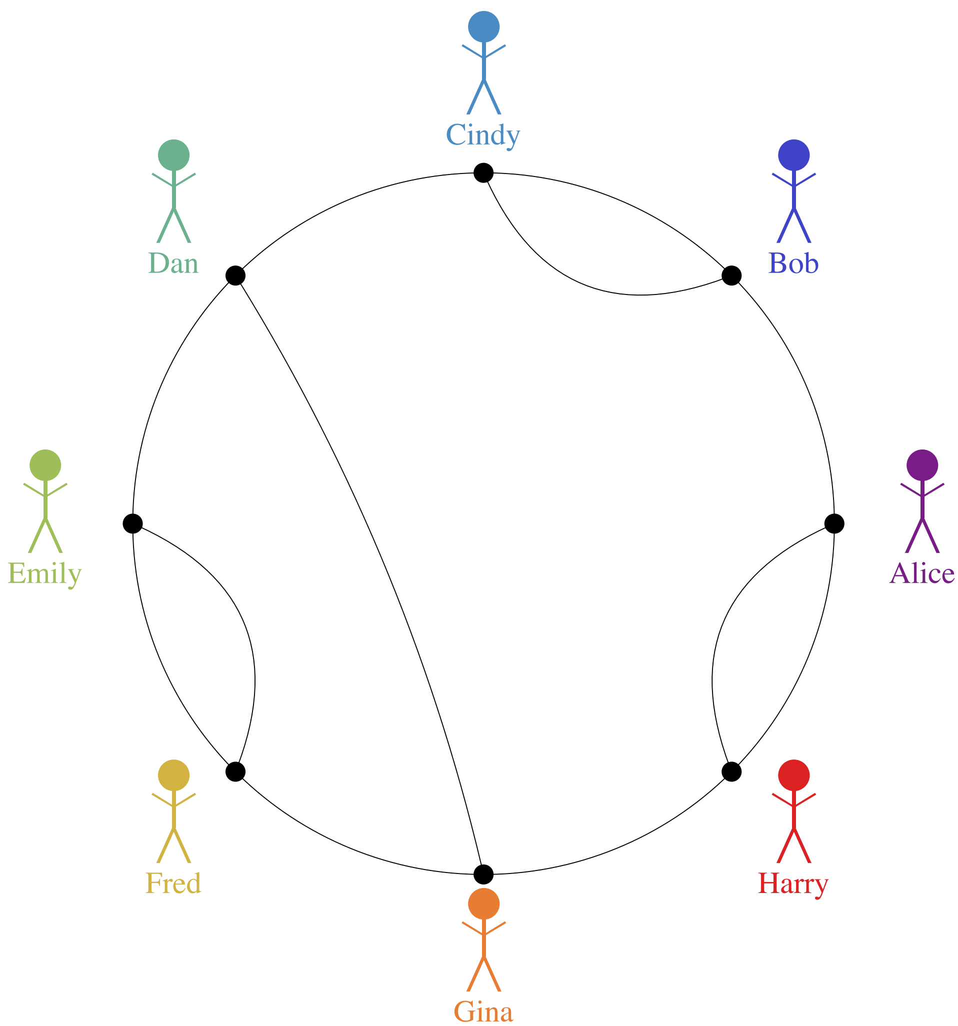
Noncrossing matchings

This talk will feature several families of combinatorial objects; some well-known, others less so. Let's start with the most well-known one.

A **noncrossing matching** is, informally, a "handshaking pattern," that is, a way for an even number of people sitting around a table to pair off into pairs, with each pair shaking hands across the table, so that their arms don't have to cross over or under each other.

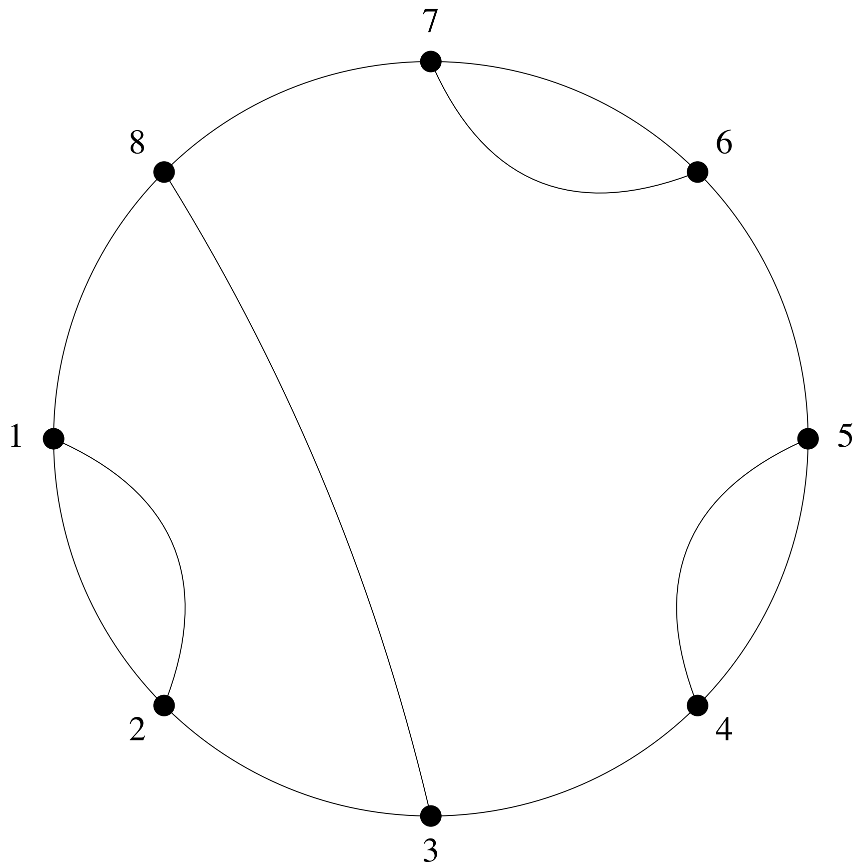
An example

```
Graphics[{NoncrossingMatchingGraphicsCircular[{2, 1, 8, 5, 4, 7, 6, 3}, False][[1]],
  Table[{ColorData["Rainbow"][(j - 1) / 7],
    {stickfigure[1.25 Cos[2 Pi (j - 1) / 8], -0.03 + 1.25 Sin[2 Pi (j - 1) / 8], 0.5],
     Text[Style[{"Alice", "Bob", "Cindy", "Dan", "Emily", "Fred", "Gina", "Harry"}[[j]],
       16], {1.25 Cos[2 Pi (j - 1) / 8],
       -0.15 + 1.25 Sin[2 Pi ( j - 1) / 8]}]}}, {j, 1, 8}]}, ImageSize → 500]
```



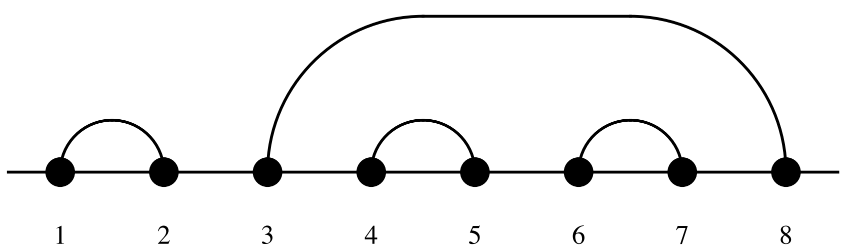
Typically we'll forget about the people and their social interactions and talk about a "noncrossing matching of order n", meaning a matching of $2n$ abstract objects, or points, arranged around a circle:

```
Graphics[
 NoncrossingMatchingGraphicsCircular[{2, 1, 8, 5, 4, 7, 6, 3}, True][[1]], ImageSize → 450]
```



We can decide to cut the circle open at an arbitrary place, to get a matching of points arranged on a *line*, giving a slightly different (but equivalent) graphical representation of the matching:

```
Graphics[
 NoncrossingMatchingGraphicsLinear[{2, 1, 8, 5, 4, 7, 6, 3}, True][[1]], ImageSize → 450]
```



▪ Enumeration of noncrossing matchings

How many noncrossing matchings are there for a given order *n*? Let’s investigate:

```
Table[Length[NoncrossingMatchings[n]], {n, 1, 9}]
```

```
{1, 2, 5, 14, 42, 132, 429, 1430, 4862}
```

Hmm. These numbers look familiar... Maybe the internet can help?

```
OnlineEncyclopediaOfIntegerSequencesLookup[%]
```

Indeed, these are the famous **Catalan numbers**:

```
catalan[n_] := 1/(n + 1) Binomial[2 n, n];
Table[catalan[n], {n, 1, 9}]
```

$$\text{catalan}[n\_] := \frac{1}{n+1}\binom{2n}{n};$$

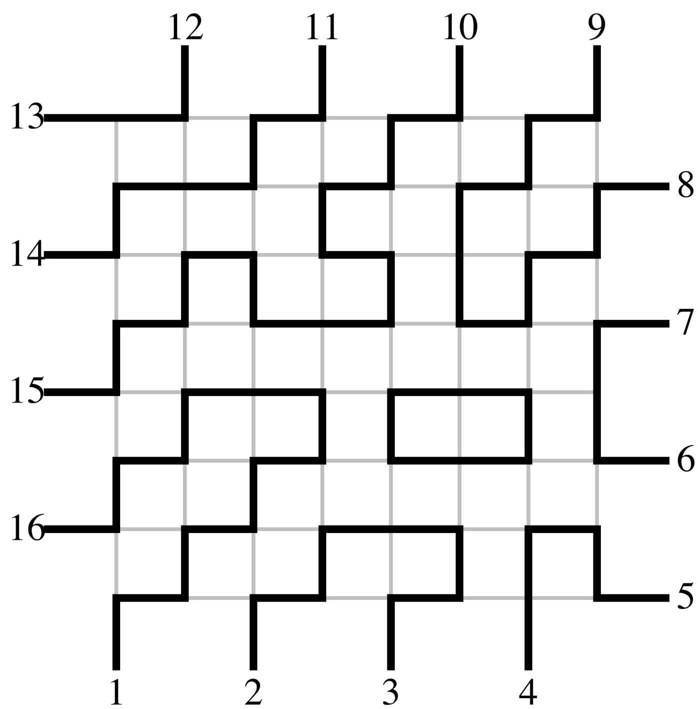
```
{1, 2, 5, 14, 42, 132, 429, 1430, 4862}
```

The enumeration of noncrossing matchings is one of several hundred interpretations (to be listed in a forthcoming book by Richard P. Stanley) of these magical numbers.

Fully Packed Loop arrangements

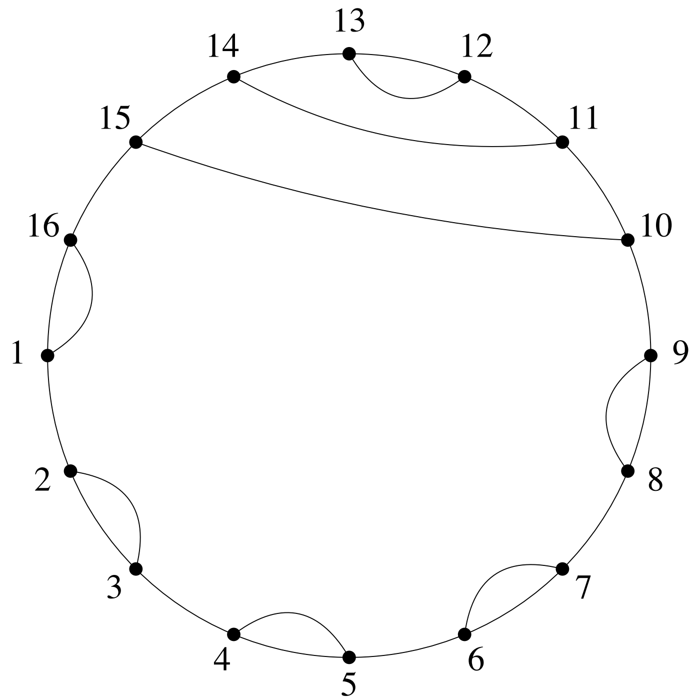
A **Fully Packed Loop (FPL)** arrangement of order n is a certain arrangement of paths and loops on an $(n-1)\times(n-1)$ square lattice to which are added "stubs" which are $2\,n$ alternating edges around the boundary of the square. Here's an example:

```
fpl = RandomFPL[8];
Graphics[FPLgraphics[fpl]]
```



Given an FPL arrangement, we can associate with it in an obvious way a noncrossing matching of order n called its **connectivity pattern:**

```
NoncrossingMatchingGraphicsCircular[FPLmatching[fpl], True]
```



Enumeration of Fully Packed Loop arrangements

How many FPL arrangements are there for a given order *n*? Let's investigate.

```
Table[Length[AllFPLs[n]], {n, 1, 6}]
```

```
{1, 2, 7, 42, 429, 7436}
```

```
OnlineEncyclopediaOfIntegerSequencesLookup[%]
```

A formula for this sequence of numbers was guessed by Mills, Robbins and Rumsey in the early 1980s:

$$\mathrm{FPL}_n = \frac{1!\cdot 4!\cdot 7!\cdots(3n-2)!}{n!\cdot(n+1)!\cdots(2n)!} = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

This was proved in 1994 by Doron Zeilberger, and later by Greg Kuperberg and others.

```
numfpls[n_] := Product[(3 j + 1) ! / (n + j) !, {j, 0, n - 1}];
Table[numfpls[n], {n, 1, 8}]
```

```
{1, 2, 7, 42, 429, 7436, 218 348, 10 850 216}
```

A story for another day

FPL arrangements are in (fairly simple) bijective correspondences with several other classes of combinatorial objects:

- **Alternating sign matrices (ASMs)**
- **Square ice configurations** with "domain wall" boundary conditions
- **3-colorings of a square lattice** with certain boundary conditions
- **Complete monotone triangles**

They are also equinumerous with (but not known to be in natural bijection with):

- **Totally symmetric self complementary plane partitions**
- **Descending plane partitions**

These bijections and the remarkable enumeration results mentioned above are described in several readable accounts:

- “The story of 1, 2, 42, 429, 7436, ...” by David P. Robbins *(Mathematical Intelligencer 1991)*
- “The many faces of alternating sign matrices” by James Propp *(Discrete Mathematics and Theoretical Computer Science,* 2001)
- “Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture” by David Bressoud (Mathematical Association of America, 1991)

The rotational symmetry of FPLs

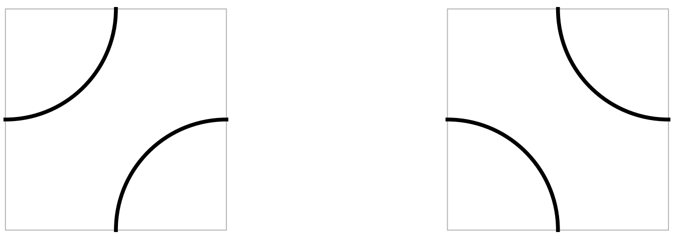
The set of noncrossing matchings of order n has an obvious rotational symmetry -- matchings can be rotated by an angle $2\pi j/(2n)$ for integer j. The square lattice has no such obvious rotational symmetry, but it turns out FPLs do have a natural operation whose effect is to rotate the associated connectivity pattern. This operation, discovered by Ben Wieland in 2001, is called **Wieland gyration**. Let’s see how it works.

```
CreatePalette[FPLsInteractiveDemo[], WindowTitle → "Fully Packed Loops"];
```

Loop percolation

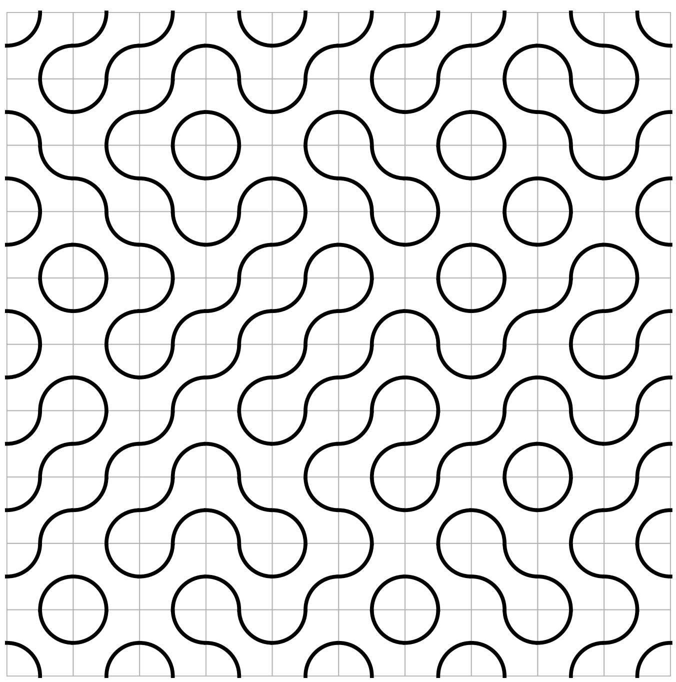
Loop percolation (a.k.a. the "**dense *O*(1) loop model**") is another combinatorial structure that gives rise to a noncrossing matching.
The idea is to tile a planar region with the following two kinds of square tiles (known by the Frenchism "plaquettes"):

```
Graphics[{plaquette[0, {0, 0}], plaquette[1, {2, 0}]}]
```



Putting together many of these plaquettes produces a collection of loops and paths (reminiscent of Fully Packed Loops, but not quite the same):

```
myplaquettes = randombits[10, 10];
Graphics[plaquettes[myplaquettes]]
```

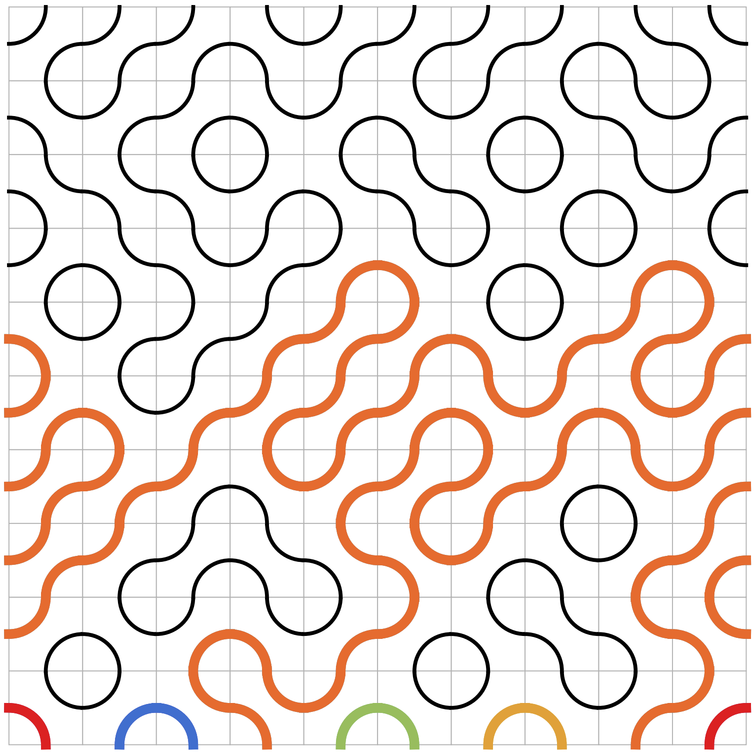


(Connoisseurs of percolation theory will recognize that this is a disguised version of **critical bond percolation on the square lattice.**)

■ The connectivity pattern of loop percolation

What we'll do is to consider this as an arrangement of plaquettes on a *cylinder*, by identifying the left and right boundary edges of the big square. The paths then connect boundary points on the bottom and top edges (the "lids" of the cylinder) to each other in some complicated pattern.

```
plaquettegr =
  Graphics[{plaquettes[myplaquettes], Table[plaquettesendpointpathgraphics[myplaquettes,
      j, ColorData["Rainbow"][(j - 1) / 9]], {j, 1, 10}]}, ImageSize → 400];
Row[{plaquettegr, CylinderGraphics[plaquettegr]}]
```



We're interested in the connectivity pattern that emerges between the *bottom* boundary endpoints. Imagine now the cylinder to be infinitely long, extending upwards all the way to infinity. The bottom boundary endpoints become more and more likely to be interconnected among themselves -- in other words to define a noncrossing matching.

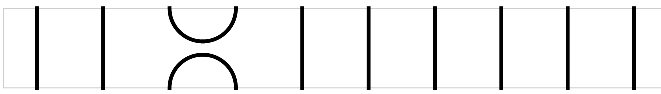
```
CreatePalette[LoopPercolationInteractiveDemo[], WindowTitle → "Loop Percolation"];
```

Notice what happens in the randomized setting when the plaquette bias becomes very close to 0 or 1. In this case the model converges to a limiting process called **pipe percolation**.

Pipe percolation

In **pipe percolation** (a.k.a. the **Temperley-Lieb random walk**, **Temperley-Lieb stochastic process**), we lay down a sequence of graphical **pipe operators** that act on noncrossing matchings.

```
Graphics[{AbsoluteThickness[2], PipeOperatorGraphics[10, 3, 0]}]
```



In this representation, the jth operator has the effect of matching points j and $j+1$, and rerouting the point previously matched to j to the point that was matched to $j+1$. For each operator, j is chosen uniformly at random from the integers $1,...,2\,n$.

```
CreatePalette[PipePercolationInteractiveDemo[], WindowTitle → "Pipe Percolation"];
```

All loops lead to Rome

We defined random noncrossing matchings obtained as the connectivity patterns of several different processes: Fully Packed Loop arrangements, loop percolation, and pipe percolation. There are several amazing connections between these different random objects.

The connectivity pattern of loop percolation is independent of the plaquette bias $0 < p < 1$

- This property observed by Di Francesco, Zinn-Justin (2006) and possibly others. Catch phrase: “Loop percolation is integrable”
- Technically, the row-transfer matrices $T_n^{(p)}$ all commute with each other, therefore share the same Frobenius-Perron eigenvector
- Proof using a beautiful algebraic technique -- the **Yang-Baxter equation**
- New proof (R.+Peled 2014) using an explicit combinatorial bijection

The connectivity pattern of pipe percolation is the same as that of loop percolation

- Follows easily from the above invariance result by taking the limit as $p \to 0$

The Cantini-Sportiello-Razumov-Stroganov theorem

- In 2001, Alexander Razumov and Yuri Stroganov numerically computed the distribution of the connectivity pattern of pipe percolation (equivalently: the stationary distribution of the Temperley-Lieb random walk). Let’s follow in their footsteps:

```
Manipulate[Column[{Apply[Plus, #], #}] &@RazumovStroganovEigenvector[n], {n, 1, 7, 1}]
```

```
n

429
{42, 17, 17, 14, 4, 17, 6, 14, 4, 17, 6, 6, 4, 1, 17, 6, 6, 4, 1, 14,
 4, 4, 1, 17, 6, 6, 4, 1, 42, 17, 17, 14, 4, 17, 6, 14, 4, 17, 6, 6, 4, 1}
```

- They recognized that the coordinates of the vector enumerated Fully Packed Loops, and formulated a conjecture, which was proved in 2010 by Luigi Cantini and Andrea Sportiello:

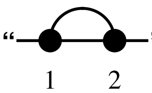
  The Cantini-Sportiello-Razumov-Stroganov theorem. The probability in pipe percolation/loop percolation (of order n) for a given connectivity pattern α is equal to the number of FPL arrangements (of order n) whose connectivity pattern is equal to α divided by the total number FPL_n of arrangements.

  Equivalently: the connectivity pattern of a uniformly random FPL arrangement is *equal in distribution* to that of loop percolation (with any bias p) and pipe percolation.

- Cantini and Sportiello’s proof is ingenious and highly nontrivial. A variant of Wieland’s gyration map plays a key role.

Rational probabilities of connectivity events

In numerical work around 2001-2004, D. Wilson, J.-B. Zuber and Mitra-Nienhuis-de Gier-Batchelor found simple formulas for probabilities of simple events in the Razumov-Stroganov distribution on noncrossing matchings (the distribution of the connectivity pattern of loop percolation, Fully Packed Loops etc.).

- **Example: The submatching event "** 

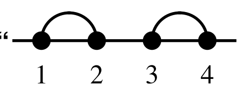
"

```
Table[ProbabilityForSubmatchingEvent[n, {2, 1}], {n, 1, 6}]
```

$\left\{1, \frac{1}{2}, \frac{3}{7}, \frac{17}{42}, \frac{13}{33}, \frac{111}{286}\right\}$

```
GuessFormula[%, {n, 1, 6}, 1]
```

$$\frac{3\,(n^2+1)}{8\,n^2-2}$$

- **Example: The submatching event "** 

"

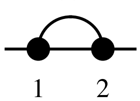
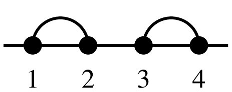
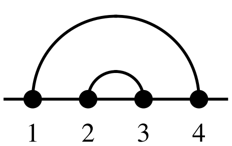
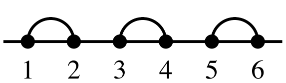
```
Table[ProbabilityForSubmatchingEvent[n, {2, 1, 4, 3}], {n, 2, 6}]
```

$\left\{\frac{1}{2}, \frac{2}{7}, \frac{5}{21}, \frac{94}{429}, \frac{779}{3718}\right\}$

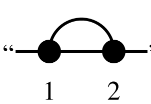
```
GuessFormula[%, {n, 2, 6}, 2]
```

$$\frac{97\,n^6+82\,n^4-107\,n^2-792}{32\,(1-4\,n^2)^2\,(n^2-1)}$$

- **Other examples**

Event	Probability
	$\frac{3}{2}\times\frac{n^2+1}{4\,n^2-1}$
	$\frac{1}{8}\times\frac{97\,n^6+82\,n^4-107\,n^2-792}{\left(4\,n^2-1\right)^2\,\left(4\,n^2-9\right)}$
	$\frac{1}{16}\times\frac{59\,n^6+299\,n^4+866\,n^2+576}{\left(4\,n^2-1\right)^2\,\left(4\,n^2-9\right)}$
	$\frac{1}{512}\times\frac{214\,093\,n^{12}-980\,692\,n^{10}-584\,436\,n^8-1\,887\,916\,n^6+1\,361\,443\,n^4-17\,432\,892\,n^2-316\,353\,600}{\left(4\,n^2-1\right)^3\,\left(4\,n^2-9\right)^2\,\left(4\,n^2-25\right)}$

- **Theoretical developments**

  - The formula for the probability of the simplest submatching event " 

" was proved in 2009 (Fonseca-Zinn-Justin 2009).

- Using the “qKZ equation” techniques developed by Di Francesco-Zinn Justin (2006) and Fonseca-Zinn-Justin, I proved a couple of the other conjectured formulas (R. 2014).
- In the same paper, I also proved a general formula expressing the probability of an arbitrary submatching event as a constant term of a certain multivariate Laurent polynomial. It’s not clear yet how to see that this complicated algebraic expression is always a rational function of n (the obvious empirical pattern one observes in the simple cases). This is an interesting conjecture in algebraic combinatorics with beautiful probabilistic consequences (especially in the limit as $n \to \infty$).
- My constant term formula was discovered experimentally using *Mathematica*. In fact, I wrote *Mathematica* code specifically looking for a certain kind of algebraic expansion (relating a sum over a certain class of “wheel polynomials” to another sum involving a different basis of the same vector space; see my paper). Thus, using a combination of intuition, patience and coding skills I was able to discover (and eventually prove) a beautiful mathematical theorem.
- (Moral of the story: good programming skills can make you look smarter than you are!)

Razumov-Stroganov conjectures and symmetry classes of FPLS

Razumov and Stroganov also studied several natural variants of pipe percolation and the associated Temperley-Lieb random walk with different boundary conditions, and discovered numerically a host of similar "Razumov-Stroganov conjectures" relating the associated eigenvector with classes of FPL arrangements. A few of them were also proved by Cantini and Sportiello using their generalized gyration; others are still unsolved.

Example: aperiodic boundary conditions and Vertically Symmetric FPLs

Thinking of noncrossing matchings as objects existing on a line rather than a circle, it is natural to disallow the "wraparound" pipe operator connecting the points 1 and $2n$. The associated stationary distribution no longer has rotational symmetry. Let's explore this.

```
eigenvectors = Table[RazumovStroganovAperiodicEigenvector[n], {n, 1, 7, 1}];
Manipulate[Column[{Apply[Plus, #], #}] &@eigenvectors[[n]], {n, 1, 7, 1}]
```

```
n

45 885
{7429, 3235, 3021, 2346, 580, 3021, 1199, 2234, 561, 1995, 560, 565,
 246, 42, 3235, 1375, 1199, 890, 210, 2346, 890, 580, 210, 1995, 565, 560,
 246, 42, 1862, 539, 588, 267, 48, 539, 154, 267, 48, 133, 27, 27, 8, 1}
```

```
Map[Apply[Plus, #] &, eigenvectors]
```

```
{1, 3, 26, 646, 45 885, 9 304 650, 5 382 618 660}
```

```
OnlineEncyclopediaOfIntegerSequencesLookup[%]
```

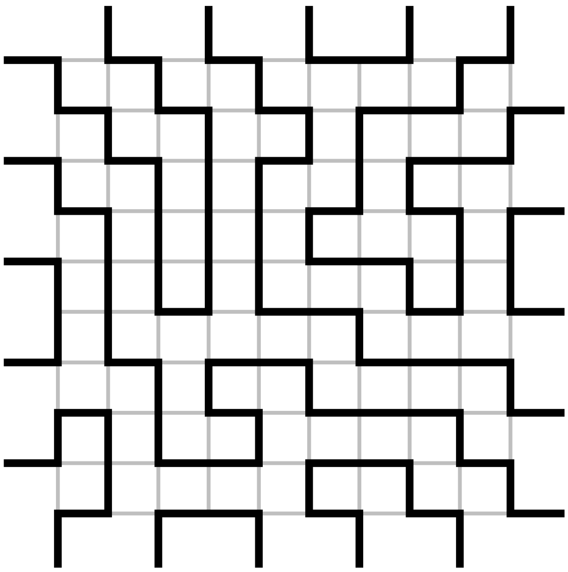
It turns out that the stationary probabilities are related to the connectivity patterns of uniformly random FPL arrangements chosen from the set of **Vertically Symmetric FPLs** (related to Vertically Symmetric Alternating Sign Matrices, known as VSASMs).

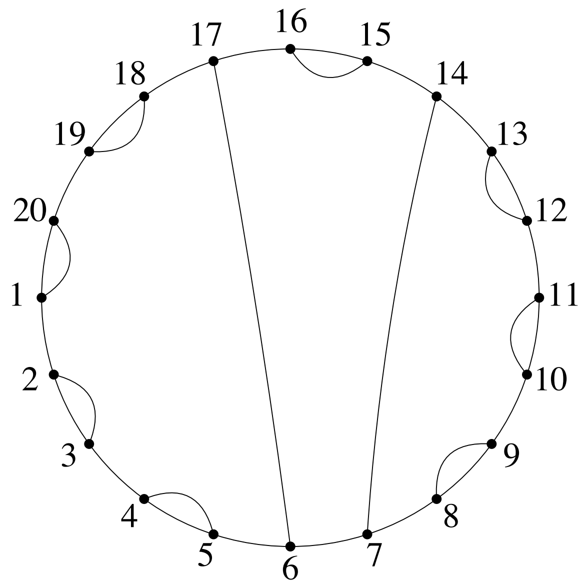
Other Razumov-Stroganov variants

The Razumov-Stroganov conjectures involve symmetry classes of FPLs, or equivalently of ASMs. Here our story intersects with another story -- the enumeration of these symmetry classes (mostly solved in two papers by Greg Kuperberg and Soichi Okada). Some of the symmetry classes that show up are:

- Half-Turn Symmetric ASMs (HTASM)
- Quarter-Turn Symmetric ASMs (QTASM)
- Diagonally and Antidiagonally Symmetric ASMs (DASASM)
- ... (?)

The End - Thank You!





References

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- Alexander Razumov, Yuri Stroganov. Combinatorial nature of ground state vector of $O(1)$ loop model. *Theor. Math. Phys.* **138** (2004), 333-337.
- Luigi Cantini, Andrea Sportiello. Proof of the Razumov-Stroganov conjecture. *J. Comb. Theory Ser. A* **118** (2011), 1549-1574.
- Dan Romik. Connectivity patterns in loop percolation I: the rationality phenomenon and constant term identities. *Commun. Math. Phys.* **330** (2014), 499-538.
- Dan Romik, Ron Peled. Bijective combinatorial proof of the commutation of transfer matrices in the dense $O(1)$ loop model. Preprint, 2014.