Homework #1

Due: Wednesday April 7 by 8 PM.[1] Upload your work through Canvas.

1. Answer problems 1, 4, 11 in Section 2.4 of the textbook.

2. **Shape of the Eiffel Tower**

   In this problem you will use differential equations to derive the shape of the Eiffel Tower, the iconic Parisian landmark built in 1887–9, which stands 984 feet tall. As discussed in class, we approximate the shape of the tower as a solid body with square cross-sections, such that at an elevation of x feet above ground level, the square cross-section at that elevation has side length $L = 2f(x)$ for some unknown function $f(x)$, and therefore an area of $4f(x)^2$ square feet. See the figures on the next page.

   The assumption in our derivation is that the tower makes the most efficient use of the steel used in its construction. This means that for any elevation x, the weight of the part of the tower lying above that elevation, divided by the area of the cross-section at elevation x which bears this weight, remains constant as a function of x. (The idea is that if this ratio changed as x varied, that would mean either that some parts of the tower are bearing a load that's too high, making the tower unsafe, or that some parts of the tower were bearing a load that's less than they could safely bear, making it inefficient in its use of construction material).

   Thinking what this assumption means mathematically, we see that it translates to the equation

   $$C = \frac{\int_x^H 4Af(u)^2\,du}{4f(x)^2} \qquad (0 < x < H),$$

   where A is a constant (related to the density of the tower), C is the quantity "weight above x divided by area of cross section at x", which as mentioned above is assumed to be constant, and $H = 984$ feet is the height of the tower.

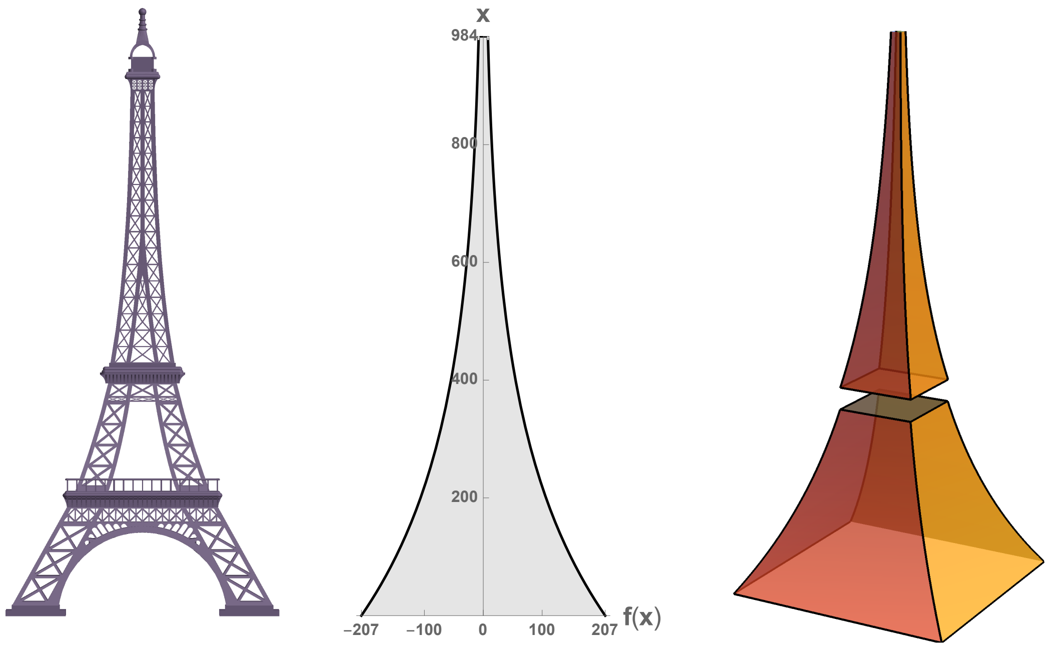
   To simplify things a bit more, rewrite this in a way that eliminates the constant A, as

   $$C = \frac{\int_x^H f(u)^2\,du}{f(x)^2}, \tag{*}$$

   where the constant C has been replaced by some other constant, which we relabel as C (think why we don't lose any generality when doing that — no need to write an explanation for this part).

   (a) The above equation (*) is an *integral* equation. Convert it to a *differential* equation through some simple operations. Explain each of the steps (in words, not just with symbols or equations).

[1] A grade penalty of increasing intensity will be imposed for late homework submissions up to 48 hours following the deadline. No submissions will be accepted after Friday April 9 at 8 PM.



(a) (b) (c)

Figure 1: (a) The Eiffel Tower (b) A schematic representation of the Eiffel Tower in terms of the cross-section half-length function $f(x)$. (Note that in this graph, the independent coordinate, labeled x, is the *vertical* coordinate.) (c) Our three-dimensional model of the tower with one of its square cross-sections highlighted.

(b) Solve the differential equation you obtained, using the method of separation of variables or any other method you know. Your solution will involve some unknown constants.

(c) Using the additional facts that $f(0) = 207$ (feet) and $f(410) = 53$ (feet),[2] find the values of the unknown constants to get an explicit formula for $f(x)$.

(d) Using the formula you found, evaluate $f(906)$, the half-width of the tower at the elevation $x = 906$ feet of the tower's public observation deck.[3]

[2]These values come from knowing the dimensions of the cross section at the base of the tower (height $x = 0$) and at the height $x = 410$ of the tower's restaurant.

[3]Note: this number is probably not representative of the width of the actual, real-life tower at that elevation — the approximation is not sufficiently accurate as one approaches the top of the tower, as Gustave Eiffel was guided by additional considerations besides the elegance of differential equations when designing his tower.