1. (a) Show that the Laurent expansion of $\Gamma(s)$ around $s = 0$ is of the form

$$\Gamma(s) = \frac{1}{s} - \gamma + O(s)$$

(where $\gamma$ is the Euler-Mascheroni constant). If you’re feeling especially energetic, derive the more detailed expansion

$$\Gamma(s) = \frac{1}{s} - \gamma + \left(\frac{\gamma^2}{2} + \frac{\pi^2}{12}\right) s + O(s^2)$$

and proceed to derive (by hand or using a software package such as Mathematica) as many additional terms in the expansion as you have the patience to do.

(b) Show that the Laurent expansion of $\zeta(s)$ around $s = 1$ is of the form

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1).$$

2. Show that the symmetric version of the functional equation for the zeta function

$$\zeta^*(1-s) = \zeta^*(s),$$

where $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, can be rewritten in the equivalent form

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

3. Show that the Taylor expansion of the digamma function $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ around $s = 1$ is given by

$$\psi(s) = -\gamma - \sum_{n=1}^{\infty} (-1)^{n-1} \zeta(n+1)(s-1)^n \quad (|s-1| < 1).$$

4. Define a function $D(s)$ of a complex variable $s$ by

$$D(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} s^{-n}}{n} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots.$$
(a) Prove that the series defining $D(s)$ converges uniformly on any half-plane of the form $\text{Re}(s) \geq \alpha$ where $\alpha > 0$, and conclude that $D(s)$ is defined and holomorphic in the half-plane $\text{Re}(s) > 0$.

(b) Show that $D(s)$ is related to the Riemann zeta function by the formula

$$D(s) = (1 - 2^{1-s})\zeta(s) \quad (\text{Re}(s) > 1).$$

(c) Using this relation, deduce a new proof that the zeta function can be analytically continued to a meromorphic function on $\text{Re}(s) > 0$ that has a simple pole at $s = 1$ with residue 1 and is holomorphic everywhere else in the region.

5. Let $\psi(x) = \sum_{p^k \leq x} \log p$ denote von Mangoldt’s weighted prime counting function. Show that $\psi(n) = \log \text{lcm}(1, 2, \ldots, n)$, where for integers $a_1, \ldots, a_k$, $\text{lcm}(a_1, \ldots, a_k)$ denotes the least common multiple of $a_1, \ldots, a_k$.

Note that this implies that an equivalent formulation of the prime number theorem is the interesting statement that

$$\text{lcm}(1, \ldots, n) = e^{(1+o(1))n} \quad \text{as } n \to \infty.$$  

6. (a) Reprove the “toy Riemann hypothesis” — the theorem that the Riemann zeta function has no zeros on the line $\text{Re}(s) = 1$ by considering the behavior of

$$Y = \text{Re} \left[ -3 \frac{\zeta'(\sigma)}{\zeta(\sigma)} - 4 \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} - \frac{\zeta'(\sigma + 2it)}{\zeta(\sigma + 2it)} \right]$$

for $t \in \mathbb{R} \setminus \{0\}$ fixed and $\sigma \searrow 1$, instead of the quantity

$$X = \log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)|.$$  

Use the series expansion

$$-\frac{\zeta(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s},$$

where $\Lambda(n)$ is von Mangoldt’s function (equal to $\log p$ if $n = p^k$ is a prime power, and 0 otherwise).
(b) Try to reprove the same theorem in yet a third way by considering

\[ Z = \log |\zeta(\sigma)|^{10} \zeta(\sigma + it)^{15} \zeta(\sigma + 2it)^{6} \zeta(\sigma + 3it)|, \]

and attempting to repeat the argument involving expanding the logarithm in a power series and deducing that \( Z \geq 0 \). Does this give a proof of the theorem? If not, what goes wrong?

**Hint.** \((a+b)^6 = a^6 + 6a^5b + 10a^4b^2 + 15a^3b^3 + 10a^2b^4 + 6ab^5 + b^6\).

7. (a) Prove that for all \( x \geq 1 \),

\[ \prod_{p \leq x} \frac{1}{1 - \frac{1}{p}} \geq \log x \]

(where the product is over all prime numbers \( p \) that are \( \leq x \)).

(b) Pass to the logarithm and deduce that for some constant \( K > 0 \) we have the bound

\[ \sum_{p \leq x} \frac{1}{p} \geq \log \log x - K \quad (x \geq 1). \]

That is, the *harmonic series of primes* \( \sum_{p} \frac{1}{p} \) diverges as \( \log \log x \), in contrast to the usual harmonic series which diverges as \( \log x \).

8. Define arithmetic functions taking an integer argument \( n \), as follows:

\[ \mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \cdots p_k \text{ is a product of } k \text{ distinct primes}, \\ 0 & \text{otherwise}, \end{cases} \]

(the Möbius \( \mu \)-function),

\[ d(n) = \sum_{d|n} 1, \]

(the number of divisors function),

\[ \sigma(n) = \sum_{d|n} d, \]

(the sum of divisors function),

\[ \phi(n) = \# \{1 \leq k \leq n - 1 : \gcd(k,n) = 1\}, \]

(the Euler totient function),

\[ \Lambda(n) = \begin{cases} \log p & \text{if } n = p^k, \text{ } p \text{ prime}, \\ 0 & \text{otherwise}, \end{cases} \]

(the von Mangoldt \( \Lambda \)-function).
We saw that the zeta function and its logarithmic derivative have the Dirichlet series representations

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \]

\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}. \]

Use the Euler product formula for the zeta function or other elementary manipulations to prove the following identities:

\[ \zeta'(s) = -\sum_{n=1}^{\infty} \log n \cdot n^{-s}, \]

\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n)n^{-s}; \]

\[ \frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} |\mu(n)|n^{-s}. \]

Other famous Dirichlet series representations you may want to think about or look up are

\[ \zeta(s)^2 = \sum_{n=1}^{\infty} d(n)n^{-s}, \]

\[ \frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \phi(n)n^{-s}, \]

\[ \zeta(s)\zeta(s-1) = \sum_{n=1}^{\infty} \sigma(n)n^{-s}. \]