A proof of the Hardy-Ramanujan formula

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1. Introduction

Let $p(n)$ be the number of unordered partitions of n. Our aim in this note is to prove the Hardy-Ramanujan formula

$$(1) \qquad p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}$$

where $a_n \sim b_n$ means $\lim_{n\to\infty} a_n/b_n = 1$. Our method, like most existing proofs of this formula, uses contour integration in the complex plane. However, we will show that the proof of (1) reduces to proving a local limit theorem in a probabilistic model for random partitions introduced by Fristedt [1]. This will result in a proof that is nicely structured and conceptually quite simple.

Throughout, we use the following notation:

$$F(z) = \sum_{n=0}^{\infty} p(n) z^n = \prod_{k=1}^{\infty} \frac{1}{1-z^k} \qquad (|z| < 1)$$

is the generating function for $p(n)$. Denote

$$c = \frac{\pi}{\sqrt{6}}\ , \qquad s_n = \frac{c}{\sqrt{n}}\ , \qquad x_n = e^{-s_n} = 1 - \frac{c}{\sqrt{n}} + O\left(\frac{1}{n}\right)$$

We now describe Fristedt's probabilistic model for partitions: Let $0 < x < 1$. Define independent random variables $R_1, R_2, R_3, ...$ such that $R_k + 1$ has geometric distribution with parameter $1 - x^k$. More precisely:

$$P_x(R_k = j) = (1 - x^k)x^{kj} \qquad j = 0, 1, 2, ...$$

where P_x denotes probability (the subscript x denotes the choice of parameter). Let $N = \sum_{k=1}^{\infty} kR_k$. Then $(R_1, R_2, R_3, ...)$ can be thought of as the frequential coding of a random partition of the (random) integer N, i.e. the partition in which 1 appears R_1 times, 2 appears R_2 times etc. Then for any (nonrandom) partition

$$n = 1 \cdot r_1 + 2 \cdot r_2 + 3 \cdot r_3 + ...$$

of n, given in frequential coding, the probability of it appearing in the random model is

$$P_x(R_1 = r_1, R_2 = r_2, R_3 = r_3, ...) = \prod_{k=1}^{\infty} P_x(R_k = r_k) =$$

$$= \prod_{k=1}^{\infty} \left((1 - x^k) x^{k r_k} \right) = \frac{x^n}{F(x)}$$

Therefore the probability that $N = n$ is a sum over all $p(n)$ different partitions of n of this quantity, namely

$$P_x(N = n) = \frac{p(n) x^n}{F(x)}$$

This is the key observation that we will require for our proof; we have constructed a random variable whose value probabilities are related to $p(n)$ in a simple way. Furthermore, this random variable is a sum of lattice random variables, and thus we can expect it to be an approximately normal lattice random variable and satisfy a local limit theorem.

The proof of (1) will now follow from the following facts:

Fact 1. For positive real s, we have

$$\log F\left(e^{-s}\right) = \frac{\pi^2}{6s} + \frac{1}{2} \log s - \frac{1}{2} \log(2\pi) + o(1)$$

as $s \searrow 0$.

Fact 2. For choice of parameter x_n, N is a random variable with expectation

$$E_{x_n}(N) = n(1 + O(1/\sqrt{n}))$$

and variance

$$\sigma_{x_n}^2(N) \sim \frac{2\sqrt{6}}{\pi} n^{3/2}$$

Fact 3. The random variable N "satisfies a local limit theorem at 0", that is

$$P_{x_n}(N = n) \sim \frac{1}{\sqrt{2\pi}\sigma_{x_n}(N)}$$

as $n \to \infty$.

Fact 1 is well-known. Facts 2 and 3 were proved by Fristedt [1]. We give complete proofs below, but first, let us show how they imply (1):

Deduction of (1) from Facts 1,2,3:

$$p(n) = x_n^{-n} \cdot F(x_n) \cdot P_{x_n}(N = n) = e^{n s_n} \cdot F\left(e^{-s_n}\right) \cdot P_{x_n}(N = n) \sim$$

$$\sim e^{c\sqrt{n}} \cdot \left(e^{c\sqrt{n}} \sqrt{\frac{c}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}} \right) \cdot \frac{1}{\sqrt{2\pi}\; n^{3/4} \sqrt{2\sqrt{6}/\pi}} =$$

$$= \frac{1}{2\sqrt{2}\sqrt{6}\; n} e^{2c\sqrt{n}} = \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{2n/3}}$$

2. Proof of Fact 1

We follow Newman [2,3]:

$$\log F\left(e^{-s}\right) = -\sum_{k=1}^{\infty}\log(1-e^{-ks}) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{e^{-kjs}}{j} =$$

$$= \sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=1}^{\infty}e^{-jks} = \sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{e^{js}-1} =$$

$$= \sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{1}{js}-\frac{e^{-js}}{2}\right) + \sum_{j=1}^{\infty}\frac{1}{j}\left(\frac{1}{e^{js}-1}-\frac{1}{js}+\frac{e^{-js}}{2}\right) =$$

$$= \frac{\pi^2}{6s} + \frac{1}{2}\log(1-e^{-s}) + \sum_{j=1}^{\infty}\frac{s}{js}\left(\frac{1}{e^{js}-1}-\frac{1}{js}+\frac{e^{-js}}{2}\right) =$$

$$= \frac{\pi^2}{6s} + \frac{1}{2}\log s + \int_0^{\infty}\frac{1}{x}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{e^{-x}}{2}\right)dx + o(1)$$

It remains therefore to prove that

$$\int_0^{\infty}\frac{1}{x}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{e^{-x}}{2}\right)dx = -\frac{1}{2}\log(2\pi).$$

But this integral is the limit, as $m\to\infty$, of the integral

$$\int_0^{\infty}\frac{1-e^{-mx}}{x}\left(\frac{1}{e^x-1}-\frac{1}{x}+\frac{e^{-x}}{2}\right)dx =$$

$$= \int_0^{\infty}\frac{1-e^{-mx}}{x}\left(\frac{1}{e^x-1}-\frac{1}{x}\right)dx + \frac{1}{2}\int_0^{\infty}\frac{1-e^{-mx}}{x}e^{-x}dx =$$

$$= \sum_{k=1}^{m}\int_0^{\infty}e^{-kx}\frac{1+x-e^x}{x^2}dx + \frac{1}{2}\int_0^{\infty}\frac{e^{-x}-e^{-(m+1)x}}{x}dx$$

These integrals can be evaluated by noticing that

$$\frac{e^{-x}-e^{-(m+1)x}}{x} = \int_1^{m+1}e^{-ux}du$$

$$\frac{1+x-e^x}{x^2} = \int_0^1 -ue^{(1-u)x}du$$

and then interchanging the order of integration, to get

$$\sum_{k=1}^{m}\int_0^1\frac{-u}{k+u-1}du + \frac{1}{2}\int_1^{m+1}\frac{du}{u} =$$

$$= \sum_{k=1}^{m}\left((k-1)\log\left(\frac{k}{k-1}\right)-1\right) + \frac{1}{2}\log(m+1) =$$

$$= m\log m - \log(m!) - m + \frac{1}{2}\log(m+1),$$

which by Stirling's formula indeed converges to $-\log(2\pi)/2$.

3. Proof of Fact 2

We use the simple probabilistic facts that if X is a random variable such that $X+1$ has geometric distribution with parameter $0<p<1$, that is

$$P(X=j)=p(1-p)^j \quad j=0,1,2,3,...$$

then

$$E(X)=\sum_{j=0}^{\infty} jp(1-p)^j = \frac{1-p}{p}$$

$$\sigma^2(X)=\sum_{j=0}^{\infty} j^2 p(1-p)^j - \left(\frac{1-p}{p}\right)^2 = \frac{1-p}{p^2}$$

Now $N=\sum_{k=1}^{\infty} kR_k$, so

$$E_{x_n}(N)=\sum_{k=1}^{\infty} k\frac{x_n^k}{1-x_n^k} = n\sum_{k=1}^{\infty}\frac{1}{\sqrt{n}}\frac{k}{\sqrt{n}}\frac{e^{-ck/\sqrt{n}}}{1-e^{-ck/\sqrt{n}}}$$

The sum is a Riemann sum, with $\Delta u = 1/\sqrt{n}$, for the integral

$$\int_0^{\infty}\frac{ue^{-cu}}{1-e^{-cu}}du = \frac{\mathrm{Li}_2(1)}{c^2} = \frac{\pi^2/6}{c^2} = 1,$$

where $\mathrm{Li}_2(x) = -\int_0^x \log(1-t)dt/t = \sum_{m=1}^{\infty} x^m/m^2$ is the dilogarithm function. The difference between the Riemann sum and the integral is easily seen to be $O(1/\sqrt{n})$, so

$$E_{x_n}(N) = n(1+O(1/\sqrt{n}))$$

Similarly, the variance

$$\sigma^2_{x_n}(N)=\sum_{k=1}^{\infty} k^2\frac{x_n^k}{(1-x_n^k)^2} = n^{3/2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{k}{\sqrt{n}}\right)^2\frac{e^{-ck/\sqrt{n}}}{(1-e^{-ck/\sqrt{n}})^2} \sim$$

$$\sim n^{3/2}\int_0^{\infty}\frac{u^2e^{-cu}}{(1-e^{-cu})^2}du$$

The integral can be evaluated to be

$$\int_0^{\infty}\frac{u^2e^{-cu}}{(1-e^{-cu})^2}du = \frac{1}{c^3}\int_0^1\frac{\log^2(1-x)}{x^2}dx =$$

$$= \frac{1}{c^3}\left[2\mathrm{Li}_2(x) - \frac{1-x}{x}\log^2(1-x)\right]\Bigg|_{x=0}^{x=1} = \frac{\pi^2/3}{c^3} = \frac{2\sqrt{6}}{\pi}$$

4. Proof of Fact 3

We now reach the "delicate" part of the analysis, namely the proof of the claim that N satisfies a local limit theorem at 0. We proceed by the standard methodology of probability theory, which is to represent the probabilities as inverse Fourier integrals of the characteristic function. But this is exactly a parametrized contour integral! So the probabilistic approach leads to the same analytic ideas that appear in the traditional proofs of (1). However, the probabilistic thinking assigns meanings to the various quantities that appear in the analysis. This puts the analysis on a solid conceptual framework, and makes it easier to find the correct estimates and manipulations, as well as enabling one to "guess" formula (1) before actually proving it.

Denote by $\phi_x(t) = E_x(e^{itN})$ the characteristic function of N for parameter choice x. Then

$$\phi_x(t) = \sum_{n=0}^{\infty} P_x(N=n)e^{int} = \sum_{n=0}^{\infty} \frac{p(n)x^n}{F(x)} e^{int} = \frac{F(xe^{it})}{F(x)}$$

And using Fourier inversion we get the disguised contour integral

$$P_{x_n}(N=n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_{x_n}(t)e^{-int}dt =$$

$$= \frac{1}{2\pi\sigma_{x_n}(N)} \int_{-\pi\sigma_{x_n}(N)}^{\pi\sigma_{x_n}(N)} \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}du$$

So it is enough to prove that

$$\text{(2)} \qquad \int_{-\pi\sigma_{x_n}(N)}^{\pi\sigma_{x_n}(N)} \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}du \xrightarrow[n\to\infty]{} \sqrt{2\pi}$$

Indeed, probabilistic thinking leads us to expect that for any $u \in \mathbb{R}$,

$$\text{(3)} \qquad \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)} \xrightarrow[n\to\infty]{} e^{-u^2/2},$$

which will give us (2) if we can prove some additional boundedness estimates. Note that (3) is equivalent to the claim that N satisfies a (non-local) central limit theorem, i.e. that $(N-n)/\sigma_{x_n}(N) \to N(0,1)$ in distribution as $n \to \infty$. This can be deduced e.g. by using the Lindeberg central limit theorem for triangular arrays. Instead, we give a direct proof. First, we need a technical lemma:

Lemma. For $0 < x < 1, t \in \mathbb{R}$, let

$$f_x(t) = \log\left(\frac{1-x}{1-e^{it}x}\right) - i\frac{x}{1-x}t + \frac{1}{2}\frac{x}{(1-x)^2}t^2$$

Then there exists a constant $C > 0$ such that

$$\text{(4)} \qquad |f_x(t)| \le C\frac{x|t|^3}{(1-x)^3} \qquad (0 < x < 1, t \in \mathbb{R})$$

Proof. First, consider the case $|t| \le (1-x)/2$:

$$\log\left(\frac{1-x}{1-e^{it}x}\right) = \sum_{j=1}^{\infty} \frac{x^j}{j}(e^{ijt}-1) = \sum_{j=1}^{\infty} \frac{x^j}{j} \sum_{k=1}^{\infty} \frac{i^k j^k t^k}{k!} =$$

$$= \sum_{k=1}^{\infty} \frac{i^k}{k!} \left(\sum_{j=1}^{\infty} j^{k-1} x^j \right) t^k = i\frac{x}{1-x}t - \frac{1}{2}\frac{x}{(1-x)^2}t^2 + \sum_{k=3}^{\infty} i^k \left(\frac{1}{k!} \sum_{j=1}^{\infty} j^{k-1} x^j \right) t^k$$

So that

$$|f_x(t)| \le \sum_{k=3}^{\infty} \left(\frac{1}{k!} \sum_{j=1}^{\infty} j^{k-1} x^j \right) |t|^k \le \sum_{k=3}^{\infty} \left(\frac{1}{k!} \sum_{j=1}^{\infty} j(j+1)...(j+k-2) x^j \right) |t|^k =$$

$$= \sum_{k=3}^{\infty} \frac{x}{k} \frac{|t|^k}{(1-x)^k} \le \sum_{k=3}^{\infty} \frac{x}{3} \left(\frac{|t|}{1-x} \right)^k = \frac{x}{3} \frac{|t|^3/(1-x)^3}{1-t/(1-x)}$$

When $|t| \le (1-x)/2$ this gives us $|f_x(t)| \le 2x|t|^3/3(1-x)^3$. Next, for $|t| > (1-x)/2$ we have

$$\left| -i\frac{x}{1-x}t + \frac{1}{2}\frac{x}{(1-x)^2}t^2 \right| \le \frac{x|t|^3}{1-x}\frac{1}{t^2} + \frac{x|t|^3}{(1-x)^2}\frac{1}{|t|} \le (4+2)\frac{x|t|^3}{(1-x)^3},$$

so it remains to prove

$$\left| \log\left(\frac{1-x}{1-xe^{it}} \right) \right| \le C \frac{x|t|^3}{(1-x)^3} \quad (|t| > (1-x)/2)$$

For $|t| \ge 1/4$, clearly

$$\left| \log\left(\frac{1-x}{1-xe^{it}} \right) \right| \le \sum_{j=1}^{\infty} \frac{1}{j} |e^{ijt} - 1| \le -2\log(1-x) \le$$

$$\le C' \frac{x}{(1-x)^3} \le 64\ C' \frac{x|t|^3}{(1-x)^3}$$

Finally, for $0 \le (1-x)/2 \le |t| \le 1/4$ (which means in particular $1/2 \le x \le 1$,)

$$\log\left(\frac{1-x}{1-xe^{it}} \right) = -\log\left(1 + \frac{x}{1-x}(e^{it}-1) \right) =$$

$$= -\log\left(1 + \frac{x}{1-x} 2ie^{it/2}\sin(t/2) \right) = -\log(1 + ie^{it/2}T),$$

where we denote $T = 2\sin(t/2)x/(1-x)$. We have

$$\frac{1}{40} \le \frac{x}{1-x}\frac{|t|}{10} \le |T| \le \frac{x}{1-x}|t|$$

and therefore, since $\pi/2 - 1/8 \le \arg(iTe^{it/2}) \le \pi/2 + 1/8$,

$$\left| \log\left(\frac{1-x}{1-xe^{it}} \right) \right| = |\log(1 + iTe^{it/2})| \le C''|T|^3 \le C'' \left(\frac{|t|}{1-x} \right)^3 \le 2C'' \frac{x|t|^3}{(1-x)^3}$$

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Proof of (3).

$$\log\left(\phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}\right) = \log F(xe^{it}) - \log F(x) - \frac{inu}{\sigma_{x_n}(N)} =$$

$$= \sum_{k=1}^{\infty} \log\left(\frac{1-x_n^k}{1-x_n^k e^{iku/\sigma_{x_n}(N)}}\right) - \frac{inu}{\sigma_{x_n}(N)} =$$

$$= \sum_{k=1}^{\infty} f_{x_n^k}(ku/\sigma_{x_n}(N)) + i\left(\sum_{k=1}^{\infty}\frac{kx_n^k}{1-x_n^k} - n\right)\frac{u}{\sigma_{x_n}(N)} - \frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{k^2x_n^k}{(1-x_n^k)^2}\right)\frac{u^2}{\sigma_{x_n}^2(N)} =$$

$$= i(E_{x_n}(N)-n)\frac{u}{\sigma_{x_n}(N)} - \frac{u^2}{2} + R_n(u) = O(n^{-1/4})u - \frac{u^2}{2} + R_n(u),$$

where

$$|R_n(u)| = \left|\sum_{k=1}^{\infty} f_{x_n^k}(ku/\sigma_{x_n}(N))\right| \le C\frac{|u|^3}{\sigma_{x_n}^3(N)}\sum_{k=1}^{\infty}\frac{k^3x_n^k}{(1-x_n^k)^3} = |u|^3 O(n^{-1/4}),$$

since

$$\sum_{k=1}^{\infty}\frac{k^3x_n^k}{(1-x_n^k)^3} = \frac{1}{n^2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{k}{\sqrt{n}}\right)^3\frac{e^{-ck/\sqrt{n}}}{(1-e^{-ck/\sqrt{n}})^3} \sim$$

$$\sim n^{-2}\int_0^{\infty}\frac{v^3e^{-cv}}{(1-e^{-cv})^3}dv$$

so altogether we have shown that for all $u \in \mathbb{R}$

$$\log\left(\phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}\right) \xrightarrow[n\to\infty]{} -\frac{u^2}{2}$$

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Proof of (2). To prove that (2) follows from (3), note first that for $z = xe^{it}$,

$$F(z) = \exp\left(-\sum_{k=1}^{\infty}\log(1-z^k)\right) = \exp\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{z^{kj}}{j}\right) = \exp\left(\sum_{j=1}^{\infty}\frac{1}{j}\frac{z^j}{1-z^j}\right),$$

so

$$\left|\frac{F(z)}{F(x)}\right| = \exp\left(\mathrm{Re}\frac{1}{1-z} - \frac{1}{1-x} + \sum_{j=2}^{\infty}\frac{1}{j}\left(\mathrm{Re}\frac{z^j}{1-z^j} - \frac{x^j}{1-x^j}\right)\right) \le$$

$$\le \exp\left(\frac{1}{|1-z|} - \frac{1}{1-x}\right) \le \exp\left(\frac{1}{\sqrt{(1-x)^2+\sin^2 t}} - \frac{1}{1-x}\right)$$

This implies that for $|\sin t| > 1 - x$, we have the estimate

$$|\phi_x(t)| \le \exp\left(\left(\frac{1}{\sqrt{2}} - 1\right)\frac{1}{1-x}\right)$$

And for $|\sin t| \le 1 - x$,

$$|\phi_x(t)| \le \exp\left(\frac{1}{1-x}\left(\frac{1}{\sqrt{1+\sin^2 t/(1-x)^2}} - 1\right)\right) \le \exp\left(-\frac{1}{16}\frac{t^2}{(1-x)^3}\right),$$

where we have used the elementary inequalities

$$\sin t \ge \frac{t}{2} \quad (0 \le t \le 1), \qquad \frac{1}{\sqrt{1+u}} \le 1 - \frac{u}{4} \quad (0 \le u \le 1)$$

Now (2) follows immediately, because

$$\int_{-\pi\sigma_{x_n}(N)}^{\pi\sigma_{x_n}(N)} \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}du =$$

$$= \int_{|\sin(u/\sigma_{x_n}(N))| \le 1-x_n} \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}du +$$

$$+ \int_{|\sin(u/\sigma_{x_n}(N))| > 1-x_n} \phi_{x_n}(u/\sigma_{x_n}(N))e^{-inu/\sigma_{x_n}(N)}du$$

In the first term, the integrand is bounded in absolute value by $\exp(-O(u^2))$, therefore this term converges to $\sqrt{2\pi}$ by the dominated convergence theorem (note that $\sigma_{x_n}(N)\arcsin(1-x_n) \sim An^{1/4} \to \infty$). The second term is bounded in absolute value by

$$2\pi\sigma_{x_n}(N)\exp\left(-\left(1 - \frac{1}{\sqrt{2}}\right)\frac{\sqrt{6n}}{\pi}\right) \xrightarrow[n\to\infty]{} 0$$

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