**Problem #1 (20pts):** (a) Sketch and find the AREA of the region of integration $R_{xy}$ determined by the following iterated integral. (You do not need to evaluate this integral.)

\[
\int_{0}^{\pi} \int_{-\sin x}^{\sin x} \ln \left(x^2 y^4\right) \, dy \, dx. \tag{1}
\]

**Solution.** $R_{xy}$ is the region is bounded between the curves $y = -\sin x$ and $y = \sin x$, $0 \leq x \leq \pi$:

![Diagram of region $R_{xy}$]

The area of $R_{xy}$ is

\[
\iint_{R_{xy}} dA = \int_{0}^{\pi} \int_{-\sin x}^{\sin x} dy \, dx = \int_{0}^{\pi} (\sin x - (-\sin x)) \, dx
\]

\[
= \int_{0}^{\pi} 2 \sin x \, dx = 2(-\cos x) \bigg|_{x=0}^{x=\pi} = 2(1 - (-1)) = 4.
\]

(b) Rewrite this iterated integral as an iterated integral with the order of integration reversed that produces the same value. (Again, you do not need to evaluate the integrals.)

\[
\int_{0}^{1} \int_{2x^3}^{2x} \ln \left(x^2 y^4\right) \, dy \, dx. \tag{2}
\]

**Solution.** The region of integration consists of pairs $(x, y)$ that satisfy $0 \leq x \leq 1$, $2x^3 \leq y \leq 2x$. Thus as $x$ ranges between 0 and 1, $y$ ranges between 0 and 2, and the inequalities $2x^3 \leq y \leq 2x$ can be rewritten as $y/2 \leq x \leq (y/2)^{1/3}$, so the integral is equal to

\[
\int_{0}^{2} \int_{y/2}^{(y/2)^{1/3}} \ln \left(x^2 y^4\right) \, dx \, dy.
\]
Problem #2 (20pts): (a) Use polar coordinates to evaluate the integral

\[ \int \int_{R_{xy}} e^{-r^2} \, dx \, dy, \]

where \( R_{xy} \) is the region inside the circle of radius \( R \) centered at \((0, 0)\).

Solution.

\[
\int \int_{R_{xy}} e^{-r^2} \, dx \, dy = \int_0^{2\pi} \int_0^R e^{-r^2} \, r \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^R e^{-u} \frac{du}{2} \, d\theta \quad \text{(substitution: } u = r^2) \]

\[
= \frac{1}{2} \int_0^{2\pi} (-e^{-u}) \bigg|_{u=0}^{u=R^2} \, d\theta
\]

\[
= \frac{1}{2} \int_0^{2\pi} \left(1 - e^{-R^2} \right) \, d\theta = \pi \left(1 - e^{-R^2} \right).
\]

(b) Since in the limit \( R \to \infty \) you are integrating over all of \( R^2 \), take the limit \( R \to \infty \) in (a) to obtain a value for \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy \).

Solution.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int \int_{R^2} e^{-(x^2+y^2)} \, dx \, dy
\]

\[
= \lim_{R \to \infty} \pi \left(1 - e^{-R^2} \right) = \pi(1 - 0) = \pi.
\]
(c) Iterate the integral to show \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 \).

**Solution.**

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} \, dx \, dy \quad \text{(using } e^{a+b} = e^a e^b) \]

\[
= \int_{-\infty}^{\infty} e^{-y^2} \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \, dy \\
\quad \text{(moving a scalar } e^{-y^2} \text{ outside the integral } dx) \\
= \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \int_{-\infty}^{\infty} e^{-y^2} \, dy \\
\quad \text{(moving a scalar } \int_{-\infty}^{\infty} e^{-x^2} \, dx \text{ outside the integral } dy) \\
= \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2.
\]

(d) Use (b) and (c) to evaluate the famous integral \( \int_{-\infty}^{\infty} e^{-x^2} \, dx \) (the *Gaussian*).

**Solution.** Denote \( G = \int_{-\infty}^{\infty} e^{-x^2} \, dx \). We showed that

\[
G^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right)^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy = \pi.
\]

Also clearly \( G \) is nonnegative as the integral of a nonnegative function. It follows that

\[
G = \sqrt{\pi}.
\]
**Problem #3 (20pts):** Consider a triangular metal plate with three corners 
(0, 0), (2, 0), (0, 2) meters and density \(\delta(x, y) = xy \text{ kg/m}^2\).

(a) Find the mass of the plate. (Put in units.)

Solution.

\[
M = \int_0^2 \int_0^{2-x} \delta(x, y)\, dy\, dx = \int_0^2 \int_0^{2-x} xy\, dy\, dx
\]

\[
= \int_0^2 \left. x \cdot \frac{y^2}{2} \right|_{y=0}^{y=2-x}\, dx = \frac{1}{2} \int_0^2 x(2-x)^2\, dx
\]

\[
= \frac{1}{2} \left( 4x - 4x^2 + x^3 \right)\bigg|_{x=0}^{x=2} = \frac{1}{2} \left( 2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4 \right)\bigg|_{x=0}^{x=2}
\]

\[
= \frac{1}{2} \left( 2 \times 4 - \frac{4}{3} \times 8 + \frac{1}{4} \times 16 \right)\, kg = \frac{2}{3}\, kg.
\]

(b) Find the center of mass \((\bar{x}, \bar{y})\). (Put in units. Note by symmetry \(\bar{x} = \bar{y}\).)

Solution.

\[
M_x = \int_0^2 \int_0^{2-x} x\delta(x, y)\, dy\, dx = \int_0^2 \int_0^{2-x} x^2y\, dy\, dx
\]

\[
= \int_0^2 \left. x^2 \cdot \frac{y^2}{2} \right|_{y=0}^{y=2-x}\, dx = \frac{1}{2} \int_0^2 x^2(2-x)^2\, dx
\]

\[
= \frac{1}{2} \left( 4x^2 - 4x^3 + x^4 \right)\bigg|_{x=0}^{x=2} = \frac{1}{2} \left( \frac{4}{3}x^3 - x^4 + \frac{1}{5}x^5 \right)\bigg|_{x=0}^{x=2}
\]

\[
= \frac{1}{2} \left( 4 \times 3 \times 8 - 16 + \frac{1}{5} \times 32 \right)\, kg \cdot \text{meter} = \frac{8}{15}\, kg \cdot \text{meter}.
\]

The value \(\bar{x}\) is therefore given by

\[
\bar{x} = \frac{M_x}{M} = \frac{8/15}{2/3} = \frac{4}{5}\, \text{meters}.
\]

Since \(\bar{x} = \bar{y}\) as was noted in the question, the center of mass is at

\[
(\bar{x}, \bar{y}) = \left( \frac{4}{5}, \frac{4}{5} \right)\, \text{meters}.
\]
Problem #4 (20pts): Evaluate \( \iiint_D x \, dV \), where \( D \) is the region bounded by the planes \( x = 0, \ y = 0 \) and \( z = 2 \), and the surface \( z = x^2 + y^2 \) and lying in the quadrant \( x \geq 0, \ y \geq 0 \). Sketch the region.

Solution. \[
\begin{align*}
\iiint_D x \, dV &= \int_0^{\sqrt{2}} \left[ \int_0^{\sqrt{2-x^2}} \left( \int_0^{2} x \, dz \right) \, dy \right] \, dx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} x(2-x^2-y^2) \, dy \, dx \\
&= \int_0^{\sqrt{2}} x \left[ (2-x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3} \right] \, dx \\
&= \int_0^{\sqrt{2}} \frac{2x}{3} (2-x^2)^{3/2} \, dx = \left. -\frac{2(2-x^2)^{5/2}}{15} \right|_0^{\sqrt{2}} \\
&= 2 \cdot \frac{2^{5/2}}{15} = \frac{8\sqrt{2}}{15}.
\end{align*}
\]

Geometrically, the region is bounded between three planes and the paraboloid \( z = x^2 + y^2 \). The intersection of the paraboloid with the plane \( z = 2 \) (in the region \( x \geq 0, \ y \geq 0 \)) is a quarter-circle \( x^2 + y^2 = (\sqrt{2})^2 \). Here is a picture:
Problem #5 (20pts): Let \( R_{xy} \) denote the rectangle 
\[ a \leq x \leq b, \ c \leq y \leq d. \]
Consider \( R_{xy} \) as a thin plate with uniform density \( \delta \). Derive the formula expressing the kinetic energy obtained by rotating \( R_{xy} \) at an angular velocity of \( \omega \) radians per second about the \( x \)-axis as a double integral, by using the definition of the integral as a limit of Riemann sums.

**Guidance.** Approximate the plate as a collection of \( N \) (=some large integer) point masses each having mass \( m = \delta \times \Delta A_k \) centered at points \((x_1, y_1), \ldots, (x_N, y_N)\) (where \( \Delta A_k \) is the area of a small rectangle near \((x_k, y_k)\)). Use the Newtonian relation \( KE = \frac{1}{2}mv^2 \) for the kinetic energy of each point mass, and interpret the total energy as a Riemann sum. Explain each step. You do not need to evaluate the double integral.

**Solution.** As indicated in the question, the kinetic energy of each point mass is \( \frac{1}{2}mv^2 = \frac{1}{2}\delta \times \Delta A_k v^2 = \frac{1}{2}\delta \Delta A_k (\omega r_k)^2 \), where \( r_k \) is the distance of the point from the axis of rotation, or in this case \( r_k = y_k^2 \). Thus the total kinetic energy in this approximation is
\[
KE \approx \sum_{k=1}^{N} \frac{1}{2}\delta \Delta A_k \omega^2 y_k^2 = \frac{1}{2}\delta \omega^2 \left( \sum_{k=1}^{N} y_k^2 \Delta A_k \right).
\]
The sum is a Riemann sum, and in the limit where the number of points grows large it converges to the limiting value
\[
\iint_{R_{xy}} y^2 \, dA.
\]
The total kinetic energy is therefore equal to
\[
KE = \frac{1}{2}\delta \omega^2 \iint_{R_{xy}} y^2 \, dA.
\]
As we discussed in class, this can also be written as
\[
KE = \frac{1}{2}I_x \omega^2,
\]
where \( I_x \) is the moment of inertia about the \( x \)-axis, given by
\[
I_x = \iint_{R_{xy}} y^2 \delta \, dA.
\]