MAT235C Problem Sheet #1

1. Let $(\Omega, \mathcal{F}, P, T)$ be a measure preserving system. Prove that a random variable X is invariant, i.e., $X \circ T = X$ almost surely, if and only if X is measurable with respect to the σ-algbera $\mathcal{I}$ of invariant subsets.

2. Let $(\Omega, \mathcal{F}, P, T)$ be a measure preserving system. Prove that the following two conditions are equivalent:

   (a) For any invariant set A, $P(A)$ is 0 or 1.

   (b) For any invariant random variable X, X is almost surely constant.

3. Let $T : [0,1] \to [0,1]$ be a piecewise continuously differentiable unit interval map. Given a measure $\mu(dx) = f(x)dx$ (an absolutely continuous measure with density f) on $[0,1]$, show that in order to test whether μ is preserved under the map T, it is necessary and sufficient to check whether the following identity holds:

$$f(x) = \sum_{y \in T^{-1}(x)} \frac{1}{|T'(y)|} f(y) \quad \text{for almost every } x \in [0,1].$$

   Here, the sum ranges over preimages y of x.

   (To think about this, it helps to frame the property of being measure preserving in the following terms: T preserves the measure μ if and only if, given a random variable X with distribution μ, the random variable $Y = X \circ T$ also has distribution μ.)

4. Apply the criterion in the previous question to check these claims that were discussed in class:

   (a) The doubling map $x \mapsto 2x \bmod 1$ and the rotation map $x \mapsto x + \alpha \bmod 1$ both preserve Lebesgue measure.

   (b) The logistic map $x \mapsto 4x(1-x)$ preserves the measure $\mu(dx) = \frac{1}{\sqrt{x(1-x)}}dx$.

   (c) The continued fraction map $x \mapsto 1/x \bmod 1$ preserves the Gauss measure $\mu(dx) = \frac{1}{\log 2}\frac{dx}{1+x}$.

5. Along similar lines as the previous question, check that the following dynamical systems are measure preserving:

   (a) The **Gauss map** $G(x) = \begin{cases} \frac{x}{1-x} & \text{if } 0 < x < 1/2, \\ \frac{1-x}{x} & \text{if } \frac{1}{2} < x < 1. \end{cases}$ acting on $(0,1)$, with the Gauss measure $\mu(dx) = \frac{1}{\log 2}\frac{dx}{1+x}$.

   (b) The map $D(x) = \begin{cases} \frac{x}{1-2x} & \text{if } 0 < x < \frac{1}{3}, \\ \frac{1-2x}{x} & \text{if } \frac{1}{3} < x < \frac{1}{2}, \\ \frac{2x-1}{x} & \text{if } \frac{1}{2} < x < 1 \end{cases}$ acting on $(0,1)$, with the measure $d\nu(x) = \frac{1}{x(1-x)}\,dx$.

   Remarks. 1. This is an example of a measure preserving system with an *infinite measure*; the branch of ergodic theory studying such systems is a somewhat specialized area of research called **infinite ergodic theory**.

   2. The map $D(x)$ is related to a number-theoretic expansion for Pythagorean triples, discussed in the paper "The dynamics of Pythagorean triples" (D. Romik, *Trans. Amer. Math. Soc.* 360 (2008), 6045–6064).

   (c) The map $x \mapsto x+1$ acting on $\mathbb{R}$ with Lebesgue measure (another infinite ergodic theory example).

   (d) The **Boole map** $B(x) = x - \frac{1}{x}$ acting on $\mathbb{R}$ with Lebesgue measure (another infinite ergodic theory example).

   (e) The tangent map $\tau(x) = \tan(x)$ acting on $\mathbb{R}$ with the measure $d\sigma(x) = \frac{1}{x^2}\,dx$ (another infinite ergodic theory example).

6. With the notation of problem 3, it seems worthwhile to define an operator $\widehat{T}$ acting on functions on $(0,1)$ by

$$(\widehat{T}f)(x) = \sum_{y \in T^{-1}(x)} \frac{1}{|T'(y)|} f(y).$$

   This operator is called the **transfer operator** associated with the interval map T.

   (a) Convince yourself that the following statement is true (and also helps explain why transfer operators are an interesting object to think about): if X is a random variable with distribution $\mu(dx) = f(x)\,dx$, then the random variable $Y = X \circ T$ has distribution $\nu(dx) = g(x)\,dx$, where $g = \widehat{T}(f)$.

Note. Once you understand the above statement, the condition of T preserving the measure $\mu(dx) = f(x)\,dx$ is now seen to be equivalent to the statement that $\widehat{T}f = f$; that is, the function f has to be an eigenfunction of $\widehat{T}$ associated with the eigenvalue 1. Such an eigenfunction is sometimes referred to as the **Perron-Frobenius eigenfunction** (it is an infinite dimensional version of the Perron-Frobenius eigenvector for nonnegative matrices, studied in the theory of Markov chains).

(b) It is also interesting to study the eigenfunctions of the transfer operator associated with eigenvalues other than 1, i.e., functions f for which $\widehat{T}f = \lambda f$ for some λ. These are sometimes referred to as "resonances" of the dynamical system, and encode useful information about the behavior of iterates $\widehat{T}^k f$ when f is a density that is not invariant under the map T, that is, when the dynamical system has an initial state that is not statistically in equilibrium.

The resonances are usually extremely difficult to analyze. (As a famous example, in the case of the continued fraction map, the transfer operator is known as the Gauss-Kuzmin-Wirsing operator, and there is an extensive literature dedicated to analyzing its spectral structure; see this Wikipedia page.)

Show that in the case of the doubling map $T(x) = 2x \bmod 1$, the resonances can be analyzed precisely. Specifically, prove that for this map, its eigenfunctions are precisely the Bernoulli polynomials $(B_n(x))_{n=0}^{\infty}$, defined in terms of the generating function

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{te^{xt}}{e^t - 1},$$

with the eigenfunction relation for the nth eigenfunction in the series taking the form

$$\widehat{T}(B_n) = \frac{1}{2^n} B_n.$$