1. Show that Liouville’s theorem (“a bounded entire function is constant”) can be proved directly using the “simple” \((n = 0)\) case of Cauchy’s integral formula, instead of using the case \(n = 1\) of the extended formula as we did in the lecture.

**Hint.** For an arbitrary pair of complex numbers \(z_1, z_2 \in \mathbb{C}\), show that \(|f(z_1) - f(z_2)| = 0|\).

2. Show that Liouville’s theorem can in fact be deduced even just from the mean value property of holomorphic functions, which is the special case of Cauchy’s integral formula in which \(z\) is taken as the center of the circle around which the integration is performed.

**Hint.** Here it makes sense to consider a modified version of the mean value property (that follows easily from the original version) that says that \(f(z)\) is the average value of \(f(w)\) over a disc \(D_R(z)\) (instead of a circle \(C_R(z)\)). That is,

\[
f(z) = \frac{1}{\pi R^2} \int \int_{D_R(z)} f(x + iy) \, dx \, dy,
\]

where the integral is an ordinary two-dimensional Riemann integral. Explain why this formula holds, then use it to again bound \(|f(z_1) - f(z_2)|\) from above by a quantity that goes to 0 as \(R \to 0\).

3. Prove the following generalization of Liouville’s theorem: let \(f\) be an entire function satisfying the inequality

\[
|f(z)| \leq A + B|z|^n \quad (z \in \mathbb{C})
\]

for some constants \(A, B > 0\) and integer \(n \geq 0\). Then \(f\) is a polynomial of degree \(\leq n\).

4. It is important to have computational facility with contour integrals in addition to a good grasp of the theory. To practice your contour integral wizardry skills, solve exercises 1–4 (pages 64–65) in Chapter 2 of [Stein-Shakarchi].
5. Spend at least 5–10 minutes thinking about the concept of a **toy contour**. Specifically, for the case of a **keyhole contour** we discussed in the context of the proof of Cauchy’s integral formula, think carefully about the steps that are needed to get a proof of Cauchy’s theorem for the region enclosed by such a contour. Even better, sketch a proof of the key result that a function holomorphic in such a region (and therefore having the property that its contour integral along triangles and rectangles vanish) has a primitive.

6. The Cauchy integral formula is intimately connected to an important formula from the theory of the Laplace equation and harmonic functions called the **Poisson integral formula**. Solve exercises 11–12 (pages 66–67) in Chapter 2 of [Stein-Shakarchi], which explore this connection, and more generally the connection between holomorphic and harmonic functions.