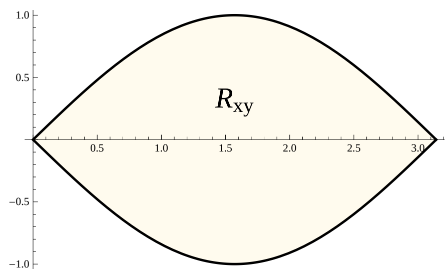
# Midterm Exam 1–Solutions

MAT 21D, Temple/Romik, Spring 2016

**Problem #1 (20pts): (a)** Sketch and find the AREA of the region of integration $\mathbf{R}_{xy}$ determined by the following iterated integral. (You do not need to evaluate this integral.)

$$\int_0^{\pi} \int_{-\sin x}^{\sin x} \ln\left(x^2 y^4\right) dy\, dx. \tag{1}$$

**Solution.** $\mathbf{R}_{xy}$ is the region is bounded between the curves $y = -\sin x$ and $y = \sin x$, $0 \le x \le \pi$:




The area of $\mathbf{R}_{xy}$ is

$$\iint_{\mathbf{R}_{xy}} dA = \int_0^{\pi} \int_{-\sin x}^{\sin x} dy\, dx = \int_0^{\pi} (\sin x - (-\sin x))\, dx$$
$$= \int_0^{\pi} 2\sin x\, dx = 2(-\cos x)\Big|_{x=0}^{x=\pi} = 2(1-(-1)) = 4.$$

**(b)** Rewrite this iterated integral as an iterated integral with the order of integration reversed that produces the same value. (Again, you do not need to evaluate the integrals.)

$$\int_0^{1} \int_{2x^3}^{2x} \ln\left(x^2 y^4\right) dy\, dx. \tag{2}$$

**Solution.** The region of integration consists of pairs $(x, y)$ that satisfy $0 \le x \le 1$, $2x^3 \le y \le 2x$. Thus as $x$ ranges between 0 and 1, $y$ ranges between 0 and 2, and the inequalities $2x^3 \le y \le 2x$ can be rewritten as $y/2 \le x \le (y/2)^{1/3}$, so the integral is equal to

$$\int_0^{2} \int_{y/2}^{(y/2)^{1/3}} \ln\left(x^2 y^4\right) dx\, dy.$$

**Problem #2 (20pts): (a)** Use polar coordinates to evaluate the integral

$$\iint_{\mathcal{R}_{xy}} e^{-r^2} dx\, dy,$$

where $\mathcal{R}_{xy}$ is the region inside the circle of radius $R$ centered at $(0,0)$.

**Solution.**

$$\begin{aligned}
\iint_{\mathcal{R}_{xy}} e^{-r^2} dx\, dy &= \int_0^{2\pi} \int_0^R e^{-r^2} r\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^{R^2} e^{-u} \frac{du}{2}\, d\theta \quad (\text{substitution: } u = r^2) \\
&= \frac{1}{2} \int_0^{2\pi} \left(-e^{-u}\right) \Big|_{u=0}^{u=R^2} d\theta \\
&= \frac{1}{2} \int_0^{2\pi} \left(1 - e^{-R^2}\right) d\theta = \pi \left(1 - e^{-R^2}\right).
\end{aligned}$$

**(b)** Since in the limit $R \to \infty$ you are integrating over all of $\mathcal{R}^2$, take the limit $R \to \infty$ in $(a)$ to obtain a value for $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy$.

**Solution.**

$$\begin{aligned}
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy &= \iint_{\mathcal{R}^2} e^{-(x^2+y^2)}\, dx\, dy \\
&= \lim_{R\to\infty} \pi \left(1 - e^{-R^2}\right) = \pi(1-0) = \pi.
\end{aligned}$$

**(c)** Iterate the integral to show $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2$.

**Solution.**

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \quad (\text{using } e^{a+b}=e^a e^b)\\
&= \int_{-\infty}^{\infty} e^{-y^2}\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)dy\\
&\qquad\qquad (\text{moving a scalar } e^{-y^2} \text{ outside the integral } dx)\\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\int_{-\infty}^{\infty} e^{-y^2}\,dy\\
&\qquad\qquad (\text{moving a scalar } \int_{-\infty}^{\infty} e^{-x^2}\,dx \text{ outside the integral } dy)\\
&= \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2.
\end{aligned}$$

**(d)** Use (b) and (c) to evaluate the famous integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ (the *Gaussian*).

**Solution.** Denote $G = \int_{-\infty}^{\infty} e^{-x^2}\,dx$. We showed that

$$G^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi.$$

Also clearly $G$ is nonnegative as the integral of a nonnegative function. It follows that

$$G = \sqrt{\pi}.$$

**Problem #3 (20pts):** Consider a triangular metal plate with three corners $(0,0)$, $(2,0)$, $(0,2)$ meters and density $\delta(x,y) = xy\ kg/m^2$.

**(a)** Find the mass of the plate. (Put in units.)

**Solution.**

$$\begin{aligned}
M &= \int_0^2 \int_0^{2-x} \delta(x,y)\,dy\,dx = \int_0^2 \int_0^{2-x} xy\,dy\,dx \\
&= \int_0^2 x \cdot \frac{y^2}{2}\bigg|_{y=0}^{y=2-x} dx = \frac{1}{2}\int_0^2 x(2-x)^2\,dx \\
&= \frac{1}{2}\int_0^2 \left(4x - 4x^2 + x^3\right)\,dx = \frac{1}{2}\left(2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right)\bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left(2 \times 4 - \frac{4}{3} \times 8 + \frac{1}{4} \times 16\right)\ kg = \frac{2}{3}\ kg.
\end{aligned}$$

**(b)** Find the center of mass $(\bar{x}, \bar{y})$. (Put in units. Note by symmetry $\bar{x} = \bar{y}$.)

**Solution.**

$$\begin{aligned}
M_x &= \int_0^2 \int_0^{2-x} x\delta(x,y)\,dy\,dx = \int_0^2 \int_0^{2-x} x^2 y\,dy\,dx \\
&= \int_0^2 x^2 \cdot \frac{y^2}{2}\bigg|_{y=0}^{y=2-x} dx = \frac{1}{2}\int_0^2 x^2(2-x)^2\,dx \\
&= \frac{1}{2}\int_0^2 \left(4x^2 - 4x^3 + x^4\right)\,dx = \frac{1}{2}\left(\frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5\right)\bigg|_{x=0}^{x=2} \\
&= \frac{1}{2}\left(\frac{4}{3} \times 8 - 16 + \frac{1}{5} \times 32\right)\ kg \cdot \text{meter} = \frac{8}{15}\ kg \cdot \text{meter}.
\end{aligned}$$

The value $\bar{x}$ is therefore given by

$$\bar{x} = \frac{M_x}{M} = \frac{8/15}{2/3} = \frac{4}{5}\ \text{meters}.$$

Since $\bar{x} = \bar{y}$ as was noted in the question, the center of mass is at

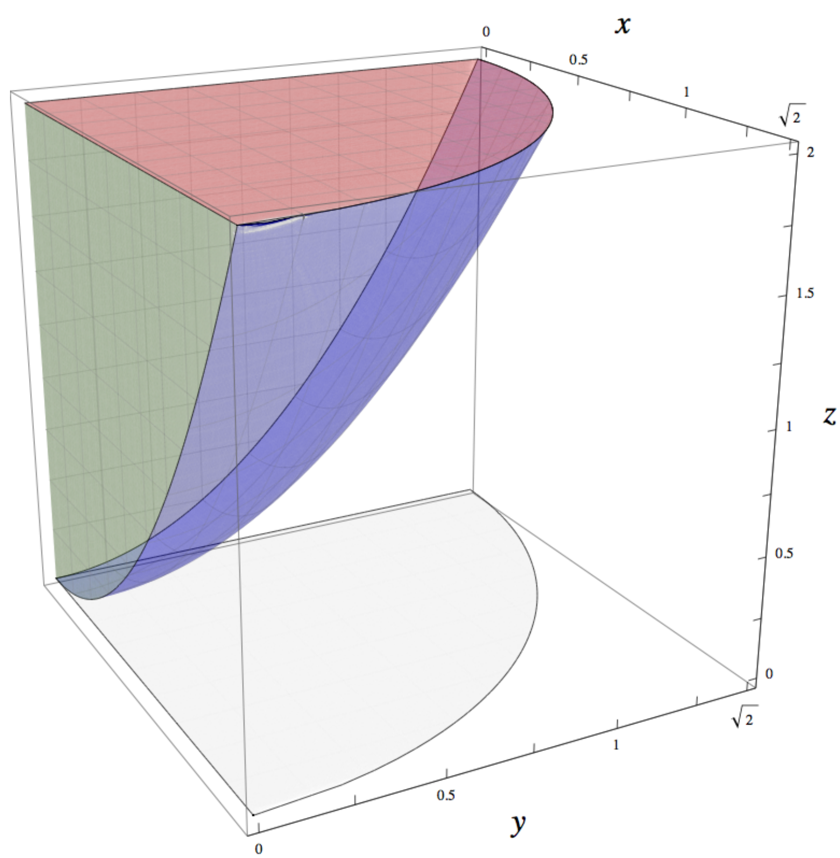
$$(\bar{x}, \bar{y}) = \left(\frac{4}{5}, \frac{4}{5}\right)\ \text{meters}.$$

**Problem #4 (20pts):** Evaluate $\iiint_D x\,dV$, where $D$ is the region bounded by the planes $x = 0$, $y = 0$ and $z = 2$, and the surface $z = x^2 + y^2$ and lying in the quadrant $x \geq 0$, $y \geq 0$. Sketch the region.

**Solution.**

$$\begin{aligned}
\iiint_D x\,dV &= \int_0^{\sqrt{2}} \left[ \int_0^{\sqrt{2-x^2}} \left( \int_{x^2+y^2}^{2} x\,dz \right) dy \right] dx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2 - x^2 - y^2)dydx \\
&= \int_0^{\sqrt{2}} x \left[ (2-x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3} \right] dx \\
&= \int_0^{\sqrt{2}} \frac{2x}{3}(2-x^2)^{3/2}dx = \left. \frac{-2(2-x^2)^{5/2}}{15} \right|_0^{\sqrt{2}} \\
&= 2 \cdot \frac{2^{5/2}}{15} = \frac{8\sqrt{2}}{15}.
\end{aligned}$$

Geometrically, the region is bounded between three planes and the paraboloid $z = x^2 + y^2$. The intersection of the paraboloid with the plane $z = 2$ (in the region $x \geq 0, y \geq 0$) is a quarter-circle $x^2 + y^2 = (\sqrt{2})^2$. Here is a picture:



**Problem #5 (20pts):** Let $R_{xy}$ denote the rectangle

$$a \leq x \leq b, \ c \leq y \leq d.$$

Consider $R_{xy}$ as a thin plate with uniform density $\delta$. Derive the formula expressing the kinetic energy obtained by rotating $R_{xy}$ at an angular velocity of $\omega$ radians per second about the $x$-axis as a double integral, by using the definition of the integral as a limit of Riemann sums.

**Guidance.** Approximate the plate as a collection of $N$ (=some large integer) point masses each having mass $m = \delta \times \Delta A_k$ centered at points $(x_1, y_1), \ldots, (x_N, y_N)$ (where $\Delta A_k$ is the area of a small rectangle near $(x_k, y_k)$). Use the Newtonian relation $KE = \frac{1}{2}mv^2$ for the kinetic energy of each point mass, and interpret the total energy as a Riemann sum. Explain each step. You do not need to evaluate the double integral.

**Solution.** As indicated in the question, the kinetic energy of each point mass is $\frac{1}{2}mv^2 = \frac{1}{2}\delta \times \Delta A_k v^2 = \frac{1}{2}\delta \Delta A_k (\omega r_k)^2$, where $r_k$ is the distance of the point from the axis of rotation, or in this case $r_k = y_k^2$. Thus the total kinetic energy in this approximation is

$$KE \approx \sum_{k=1}^{N} \frac{1}{2}\delta \Delta A_k \omega^2 y_k^2 = \frac{1}{2}\delta\omega^2 \left( \sum_{k=1}^{N} y_k^2 \Delta A_k \right).$$

The sum is a Riemann sum, and in the limit where the number of points grows large it converges to the limiting value

$$\iint_{R_{xy}} y^2 \, dA.$$

The total kinetic energy is therefore equal to

$$KE = \frac{1}{2}\delta\omega^2 \iint_{R_{xy}} y^2 \, dA.$$

As we discussed in class, this can also be written as

$$KE = \frac{1}{2} I_x \omega^2,$$

where $I_x$ is the moment of inertia about the $x$-axis, given by

$$I_x = \iint_{R_{xy}} y^2 \delta \, dA.$$